

Reply to Referee Report

Dear Editor,

First of all, I would like to express my sincere gratitude to two referees for providing detailed and helpful feedback. I carefully considered their referee reports and revised the paper with changes highlighted accordingly (blue for the first referee, red for the second referee). Attached is my detailed answer to the referee report given by the second referee. I have combined the revised version with my response to the referee's comments into a single PDF file. I hope this paper is now suitable for publication in Scipost Physics.

Best regards,
Jin Hu.

Reply to the Referee —

1) Comment: in the introduction, second paragraph, the author mentions some results on the holographic approach to the thermalization/hydrodynamization. It is also worth noting more recent work such as <https://arxiv.org/pdf/2406.06685>, where the authors suggest the existence of initial states that take arbitrary long times to thermalize, just as in the manuscripts references 13, 14 or 36.

2) Also in the introduction, second paragraph of page 2: the author mentions hard sphere potential, but, as far as I have seen does not mention <https://arxiv.org/abs/2309.09335>, which deals with the linearized collision term spectrum for this interaction.

Reply: I thank the referee for helpful comments. I have cited the helpful papers provided by the referee at the relevant location in the manuscript and included an appropriate discussion.

3) Is there a physical intuition for employing the variable g (introduced in the first paragraph of page 3) instead of the usual Mandelstam variables, such as s 's? Or is it just convenient so that the results in ref.33 (displayed in page 7, at the end of sec.III) are employed?

Reply: I thank the referee for helpful comments. As the referee correctly pointed out, the variable g is introduced precisely for convenience, as it will be used in the subsequent discussion.

4) In Eq.(9), the author in the paragraph containing "This can be achieved by redefining the weight function within the inner product definition as (...)", it should also be mentioned that the corresponding integrals should converge. This may exclude very large values for α .

Reply: I am very grateful to the referee for raising this point. I have now incorporated your suggestion regarding the convergence into the revised manuscript at the appropriate place.

5) In Sec.IIIB, last paragraph. Couldn't it be that the eigenvalue γ_6 also grows very large, since it also possesses an α dependence? From ref.36, the interplay between E_p and the linearized collision operator can lead to a very different behavior regarding the spectrum.

Reply: I agree on the statement of the referee "the interplay between E_p and the linearized collision operator can lead to a very different behavior regarding the spectrum", as also mentioned in my script. Also, as the referee correctly points out, γ_6 can indeed vary with α . However, in the discussion presented in Section IIIB, I consider the behavior of the system for a fixed value of α , under which condition the only variable quantity is E_p . Therefore, while the possibility that γ_6 grows with α is certainly interesting, it lies outside the scope of the current analysis.

6) In the end of Sec IIIC: the gapless continuous eigenspectrum of what the author calls \mathcal{L}_0 was also confirmed in Ref.36 and generalized in Ref.23. Then, the interplay of E_p with what the author calls \mathcal{L} (whose spectrum is known for ϕ^4 theory) is what is crucial, but I would regard their argument as an equivalent justification.

Reply: I thank the referee for helpful comments. I have added a note in the appropriate section of the manuscript, clarifying that the analysis and calculations presented here provide an equivalent justification.

7) In Eq.22, I would recommend using $\tilde{\chi}$, for the sake of clarity

8) In theorem 2: should v_0 be a ν_0 ? Is it a typo?

Reply: I thank the referee for helpful comments. I have change χ into $\tilde{\chi}$ in Sec.III.D for clarity. Also, v_0 is a typo and has been corrected.

9) In page 8, is $\cos\theta$, the angle between "boltzmann" momentum 3-vector p and "fourier" momentum 3-vector k ? If so, I wouldn't expect it to commute with the linearized collision term, since $E_p = u.p$ seen as an operator in Hilbert space also doesn't. Is there any physical intuition behind, or is an assumption to simplify the analysis?

Reply: I thank the referee for helpful comments. I believe that my original wording may have led to a misunderstanding on the referee's part. In the text, I first assume that the linearized collision operator commutes with $\cos\theta$, under which assumption the resulting branch cut structure takes the form given in Eq.(30). However, in the following paragraph, I explicitly point out that the linearized collision operator does not actually commute with $\cos\theta$, and therefore the true analytic structure is not as regular as the one suggested by Eq.(30). I have revised the relevant part in the manuscript to clarify this point and hope that the revised version adequately resolves this confusion.

10) The derivation made in the end of Sec. V seems to be quite similar to the one in ref 19, but ref 19 uses \mathcal{L}_1 in the current authors notation, whereas I the current author, as far as I understood, wants to justify a constant relaxation time RTA, which becomes that of Anderson-Witting for Landau matching conditions. Anyways, ref.19 should be mentioned in this section, even so if the author means something different.

Reply: I thank the referee for helpful comments. Indeed, although different forms of the collision operator are discussed, the methodology employed is consistent. Following the referee's suggestion, I have now mentioned Ref.21 (corresponding to Ref.19 in the original version) in this section.

I thank the referee for the careful revision and thoughtful comments for improving the manuscript. Upon reading the referee's remarks, I realized that my literature review was incomplete, and I am truly grateful for the referee bringing these important references to my attention. I have now added the relevant citations to the manuscript. Also, I carefully considered the referee report and revised the paper with changes highlighted (in red) accordingly, see the attached files. If the referee has any further comments, please kindly let me know. I also hope that the updated version is now suitable for publication in Scipost Physics.

Relaxation time approximation revisited and non-analytical structure in retarded correlators

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In this paper, we give a rigorous mathematical justification for the relaxation time approximation (RTA) model. We find that only the RTA with an energy-independent relaxation time can be justified in the case of hard interactions. Accordingly, we propose an alternative approach to restore the collision invariance lacking in traditional RTA. Besides, we provide a general statement on the non-analytical structures in the retarded correlators within the kinetic description. For hard interactions, hydrodynamic poles are the long-lived modes. Whereas for soft interactions, commonly encountered in relativistic kinetic theory, the gapless eigenvalue spectrum of linearized collision operator leads to gapless branch-cuts. We note that particle mass and inhomogeneous perturbations would complicate the above-mentioned non-analytical structures.

I. INTRODUCTION

Over the past several decades, relativistic hydrodynamics has notably contributed to characterizing the dynamic evolution of Quark-Gluon Plasma (QGP), as observed in heavy-ion collision experiments at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) [1]. Within the framework of phenomenological modeling for these collisions, the initialization time of hydrodynamic simulation is usually set to be less than 1 fm/c to match experimental observations. This suggests that the system may rapidly equilibrate. Moreover, the applicability of hydrodynamics has been noted to extend into less expected domains, including small collision systems like nucleus-nucleon and proton-proton collisions [2, 3], which implies that these small collision systems could possibly exhibit fluid-like behaviors. What is the dynamic mechanism that triggers rapid equilibration? Why does relativistic hydrodynamics also work well for small systems? Regarding the first question, it has been proposed that hydrodynamics can be viewed as an attractor governing the late-time behavior of systems as they approach equilibrium [4]. At the early stage of evolution, the system tends to flow towards the hydrodynamic attractor, even when far from local equilibrium, which may account for rapid hydrodynamization. Thus, hydrodynamization may have a broader application range than thermalization, contrasting with the conventional view that hydrodynamics is a truncated gradient expansion near local equilibrium. Despite significant advancements, it remains an open question how hydrodynamization emerges from a general dynamic system with diverse microscopic interactions.

As the first step towards understanding how relativistic nonequilibrium systems reach thermal equilibrium, the properties of retarded correlation functions have recently garnered extensive research interests. Two-point retarded correlation functions are pivotal, as they provide a wealth of insights into the transport characteristics of multi-particle systems, particularly how an equilibrium system responds to off-equilibrium disturbances within the linear regime. Moreover, non-analytical structures—such as poles or discontinuities in Fourier space—are crucial for determining the system’s evolutionary patterns. Poles are indicative of collective excitations that evolve towards equilibrium, corresponding to the persistent hydrodynamic modes, whereas the presence of non-hydrodynamic modes, which could be cuts or poles, is intrinsically linked to the emergence of hydrodynamic phenomena and the applicability of hydrodynamics. Research has shown that the correlators contain only poles at infinite ’t Hooft coupling in large N thermal $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory. In the holographic description, the spectra of correlation functions correspond to the ring-down spectra of dual linearly perturbed black holes: the quasinormal modes [5, 6]. However, in Ref. [5], the authors argue that the branch-cut structure emerges in the regime of weak, but finite ’t Hooft coupling, indicating a possible transition behavior controlled by the ’t Hooft coupling. Motivated by this illuminating finding, Paul Romatschke calculated thermal correlators of large N gauge theories in effective kinetic theory [7] (see also [8, 9]), reporting the onset of transition behavior for hydrodynamic poles. In other related studies, including analytical estimates [10], qualitative models [11, 12], and numerical calculations [13, 14], the dominant non-analytical structure is found to be the branch-cut rather than poles, thus posing the ‘poles or cuts?’ dilemma. Specifically, this dilemma initially refers to the mathematical essence of non-hydrodynamic excitations. In this work, we also use it to denote the comparison of the lifetime of hydrodynamic/non-hydrodynamic excitations. Also, it is worth noting that in a recent study [15] on the transient dynamics of quasinormal modes within a holographic framework, the authors demonstrated the existence of arbitrarily long-lived excitations arising from summation of short-lived quasinormal modes. These correspond to initial conditions in which a packet of energy is localized near the future horizon, a feature that bears a resemblance to the numerical results presented in [13, 14]. This may suggest a deeper and more direct connection between the linear response behaviors described by weakly-coupled kinetic theory and strongly-coupled holographic theories.

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As a universal low-energy effective theory, hydrodynamics describes the collective, macroscopic dynamics over large distances and time scales. As previously mentioned, elucidating the relationship between the microscopic dynamics of matter's fundamental constituents and the macroscopic dynamics of coarse-grained degrees of freedom is a fundamental question. Typically, collective macroscopic dynamics heavily relies on the intricate microscopic dynamic details involving a vast number of degrees of freedom: for instance, a microscopic theory is required to determine the relevant coefficients in the macroscopic description. In principle, an accurate description of a dynamic system necessitates a comprehensive treatment of all particles at the microscopic level. However, tracking the evolution of all particles is impractical; thus, a reduced description of microscopic degrees of freedom is essential. In scenarios of weak coupling, where the concept of quasiparticles is applicable, kinetic theory serves as an effective tool for describing many-body systems, with the relevant degrees of freedom being statistical distribution functions. For example, the BBGKY hierarchy equation describes the coupled dynamic evolution of n -particle distribution functions [16–19]. The BBGKY hierarchy equation is an infinite tower of integro-differential equations, making it challenging to solve without truncation. By disregarding n -particle correlations (for $n \geq 2$) and truncating this infinite series of integro-differential equations to the lowest order, the renowned Boltzmann equation is recovered, which describes the evolution of the one-particle distribution function. Despite its omission of nearly all particle correlations, the Boltzmann equation maintains profound physical significance and is capable of characterizing various non-equilibrium phenomena within kinetic theory. Notably, hydrodynamic behavior can be derived from the long-wavelength limit of the Boltzmann equation. Hence, the Boltzmann equation is an excellent candidate, particularly for weakly coupled cases, for elucidating how hydrodynamization is achieved in a dynamic system and how the multitude of microscopic degrees of freedom condense into coarse-grained hydrodynamic degrees of freedom.

However, the Boltzmann equation still contains an intractable collision integral, rendering its analytical or numerical solution challenging. Even with the simplest interactions, such as the hard-sphere potential, the linearized Boltzmann equation's collision operator retains a complex structure, making its analysis extremely difficult, [see \[20\] for related discussions on the linearized collision operator spectrum for hard-sphere interaction](#). Recently, the eigenspectrum of the linearized Boltzmann collision operator in massless scalar ϕ^4 theory was analytically determined [21]. However, extending this to other realistic interactions remains a significant challenge¹. [Marle \[23, 24\], and Anderson and Witting \(AW\) \[25\] proposed an approximation of the relativistic Boltzmann equation using a simplified collision operator, effectively extending the BGK \(Bhatnagar, Gross, and Krook\) model to the relativistic domain \[26\]. In the case of Marle, the proportionality factor is given by \$m/\tau_R\$, with \$m\$ the mass of the particles and \$\tau_R\$ the relaxation time, whereas for AW model, it is \$u \cdot p/\tau_R\$. We focus on the latter case in this work.](#) The AW model, also known as the relaxation time approximation, omits much of the dynamic information in the full collision operator. Firstly, it disregards nonlinearity. Secondly, its validity hinges on a clear separation between the eigenvalue representing the slowest relaxation and the others. Despite these limitations, the relaxation time approximation addresses practical challenges, enabling analytic and semi-analytic solutions for the simplified Boltzmann equation. This approach paves the way for an insightful and instructive analysis. Furthermore, the AW RTA model has been extended to a more general form with nontrivial energy dependence. However, the justification of this extended RTA model within kinetic theory awaits rigorous validation, which is another topic of interest.

The paper is organized as follows. In Sec. II, we briefly review the basic aspects of the linearized Boltzmann equation. In Sec. III, we revisit the formulation of relaxation time approximation within the linearized Boltzmann equation. In this section, we provide a justification for the RTA, based on general mathematical considerations regarding the eigenspectrum structure of the linearized collision operator. It turns out that the RTA is well-justified exclusively in scenarios involving relativistic hard interactions. Sec. IV serves as an application of the findings from Sec. III to address the non-analytical structures contained in the retarded correlation functions. In Sec. V, we introduce a novel relaxation time approximation by truncating the full linearized operator, anticipating a broader range of applicability. Summary and outlook are given in Sec. VI.

Natural units $k_B = c = \hbar = 1$ are employed. The metric tensor is given by $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$, while $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ is the projection tensor orthogonal to the four-vector fluid velocity u^μ . The abbreviation dP stands for $\int dP \equiv \frac{2}{(2\pi)^3} \int d^4p \theta(p^0) \delta(p^2 - m^2)$.

Note — Upon finalizing our manuscript, we became aware of a concurrent and highly pertinent study by L. Gavassino [27], which also focuses on the discussions about gapless modes, and has some overlap with our results.

¹ The first analytical eigenspectrum was obtained by C.S.Wang Chang and U.E.Uhlenbeck in the context of monatomic gases in the non-relativistic case (see chapter IV of [22]).

II. LINEARIZED BOLTZMANN EQUATION

As the lowest order truncation of the relativistic BBGKY hierarchy, the on-shell relativistic Boltzmann equation describes the non-equilibrium evolution of a weakly coupled system,

$$p \cdot \partial f(x, p) = C[f], \quad (1)$$

$$C[f] \equiv \int dP' dP_1 dP_2 (f(x, p_1) f(x, p_2) - f(x, p) f(x, p')) W_{p, p' \rightarrow p_1, p_2}, \quad (2)$$

where $f(x, p)$ is the one-particle distribution function in phase space, and $C[f]$ represents the collision kernel. Here we neglect the external force and focus on local two-body collisions and the classical statistics. Furthermore, $W_{p, p' \rightarrow p_1, p_2} = (2\pi)^6 s \sigma(s, \Theta) \delta^{(4)}(p + p' - p_1 - p_2)$ with the differential cross section $\sigma(s, \Theta)$ encoding the interaction information. In subsequent discussions, the differential cross-section is alternatively expressed as $\sigma(g, \Theta)$ depending on $g \equiv \sqrt{-(p - p') \cdot (p - p')}$. Here s represents the total center-of-momentum energy squared and Θ is the scattering angle in the center-of-momentum frame. The detailed balance property, $W_{p, p' \rightarrow p_1, p_2} = W_{p_1, p_2 \rightarrow p, p'}$, inherent to the transition rates, is implicitly considered in the above expression. Note that Eq.(1) is specific to one-component systems; for generalizations to multi-component systems, refer to [28, 29].

As mentioned before, the complicated Boltzmann equation is often linearized to facilitate a more straightforward analysis. Following the linearization procedure, we expand the distribution function around the local equilibrium state $f(x, p) = f_0(x, p)(1 + \chi(x, p))$, then Eq.(1) transforms into the following form,

$$Df(x, p) + E_p^{-1} p^{(\nu)} \partial_\nu f(x, p) = -f_0(x, p) \mathcal{L}_0[\chi], \quad (3)$$

with the linearized collision operator

$$-\mathcal{L}_0[\chi] \equiv E_p^{-1} \int dP' dP_1 dP_2 f_0(x, p') W_{p, p' \rightarrow p_1, p_2} (\chi(x, p_1) + \chi(x, p_2) - \chi(x, p) - \chi(x, p')), \quad (4)$$

where $D \equiv u \cdot \partial$, $E_p = u \cdot p$ and $p^{(\nu)} \equiv \Delta^{\nu\rho} p_\rho$. The local equilibrium distribution is defined as

$$f_0(x, p) = \exp[\xi(x) - \beta(x) \cdot p], \quad (5)$$

where $\beta^\mu \equiv \frac{u^\mu}{T}$, $\xi \equiv \frac{\mu}{T}$, $\beta \equiv \frac{1}{T}$ with the local temperature $T(x)$, and the chemical potential $\mu(x)$ associated with the conserved particle number. Evidently, the collisional invariance of 1 and p_μ is respected by construction. From a mathematical standpoint, it can be verified that the linearized collision operator \mathcal{L}_0 is self-adjoint and positive semidefinite within the square-integrable Hilbert space [28],

$$\begin{aligned} \int dP f_0(p) E_p \psi(p) \mathcal{L}_0 \phi(p) &= \int dP f_0(p) E_p \phi(p) \mathcal{L}_0 \psi(p), \\ \int dP f_0(p) E_p \psi(p) \mathcal{L}_0 \psi(p) &\geq 0, \end{aligned} \quad (6)$$

where the spacetime dependence is neglected for simplicity.

In certain special cases, an additional simplification proves particularly useful. When it comes to the normal mode solution of the kinetic equation, the background around which the distribution function is expanded is typically assumed to be a homogeneous and static equilibrium configuration $f_{eq}(p)$. For the sake of subsequent discussions, we introduce this simplification as well. Adopting the expansion $f(x, p) = f_{eq}(p)(1 + \chi(x, p))$, Eq.(3) can be further simplified to

$$\partial_t \chi(x, p) + \mathbf{v} \cdot \nabla \chi(x, p) = -\mathcal{L}_0[\chi], \quad (7)$$

where $\mathbf{v} \equiv \frac{\mathbf{p}}{p_0}$, $\nabla_\alpha \equiv \Delta_\alpha^\beta \partial_\beta$, and we should change $f_0(x, p)$ into $f_{eq}(p)$ in the above expressions accordingly.

III. REVISITING THE RELAXATION TIME APPROXIMATION

The relaxation time approximation of the Boltzmann equation, known as the BGK model in the nonrelativistic case and the AW model in the relativistic case, offers an effective description within kinetic theory. Notably, its simple mathematical structure facilitates the analytical extraction of underlying physics, albeit at the cost of precision. In this section, we examine the relationship between the RTA model and its complete form, the (linearized) Boltzmann equation, to elucidate the approximation's limitations, specifically when the model is valid within the framework of linearized kinetic theory. Note as an aside, the relaxation time is allowed to possess power-law energy dependence, conveniently parameterized as follows [11, 30, 31]

$$\tau_R = (\beta E_p)^\alpha t_R. \quad (8)$$

Here, α is an arbitrary constant controlling the energy dependence of the relaxation time, while t_R is independent of momentum. The specific value of α is believed to depend on the dynamic details and corresponds to various physical scenarios: $\alpha = 0$ corresponds to the traditional AW RTA [25]; $\alpha = 0.38$ is argued to well approximate the effective kinetic descriptions of quantum chromodynamics [30–32]; while $\alpha = 0.5$ is a good modeling in extreme out-of-equilibrium perturbations, e.g., jets, in this case τ_R is related to the famous jet stopping time [33, 34]. The successes achieved through flexible parameterization suggest that the RTA may effectively model realistic scenarios in practical applications. Irrespective of phenomenological considerations from practical simulations, we concentrate solely on the mathematical aspects of the RTA model. We demonstrate below that the energy dependence is ascribed to the redefinition of the linearized collision operator

$$Df(x, p) + E_p^{-1} p^{(\nu)} \partial_\nu f(x, p) = -f_0(x, p) E_p^{-\alpha} \mathcal{L}_\alpha[\chi], \quad (9)$$

where $\mathcal{L}_\alpha \equiv E_p^\alpha \mathcal{L}_0$ represents the redefined linearized collision operator with \mathcal{L}_1 and \mathcal{L}_0 being specific instances. It is easily proved that \mathcal{L}_α inherits the positive semidefinite and self-adjoint properties from \mathcal{L}_0 . This can be achieved by redefining the weight function within the inner product definition as $f_0(x, p) E_p^{1-\alpha}$ in Eq.(6), which means the square-integrable function space should be also altered accordingly. **It should be noted that the corresponding integrals within the definitions should converge, which excludes very large values for α .** In the following subsections III B and III C, we illustrate that naively truncating \mathcal{L}_α leads to a form resembling the energy-dependent RTA; however the resulting RTA-like model is not well-defined.

A. Anderson and Witting model

The traditional RTA proposed by Anderson and Witting is a relativistic generalization of the BGK model

$$p \cdot \partial f(x, p) = -\frac{E_p}{\tau_R} (f(x, p) - f_0(x, p)), \quad (10)$$

where τ_R is energy-independent, corresponding to $\alpha = 0$ in Eq.(8). The above equation can be rewritten as

$$Df(x, p) + E_p^{-1} p^{(\nu)} \partial_\nu f(x, p) = -f_0(x, p) \frac{1}{\tau_R} \chi(x, p). \quad (11)$$

By substituting $\mathcal{L}_0[\chi]$ with $\frac{1}{\tau_R} \chi$ in Eq.(3), effectively treating \mathcal{L}_0 as an identity operator (up to a constant factor), Eq.(3) simplifies to Eq.(11). Note we have specified the linearized collision operator \mathcal{L}_0 as our focus in this subsection. As will be manifest, $\frac{1}{\tau_R}$ can be identified as the smallest eigenvalue of \mathcal{L}_0 .

To elucidate this, by linearizing the Boltzmann equation around a stationary homogeneous distribution $f_{eq}(p)$ and focusing on spatially uniform transport, we arrive at the equation

$$\partial_t \chi(t, p) = -\mathcal{L}_0[\chi], \quad (12)$$

which can be formally solved to get $\chi(t) = e^{-\mathcal{L}_0 t} \chi(0)$. Since \mathcal{L}_0 is self-adjoint and semipositive, the linear perturbation from equilibrium decays exponentially, which is consistent with stability requirements. The same working logic applied to Eq.(11) leads to $\chi'(t) = e^{-t/\tau_R} \chi'(0)$. By comparison, it is natural to interpret τ_R as the relaxation time for the linear perturbation from equilibrium, and \mathcal{L}_0^{-1} as the infinite-dimensional relaxation time matrix (we adopt the terminology “matrix” although it may not be appropriate to use it when the eigenvalue spectrum is continuous).

Recognized as a self-adjoint and semipositive operator in Hilbert space, \mathcal{L}_0 may contain both discrete and continuous eigenvalues. By expressing χ as a linear combination of the eigenfunctions ψ_n of \mathcal{L}_0 ,

$$\chi(t, p) = \sum_n c_n(t) \psi_n(p), \quad \mathcal{L}_0[\chi] = \sum_n \gamma_n c_n(t) \psi_n(p) \quad (13)$$

where \sum_n implicitly includes the integral over the continuous eigenvalue spectrum, and we use γ_n to denote the eigenvalues of \mathcal{L}_0 . With the eigenfunction expansion, Eq.(12) can be resolved to reach

$$\chi(t, p) = \sum_n c_n(0) \psi_n(p) e^{-\gamma_n t} + C(p) = \sum_{n>5} c_n(0) \psi_n(p) e^{-\gamma_n t} + \sum_{n=1}^5 c_n(0) \psi_n(p) + C(p), \quad (14)$$

$$\chi'(t, p) = \sum_n c'_n(0) \psi_n(p) e^{-t/\tau_R} - \sum_{n=1}^5 c'_n(0) \psi_n(p) e^{-t/\tau_R} + C'(p) = \sum_{n>5} c'_n(0) \psi_n(p) e^{-t/\tau_R} + C'(p). \quad (15)$$

For comparative purposes, we also present the expansion form of $\chi' = \sum_n c'_n(t) \psi_n(p)$. Additionally, the counter term $-\sum_{n=1}^5 c'_n(0) \psi_n(p) e^{-t/\tau_R}$ is introduced to account for the absence of collision invariance in the traditional

RTA, as further discussed in Sec.V. Note $C(p), C'(p)$ are integration constants. In the above equations, the eigenvalues are sequenced in ascending order when increasing n . As clearly seen in Eq.(14), the collision invariants, $\psi_n (n = 1, \dots, 5)$ with zero eigenvalues $\gamma_n = 0$, contribute to the deviation function χ but remain unchanged with time. In the solution of Eq.(14), the second term, reflecting the contribution from collision invariants, can also be regarded as an integration constant. By comparing Eqs.(14) and (15), we can deduce that the traditional RTA simplifies the model by condensing the entire nonzero eigenvalue spectrum into a single representative eigenvalue, $\gamma_n \rightarrow \frac{1}{\tau_R}$. Given the exponentially decaying form, $1/\tau_R$ should be identified as the smallest eigenvalue, where the mode ψ_6 persists until the late stage of evolution. In other words, other modes with a larger eigenvalue, which should have been absent from an earlier time, extend their lifetime to the final stage in this approximation.

However, the RTA model is justified only when the eigenvalue sequence γ_n is distinctly separated from the origin. If this is not the case, such as when the eigenvalue spectrum extends continuously from the origin to other nonzero points, the relaxation time cannot be defined as the inverse of the smallest eigenvalue, as it might diverge, leading to the elimination of collisional effects. Thus, the justification of the RTA heavily depends on the eigenspectrum structure of the linearized Boltzmann collision operator, particularly the characteristics of the eigenspectrum near the origin. As shown in III D of this section, the traditional RTA is permissible only in scenarios of 'hard-interaction' collisions, where the eigenspectrum of the linearized collision operator features a continuous range from ν_0 ($\nu_0 > 0$) to ∞ , along with possibly some discrete points. In such scenarios, a gap exists between the origin and other nonzero eigenvalues, providing a basis for constructing the RTA.

B. Energy-dependent relaxation time approximation ($\alpha > 0$)

As noted in the convenient parameterization Eq.(8), the relaxation time, the only parameter reflecting the microscopic dynamic details within the RTA model, is allowed to exhibit energy dependence. It has long been believed that incorporating various energy dependence can reveal characteristic features of bottom-up thermalization and uncover hidden aspects of the full kinetic description [11]. Furthermore, the linearized kinetic equation within the energy-dependent relaxation time approximation has been successfully applied to jet physics [33, 34], hydrodynamic simulations [30, 31], and Weyl semimetals [35]. Does this imply that the model or approximation is well-justified? One caution should be taken: an approximated model may capture some aspects of the underlying physics relevant to specific issues but may fail to do so for others. Nevertheless, elucidating how an approximation is derived from a complete theory is always highly significant. In this subsection, we show mathematically how this model can be similarly derived from the linearized Boltzmann equation, and how an inconsistency appears hindering the sound justification of the model.

Before delving into a more general discussion, let's consider the specific case where $\alpha = 1$ in Eq.(8) for concreteness. The model thus becomes

$$p \cdot \partial f(x, p) = -\frac{1}{\beta t_R} (f(x, p) - f_0(x, p)). \quad (16)$$

This can be further rewritten as

$$Df(x, p) + E_p^{-1} p^{(\nu)} \partial_\nu f(x, p) = -f_0(x, p) \frac{1}{\beta E_p t_R} \chi(x, p) \quad (17)$$

which can be compared with Eq.(9) but now the replacement of $\mathcal{L}_1 \rightarrow \frac{1}{\beta t_R}$ should be invoked. This specific example aligns precisely with the energy-dependent RTA utilized in [11, 21].

Without delving into repetitive details, we now proceed to a general discussion concerning \mathcal{L}_α with positive α , termed as $\mathcal{L}_{\alpha>0}$

$$\chi(t, p) = \sum_n d_n(0) \Psi_n(p) e^{-\gamma'_n t / E_p^\alpha} + D(p) = \sum_{n>5} d_n(0) \Psi_n(p) e^{-\gamma'_n t / E_p^\alpha} + \sum_{n=1}^5 d_n(0) \Psi_n(p) + D(p), \quad (18)$$

$$\begin{aligned} \chi'(t, p) &= \sum_n d'_n(0) \Psi_n(p) e^{-t / (\beta E_p^\alpha t_R)} - \sum_{n=1}^5 d'_n(0) \Psi_n(p) e^{-t / (\beta E_p^\alpha t_R)} + D'(p) \\ &= \sum_{n>5} d'_n(0) \Psi_n(p) e^{-t / (\beta E_p^\alpha t_R)} + D'(p), \end{aligned} \quad (19)$$

where $\{\gamma'_n, \Psi_n\}$ represents the eigensystem of $\mathcal{L}_{\alpha>0} = E_p^\alpha \mathcal{L}_0$, and $D(p), D'(p)$ are integration constants. Given that $E_p^\alpha \geq 0$, it still holds that the eigen modes with a larger eigenvalue fade away more quickly. Depending on whether the eigenspectrum continuously extends to the origin, we decide to retain the smallest eigenvalue but zero or not. It seems that the story remains no changes at all compared to what has been done in the previous subsection.

However, that is not the case, as the relaxation timescale for eigenfunctions is E_p^α / γ'_n : the perturbation contributed by the hard modes takes longer time to relax towards equilibrium. With a fixed E_p , approximating the

linearized collision kernel with a single eigenvalue is valid, as it suffices to identify the slowest mode. If varying E_p , we may always encounter the following situation $E_p^\alpha/\gamma'_6 < E_{p'}^\alpha/\gamma'_7$ ($p \ll p'$), indicating that slower modes are excluded in the energy-dependent RTA. Hence, it's insufficient to consider only γ'_6 ; we must account for an infinite series of γ'_n . This renders the RTA derivation process questionable from a physical standpoint. Unlike AW RTA, the proposed relaxation time, expressed as E_p^α/γ'_6 , lacks an upper bound. Indeed, as E_p^α grows large enough, the hierarchy among the eigenvalues γ'_n becomes irrelevant, suggesting that $\mathcal{L}_{\alpha>0}$ can not support an energy-dependent RTA, regardless of the interactions involved: we don't even have to discuss the eigenvalue spectrum structure of $\mathcal{L}_{\alpha>0}$.

C. Energy-dependent relaxation time approximation ($\alpha < 0$)

The lack of a well-defined relaxation time is due to the lack of a gap: As illustrated in Section III B, a bounded energy-dependent relaxation time cannot be identified therein, but we can take a lesson from Eq.(8). It can be proved that such a gap can exist if the following three conditions are met:

- 1. The redefined linearized collision operator has nonzero eigenvalues separated from the origin.
- 2. The particles in consideration are massive.
- 3. α takes the negative value in the parameterization Eq.(8).

The necessity of the third condition can be relaxed if we deviate from the widely used parameterization in Eq.(8). Additionally, the second condition may also be loosened by employing alternative parameterizations. In this analysis, we focus our attention on the parameterization specified in Eq.(8).

Solving the corresponding linearized Boltzmann equation

$$\partial_t \chi(t, p) = -E_p^{-\alpha} \mathcal{L}_{\alpha<0}[\chi], \quad (20)$$

yields the formal solution effortlessly

$$\chi(t) = e^{-E_p^{-\alpha} t \mathcal{L}_{\alpha<0}} \chi(0). \quad (21)$$

We “naively” truncate $\mathcal{L}_{\alpha<0}$ to the smallest nonzero eigenvalue γ'_6 (or the nonzero infimum if the spectrum is continuous) to reproduce Eq.(8). Thus, the relaxation timescale for the perturbation decay is precisely $\frac{E_p^\alpha}{\gamma'_6}$ bounded from above by $\frac{m^\alpha}{\gamma'_6}$. This indicates that all modes attenuate at a finite rate with the slowest rate interpreted as the inverse relaxation time, $1/\tau_R = \frac{\gamma'_6}{m^\alpha}$, in contrast to what we encounter in III B. But if the particles are massless, then it follows that $\frac{E_p^\alpha}{\gamma'_6}$ is unbounded, and the well-defined RTA is also lacking for the similar reason shown in III B.

However, we confirm that $\tau_R = \frac{m^\alpha}{\gamma'_6}$ equates to the scenario where $\alpha = 0$, i.e., the AW RTA, and does not introduce a new class of RTAs. This verification is ruled by the following physical argument. Suppose an observer perturbs the system in equilibrium with a small-amplitude spatially homogeneous disturbance at $t = 0$, then the observer records when the disturbance dissipates to none. It is reasonable to require that Eqs.(3), (9), and their corresponding RTAs (if any) give the same relaxation timescale.

Assuming that the eigenvalue spectrum of \mathcal{L}_0 is gapped, the relaxation scale is given by $\frac{1}{\gamma_6}$ according to Eq.(3). If we try to describe the system using Eq.(9) with $\alpha < 0$, the relaxation scale is then dictated by the mode with the longest lifetime, and reads $\tau_R(p \rightarrow 0) = \frac{m^\alpha}{\gamma_6}$. Hence at $t = \frac{m^\alpha}{\gamma_6}$, the observer announces that the system returns to its equilibrium state. Given that the same experiment is conducted for the same system, the observer should give identical observation results, as Eq.(9) is an equivalent transformation of Eq.(3), i.e., $\frac{1}{\gamma_6} \simeq \frac{m^\alpha}{\gamma_6}$. For the $\alpha < 0$ case, even if there is an upper bound for relaxation scales and one can manufacture an energy-dependent (or precisely mass-dependent) RTA, it is still a redundant description compared to AW RTA. In summary, an energy-dependent RTA should not be regarded as a well-defined truncation to its UV complete theory, i.e., the linearized Boltzmann equation, irrespective of the interaction details. It is suggestive to keep only the description provided by \mathcal{L}_0 , because it is simplest without energy-dependent factor, and the mathematical analysis specifically for \mathcal{L}_0 is readily available [36, 37].

The observer argument above can provide an equivalent justification for reconciling the seemingly contradictory aspects between two detailed calculations for massless scalar ϕ^4 theory with leading order interaction [14, 21]. Denicol and Noronha analytically calculate the eigenspectrum of \mathcal{L}_1 and obtain a series of discrete eigenvalues [21]. By utilizing the observer argument again, we can conclude that the eigenspectrum of \mathcal{L}_0 must extend continuously towards the origin, and thus gapless. This conclusion is consistent with the mathematical aspect: one can show that for leading order scalar ϕ^4 theory, $\sigma(g, \Theta) \sim \frac{1}{s} = \frac{1}{g^2 + 4m^2} < \frac{2}{g^2}$, so the interaction falls within the soft interaction category according to the criteria given in [36, 37]. Furthermore, the gapless continuous eigenspectrum has also been confirmed in recent calculations [14, 27, 38].

D. Hard interaction VS soft interaction

Based on the above discussions, the Anderson and Witting model seems to be the only candidate with a potentially well-defined justification. However, it's important to note that this justification is contingent upon the eigenspectrum properties near the origin, which are, in turn, dependent on the specifics of the interactions. In this subsection, we will demonstrate that the aforementioned interaction details pertain to the softness of the interaction: the interaction is classified as soft or hard based on the differential cross-section's form, as encapsulated in the following two mathematical theorems [36, 37]. Before presenting these, let's briefly review the basic aspects. Following [36, 37], we linearize the Boltzmann equation around the global equilibrium distribution. But quite differently, the distribution function is written as

$$f(x, p) = f_{eq}(p) + f_{eq}^{1/2}(p)\tilde{\chi}(x, p), \quad (22)$$

and the linearized Boltzmann equation is cast into

$$\partial_t \tilde{\chi}(x, p) + \mathbf{v} \cdot \nabla \tilde{\chi}(x, p) = -\mathcal{L}_0[\tilde{\chi}] \equiv -v(p)\tilde{\chi}(x, p) + K[\tilde{\chi}], \quad (23)$$

with

$$\nu(p) \equiv f_{eq}^{1/2}(p)p_0^{-1} \int dP' dP_1 dP_2 f_{eq}(p') W_{p,p' \rightarrow p_1, p_2} \frac{1}{f_{eq}^{1/2}(p)}, \quad (24)$$

$$K[\tilde{\chi}] \equiv f_{eq}^{1/2}(p)p_0^{-1} \int dP' dP_1 dP_2 f_{eq}(p') W_{p,p' \rightarrow p_1, p_2} \left(\frac{\tilde{\chi}(x, p_1)}{f_{eq}^{1/2}(p_1)} + \frac{\tilde{\chi}(x, p_2)}{f_{eq}^{1/2}(p_2)} - \frac{\tilde{\chi}(x, p')}{f_{eq}^{1/2}(p')} \right), \quad (25)$$

where $\nu(p)$ represents the collision frequency and we introduce a compact operator K . Here we maintain the same notations as in the previous context to avoid complicating the overall notation system, although the expansion Eq.(22) looks quite different. Furthermore, it is immediately observable that Eq.(23) shares the similar form as Eq.(7). As a result, they should also share the similar formal solution $\tilde{\chi}(t) = e^{-\mathcal{L}_0 t} \tilde{\chi}(0)$ when the system is homogeneously perturbed. In both cases, whether considering Eq.(23) or Eq.(7), χ or $\tilde{\chi}$ should be regarded as the perturbation above the same equilibrium state. Given that both Eqs.(23) and (7) are derived from the same Boltzmann equation through linearization, we would not expect the eigenspectrum for \mathcal{L}_0 in Eq.(23) to differ from that in Eq.(7). If this were the case, the decay behavior of the perturbation would exhibit significant differences, signifying the physical inconsistency. This is another successful application of the observer argument from the previous subsection. Therefore, we argue that the eigenspectrum of \mathcal{L}_0 should be identical in Eqs.(23) and (7), and this is why we maintain the same notations. Then let's come to the core of this subsection:

Theorem 1 • Assume that $\exists \gamma > -2$, $0 \leq \beta < \gamma + 2$, $B > 0$ and $c_0 > 0$, so that $\sigma(g, \Theta) > B \frac{g^{\beta+1}}{c_0+g} \sin^\gamma \Theta$, then $\nu(p) > \nu_0(p_0/m)^{\beta/2}$ where ν_0 is a constant, the interaction is hard.

Theorem 2 • Assume that $\exists 0 < \alpha < 4$, $\gamma > -2$ and $B' > 0$, so that $\sigma(g, \Theta) < B' g^{-\alpha} \sin^\gamma \Theta$, then $\nu(p) < \nu_0(p_0/m)^{-\epsilon/2} \leq \nu_0$, the interaction is soft, where

$$\epsilon = \begin{cases} \alpha, & \text{for } 0 < \alpha < 3, \\ \alpha - 2, & \text{for } 3 < \alpha < 4, \\ \delta + 1, & \text{for } \alpha = 3, \text{ and } 0 < \delta < 1, \end{cases} \quad (26)$$

and ν_0 is a constant,

where the proofs of these theorems can be found in [37].

Analyzing the collision frequency $\nu(p)$ enables us to distinguish between relativistic soft and hard interactions. Mathematical analysis indicates that relativistic interactions tend to be softer than their nonrelativistic counterparts [36, 37]. It can be also proved that for soft interactions $-\mathcal{L}_0$ is a bounded operator with the eigenspectrum $[-\nu_{max}, 0]$, whereas for hard interactions, $-\mathcal{L}_0$ is unbounded, featuring an eigenspectrum of $[-\infty, -\nu_{min}]$, where ν_{max} and ν_{min} can be estimated by the extrema of $\nu(p)$. Given the eigenspectrum properties presented here, we conclude that the AW RTA is only well-justified in the case of hard interactions. For example, in the Weinberg-Salam theory at low energies [28], e.g., four fermions interaction in the electroweak sector well below the gauge boson masses, the typical differential cross section behaves as $\sigma(g, \Theta) \sim s = g^2 + 4m^2 > \frac{g^3}{c_0+g} \geq \frac{g^3}{c_0+g} \sin \Theta$ with the conditions $\gamma = 1, \beta = 2, B = 1$ in Theorem 1. Consequently, the model of a gas consisting of elastically colliding neutrinos in the low-energy limit permits an RTA description of transport phenomena.

As most relativistic interactions are soft according to Theorem 2, the AW RTA lacks a solid foundation in the majority of cases. For instance, in the extensively studied scalar ϕ^4 theory, the interaction is soft as exhibited in the previous section. This softness results in a branch-cut structure that extends across the entire negative imaginary axis in the retarded correlators [13, 39].

There is one comment left. When interactions are too complicated so that their classification as hard or soft cannot be easily known, we recommend employing the finite-element method described in [14] for analyzing the eigenspectrum properties. By implementing the outlined procedures therein, we can establish the dictionary between various field theories and the eigenspectrum structure within their linearized kinetic description. In this sense, we can exhaust all commonly used interactions admitting an RTA description.

IV. POLE OR CUT — THE NON-ANALYTICAL STRUCTURES IN RETARDED CORRELATORS

Two-point retarded correlation functions are crucial and insightful as they encapsulate rich information about the transport properties of many-body systems. Their analytical structures can reflect the characteristic properties of how thermal equilibrium is reached. For instance, the non-analytical structures — poles or cuts in Fourier space, govern the evolution behavior of the system: poles describe collective excitations evolving towards equilibrium corresponding to hydrodynamic modes, while the presence of cuts or non-hydrodynamic modes is closely linked to the emergence of hydrodynamic behavior and the applicability of hydrodynamics. Thus, the research into the analytical properties of retarded correlators is profound, which is initially explored by Romatschke in the weakly-coupled kinetic theory [7]. Romatschke's findings highlight two key features in the analytical properties of the retarded correlators: the cuts are gapped corresponding to non-hydrodynamic modes, below which hydrodynamic poles dominate as long-lived degrees of freedom; the hydrodynamic poles cease to exist for some critical value of the wavenumber reminiscent of the phenomenon of onset transitions, which are successfully reproduced within the mutilated RTA model detailed in [40]. As a supplement, we note that the universal behavior of onset transitions has been reported for a long time in the context of the nonrelativistic kinetic theory using the mutilated model [22]. However, the AW RTA adopted in [7] is less well-founded than its nonrelativistic counterpart, given that interactions tend to be softer in relativistic scenarios: the universal onset transitions in nonrelativistic systems may be rarely observed in relativistic systems due to varying degrees of interaction softness.

Later on, Kurkela and Wiedemann reexamined the behavior of the retarded correlators, beginning with the parametrized energy-dependent RTA as described in Eq. (8), with a particular focus on the case when $\alpha = 1$ [11]. Their conclusions, however, stand in contrast to those in [7]: there is no sharp onset of hydrodynamic behavior; the structure of cuts turns into the entire strip $\text{Im } \omega < 0, -k \leq \text{Re } \omega \leq k$ from a gapped line given in [7]. They also state that the appearance of poles in the first (physical) Riemann sheet of retarded correlation functions is a matter of choosing a particular analytical continuation and thus cannot be related unambiguously to the onset of fluid dynamic behavior. However, as demonstrated in III B, their model corresponding to Eq.(8) with $\alpha = 1$ exhibits inconsistencies: it can only be seen as an incomplete truncation to its ultraviolet (UV) completion — the linearized Boltzmann equation, because an infinite number of slow modes are excluded. Although the analysis given in [11] is still illuminating, we choose to work with RTA only in the hard interaction case for theoretical consistency.

As observed, these two studies introduced above correspond to the models discussed in III A and III B. In this place, we want to give a general statement on the topic of non-analytical structures in the (stress-stress) retarded correlators. Before proceeding, let's elaborate on how to derive the non-analytical structure if the interactions are soft. In this case, the eigenvalue spectrum of \mathcal{L}_0 is gapless and the typical stress-stress retarded correlation function is given by

$$G_R(\omega) = \int_0^{\nu_{max}} d\gamma \frac{\rho(\gamma)}{\omega + i\gamma} \quad (27)$$

where we use γ to denote the continuously distributed eigenvalues, and the weight function $\rho(\gamma)$ is nonzero in the integration range. This expression can be derived from the Fourier transform of the linearized Boltzmann equation Eq.(3) in the limit of vanishing k [13, 39]. This expression clearly exhibits discontinuity as we shift from $\omega = -i\gamma + \epsilon$ to $\omega = -i\gamma - \epsilon$, indicating the presence of a branch-cut line extending from $-i\nu_{max}$ to 0.

In the limit of vanishing mass and nonzero k , extracting the non-analytical structure becomes unmanageable. **We first assume that $\cos \theta$, regarded as an operator in momentum-space functions, commute with \mathcal{L}_0** , then we can replace ω by $\omega - k \cos \theta$, where θ is the angle between spatial components of p and k . Then we can cast $G_R(\omega)$ into

$$G_R(\omega, k) \sim \int_{-1}^1 d\cos \theta \int_0^{\nu_{max}} d\gamma \frac{\rho(\gamma)}{\omega - k \cos \theta + i\gamma}, \quad (28)$$

where the integration over $\cos \theta$ must be performed in the momentum integral. The resulting expression is

$$G_R(\omega, k) \sim \frac{1}{k} \int_0^{\nu_{max}} d\gamma \rho(\gamma) (\log(\omega + k + i\gamma) - \log(\omega - k + i\gamma)). \quad (29)$$

This results in the branch-cut structure

$$\text{Im } \omega = -\gamma, \quad -k \leq \text{Re } \omega \leq k, \quad 0 < \gamma < \nu_{max}, \quad (30)$$

where $\nu_{max} = \infty$ for a massless theory (see Eq.(31)). Ultimately, **the resulting branch-cut structure seemingly reproduces the result for nonzero k given in [11]**².

However, a reminder should be given that Eq.(30) relies heavily on the commutation approximation for $\cos\theta$ and \mathcal{L}_0 , which breaks down when \mathcal{L}_0 is the complete linearized Boltzmann collision operator. In other words, the structure in Eq.(30) should be modified. For situations involving hard interactions, the extent of modification should be minimal as the RTA can be seen as a good approximation to the complete linearized collision operator in that case.

Based on the above statement, determining the non-analytical structures in the retarded correlation functions equates to solving for the eigenspectrum of \mathcal{L}_0 . However, a contradictory question arises: both [13] and [14] demonstrate that the cut extends across the entire negative imaginary frequency axis, contrasting with the bounded region $[-i\nu_{max}, 0]$ inferred from mathematical analysis. The discrepancy vanishes because, in the context of a massless theory with soft interactions, the collision frequency becomes unbounded [37]

$$\nu(p) \longrightarrow \infty, \quad \text{when } p \rightarrow 0. \quad (31)$$

Therefore, $\nu_{max} = \infty$, and these results are consistent with each other.

For clarity, we give a summary of this section by reconsidering the interplay of nonzero particle mass m and wavenumber k , and our statement can be summarized as follows:

- Hard interactions: the RTA is a well-defined approximation relative to its UV complete theory.
 - $m = 0$ or $k = 0$: Romatschke's analysis applies: the retarded correlators (including stress-stress correlator) exhibit a gapped branch-cut line (only two endpoints are branch points)

$$\text{Im } \omega = -\frac{1}{\tau_R}, \quad -k \leq \text{Re } \omega \leq k, \quad (32)$$

associated with nonhydrodynamic modes. Below this gap, there is a window where hydrodynamic modes become the long-lived degrees of freedom.

- $m \neq 0, k \neq 0$: The non-analytical structures of the retarded correlators become complicated due to the interplay between the nonzero particle mass m and wavenumber k . Upon inspecting the derivation detailed in [7], we find that if the particles are massive, the free-streaming term is proportional to $\mathbf{v} \cdot \nabla \sim \frac{pk \cos \theta}{\sqrt{p^2 + m^2}}$ depending on p . This could introduce additional complex structures into correlators after performing momentum integral, beyond the poles or gapped cuts predicted by Romatschke. We leave this discussion to future work. For now, we can only conclude that the non-analytical structure remains in the form of Eq.(32). The remarkable distinction is that the points lying within the branch-cut line are all branch points.
- Soft interactions. The RTA does not constitute a well-defined model. The simplifying assumptions should at least retain the eigenvalues and eigenfunctions of \mathcal{L}_0 near the origin, specifically within the region $[0, \nu_{max}]$.
 - $k = 0$: In this case, the stress-stress correlator possesses a branch cut described by

$$-\nu_{max} < \text{Im } \omega < 0, \quad \text{Re } \omega = 0, \quad (33)$$

which in the limit of vanishing mass matches the conclusion drawn in [13]. Strictly speaking, the author focuses on the discussions on the symmetrized 2-point function therein, but the retarded correlation function can be related to it through KMS relation. At weak coupling where the relevant frequencies will be suppressed by powers of the coupling, the result aligns with Eq.(33), which is further confirmed in a related study discussing \mathcal{L}_1 [38].

- $k \neq 0$: The complication arising from $\frac{pk \cos \theta}{\sqrt{p^2 + m^2}}$ also exists in this scenario, and the issue should be trickier than the case of massive RTA. Even if $m = 0$, it is still difficult to reach a concise-form conclusion due to the non-commutativity of $\cos\theta$ and \mathcal{L}_0 . Therefore, the non-analytical structures don't exhibit regular shapes like Eq.(30).

In the end of this section, we would like to add several comments as follows:

— Although failing to reach a definite conclusion in the cases involved with nonzero k or/and m , we plan to numerically solve the issue following finite-element analysis given in [14], which is left to future work. For instance, for soft interactions, we can examine the simplest interaction case, scalar ϕ^4 theory with leading-order interaction, which is expected to reveal the property of non-analytical structures qualitatively.

² [11] is based on the choice of \mathcal{L}_1 , or more precisely, an incomplete truncation of \mathcal{L}_1 , as shown in III B. For completeness and consistency, a thorough study extending [38] to spatially inhomogeneous perturbations is warranted.

— If Eq.(30) holds, hydrodynamic modes are completely embedded in this strip, leading to the conclusion that nonhydrodynamic modes related to free-streaming dynamics are long-lived degrees of freedom compared to hydrodynamic modes. According to our proposed non-commutativity between $\cos\theta$ and \mathcal{L}_0 , Eq.(30) would be modified so that there could be a window where hydrodynamic modes dominate as long-lived modes. In this sense, the physical picture may change by reconsidering the impact of the non-commutativity.

— In this script (likewise in [27]), we focus on the gapped/gapless property of nonhydrodynamic modes. By using the terminology “dominant” or “dominate”, we refer to the comparison of their lifetime. In practical cases, the dominant role should be assessed by combining their corresponding residues or discontinuity, which encode their contribution to the retarded correlation functions. This issue can be addressed using numerical calculations outlined in the first comment.

V. THE NOVEL RELAXATION TIME APPROXIMATION

This section focuses on the case of hard interactions, where the relaxation time approximation is validated. Despite its validation, the relaxation time approximation by Anderson and Witting still has inherent flaws as it fails to respect the collision invariance of the Boltzmann collision operator, which is tantamount to microscopic conservation laws. To address this issue, the counter terms to restore collision invariance are introduced,

$$-\mathcal{L}_0 \simeq \left(-\gamma_6 + \gamma_6 \sum_{n=1}^5 |\psi_n\rangle\langle\psi_n| \right), \quad (34)$$

which is referred to as the novel RTA in [21, 32], and is also known as the mutilated operator in [40–42] (see chapter V of [22] for an earlier discussion). Here $|\psi_n\rangle$ denotes the orthonormal eigenfunctions of \mathcal{L}_0 with zero eigenvalues, and γ_6 represents the smallest positive eigenvalue with the dimension of $[E]$. Thus we recover the collision invariance: $\mathcal{L}_0|\psi_n\rangle = 0, n = 1, \dots, 5$. To relate it to the traditional RTA, one needs to identify the relaxation time as $\tau_R \equiv \frac{1}{\gamma_6}$. When \mathcal{L}_0 acts on other eigenfunctions, it results in $\mathcal{L}_0|\psi_n\rangle = \gamma_6|\psi_n\rangle$ for $n > 5$, collapsing all positive eigenvalues into the smallest positive one. This is why the model in Eq.(34) is sometimes referred to as “mutilated” [22].

Macroscopic conservation laws are inherently restored when microscopic conservation laws, specifically the collision invariance of the collision operator, are respected in the model’s construction, thereby simultaneously fixing the basic flaws. The novel RTA provides the flexibility to adjust the matching conditions used in kinetic theory, a feature not present in the traditional RTA as discussed in [32, 40, 43–45]. This flexibility is particularly advantageous when discussing hydrodynamic frame dependence, such as in the first-order causal theory of the BDNK type [46–49].

By specifying the inner product definition, we can cast Eq.(34) into a less abstract form. For brevity, we omit the x dependence in this section. Omitting the derivation details (for which we refer readers to [32]), the novel RTA can be formulated as follows

$$-\mathcal{L}_0\chi(p) = -\frac{1}{\tau_R} \left[\chi(p) - \frac{(1, \chi(p))}{(1, 1)} - P_1^{(0)} \frac{(1, P_1^{(0)}\chi(p))}{(1, P_1^{(0)}P_1^{(0)})} - p^{\langle\mu\rangle} \frac{(1, p_{\langle\mu\rangle}\chi(p))}{(1/3)(1, p_{\langle\nu\rangle}p^{\langle\nu\rangle})} \right]. \quad (35)$$

Here the orthogonal basis is constructed from the collision invariants 1 and p^μ , and given by

$$P_0^{(0)} \equiv 1, \quad P_1^{(0)} \equiv 1 - \frac{(1, 1)}{(1, E_p)} E_p, \quad p^{\langle\mu\rangle} \equiv \Delta^{\mu\nu} p_\nu, \quad (36)$$

and the definition for the inner product bracket is invoked

$$(B, C) \equiv \int dP w(p) B(p) C(p), \quad (37)$$

with the weight function

$$w(p) \equiv f_0(p) E_p, \quad (38)$$

note that the weight function is the same as the one in Eq.(6), as it should be. In Eq.(35), the novel RTA is constructed by adding counter terms to the RTA, but it can also be derived through an alternative method, as shall be given below. To proceed, we expand $\chi(p)$ as the linear combination of the eigenfunctions of \mathcal{L}_0 as

$$\chi(p) = \sum_n a_n \psi_n, \quad \text{with} \quad a_n \equiv \frac{(\chi(p), \psi_n)}{(\psi_n, \psi_n)}, \quad (39)$$

where the summation can also denote the integral for continuous spectra. Then the action of \mathcal{L}_0 on χ leads to

$$\begin{aligned}\mathcal{L}_0\chi(p) &= \sum_{n=1}^{\infty} a_n \gamma_n \psi_n \simeq \sum_{n=1}^N a_n \gamma_n \psi_n + \gamma_N \sum_{n=N+1}^{\infty} a_n \psi_n \\ &= \sum_{n=1}^N a_n (\gamma_n - \gamma_N) \psi_n + \gamma_N \chi(p),\end{aligned}\tag{40}$$

where as prescribed previously γ_n is sequenced in ascending order, and we make the approximation in the first line [50]. If $N = 6$, then the second term in the last line matches the RTA collision kernel. Given that $\gamma_i = 0$ for i ranging from 1 to 5, the above equation can be cast into

$$\mathcal{L}_0\chi(p) \simeq \gamma_6 \chi(p) - \gamma_6 \sum_{n=1}^5 a_n \psi_n\tag{41}$$

where the second term on the right-hand side precisely constitutes the counter term required to restore collision invariance. Hence, Eq.(40) encompasses Eq.(35) as a particular case. One can also show that the collision invariance is restored for $N > 6$. Compared to the constructions in [32, 40, 42], the method introduced here embodies the essence of truncation that leads to RTA and is capable of accommodating a broader spectrum of modified RTAs. By increasing N , we can incorporate additional eigenvalues and eigenfunctions into the original RTA if available. This approach is beneficial when our knowledge is limited to a finite set of eigenvalues and eigenfunctions (these can be determined numerically at least [14]), when we attempt to integrate the relevant information into the RTA. Furthermore, the model in Eq.(40), serving as an intermediate between the RTA and the full linearized collision operator, can be fine-tuned to balance accuracy and simplicity, allowing for a judicious compromise.

VI. SUMMARY AND OUTLOOK

In this paper, we revisit the widely used relaxation time approximation within the linearized Boltzmann equation. According to the mathematical analysis on the eigenspectrum of linearized Boltzmann collision operator \mathcal{L}_0 , the RTA model is justified only for hard interactions, thereby ruling out the energy-dependent parametrization in Eq.(8). The consideration is grounded on mathematical aspects, and Eq.(8) can effectively serve as a convenient parameterized model. Furthermore, Eq.(8) sheds light on the redefinition of linearized collision operator denoted as \mathcal{L}_α . We also provide a derivation of the novel RTA to restore the collision invariance by truncating the linearized collision operator, instead of by adding counter terms.

For discussing the non-analytical structures within the retarded correlators, we find that focusing on \mathcal{L}_0 and its eigenspectrum properties is the simplest approach. When interactions are hard, the RTA is well-defined, and the analytical properties of retarded correlators, as detailed in [7], are applicable. There is a gap between the branch-cut lines and the real axis $\text{Im}\omega = 0$. Therefore, the gapless hydrodynamic modes are well-defined low-energy degrees of freedom when k is small. However, according to mathematical derivations, relativistic interactions are often soft, leading us to focus on the alternative scenario in most cases. In scenarios with soft interactions, the RTA is no longer well-justified, and the dominating long-lived non-analytical structure turns into the branch-cut or the non-hydrodynamic modes. Our conclusion is consistent with the previous related studies with the comparison details elaborated in the main text. Note that if the particles constituting the system are massive or the perturbations are inhomogeneous, the non-analytical structures would be more complicated and richer.

There are possible extensions to the present research. As mentioned in III D, we can establish a dictionary between various field theories and the eigenspectrum structure within their linearized kinetic description. This can help us determine the dominant non-analytical structure and derive all possible RTA models. The former concerns the properties of the retarded correlation function of many-body systems, while the latter provides a solid theoretical basis for RTA, if any. We believe that applying RTA to a justified system is theoretically more consistent than doing so without justification, which is one of the motivations for this work. Additionally, almost all related research on the properties of retarded correlators is based on the linearized kinetic description. It would be interesting to explore the impacts of the nonlinear structure contained in the complete kinetic description, such as the Boltzmann equation, on the present conclusions.

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