Dear Editor of SciPost Physics,

Thank you for your editorial recommendation from September 14th with the reports of the two Reviewers. In the following, we provide a detailed point-by-point response to the comments of the reviewers. The revisions are also highlighted in blue in the revised manuscript. We hope that our revised manuscript will be accepted for publication in SciPost Physics.

Sincerely,
Kristian Blom, Uwe Thiele, and Aljaž Godec

General Remark

In addition to addressing the Reviewers' helpful comments, we made several minor, non-comment-related edits throughout the manuscript which are marked in blue. In particular, Sections 3.5.1 and 3.5.2 were substantially rewritten to improve readability and clarify the linear stability analysis. Complementing the newly added Section 3.4 (added in response to Reviewer 2), we have also included Section 5.4, which presents a series expansion about the stationary state for the reactive nonreciprocal Cahn–Hilliard model. Furthermore, we realized that Table I, which lists the different models for two nonreciprocally coupled fields, was incomplete; we have therefore extended it to include the full set of models. We hope the reviewers find these changes helpful.

Reviewer 1

Comment: *In their article „Dynamic Models for Two Nonreciprocally Coupled Fields", the authors present a microscopic derivation and detailed theoretical investigation of several continuum models representing the dynamics of lattice models with nonreciprocal interactions. They consider different scenarios with zero, one, and two conservation laws.*

The article presents a very rich and valuable contribution to a timely topic. It contains interesting results from both a statistical mechanics and a nonlinear dynamics perspective. I particularly like the way that different scenarios (single-spin flip, intralattice and intralattice+interlattice exchange) are presented next to each other in a way that allows to appreciate their relations and differences. Also, the availability of a microscopic derivation for commonly used nonreciprocal field theories that connects them to widely used lattice models is clearly valuable for the study of both the continuum and the lattice community. The authors present their results carefully and in a way that makes it clear what is the added benefit of this work in the existing research landscape.

Response: We thank the Reviewer for the positive assessment and insightful comments.

Comment: *The result (11), does, even in the passive case, differ in an interesting way from the familiar Allen-Cahn form that is only obtained when linearizing the tanh. (The authors confirm that the Lyapunov condition still holds.) However, I wonder what would happen with this equation in the stochastic case. Would $\mathcal{M}$ still a mobility in the sense that it sets, via an FDT, the correlation of the noise? Would the equilibrium distribution be $\exp(-\beta F)$? And if the answer is "no", would these still count as "free energy" and "mobility" in the usual thermodynamic sense?*

Response: This is a very intriguing question and we thank the Reviewer for bringing this to our attention. We can show with a simple trick that via an FDT the equilibrium distribution would still be $\exp(-\beta F)$ in the stochastic case. Note that throughout this derivation we use the same notation as in our manuscript and therefore set $\beta = 1$. Since the FDT only applies to equilibrium systems, we first set the interlattice couplings equal, i.e. $K_a = K_b$, so that there is no nonreciprocal interaction. Then we have the following equations

$$\partial_t m^\mu(\mathbf{x}, t) = -\mathcal{M}^\mu(m^a, m^b) \tanh\left(\frac{\delta F[m^a, m^b]}{\delta m^\mu}\right). \tag{1}$$

We can now apply a trick and rewrite this equation as

$$\partial_t m^\mu(\mathbf{x}, t) = -\hat{\mathcal{M}}^\mu(m^a, m^b) \frac{\delta F[m^a, m^b]}{\delta m^\mu}, \tag{2}$$

where we have defined a new mobility that reads

$$\hat{\mathcal{M}}^\mu(m^a, m^b) \equiv \mathcal{M}^\mu(m^a, m^b) \tanh\left(\frac{\delta F[m^a, m^b]}{\delta m^\mu}\right) / \frac{\delta F[m^a, m^b]}{\delta m^\mu}. \tag{3}$$

This new (effective) mobility remains thermodynamically consistent since $\tanh(x)/x \geq 0 \ \forall x \in \mathbb{R} \setminus 0$, and, furthermore,

$$\lim_{x \to 0} \frac{\tanh(x)}{x} = 1, \tag{4}$$

hence, the fraction in Eq. (3) exists (even when $\delta F[m^\mu]/\delta m^\mu = 0$). With this trick we have now transformed Eq. (1) into the standard Allen-Cahn form, and therefore one can immediately anticipate that for the stochastic case the equilibrium distribution would be $\exp(-F[m^a, m^b])$ as long as some FDT is obeyed. So, coming back to the questions of the Reviewer: When the effective mobility $\hat{\mathcal{M}}^\mu(m^a, m^b)$ sets the correlation of the noise via an FDT, the equilibrium distribution would be given by $\exp(-F[m^a, m^b])$.

Comment: *I think the gradient expansion used in the derivation of Eq. (15) could be spelled out in more detail, at present it is a bit hard to follow. How would one generate the $O(\nabla^4)$ corrections? What would the thermodynamic limit look like without the gradient expansion (presumably it would have a convolution integral)?*

Response: We thank the Reviewer for this comment. We have now added the following explanation in section 3.3 on page 9 to clarify how the gradient expansion works:

To implement this, let $\hat{\mathbf{e}}_x = (1, 0)^\mathrm{T}$ and $\hat{\mathbf{e}}_y = (0, 1)^\mathrm{T}$ denote the shift vectors in the x and y directions (in units of ℓ), respectively. In the thermodynamic limit where $\ell \to 0$ we can write the shifted fields as a Taylor expansion

$$m^\mu(\mathbf{x} \pm \hat{\mathbf{e}}_x, t) = \sum_{k=0}^{\infty} \frac{(\pm\partial_x)^k}{k!} m^\mu(\mathbf{x}, t) = \mathrm{e}^{\pm\partial_x} m^\mu(\mathbf{x}, t), \tag{15}$$

$$m^\mu(\mathbf{x} \pm \hat{\mathbf{e}}_y, t) = \sum_{k=0}^{\infty} \frac{(\pm\partial_y)^k}{k!} m^\mu(\mathbf{x}, t) = \mathrm{e}^{\pm\partial_y} m^\mu(\mathbf{x}, t), \tag{16}$$

where the second equality follows from the definition of the Taylor series of the exponential. Using the Taylor series gives the following gradient expansion for the sum of nearest neighbors

$$\lim_{\substack{\{N_x,N_y\}\to\infty}}^{\{L_x,L_y\}=\text{const.}} \sum_{\langle ij\rangle} m_j^\mu(t) = 2[\cosh(\partial_x) + \cosh(\partial_y)]m^\mu(\mathbf{x},t) = 4m^\mu(\mathbf{x},t) + \nabla^2 m^\mu(\mathbf{x},t) + \mathcal{O}(\nabla^4 m^\mu), \quad (17)$$

where ∇^2 is the Laplace operator.

From Eq. (17) the Reviewer can now also see what happens when the gradient expansion is not applied, which results in the operators $2[\cosh(\partial_x) + \cosh(\partial_y)]$ acting on the field. Hopefully this clarifies for all readers how the gradient expansion works.

Comment: *How confident are the authors in the validity of the mean-field approximation, which is a crucial element in this work? It is known from reciprocal lattice models (see, e.g., 1D Ising) that this assumption induces phase transitions that the underlying model does in fact not have.*

Response: It is indeed well known that, for the 1D classical Ising model, the MF approximation incorrectly predicts a phase transition where none exists. The same issue is encountered in our work: in one dimension the local free energy $f(m^a, m^b)$ takes the form

$$f(m^a, m^b) \equiv \sum_\mu \left[\Phi(m^\mu) - 2H_\mu m^\mu - 2J_\mu (m^\mu)^2 - K_\mu m^a m^b\right], \quad (18)$$

which, for $H_\mu = 0$ and $K_\mu = 0$ (i.e., the standard 1D Ising model), exhibits an MF transition at $J_\mu = 1/2$, contrary to the exact result.

Consequently, regarding the *quantitative* validity of MF for the 2D nonreciprocal Ising model, we remain cautious: MF may capture qualitative trends but can misestimate critical behavior. However, at present, it is the most tractable analytical tool available to us to derive the nonreciprocal field theories. A natural next step is to improve upon MF via the *pair (Bethe) approximation*, which incorporates short-range correlations. Specifically for the nonreciprocal Ising model this has already been emplyed for the spatially averaged magnetization (i.e. without any spatial aspect) in our previous work [14], where a comparison between the phase diagrams obtained with the MF and Bethe approximation shows strong quantitative differences (see Fig. 5 in [14]). For the spatially extended systems discussed in our current manuscript, we indeed plan to apply the Bethe approximation in the follow-up work. According to our experience, the “typical” reader in the field finds the results on the Bethe-level substantially more difficult to digest, this is why we decided to develop the theory in two steps, whereby the present MF treatment is the first step. We have expanded on these points in the concluding section on page 29-30:

While the MF approximation relies on Eqs. (8) and (40), which are not strictly valid for the square lattice and in fact are known to induce an erroneous phase transition in 1D [81], the pair approximation explicitly accounts for correlation effects and does not approximate the local energetic field. Although this more accurate method has been successfully applied to describe the overall average magnetization and nearest neighbor spin correlations under single spin-flip dynamics in the nonreciprocal Ising model [14], it has yet to be extended to such systems with spatially varying fields. Such an extension could shed light on whether a Turing-type instability is truly absent in the nonreciprocal Ising model, since it is known that the gradient-energy coefficient in the pair approximation is a nonlinear function of the coupling strength [82].

Comment: *The Swift-Hohenberg model in Section 7 is motivated by the need to avoid an UV instability in a regime where the underlying lattice model wants to have antiferromagnetic coupling. Does that mean that I should think of the characteristic length scale of the SH model as the lattice spacing? Is that compatible with a thermodynamic limit where this length scale should be zero? (See also my question on the gradient expansion.)*

Response: We thank the Reviewer for this comment. Please note that the characteristic length scale of the SH model has nothing to do with the lattice spacing itself, as any length scale in our theory is inherently defined w.r.t. the lattice spacing. Therefore, the characteristic length scale set by the SH model is also defined w.r.t. the lattice spacing, and therefore is *not* the lattice spacing itself. As this remark in our conclusion is solely thought of as an outlook onto a pathway to follow in the future, we decided to only expand on this comment in the response to the Reviewer. We hope that this clarifies the comment raised by the Reviewer.

Reviewer 2

Comment: *In this paper, the authors derive non-reciprocal field theories from microscopic dynamics, such as the non-reciprocal Ising model, discussing how zero, one, and two conservation laws for the magnetization can be implemented through kinetic rules for single spin-flip and spin-exchange dynamics. The authors first consider the case without conservation and the corresponding non-reciprocal Allen-Cahn model, which reveals unstable stationary and oscillatory modes as a result of a linear stability analysis. Second, the authors focus on the case of one and two conservation laws (nonreciprocal reactive CH model and nonreciprocal Cahn-Hilliard model, respectively).*

I believe that this paper is well-written and clear. It tackles a hot topic in the field of non-equilibrium statistical mechanics and soft matter, and, as a result, deserves publication in SciPost after the authors address my comments and doubts reported below.

Response: We are grateful to the Reviewer for the positive overall assessment of our work.

Comment: *1) I believe that adding further comments on the model (1)-(2) could be helpful: I would specify explicitly the meaning of the second and third terms in Eq.(2) – interaction with the neighboring spins in the same lattice and interactions with the same i-th spin in the other lattice. In addition, I suggest that the authors clarify the meaning of this model, making explicit examples of physical systems that can be modeled by Eqs. (1) and (2).*

In general, it would be helpful if the authors made this effort to justify the three dynamics studied (Single Spin-Flip Dynamics, Intralattice Exchange Dynamics, and Inter- and Intralattice Exchange Dynamics) in terms of experimental realizations. This discussion will provide a strong motivation for this study.

Response: The authors thank the reviewer for this comment. First, we have added further comments about our model and explicitly specified the meaning of the second and third terms in Eq. (2) on page 4:

The first term H_μ in Eq. (2) denotes an external magnetic field acting on lattice μ. The second term describes nearest-neighbor interactions within the same lattice, with J_μ the inner-lattice coupling strength. The third term accounts for the interaction of spin σ_i^μ with the corresponding *spin at position i on the opposite lattice $\nu \neq \mu$, where K_μ is the directed interlattice coupling strength.*

Second, to make explicit examples for both the model and the three dynamics that we studied, we have added the following section 2.3 to our introduction on page 6-7:

2.3 Significance of the Nonreciprocal Ising Model and Conservation Laws

For over a century, the Ising model has been the paradigmatic framework for equilibrium phase transitions [52]; analogously, the nonreciprocal Ising model can play a foundational role in understanding nonequilibrium phase transitions. It offers a minimal setting for asymmetric interactions between two many-body subsystems [28], with applications ranging from Ising machines [53,54] to collective opinion dynamics [55-57] and asymmetric Hopfield-type neural networks [58-60]. Less emphasized, however, is how the choice of dynamics (and therefore the associated conservation laws) shapes the resulting phenomenology.

To highlight one concrete setting, consider collective opinion dynamics where the up/down spins encode two opinions and the two lattices label two agent types, namely conformists and contrarians. Nonreciprocity naturally emerges in this settings: conformists prefer to align with their local neighbors, whereas contrarians tend to disalign with their neighbor on the opposing lattice and align with their neighbors within the same lattice, leading to directed cross-influences between the two groups. Different kinetic choices then probe different mechanisms: (i) single–spin flip dynamics *models the changes of opinion under local social pressure; (ii)* intralattice spin exchange dynamics *represents spatial relocation of agents while keeping their type and opinion fixed, capturing segregation of opinions within each group; and (iii)* intra- and interlattice spin-exchange dynamics *allows swaps of agents between the two groups, enabling the segregation of opinions between the conformists and contrarians. In this way, the nonreciprocal Ising model, combined with appropriate conservation laws, provides a flexible framework to investigate how asymmetric influence and kinetic constraints govern pattern formation far from equilibrium.*

Although we have illustrated these ideas with opinion dynamics, the same nonreciprocal Ising model with the appropriate kinetic constraints provides a minimal framework for other systems featuring asymmetric interactions between two subgroups [28].

Hopefully, this resolves the concerns raised by the Reviewer.

Comment: *2) The authors should make an explicit comparison between the current manuscript, their previous work (Ref.[14]) and Ref. [26], highlighting the differences which justify publication.*

Response: In the first version of our manuscript we already summarized the main outcomes of works [14] and [26] in the introduction. Based on the comment of the Reviewer we have now added an additional explicit statement that delineates our contribution and clarifies how it differs from the aforementioned works (see page 3):

Similar phenomena arise in nonreciprocal spin models [26-28], where mean-field analyses and extensive computer simulations uncover a rich phase behavior including ordered, disordered, and *oscillatory phases. Specifically for the square lattice it was found through simulations that nonreciprocity can induce the nucleation of droplets, which can subsequently lead to spiral patterns [26,28]. These results have been further analyzed beyond mean-field theory [14] and extended to three-dimensional nonreciprocal spin systems [15]. While these studies on nonreciprocal spin models focus on nonconserved dynamics, a microscopic derivation of the underlying spatially extended field theories and a direct link to the nonreciprocal Allen–Cahn or Swift–Hohenberg equations has remained elusive. Furthermore, it is natural to ask how nonreciprocal interactions give rise to distinct pattern-forming behavior in conserved systems that are governed by conservation laws and in mixed systems where conserved and nonconserved quantities interact.* (page 3)

Comment: *3) After Eq.(19), the authors state that the dynamics (16) reduce to the non-reciprocal Allen-Cahn equations up to linear order (discussed in Ref. [24]), after taking the linear approximation of the hyperbolic tangent function appearing in Eq.(16). I have the feeling that this is true only if the mobility $\mathcal{M}^\mu$ are constants as in Ref.[24]. Otherwise, the resulting dynamics would be more complicated. In the general case discussed here, where $\mathcal{M}^\mu$ is given by Eq.(19), you could obtain much richer phenomena also after linearizing Eq.(19). Please, clarify this point.*

Response: We thank the Reviewer for this critical comment. Indeed, the Reviewer correctly pointed out that our dynamics after linearizing the tanh is still more complicated than the canonical nonreciprocal Allen-Cahn equations. However, we forgot to mention that upon expanding the tanh function, one must also expand the mobility w.r.t. same expansion parameter. We have now carefully explained this in section 3.4 of the manuscript (see page 10):

3.4 Expansion Close To Stationary States

To clarify the connection between Eqs. (18) and the nonreciprocal Allen-Cahn equations [24], note that in the vicinity of stationary states the argument of the hyperbolic tangent is small. For brevity we define

$$x^\mu(m^a, m^b) \equiv \frac{\delta\mathcal{F}[m^a, m^b]}{\delta m^\mu} + (-1)^{\delta_{\mu,a}} \frac{K_a - K_b}{2} m^\nu, \quad \nu \neq \mu,$$

so that $|x^\mu| \ll 1$ close to stationarity. The hyperbolic tangent can then be expanded as

$$\tanh(x^\mu) = x^\mu + \mathcal{O}((x^\mu)^3).$$

At the same time, we expand the mobility in powers of x^μ using the first line in Eq. (21), resulting in

$$\mathcal{M}^\mu(m^a, m^b) = 1 - m^\mu \tanh(\operatorname{arctanh}(m^\mu) - x^\mu) = 1 - (m^\mu)^2 + \mathcal{O}(x^\mu).$$

Consequently, close to stationarity, Eqs. (18) reduce (to linear order in x^μ) to the nonreciprocal Allen-Cahn equations with quadratic mobilities [24]:

$$\begin{aligned} \tau \frac{\partial m^a}{\partial t} &\simeq -[1 - (m^a)^2] \left(\frac{\delta\mathcal{F}[m^a, m^b]}{\delta m^a} - \frac{K_a - K_b}{2} m^b \right), \\ \tau \frac{\partial m^b}{\partial t} &\simeq -[1 - (m^b)^2] \left(\frac{\delta\mathcal{F}[m^a, m^b]}{\delta m^b} + \frac{K_a - K_b}{2} m^a \right). \end{aligned} \tag{22}$$

In this sense, Eqs. (18) constitute a nonlinear extension of the nonreciprocal Allen-Cahn model.

Hence, upon expansion, the mobility (also) becomes less complicated, and only a quadratic dependence on the magnetizations remains. A similar expansion we have now also provided for the reaction term in section 5.4 (see page 24).

Comment: *4) In general, I would include the parameter values in the caption of each figure rather than in an additional Appendix.*

Response: As suggested by the Reviewer we have included the parameter values in the caption of each figure.

Comment: *5) I was wondering if the authors could discuss the possibility of applying their method to an active Ising model, where self-propulsion and interparticle alignment are included in the dynamics in addition to non-reciprocal couplings. The authors could speculate in the conclusion on the field theory models resulting from this method, if feasible.*

Response: We thank the Reviewer for bringing this to our attention. Let us first point out that the mean-field analysis of an active Ising model with self-propulsion and interparticle alignment has already been thoroughly done in the following work: *Solon, A. P., & Tailleur, J. (2013). Revisiting the flocking transition using active spins. Physical Review Letters, 111, 078101.* In this work, the authors apply the mean field approximation and obtained partial differential equations for the magnetization $m(x,t)$ and local density $\rho(x,t)$ for a one- and two-dimensional single-lattice active Ising model. Extending that framework to include nonreciprocal couplings (or, conversely, adding self-propulsion and alignment to our nonreciprocal model) is indeed an interesting direction. However, such an extension introduces an additional hydrodynamic field (the density), strongly alters the structure of the coarse-grained theory, and would require a careful, self-contained derivation that lies beyond the scope of the present manuscript. To avoid unsupported speculation, we therefore leave this line of inquiry to future work which we have now mentioned in the conclusion (see page 30):

Beyond its immediate theoretical contributions, this study provides a foundational framework for systematically deriving nonreciprocal field theories from the underlying microscopic dynamics. Although previous work has addressed the case of single-species systems with nonreciprocal interactions [3,83] and the dynamics of three-species active-passive particle mixtures [84], a comprehensive derivation of the partial differential equations for all possible combinations of conservation laws for two nonreciprocally coupled fields has remained elusive. Our results fill this gap, offering a microscopic route to macroscopic equations that is essential for understanding pattern formation in a wide range of nonequilibrium systems. In particular, our findings complement the exact hydrodynamic analysis presented in [84], which investigates nonreciprocal effective interactions in active-passive mixtures, by extending the scope to the two-field case and deriving the partial differential equations for all possible conservation laws. Furthermore, our framework permits asymmetric transition rates in position space, yielding an active spin model. In the absence of nonreciprocal couplings, a mean-field description of this class of models was developed in [85], which demonstrated the existence of a flocking transition. It would therefore be valuable to investigate how introducing a second lattice together with nonreciprocal couplings alters this behavior.