The authors of the paper

Entitled:

New Exotic Many-body Interference in 2D-Topological Superfluid Fermi Gases: A Non-Adiabatic SU(4) Symmetry Approach

Reply to

The Reviewer2 Comments

1- About the title: ″A Non-Adiabatic SU(4) Symmetry Approach″. As far as I understand, ″ SU(4)″is just a buzz word and plays no essential role in deriving the main results. If the word ″ SU(4)″ necessary, the authors should explain why SU(4) is crucial in their approach.

Response

We appreciate the reviewer's concern regarding the relevance of the SU(4) symmetry. As we previously demonstrated in response to an earlier comment, the Hamiltonian can indeed be expressed as a linear combination of SU(4) generators and cannot be reduced to SU(2) or SU(3). However, we acknowledge that the manuscript did not adequately elaborate on this symmetry and its specific impact on the obtained results. To better reflect the content and as the reviewer suggested, we have revised the title of our work to: **"New Exotic Many-body Interference in 2D Topological Superfluid Fermi Gases in a Non-Adiabatic Scenario."** This new title more accurately reflects the core findings of our study.

2- A lot of acronyms are used in the manuscript and I spent a hard time to decipher them. I recommend the authors minimizing the use of acronyms; when the terminologies appear only a few times or they are short enough, using acronyms for them just makes the manuscript harder to understand (we need to look for their definitions in the text). I do not think they need to use acronyms for words like ″nonadiabatic″ (NA), ″quantum computing″ (QC), ″chemical potential ″ (CP), and ″pair potential″ (PP) as they are simple enough.

Response

Thank you for this valuable feedback. We will carefully minimize the use of acronyms in the revised manuscript, spelling out terms that appear infrequently or are sufficiently short, as suggested.

3- A typo. The $\varepsilon_\sigma(k)$ [in (2)] should read as: ε_k [the Zeeman interaction is not included in $\varepsilon_\sigma(k)$].

4- Misleading notations. Just above eq. (5), the authors define the four-component Nambu spinor as a ket |ψ⟩, whereas it is in fact a vector of field operators. Also, I suspect that the third and fourth components should be interchanged. Below eq. (5), they define a new quantity (total fermion density): $n_\sigma = n_\uparrow + n_\downarrow$. However, they use σ to denote the spin components ($\uparrow / \downarrow$) and the subscript σ should be removed to avoid confusion. In eq. (5), they explicitly write the time dependence on the left-hand side while they drop it on the right-hand side. To make the time-dependent parameters clear, I recommend the authors writing, e.g., $\in_{k\sigma} - \mu \to \in_{k\sigma}(t) - \mu(t)$.

Response

Thank you for pointing out these misleading notations. We acknowledge the error in describing the spinor as a ket; it is indeed a vector of field operators. We will correct this in the revised manuscript. Additionally, we will address the other notation issues you raised, including the order of the spinor components, the spin subscript in the fermion density, and the time dependence in equation (5).

5- In section 3, the authors (implicitly) interpret the Bogoliubov-de Gennes (BdG) "Hamiltonian" (5) as the actual Hamiltonian of a four-level quantum mechanical system to investigate its time-dependence. However, the BdG Hamiltonian eq. (5) is just a kernel of the second-quantized (many-body) Hamiltonian, and its behavior as a four-level quantum mechanics has little to do with the dynamics of the original many-body fermionic superfluid (only when there is a single quasi-particle with momentum k, the BdG "Hamiltonian"' becomes the actual Hamiltonian of the superfluid). The authors should clarify what situation they assume [or what the "wave function" φ in (9) actually means] in relating their analysis to physics of a 2D fermionic superfluid. .

Response

Thank you for highlighting this crucial point regarding the interpretation of the Bogoliubov-de Gennes (BdG) Hamiltonian. We fully agree that the BdG Hamiltonian, as a kernel of the second-quantized many-body Hamiltonian, describes the dynamics of quasiparticle excitations within the mean-field approximation of the superfluid. It is not the actual Hamiltonian of a four-level quantum mechanical system governing the many-body wave function. Our analysis in Section 3 focuses specifically on the time evolution of these quasiparticle excitations in a non-adiabatic scenario. The "wave function" ϕ in equation (9) represents the **Nambu spinor** for a quasiparticle with a specific momentum *k* and spin. Its time evolution, as governed by the BdG Hamiltonian HBdG(k), describes the dynamics of this quasiparticle.

To clarify the connection to the 2D fermionic superfluid, we are investigating the **excitations and their dynamics** within the established mean-field framework of the superfluid state. While the BdG approach has limitations due to the mean-field approximation (neglecting higher-order correlations), it provides valuable insights into the low-energy behavior and response of the superfluid, particularly the dynamics of its fundamental quasiparticle excitations under non-adiabatic conditions. We will ensure this interpretation is made clearer in the revised manuscript.

6- A related point. Just below eq. (15), they assume an initial condition for the four-level system without giving the definition of the states $|00\rangle$, . . . , $|11\rangle$ in the language of the fermionic superfluid.

Response

Thank you for pointing out this lack of definition. The states $|00\rangle,|01\rangle,|10\rangle,|11\rangle$ represent the basis vectors of the four-dimensional space spanned by the Nambu spinor. In this basis, the Nambu spinor

$$\begin{aligned}\varphi=(c_{k\uparrow},c_{k\downarrow},c^{\dagger}_{-k\downarrow},-c^{\dagger}_{-k\uparrow})^{\dagger} &\equiv c_{k\uparrow}|00\rangle+c_{k\downarrow}|01\rangle+c^{\dagger}_{-k\downarrow}|10\rangle-c^{\dagger}_{-k\uparrow}|11\rangle \\ &\equiv c_{k\uparrow}|1\rangle+c_{k\downarrow}|2\rangle+c^{\dagger}_{-k\downarrow}|3\rangle-c^{\dagger}_{-k\uparrow}|4\rangle\end{aligned}.$$

Where

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We will include this definition in the revised manuscript for clarity.

7. In section 3.1, the authors consider the case of linear sweep and discuss two limiting cases $\epsilon k - \mu(t) \to 0$ and $\epsilon k - \mu(t) \gg 0$. However, ϵk is k-dependent and the conditions do not make any sense.

Response

Thank you for pointing out the potential confusion regarding the conditions $\in_k - \mu \to 0$ and $\in_k - \mu \gg$. We understand that $\in_k$ is momentum-dependent, and the the chemical potential time independent. We will clarify this in the revised manuscript.

Firstly, we acknowledge that while the reviewer is correct that ϵk depends on k, the conditions discussed in Section 3.1 are intended to describe the behavior of the energy difference for **specific momentum values** within the condensate.

Secondly, it is crucial to emphasize that in the context of this work, we are considering a **constant chemical potential** μ. This assumption is valid under the conditions we are investigating, specifically a small and slowly varying magnetic field. As detailed in foundational texts on Bose-Einstein condensation (e.g., Pethick and Smith, 2008; Kurn and Stamper-Kurn, 2012; Stringari and Pitaevskii, 2003), a constant chemical potential is a reasonable approximation in the adiabatic regime where the system can adjust to the slowly changing external field without significant fluctuations in the overall particle number. Our application of the Landau-Zener (LZ) transition framework is predicated on these conditions of a small and slowly varying magnetic field, which favors adiabatic passage.

Therefore, the conditions should be interpreted as:

- $\in_k - \mu \to 0$: This condition is achieved for specific momentum values k where the single-particle energy $\in_k$ approaches the constant chemical potential μ. This typically occurs around the center of the condensate's momentum distribution.

- $\in_k - \mu \gg$: This condition is achieved for momentum values k where the single-particle energy ϵk is significantly larger than the constant chemical potential μ, typically at the edges of the relevant momentum distribution.

Furthermore, the energy difference $\in_k - \mu$,where $\in_k \equiv \frac{E_F}{\alpha}\left(\frac{k}{k_F}\right)^2$ can be effectively tuned by the sweeping rate α of the relevant parameter can be tailor by the sweeping rate allowing us to explore different regimes. The specific values of k for which these conditions are met depend on the system parameters and the chosen sweeping rate.

8- On the whole, the figure captions are poor. At least quick summaries of the main observations must be given in the captions (even if the full explanations are found in the main text). Other problems are:

• The quantity r in Fig. 1 is not defined anywhere in the text.

• Clearly, all the plots are obtained for specific fixed values of t. However, the values are not shown.

• The initial conditions are not specified.

Response

We sincerely appreciate the reviewer suggestion concerning the figure captions, this will be improved and the values of t will be included.

• The quantity r in Fig. 1 is not r but tau (τ) which is the dimensionless time of the system ($\tau = \sqrt{\alpha}t$ for linear sweep and $\tau = \Omega t$ for periodic sweep).

• The initial conditions are clearly specify in section 4 at the end of the first paragraph were it said "we will only present results when the system is prepare in the $|1\rangle$.

We sincerely appreciate the reviewer's valuable suggestions regarding the figure captions. We agree that they should provide concise summaries of the main observations. We will revise all figure captions to include these summaries and the specific fixed values of time (t) for which the plots were generated.

Regarding the specific points raised:

- The quantity labeled "r" in Figure 1 is indeed the dimensionless time, denoted as τ. We will correct the figure caption and any instances in the text where it might be mislabeled. The definition of τ is $\tau = \sqrt{\alpha}t$ for the linear sweep and $\tau = \Omega t$ for the periodic sweep, and we will ensure this is clearly stated in the caption of Fig.1.

- We acknowledge that the specific values of time (t) for each plot were not included in the original captions. We will add these values to all relevant figure captions in the revised manuscript.

- The initial conditions for our simulations are specified in Section 4, at the end of the first paragraph, where it states, "we will only present results when the system is prepared in the $|1\rangle$ "

9-Two unnumbered figures appear between Fig. 1 and 2 (in page 12).

Response

Thank you for pointing this out. These figures are indeed numbered as (a) and (b). Please refer to the figure caption on page 12, where these sub-figure labels are provided.

10-When the authors discuss the case of periodic sweep, they point out the occurrence of “a two-stage double-passage process” seen in Fig. 1b. However, they do not give any explanation which part of Fig. 1b can be interpreted as the above process.

Response

Thank you for seeking clarification on the "two-stage double-passage process" mentioned in relation to Figure 1b. In the periodic sweep scenario, each period (for example, between dimensionless time $\tau=-2$ and $\tau=2$) exhibits two crossings of the relevant energy levels. This double crossing within a single period constitutes a "double passage."

The "two-stage" aspect we refer to is visually discernible around the energy E=0 in Figure 1b. We observe a distinct upper stage where the system evolves through these double passages (for instance, in the approximate range of E from 0 to 25). Subsequently, there is a lower stage where the system undergoes a similar double-passage process (in the approximate range of E from −25 to 0). Importantly, these two stages are coupled in the vicinity of E=0, meaning the dynamics in one stage influences the other.

11- Fix an equation overflow in page 10.

Response

Thank you for pointing out the equation overflow on page 10. We will carefully adjust the formatting and layout in the revised manuscript to ensure the equation fits within the margins and is displayed correctly.

12-The figures 3-11 and the corresponding discussions in section 4 constitute the main part of the manuscript. On the whole, the physical interpretation of the results remains speculative or superficial lacking reasonable physical explanations, which makes this part far from satisfactory. I just list some (not all) examples:

• In page 12, the authors claim: “The ripple-like interference patterns in Fig.3 indicate that the particles in the system exhibit wave-like behavior.” quoting the color-map plot of the occupation probabilities of the level-1 and 2. However, what they plot is the parameter-dependence (the Zeeman field and the spin-orbit coupling, here) of the probabilities and is not the real-space (or momentum-space) profile which might reveal the wave nature. (As they work in the momentum space, the wave nature, if any, must be seen in the k-space behavior.) They should give more detailed arguments why they arrive at this conclusion, which is not at all obvious to me, from the plot.

Response

Dear reviewer, we appreciate your critical assessment of the physical interpretation in Section 4 . We understand your concern that the connection between the presented parameter dependence of occupation probabilities and the wave-like behavior of particles is not sufficiently established. But it is clearly established that the presence of interferences in a system is a clear signature of wave-like behavior.

Regarding the "ripple-like interference patterns" in Fig. 3, our interpretation stems from the principles of Landau-Zener (LZ) interferometry [S. N. Shevchenko, S. Ashhab and F. Nori, *Landau–zener–stückelberg interferometry*, Physics Reports **492**(1), 1 (2010), / S. Ashhab, J. R. Johansson, A. M. Zagoskin, and F. Nori, Franco, Two-level systems driven by large-amplitude fields, Phys. Rev. A **75** (6) , 9(2007)]. In this framework, the observed oscillations in the transition probabilities as a function of system parameters (Zeeman field and spin-orbit coupling in this case) are indeed analogous to spatial or temporal interference patterns seen in other wave phenomena.

While Fig. 3 displays the parameter dependence of the occupation probabilities, the underlying physics involves the evolution of quantum states through avoided crossings in the energy spectrum. Each particle's quantum state accumulates a phase as it traverses these regions. This phase is sensitive to the system parameters. When we vary these parameters, the relative phases between different pathways in the LZ process change, leading to constructive and destructive interference in the final occupation probabilities. The resulting oscillations or "fringes" in the parameter space are a manifestation of this quantum interference.

Therefore, while we are not directly plotting real-space or momentum-space profiles, the observed oscillations in the occupation probabilities as a function of the Zeeman field and spin-orbit coupling are a consequence of the wave-like nature of the fermionic particles and their quantum mechanical phase evolution within the LZ interferometry setting. The variation of system parameters acts as a way to probe these underlying phase relationships and reveal the interference.

• In page 14, they say: "At high frequencies (˜ω < 1), individual multiphoton resonances are visible,...", whereas I could not figure out where I can identify in Fig. 4 the signature of the multiphoton resonances. They should add some arguments on how the multiphoton resonances lead to the structure seen in Fig. 4.

Response

Dear reviewer, we appreciate your question regarding the identification of multiphoton resonances in Fig. 4. The avoided crossings observed in the energy spectrum in Fig. 1 are indeed a fundamental signature of resonances within the system. To understand whether these resonances involve single or multiple photons, we rely on the framework of Floquet theory, which is appropriate for periodically driven systems. In each avoided crossing, a transition can occur with the effective emission or absorption of photons, provided the transition is allowed by the system's selection rules. In our case, we have shown the existence of transition probabilities between all four basis states.

The condition ˜ω < 1 for long driving times signifies a regime where the system experiences multiple crossings within each stage of its evolution. This leads to the possibility of absorbing or emitting multiple photons during the overall process.

A more rigorous mathematical treatment, often employed in this context (e.g., as described in relevant literature on Floquet theory and multiphoton transitions in driven systems [S. Ashhab, J. R. Johansson, A. M. Zagoskin, and F. Nori, Franco, Two-level systems driven by large-amplitude fields, Phys. Rev. A **75** (6) , 9(2007)]), reveals resonance conditions related to the driving frequency Ω and the system's energy scales. In this context, we obtained under strong driving condition the resonance condition : $n\hbar \quad \varepsilon_k - \mu)$ for some integer n with the oscillation frequency $\Delta\left|J_n\left(\frac{2h_0}{\Omega}\right)\right|$. $J_n(z)$ is the Bessel fonction. One directly identifies the resonance with a given value of n as describing an n-photon process since the quantity $\hbar$ account for the photon energy. One of the requirement of resonance is that the light intensity/amplitude is maximal. In Fig. 4, the bright fringes, indicating regions of maximum transition probability ($P \approx 1$), correspond to the conditions where these multiphoton resonance conditions are met.

• In page 16, they conclude that the ring-shaped structure seen in Figs. 5-7 is a manifestation of the Aharonov-Bohm (AB) effect and say: "This is a significant breakthrough in this work, as such phenomena have never been observed before in Fermi gases." However, I do not think the logic connecting this ring-like structure and the AB effect is obvious, as the AB effect needs non-zero gauge potential which is zero in the present setting (if it were non-zero, it would certainly affect ϵk). They should elaborate on this point. For the same reason, the discussion of the Landau level at the end of page 16 is not quite correct in the present setting (in which only the Zeeman field is taken into account).

Response

We agree that the traditional Aharonov-Bohm (AB) effect requires a non-zero gauge potential and that in its canonical form, this potential modifies the phase of a charged particle encircling a magnetic flux even in regions where the magnetic field is zero. However, in our setting, we are considering a *synthetic gauge field* scenario realized via spin-orbit coupling (SOC) and periodic Zeeman fields in neutral ultracold Fermi gases. In such systems, effective gauge potentials can emerge through engineered light-matter interactions, and even though the real magnetic vector potential A may be zero, *synthetic* gauge potentials and associated Berry phases can still arise in momentum space due to spin-momentum coupling and the adiabatic motion of dressed states.

To clarify our claim:

- The ring-like structure observed in Figs. 5–7 emerges as a result of quantum interference between multiple passages through avoided crossings, modulated by the SOC and the periodic driving.
- The *AB-like oscillations* in our model refer to *topological phase accumulations* around closed loops in momentum space, akin to how Berry curvature gives rise to AB-type phase interference in other cold atom systems [see, e.g., Spielman *et al.*, Nature 462, 628 (2009); Dalibard *et al.*, Rev. Mod. Phys. 83, 1523 (2011)].
- While there is no real-space magnetic vector potential directly modifying ϵk, the effective Hamiltonian—especially under the SU(4) mapping—encodes internal degrees of freedom that result in a phase structure mimicking AB interference in the adiabatic basis. These phases accumulate over closed loops due to Stückelberg interferences and spin-dependent modulation of tunneling amplitudes.
- We will revise the manuscript to explicitly frame the observed effect as a *synthetic Aharonov-Bohm-type interference* due to engineered spin-orbit coupling and dynamical control of Zeeman fields in momentum space, rather than the conventional AB effect.

To summarize, in your setup, even though the real magnetic vector potential A may be zero , the combination of SOC and periodic driving creates an effective gauge structure. This structure leads to phase accumulation around closed loops in momentum space, resulting in interference patterns reminiscent of the AB effect. Such synthetic gauge fields have been shown to induce AB-like phase shifts in neutral atom systems [see J. Mumford, Phys. Rev. A.106, 033317.]

Regarding the discussion of Landau levels, it's accurate that traditional Landau quantization arises from charged particles in a uniform magnetic field. However, in systems with synthetic gauge fields, similar quantization can occur due to the engineered band structures and effective magnetic fluxes. These lead to phenomena such as Hofstadter's butterfly spectrum and quantized energy levels, even in the absence of real magnetic fields

• In page 17, the authors interpret the results in Fig. 6 as a consequence of the resonance among different energy levels without giving any concrete physical arguments. If they argue that this sort of resonance helps us implement, e.g., gate operations, they should at least give a simple (semi-quantitative) explanation how the resonance among the levels affects the interference patterns.

Response.

We appreciate the reviewer's request for more concrete physical arguments regarding the resonance observed in Figure 6 and its potential for implementing gate operations. As established in our response to a previous question, resonance conditions lead to maximized transition probabilities, manifested as alternating bright (high probability, P≈1) and dark (low probability, P≈0) fringes due to constructive and destructive interference.

To illustrate how these resonances could be utilized for gate operations, let's consider a few examples:

- **Hadamard-like Gate:** The Hadamard gate creates an equal superposition of basis states. In our four-level system, an analogous operation would aim for approximately equal probability (around 0.25 in our representation) across all accessible states. By carefully choosing the system parameters (e.g., Zeeman field, spin-orbit coupling, and sweep parameters) to land on a region in the interferogram where the probabilities are roughly equal across the relevant output states, we could achieve an operation akin to a multi-level Hadamard gate.

- **NOT-like Gate (State Flipping):** A NOT gate performs a complete population transfer between two specific states. In our system, this would correspond to finding parameter values in the interferogram where the probability of a desired target state reaches a maximum (P≈1) while the probabilities of the initial state and other states are minimized (P≈0). By tuning the parameters to achieve such complete population transfer between specific "levels" within our effective four-level system, we could implement a NOT-like operation between those states.

Therefore, the different regions of constructive and destructive interference in our interferograms, which arise from resonances at specific parameter combinations, represent the ability to selectively control the final state probabilities. By mapping the desired outcome of a specific quantum gate (i.e., the target probabilities in the output states) to specific regions in our parameter space (the "interferograms"), we can identify the parameter settings required to implement that gate.

• I do not understand what "topological Fermi ring" in page 20 stands for.

Response

Thank you for pointing out the need for clarification. The term "topological Fermi ring" refers to a scenario in momentum space where the Fermi surface, instead of being a point or a closed surface enclosing a volume, degenerates into a ring. This ring in momentum space hosts topological properties, often characterized by non-trivial Berry curvature and associated topological invariants. The presence of such a ring can lead to unique electronic and transport properties in the material. For a more detailed explanation and the context in which it arises in our work, we direct the reader to [M. Ateuafack, L.

Wah, M. Jipdi, J. Ngana Kuetche, L. Temdie, J. Diffo and L. Fai, *Probing nonadiabatic transition dynamics in 2d topological superfluid fermi gas with spin orbit coupling*, Physics Letters A **533**, 130216 (2025),].

We will carefully review the discussions related to Figures 3-11 to strengthen the physical interpretations and provide more robust explanations for the observed phenomena. Thank you for highlighting this crucial area for improvement.

We have identified and corrected some typos in the following equation:

$$Eq.25 \rightarrow \begin{pmatrix} 0 & a & 0 & -b \\ a^{*} & 0 & c & 0 \\ 0 & c^{*} & 0 & a^{*} \\ -b^{*} & 0 & a & 0 \end{pmatrix}$$

$$Eq.38 \rightarrow T_{11}^{M} = q\left[2q\left(1-p^{2}\right)+\frac{2\lambda_{01}}{\sqrt{\lambda_{01}^{2}+\lambda_{02}^{2}}}p\sqrt{1-p^{2}}\sqrt{1-q^{2}}\right]\cos\phi\exp\left\{-i\phi\right\}$$

$$Eq.39 \rightarrow T_{12}^{M} = i\left[2pq\left(1-p^{2}\right)\cos\phi+\frac{\lambda_{01}}{\sqrt{\lambda_{01}^{2}+\lambda_{02}^{2}}}\left(2p^{2}-1\right)\sqrt{1-q^{2}}\right]$$

$$Eq.40 \rightarrow T_{13}^{M} = \frac{2i\lambda_{02}}{\sqrt{\lambda_{01}^{2}+\lambda_{02}^{2}}}p\sqrt{1-p^{2}}\sqrt{1-q^{2}}\sin\phi$$

$$Eq.41 \rightarrow T_{14}^{M} = \frac{-i\lambda_{02}}{\sqrt{\lambda_{01}^{2}+\lambda_{02}^{2}}}p\sqrt{1-p^{2}}\left[p^{2}+\left(1-p^{2}\right)\exp\left(2i\phi\right)\right]$$