Referee Responses

Cedric Lim[a,†], Kirill Matirko[b,†], Hyeongjin Kim[c],
Anatoli Polkovnikov[c], and Michael O. Flynn[c,*]
[a] *Department of Physics, University of California, Berkeley, California 94720, USA*
[b] *Department of Physics, HSE University, St. Petersburg 190008, Russia*
[c] *Department of Physics, Boston University, Boston, MA 02215, USA*

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We would like to thank Referee 2 again for taking the time to respond to us. However, we fundamentally disagree with their judgment and conclusions. In their earlier reports, the referee raised valid points, for example, by asking about the applicability of our chaos measure to mixed phase space out of concern that our measure may not apply there. We resolved this point by adding a new Section 6.4 showing that the fidelity correctly identifies individual integrable and chaotic trajectories. Then, the referee objected that the fidelity indicates maximal chaos in the regime where most of phase space is regular. We also addressed this point in Section 6.3 and found that a careful analysis places the maximum of the fidelity is exactly where the referee anticipated.

Now the referee says we have only analyzed a well-studied regime. This is simply a false statement, as we have analyzed our model in all the regimes from integrable to strongly chaotic. This was true since the first draft of our paper and all subsequent changes have only expanded our analysis. The fidelity approach was originally developed for quantum many-particle systems, where it was able to correctly identify chaos. In this work, from the very beginning, we stressed that we decided to choose this well-studied classical model to check and explain how the fidelity approach works in that context. In this way we found a way to define chaos which applies both to quantum and classical systems. We explained why a large fidelity χ indicates and quantifies instabilities of time averaged probability distributions, which are defined both in quantum and classical systems. In particular, it is not a coincidence that a single chaotic trajectory has a diverging χ, while a single regular trajectory has a finite χ. At this point, there are many systems, quantum and classical, studied by various groups, which confirmed that this approach works in general, see e.g. Refs. [1–5]. This list continues to grow. If the referee believes that the fidelity approach cannot be used to identify chaos, they should provide some kind of argument as to why the fidelity measure works many other cases and explain why, in their opinion, this measure is less general than the Lyapunov instability.

Referee 2 wants us to say that the fidelity is a probe of chaos but not a definition. In our view, this is not a reasonable request, since there is no general definition of chaos that applies to both quantum and classical systems. We cannot probe what is not even defined. The definition of chaos based on Lyapunov instabilities is purely mathematical and does not apply to *any* physical system because, at the microscopic level, all laws of nature are quantum mechanical. Therefore, by the uncertainty principle, one cannot start from two arbitrarily close initial conditions to even mathematically define Lyapunov exponents.

[*] michaelflynn13@gmail.com
[†] These authors contributed equally to this work

Consider, for example, our atmosphere, as the theory of chaos emerged historically from atmospheric studies. Even if we treat our atmosphere classically, which is already incorrect, it is impossible to recreate two identical initial conditions without reducing the entropy of the atmosphere to zero, i.e., without having a perfect Maxwell's demon. Such demons do not exist. Likewise, it is impossible to use the OTOC or echos to define chaos as there is no way of enacting time reversal. From our experience, we that chaos is recognizable within individual physical realizations. If we sit on an airplane and encounter a turbulent instability, we do not need another airplane to identify chaos; all information is contained in a single plane's trajectory.

As previously mentioned, our goal in this work was to find a definition of chaos which works equally well for quantum and classical systems, few body and many body. We believe that we have found such a definition and explained why it works. Moreover, we explained why the fidelity approach is both physical and intuitive. For the simple and indeed well understood model studied in this work, high fidelity indicates that a trajectories or eigenstates are highly unstable to infinitesimal deformations of the Hamiltonian, i.e., to the rules governing the microscopic evolution. There are unpublished results by our colleagues showing that this approach works perfectly well even in non-Hamiltonian classical examples, such as the logistic map. In that case, the fidelity works because irregular chaotic motion encoded in a single trajectory leads to very large fidelity. Thermalization (ergodicity/mixing) in this way corresponds to a very particular type of linear divergence of χ following time-averaging. In quantum systems this linear divergence agrees with the expectations from random matrix theory and in classical models it corresponds to the Markovian equilibration dynamics with short-time memory. We want to stress that the fidelity is only one characteristic of the long-time probability distribution. Clearly, there is more information about chaos in the fidelity distribution, which was studied in Refs. [1,6] for quantum systems. Similarly, it can be analyzed in classical systems.

In summary, we regret to say that while Referee 2 had very useful suggestions, which allowed us to expand and improve our manuscript, they have failed to find any mistake or inconsistency in our work or present a counterexample that we could not address. The referee essentially tied their editorial recommendation to an unjustified request where we change the wording from "defining chaos" to "probing chaos" using fidelity. While we recognize that the referee can have this opinion, we believe that it is not justified to tie acceptance of our work to such a subjective demand.

References

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