Reply to Report 2 by Referee 2:

Report: *The authors compare several entropy measures for transition matrices and conclude that the ABB entropy has a proper definition in terms of its probability interpretation and LOCC-like monotonicity. They apply these entropy measures to different models, demonstrating their ability to detect "non-Hermitian" quantum chaos and exceptional points (EPs).*
Their results are interesting and could lead to future investigations of ABB entropy. However, the provided examples show that all the entropies exhibit drastic changes near the EPs and all can detect quantum chaotic properties. This suggests that the ABB entropy does not stand out from the others, but is simply another measure that fits the standard quantum information perspective.
I think the paper is interesting enough to be published in SciPost Physics after some revisions.

Authors: We are very grateful to the referee for highlighting the core results and positive assessment of our manuscript. We have carefully revised the manuscript in response to the referee's comments.
Our goal is not to use the ABB entropy as a diagnostic of quantum chaos, although it does exhibit typical behaviors for states in chaotic systems. Instead, the key advantage of the ABB entropy is its ability to quantify information transfer and admit a clear operational interpretation based on distillation applied to transition matrices.
All changes in our manuscript have been highlighted in blue except for those addressing Question 5, which was also raised by another referee. For clarity, the changes corresponding to Question 5 are highlighted in red. Our point-by-point responses to the referee's questions are given below.

1. **Q:** *In Table 1, the normalizations for $\hat{\tau}$, $\bar{\tau}$ are based on the trace norm. while the $\tilde{\tau}$ is based on the post-selected maximally mixed state. The authors should clarify this.*
   A: Thanks for referee's suggestion. We add the clarification in the caption of Table 1.
2. **Q:** *The main reason the ABB entropy has a proper probability interpretation appears to be that it is just the entropy measure of the post-selected maximally entangled states under the transition matrix, rather than of the transition matrix itself. A crucial question is : if one choose a state other than maximally entangled state, would this property change?*
   A: If one choose a general entangled state, the property that the ABB entropy admits a proper probability interpretation remains unchanged. Any pure bipartite state $|\phi\rangle$ can be represented, up to normalization, as the result of applying a

transition matrix τ_1 to a maximally entangled state $|0\rangle$,

$$|\phi\rangle = \frac{\tau_1 |0\rangle}{||\tau_1 |0\rangle||}.$$

Applying a transition matrix τ to an arbitrary entangled state $|\phi\rangle$ is therefore equivalent to applying the composite transition matrix $\tau\tau_1$ to the maximally entangled state $|0\rangle$. Consequently, replacing the maximally entangled state by a general entangled state results in a simple transformation of the ABB entropy,

$$S_{\text{ABB}}[\tau] \to S_{\text{ABB}}[\tau\tau_1],$$

but the probability interpretation still holds.

3. **Q:** *The $n = 1/2$ Renyi ABB entropy is equal to twice the SVD entropy. Is it correct?*
   A: No, for Renyi ABB entropy $S^{(n)}_{\text{ABB}}[\tau]$, it is required $n > 1$. Despite this, if $n = 1/2$, then

$$S^{(1/2)}_{\text{ABB}}[\tau] = -2 \ln \frac{\text{Tr}\left[\left(\tau^\dagger \tau\right)^{1/2}\right]}{\left(\text{Tr}\left[\tau^\dagger \tau\right]\right)^{1/2}} = -2 \ln \text{Tr}\left[\left(\tilde{\tau}^\dagger \tilde{\tau}\right)^{1/2}\right] = \frac{1}{2} \ln \text{Tr}\, \bar{\tau}^2,$$

   which is different from the SVD entropy $S_{\text{SVD}}[\tau] = -\text{Tr}\, \bar{\tau} \ln \bar{\tau}$.

4. **Q:** *On Page 12 (Page 11 of the revised manuscript), the authors discuss the SVD entropy fails to be a Schur-concave function and also point out that the (modified) pseudo entropy coincides with the SVD entropy when the transition matrix is Hermitian. I wonder: if the modified pseudo entropy is real but the transition matrix is non-Hermitian, i.e., eigenvalues are real but can be negative, can the authors numerically demonstrate the failure of Schur-concave function along the majorization path ?*
   A: Yes, we can numerically demonstrate the modified pseudo entropy is not Schur-concave even though some eigenvalues are negative. We give the simple example of a 2×2 model for PT-broken regime ($\mu \in [0, 0.5]$) in Sec. 6.1. The trace-normalized transition matrix $\hat{\tau}$ takes the following form,

$$\hat{\tau} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2\sqrt{1-4\mu^2}} & 0 \\ 0 & \frac{1}{2} - \frac{1}{2\sqrt{1-4\mu^2}} \end{pmatrix},$$

   which is Hermitian but includes negative eigenvalues. The corresponding state-normalized matrix $\tilde{\tau}\tilde{\tau}^\dagger$ takes

$$\tilde{\tau}\tilde{\tau}^\dagger = \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{1-4\mu^2}}{2(1-2\mu^2)} & 0 \\ 0 & \frac{1}{2} - \frac{\sqrt{1-4\mu^2}}{2(1-2\mu^2)} \end{pmatrix},$$

whose spectrum is denoted by $p[\mu]$. For $0 < \mu_1 < \mu_2 < 0.5$, $p[\mu_1] \succ p[\mu_2]$. So the majorization path is the one along which μ decreases. As μ decreases, ABB and SVD entropies decrease, while the modified pseudo entropy increases with μ decreasing, as shown in Fig. 12. Therefore, the modified pseudo entropy is not Schur-concave. We also have improved our discussion on Page 11 of the revised manuscript.

In addition, we add the condition "$\hat{\tau}$ is positive semi-definite" to the sentence "The (modified) pseudo entropy coincides with SVD entropy only when $\hat{\tau}$ is Hermitian." on Page 11.

5. **Q:** *For Figs.* 5, 7, 9, 10, *is there any heuristic(?) understanding of why the ABB and SVD entropy are asymmetric ?*

   A: Thanks for referee's meaningful comment. We now explain why the ensemble-averaged SVD and ABB entropy curves are asymmetric under the exchange $d_1 \leftrightarrow d_2$ observed in Figs. 5,7,9,10. Writing the bipartite states $|\psi_1\rangle_{ab}$ and $|\psi_2\rangle_{ab}$ with subsystem dimensions (d_1, d_2) in matrix form as $\psi \in \mathbb{C}^{d_1\times d_2}$ and $\phi \in \mathbb{C}^{d_1\times d_2}$, tracing out b yields

$$\tau_a = \psi\phi^\dagger \in \mathbb{C}^{d_1\times d_1},$$

while tracing out a gives

$$\tau_b = \left(\phi^\dagger\psi\right)^T \in \mathbb{C}^{d_2\times d_2}.$$

By the Sylvester determinant theorem, it can be verified that τ_a and τ_b share the same set of nonzero eigenvalues. Consequently, the pseudo entropy and modified pseudo entropy, both of which depend only on the spectrum of τ, are invariant under exchanging the subsystem dimensions. After ensemble averaging, this leads to

$$\overline{S_{\mathrm{P}}\left[\tau_a(d_1;d_2)\right]} = \overline{S_{\mathrm{P}}\left[\tau_a(d_2;d_1)\right]}, \quad \overline{S_{\mathrm{MP}}\left[\tau_a(d_1;d_2)\right]} = \overline{S_{\mathrm{MP}}\left[\tau_a(d_2;d_1)\right]}.$$

In contrast, the SVD and ABB entropies depend on the spectra of $\tau\tau^\dagger$ and related normalized operators. In general, $\tau_a\tau_a^\dagger$ and $\tau_b\tau_b^\dagger$ do not share the same nonzero spectra, leading to that $\bar{\tau}_a$ and $\bar{\tau}_b$, as well as $\tilde{\tau}_a\tilde{\tau}_a^\dagger$ and $\tilde{\tau}_b\tilde{\tau}_b^\dagger$ generally do not have the same spectra. As a result,

$$\overline{S_{\mathrm{ABB}}\left[\tau_a(d_1;d_2)\right]} \neq \overline{S_{\mathrm{ABB}}\left[\tau_a(d_2;d_1)\right]}, \quad \overline{S_{\mathrm{SVD}}\left[\tau_a(d_1;d_2)\right]} \neq \overline{S_{\mathrm{SVD}}\left[\tau_a(d_2;d_1)\right]},$$

which explains the asymmetry of ensemble-averaged ABB and SVD entropy observed in Fig. 5. The detailed proof can be seen in App. C. In addition, we have included the explanations of why the (modified) pseudo entropy are symmetric on Page 20 and why the SVD and ABB entropies are asymmetric on Page 21 for the case of two independent Haar random states. And also, we have added a statement on the asymmtry of the SVD and ABB entropies and symmetry of the (modified)pseudo entropy on Page 24 for the case of the Ginibre ensemble.

6. **Q:** *For Fig.* 12*, the SVD and ABB entropies are flat in the PT symmetric region. Why do these entropies not vary as a function of* μ *? Is* $\tau\tau^\dagger$ *independent of* μ *in the* PT*-symmetric region ?*
A: Yes. For the PT-symmetric region of the two-qubit system in Sec. 6.1, $\tau\tau^\dagger$ is independent of μ . More precisely, $\hat{\tau}$ has the form of $\mathrm{diag}(\frac{1}{2}e^{i\alpha}, \frac{1}{2}e^{-i\alpha})$. Consequently, $\bar{\tau} = \tilde{\tau}\tilde{\tau}^\dagger = \mathbb{I}/2$. Thus, the SVD and ABB entropies do not vary as a function of μ in the PT-symmetric regime for the two-qubit system.

7. **Q:** *For Fig.* 12*, in the PT-broken region* $\mu \in [0, 0.5]$*, do the real parts of pseudo and modified pseudo entropies coincide? The colors in Fig.* 12 *appear to overlap.*
A: Yes, the real parts of the pseudo and modified pseudo entropies coincide in the PT-broken region. To be more specific, in PT-broken region, the pseudo entropy takes complex values while the modified pseudo entropy takes real values, but the real part of the pseudo entropy coincides with the modified pseudo entropy. We add the clarification in the caption of Fig. 12.

8. **Q:** *In the non-Hermitian SYK example, the authors claim that the SVD and ABB entropies are sensitive to non-Hermitian chaos. Their reasoning is that* $q = 2$ *has smaller value of than* $q = 4$*. I don't think it is a strong evidence of sensitivity to non-Hermitian chaos. Am I missing something ?*
A: Thank the referee for pointing out that our argument was vague. Our evidence of non-Hermitian chaos is based on a parallel phenomenon in the subsystem entropies of eigenstates in Hermitian SYK models: the entropy in the chaotic SYK_4 takes the Page value given by Haar random states (the eigenstates of Hermitian random matrices) and the entropy in the free SYK_2 takes a lower value governed by the Wachter law for free fermions. We have clarified this argument on Page 26.

9. **Q:** *Comparing Figs. 7 and 9, the SVD and ABB entropies for both examples look almost the same. Are they the same?*
A: Their values are not exactly identical. Nevertheless, we are not surprised by their close agreement, since the nSYK_4 model exhibits behavior consistent with non-Hermitian chaos, and its eigenstate entanglement diagnostics are well described by non-Hermitian random matrix ensembles. We have clarified this point on Page 25.

10. **Q:** *All the examples show that the ABB entropy is bounded by SVD entropy. Is it accidental, or can one prove this inequality, ABB* $\leq$ *SVD?*
A: Yes, the inequality $S_{\mathrm{ABB}}[\tau] \leq S_{\mathrm{SVD}}[\tau]$ can be proved rigorously. Recall that

$$S_{\mathrm{ABB}}[\tau] = S_{\mathrm{von}}(p), \quad S_{\mathrm{SVD}}[\tau] = S_{\mathrm{von}}(q),$$

where $S_{\mathrm{von}}(p) = -\sum_i p_i \ln p_i$, and the two probability distributions are defined as

$$q_i = \frac{z_i}{\sum_{j=1}^d z_j}, \quad p_i = \frac{z_i^2}{\sum_{j=1}^d z_j^2}, \quad i = 1, \cdots, d,$$

with z_i given in Eq.(4). We can show that q is majorized by p, i.e., $q \prec p$. Writing both distributions in non-increasing order,

$$q_1 \geq q_2 \geq \cdots \geq q_d, \quad p_1 \geq p_2 \geq \cdots \geq p_d,$$

the majorization relation taks the form

$$\sum_{i=1}^{k} q_i \leq \sum_{i=1}^{k} p_i, \quad k = 1, \cdots, d.$$

Since the von Neumann entropy is Schur-concave, the majorization $q \prec p$ directly implies $S_{\mathrm{von}}(p) \leq S_{\mathrm{von}}(q)$, namely

$$S_{\mathrm{ABB}}[\tau] \leq S_{\mathrm{SVD}}[\tau],$$

with the equality holding iff all z_i are equal.
Now we give the detailed proof of $q \prec p$. Let $Z_1 = \sum_{i=1}^{d} z_i$, $Z_2 = \sum_{i=1}^{d} z_i^2$ and arrange z_i in non-increasing order as $z_1 \geq \cdots \geq z_d$. For any $k \in \{1, \cdots, d\}$, define

$$\Delta_k := \sum_{i=1}^{k} q_i - \sum_{i=1}^{k} p_i = \frac{1}{Z_1} \sum_{i=1}^{k} z_i - \frac{1}{Z_2} \sum_{i=1}^{k} z_i^2.$$

Multiplying $Z_1 Z_2$ by Δ_k yields

$$\begin{aligned} Z_1 Z_2 \Delta_k = Z_2 \sum_{i=1}^{k} z_i - Z_1 \sum_{i=1}^{k} z_i^2 &= \sum_{j=1}^{d} \sum_{i=1}^{k} \left(z_j^2 z_i - z_j z_i^2 \right) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{k} z_i z_j \left(z_j - z_i \right) + \sum_{i=1}^{k} \sum_{j=k+1}^{d} z_i z_j \left(z_j - z_i \right), \end{aligned}$$

where we split the sum over j into the parts of $j \leq k$ and $j > k$. The contribution of $j \leq k$ vanishes because the summand is anti symmetric under the exchange of i and j. Thus, only the term with $j > k$ survives:

$$Z_1 Z_2 \Delta_k = \sum_{i=1}^{k} \sum_{j=k+1}^{d} z_i z_j \left(z_j - z_i \right) \leq 0,$$

with the equality holding when $k = d$. Thus, $\Delta_k \leq 0$, i.e.,

$$\sum_{i=1}^{k} q_i \leq \sum_{i=1}^{k} p_i, \quad k = 1, \cdots, d.$$

In addition, we add this proof in App. A, and also add one statement about this general inequality below Eq. (19). We also stress that due to the majorization $q \prec p$ ensemble-averaged SVD entropy is greater than ABB entropy on Page 21 and 23.