Response to the Report of Referee-2

Dear Editor,

First, we would like to thank the referee for carefully reading our manuscript and giving valuable feedback. Please find our reply to the comments and the details of the revisions made in the revised manuscript. All the changes are made in **violet**.

1. "*The authors are discussing scattering amplitudes involving scalars. Yet there is no mention of how the scalars couple to gravity. Presumably this is just minimal coupling. However, it would help to explicitly spell out the all the interactions of this scalar, perhaps around eq(3.1).*"

   Response: We add the matter action explicitly in Eq. (3.5) of the resubmitted manuscript.

2. *"What is γ in (2.10)? It is never defined. It seems to be the same or closely related to $\boldsymbol{\Delta} = \sum_{i=1}^{4} \Delta_i$."*

   Response: We defined the relation between γ and $\boldsymbol{\Delta}$ after Eq. (2.10).

3. *"It would be helpful to distinguish between the full amplitude $\mathbb{M}^{EFT}_{Born}$ and its eikonal limit. Eq(3.6) seems to suggest that this is the full amplitude while eq(3.8) seems to suggest that it is just it eikonal limit."*

   Response: The complete Born amplitude corresponds to the sum of contributions from the various channels as presented in Eq. (3.8). In Eq. (3.9), we express the Born amplitude directly in the eikonal limit, which serves as the working approximation throughout this manuscript. A clarifying footnote (in page 8) have been included to make this point explicit.

4. *"A related point that should be clarified is whether the authors consider the Mellin transform of the eikonal limit of the amplitude or of the full amplitude. I suspect that it is the former. This is an important point since it is unclear whether the eikonal limit commutes with the Mellin transform. While the authors do not need to address the issue of whether or not the integral commutes with the limit in this paper, it helps to explicitly mention the order of operations."*

   Response: We thank the referee for the insightful question. However, we would like to clarify that the two operations—taking the eikonal limit and performing the Mellin transformation—are fundamentally independent. This is evident from the fact that the eikonal approximation loses its clear interpretation once the amplitude is integrated over ω, as required in the Mellin transformation. The CCFT framework prescribes a specific order: one must first compute the scattering amplitude (within any chosen kinematic regime), and only then apply the Mellin transformation. This is precisely the procedure we have followed in this work. We have clarified this point in the revised version of the manuscript.

5. *"It might be useful to explicitly state what δ_1 and δ_2 are near eq(3.10)? "*

   Response: We have added the definitions explicitly in Eq. (3.12) of the revised manuscript.

6. *"How is $\boldsymbol{q}_\perp$ related to t in eq(3.12), eq(3.13)? "*

   Response: t is related to $q^2 = \boldsymbol{q}_\perp^2$ as, $t = -q^2 = -\boldsymbol{q}_\perp^2$.

7. *"Typo beneath eq(3.23). $M_{eik}^{(1,2)}$ have never been used."*

   Response: Typos have been fixed. Thank you for pointing them out.

8. *"Below eq(3.26), the authors say, "in the large ω limit". How can this approximation be justified inside the integral where ω is integrated from $(0, \infty)$?"*

   Response: We thank the referee for raising this important point. The expansion of the integrand in the large-ω limit is employed to extract the ultraviolet (UV) behavior of the corresponding celestial amplitude. Strictly speaking, the appropriate procedure is to first expand the integrand in the large-ω regime, perform the indefinite integral, and then analyze the $\omega \to \infty$ behavior of the result.

   Nonetheless, guided by intuition from the saddle point approximation, one expects that in the presence of a large parameter, the dominant contribution to the integral originates near the peak of the integrand, where it is sharply localized. Although the integration domain extends over the full range $(0, \infty)$, the leading contribution arises from the large-ω region, which justifies the use of this approximation in capturing the UV behavior. This is why we did not evaluate the integral in Eq. (3.32) explicitly; instead, we applied a saddle point analysis by choosing the parameter γ to be large, leading to a dominant contribution near the saddle point ζ_*.

   For a more rigorous justification, one could invoke the "strategy of regions," a well-established technique widely used in the computation of multi-loop Feynman integrals [1]. This method systematically *expands the integrand* in distinct regions characterized by separated scales, allowing for controlled approximations. In our context, this methodology has been applied heuristically, in line with earlier treatments in [2].

9. *"The OPE coefficients in (4.27) appear to depend on γ. If this is indeed related to $\boldsymbol{\Delta}$, seems that the OPE coefficient depends on 4 conformal dimensions. Since the OPE only involves 3 operators, it is unclear what OPE coefficient this corresponds."*

Response: Strictly, the expression given in (4.27) (in the revised version) is not the OPE coefficient rather it is the multiplication of two OPE coefficients: $\mathrm{OPE}_{12O} \otimes \mathrm{OPE}_{34O}$. We changed it accordingly.

We hope that our response has answered the queries raised by the referee.

Best,
Authors

References

[1] M. Beneke and V. A. Smirnov, *Asymptotic expansion of Feynman integrals near threshold*, Nucl. Phys. B **522** (1998) 321–344 [hep-ph/9711391].

[2] T. Adamo, W. Bu, P. Tourkine and B. Zhu, *Eikonal amplitudes on the celestial sphere*, JHEP **10** (2024) 192 [2405.15594].