Reply to Referees' comments
Freelance Holography, Part II: Moving Boundary in Gauge/Gravity Correspondence

We would like to thank the referee for carefully reading the paper and for constructive comments. We have tried to address them and improve our paper accordingly. Please see below for more details. For a better visibility, we have kept the changes in the paper in red.

Reply to Referee 2

We thank the referee for the comments.

Weaknesses 1. This work follows as a direct generalization of earlier literature on holography at finite radius, and does not add significant new conceptual results.

To address the referee's comments and critiques, we would like to highlight the new viewpoints and results our work adds to the current literature.

- We *systematically derive* holography at finite distance in the saddle-point approximation using the tools of the Covariant Phase Space Formalism (CPSF). Our approach is new, in that we begin with the asymptotic holographic dictionary and derive the holographic relation at finite cutoff in the gauge/gravity limit (see equation (3.20)). In contrast, the existing literature typically constructs the deformation flow equation from the Hamiltonian constraint, which often introduces ambiguities—particularly concerning whether matter fields modify the deformation action [1]. Other approaches, such as that in [2], begin from a different starting point (equation (2.1)) and assume the holographic dictionary as given (specifically equation (1.5) in their work).
- Our construction of finite-distance holography is valid for *arbitrary boundary conditions*. This is an important new result. It is well known that gravity with Dirichlet boundary conditions at a finite distance leads to an ill-posed problem [3, 4, 5]. Therefore, earlier attempts to construct finite-distance holography under Dirichlet conditions should confront this issue. In contrast, there is growing evidence that alternative boundary conditions (such as conformal boundary conditions) can yield a well-posed variational problem [3, 5, 6]. Thus, a framework that accommodates arbitrary boundary conditions is a significant step forward in this direction.
- Our construction is not restricted to general relativity coupled to matter; it applies to *any* gravitational theory. As an illustrative case, we explicitly demonstrate its applicability to the Lovelock class of theories. In particular, our framework makes it clear

that the identification of the $T\bar{T}$ deformation with finite-cutoff holography is specific to three-dimensional general relativity without matter. In higher dimensions or in alternative gravitational theories, different operators govern the construction of holography at finite distance. Although our approach differs in methodology, our results are consistent with those of [2].

- In the process of developing finite distance holography, we used the holographic principle to fix and interpret the ambiguities/freedoms of the CPSF, a step that, to our knowledge, had not been undertaken previously. This resolution is essential for achieving a more coherent understanding of dual theories and the interpretation of bulk-boundary correspondences at finite distance.

- We introduce a new class of boundary conditions that are, by construction, equipped with a well-defined asymptotic behavior. We then develop a finite-distance holographic framework based on these boundary conditions. It is worth emphasizing that, in contrast, the counterterms associated with the standard conformal boundary conditions are not understood.

- In Section 8, we introduced two distinct classes of hydrodynamic deformations and analyzed the $T\bar{T}$ deformation in generic dimension within a broader framework, interpreting it as a specific instance of deformation between hydrodynamic systems.

- The literature contains ambiguities regarding whether the addition of matter content to general relativity modifies the deformation associated with finite-cutoff holography [7]. As noted, our construction systematically resolves this issue.

Weaknesses 2. This paper does not address the subtleties around whether the operators defined at finite radius are actually local operators.

We define operators at finite distance via deformation flows, interpreted as renormalization group (RG) flows, which in the gravity side are nothing but a part of bulk equations of motion. For instance, in pure general relativity in arbitrary dimensions, the EMT at finite distance is determined by equation (6.7c). This equation is a first-order differential equation in the radial direction, relating the EMT at the asymptotic boundary to that at a finite cutoff surface. Importantly, it is manifestly local in the boundary coordinates x^a.

Comment 1. This work continues a programme of research exploring holography at finite radius. The main new content of this work relates to generalizing the boundary condition at finite radius from purely Dirichlet, and the consideration of interpolating boundary conditions. Both are straightforward generalisations of existing literature.

We have summarized our main results in response to Weakness Point 1.

Comment 2. The paper does not address various known technical subtleties. One subtlety is that the formula (3.11), and subsequent related formulae, do not apply when the operator dimension is $d/2+n$ where n is an integer. In such contexts the relationship between operator source and expectation value is slightly different, due to the conformal anomalies. Note that *this situation is highly generic: the most interesting operators in even spacetime dimensional theories indeed do have integral dimensions. This technical subtlety could be acknowledged by a suitable comment and reference.*

We thank the referee for pointing out this subtlety. Indeed, when the operator dimension takes the form $\Delta = \frac{d}{2} + n$ with $n \in \mathbb{Z}$, the standard relation between sources and expectation values is modified due to the appearance of logarithmic terms in the asymptotic expansion of bulk fields. These terms reflect conformal anomalies in the dual boundary theory [8]. We added footnote 4 on page 9 in the manuscript to acknowledge this well-known subtlety.

Comment 3. A second subtlety relates to the assumption implicit in (3.18) about the identification of operator source at finite radius. This is not a new assumption from these authors, but there is a longstanding question on how exactly the holographic radius relates to the renormalisation scale and it is not clear that the operator identified in (3.18) would be local. Again this issue could be acknowledged by comments and references.

You are right that this is not a new assumption, and we have added a reference to restress this fact.

Regarding the local nature of operators, as we mentioned in our response to Weakness 2, we defined the boundary operators at finite distance via RG flow equations that are local in the boundary coordinates. Moreover, as emphasized in equation (3.18), the boundary operators can depend explicitly on the cutoff radius. This is expected, since the boundary theory at finite distance is no longer conformal and thus may naturally depend on the dimensionful parameter r_c.

Reply to Referee 1

We thank the referee for the comments.

Weaknesses 1. The proposal seems to lack an independent field theory definition.

We generally agree with the referee's comment on that our main focus has been the bulk/gravity side of the story. However, we would like to emphasize that the final expression is written entirely in terms of boundary variables. Any residual dependence on the radial coordinate is interpreted as dependence on the energy scale in the boundary theory. Ultimately, we view the boundary field theory as an *effective* coarse-grained description.

Weaknesses 2. The proposal is insufficiently grounded at the quantum level as the effects of irrelevant operators in the RG flow are not considered.

We largely agree with the referee. As we have repeatedly emphasized in the manuscript, our analysis is carried out in the gauge/gravity correspondence limit, i.e. in the saddle point approximation, where the bulk is treated entirely classically and the boundary theory is restricted to the planar limit. A more detailed analysis of the RG equation is currently a work in progress.

Weaknesses 3. There are certain claims that should be clearified - see the report.

We address them - see the following response.

Comment 1. Field theory flow operator:
It is claimed in the abstract that the evolution "from the [AdS] boundary to another boundary inside AdS [...] is encoded in deformations of the holographic boundary theory." However, there does not seem to be a purely field theoretic definition of the deformation. For example, the deformation action S deform in (3.25) is given in terms of the on-shell bulk Lagrangian, which is not a boundary quantity. In particular, it requires the knowledge of how the radial derivative of the bulk field is related to its conjugate momentum. See, for example, the term $\partial_r J^i$ *in (6.9) which seems mysterious from the field theory point of view. Also in sections 6.3 and 6.4, knowledge of the bulk equations is required to define what the flow operator is.*
This stands in contrast with the $T\bar{T}$ *operator, which is defined purely from field theory ingredients. (Here one should note, however, that this operation can only be interpreted as moving the boundary in the theory dual to pure gravity.)*
To make this point a bit more concrete, one can ask this question in the specific example of the duality between type IIB string theory in $AdS_5 \times S^5$ *and* $N = 4$ *SYM: what gauge theory operator does your prescription predict one should add to the SYM action to let it flow into the bulk?*
As a related remark, it is not clear what the content of equation (3.35) is, since the "QFT" on the left-hand side, as far as I can see, has not been given a precise definition.

To address this comment, we need to clarify certain subtleties:

First, we emphasize that in reference [9], it was *proposed* that the $T\bar{T}$ deformation of a boundary theory can be interpreted as moving the dual CFT into the bulk. This proposal was later supported by a range of computations, including the deformed energy spectrum, thermodynamic properties, signal propagation speed, and other consistency checks. However, the origin of this proposal was not derived from first principles in that work, accepting the AdS/CFT framework. Subsequent studies extended this idea to higher dimensions within the saddle-point approximation, typically using arguments based on the Hamiltonian constraint—which governs radial evolution—or adopting the Hamilton–Jacobi framework to extract the corresponding flow equations. In the present work, we pursue a similar objective but provide a *systematic* derivation of the deformation flow equation directly at the saddle-point level of the standard AdS/CFT correspondence, using the tools of the CPSF, which is essentially equivalent to the Hamilton–Jacobi method. While it is true that our flow equations originate from bulk dynamics, we explicitly demonstrate how they can be expressed in terms of boundary variables, as shown in equation (3.32), and in the gravity-plus-matter case, in equation (6.9).

As discussed below equation (3.32), to fully express the boundary deformation action in terms of boundary quantities, one needs to rewrite terms like $\partial_r J^i$ in terms of the canonical momenta. For instance, in pure gravity, we express $\partial_r h_{ab}$ in terms of the boundary stress tensor $\mathcal{T}_{ab}$, thereby interpreting it as a boundary operator. A similar procedure applies to bulk matter fields. One of the key motivations for presenting various examples in Section 6 is precisely to demonstrate how the boundary deformation action can be written entirely in terms of boundary variables.

Regarding the well-known example of $\mathcal{N} = 4$ SYM, we would like to clarify that our construction operates within the *saddle-point* regime of the bulk theory, and we do not claim to identify a *microscopic*, UV-complete boundary theory, as is possible in certain special cases. Our goal is to provide an *effective* (coarse-grained) description of the dual theory at a finite radial position, rather than to construct a UV-complete formulation of the deformed boundary theory. In fact, within the Wilsonian framework, a UV-complete theory is not necessary for describing low-energy physics: effective field theory offers a systematically improvable and reliable description at energy scales below the cutoff. We have added a paragraph at the end of Section 3 to highlight this important point.

Comment 2. Classical well-posedness of Lorentzian boundary conditions in the bulk: It is mentioned that finite radial cutoff and Dirichlet boundary conditions are ill-defined in $D > 3$ spacetime dimensions. However, this statement does not seem to be addressed in the rest of the paper. Even though the conclusion section mentions that one of the objectives was to address this issue, it is not clear to me how the current proposal solves this problem, or whether it is unresolved and left as an open issue.

Let us clarify our point. While it is true that Dirichlet boundary conditions are not well-posed at finite distance, there exist other boundary conditions—such as conformal boundary conditions—that are supposedly well-posed. Our goal was not to resolve the ill-posedness of Dirichlet conditions, instead, we aim to develop a holographic framework at finite distance that accommodates arbitrary boundary conditions. This approach enables us to bypass the problematic Dirichlet case and construct a consistent holography based on a well-posed boundary data. We improved the text to clarify this point further.

Comment 3. Relevance of double-trace deformations & well-definedness of CFT: Already before flowing into the bulk, the authors should clarify to what extent their proposal for altered boundary conditions using CPSF freedoms is rigorous. For example, a generic boundary term Wbulk will contain operators that are irrelevant in the CFT. To take this proposal beyond the classical level, one should therefore restrict to the small set of multi-trace operators which are relevant and trigger a flow to an IR CFT. (One example that might be worth investigating in more detail is the free/critical $O(N)$ model in 3d.) Alternatively, one would have to argue, as it is done for the special case of the $T\bar{T}$ operator in two dimensions, that the irrelevant deformation is somehow tractable. (For example, the somewhat related prescription by Compère and Marolf presented some arguments that their construction is UV complete.)
Alternatively, the authors should make it clear that one should only expect the dual field theory to be an effective description, as was done for example for the higher-dimensional $T\bar{T}$ results by Taylor and by Hartman, Kruthoff, Shaghoulian and Tajdini.

We thank the referee and fully agree with the observation. The $T\bar{T}$ deformation is an irrelevant operator and as such is expected to modify the UV behavior of the theory, but at the same time, it is a double-trace operator whose contribution is suppressed by $1/N$ factors. To address this, we emphasize that our treatment of the boundary theory is based on an effective field theory (EFT) approach designed to capture low-energy dynamics. We have added a clarifying comment at the end of section 6 to highlight that when $\mathcal{S}_{\rm deform}$ includes irrelevant operators, adopting an EFT perspective and focusing on low-energy physics is necessary.

Regarding the referee's insightful suggestion concerning the free/critical $O(N)$ model in three dimensions, we agree that this represents a valuable and intriguing direction for future investigation. We appreciate the suggestion and have noted it as a promising avenue for further work.

We would also like to stress a key feature of the deformations considered within our framework: they are not arbitrary but constrained by the underlying gravitational dynamics. Specifically, these deformations are governed by Hamiltonian constraints—see equation (8.5) for $d = 2$ and equation (A.7) for $d = 3$ and $d = 4$. These constraints impose relations among different boundary operators. For instance, the trace of the stress tensor—associated with a marginal operator—is related through the flow equations to the $T\bar{T}$ operator, which is formally irrelevant. This interdependence underscores that the deformation in our setup fundamentally differs from conventional QFT deformations. We have added a remark in the final section on page 29 to emphasize this point.

Finally, we reiterate that our construction provides an *effective* dual description. To clarify this, we have explicitly used the term *effective* at multiple points throughout the manuscript.

Request 1. The authors should address the issues raised in the main report.

Done.

Request 2. It is claimed in the introduction that the bulk Y freedom is uniquely determined in terms of the boundary symplectic potential. However, near (4.12) of part I of this paper series, this is merely called a "convenient choice" rather than a unique determination. The status of this result should be clarified.

Thank you for pointing out this subtle aspect. Indeed, the choice is made for convenience, as we explicitly derive equation (4.10) of Part I. The transition from (4.10) to (4.12) involves the use of equation (4.11), after which we adopt a convenient form. We have revised the introduction to better reflect this reasoning.

Request 3. The authors should clarify the discussion under equation (2.10). Is the equation $d\delta\Theta_D = 0$ supposed to hold off-shell, as there is no circle above the equal sign? If so, also $d\delta\Theta = 0$ off shell. In any case, it does not seem like this condition specifies Θ_D uniquely. Also the fact that it is compatible with Dirichlet boundary conditions doesn't uniquely fix it, because any function of the source on the boundary can still be added to the boundary action.

Thank you for pointing this out. Indeed, $d\delta\Theta_D = 0$ is an on-shell condition. We have addressed the on-shell nature of this equation appropriately in the text. Additionally, we have included a comment at the end of Section 2.2.

Regarding the uniqueness of the Dirichlet symplectic potential, you are right, it is not unique. We had already pointed this out in Footnote 2 on page 5. This family of deformations is further explored in Section 8, where we refer to them as intrinsic hydrodynamic deformations. In our construction, we adopt a minimal scheme to fix the Dirichlet symplectic potential. We have revised the footnote to emphasize this point more clearly.

Request 4. I object to the term "gauge/gravity correspondence" being used to specifically indicate the AdS/CFT duality in the 't Hooft limit, as is done on page 6.

We clearly explain what we mean by the gauge/gravity correspondence: the saddle-point, large N limit of the AdS/CFT duality, at the beginning of the paper and use this notion consistently throughout the manuscript. We believe there is no room for confusion.

References

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