**Evaluation of interaction part of the free-energy: cumulant expansion**

We find the cumulant expansion in the appendix A.2.1 (see (67)) becomes

$$
\begin{aligned}
&\ln\left\langle \prod_{\mu,\blacksquare,a} \exp\left[ i\eta_{\mu,\blacksquare,a} \sum_{i=1}^{N} \frac{(J^i_{\blacksquare})^a}{\sqrt{N}} (S^{\mu}_{\blacksquare})^a (S^{\mu}_{\blacksquare(i)})^a \right] \right\rangle_{J^i, S^{\mu}} \\
&= \ln\Bigg\langle 1 + \sum_{a}\sum_{\blacksquare} \frac{1}{\sqrt{N}} \sum_{i,\mu} i\eta_{\mu,\blacksquare,a} (J^i_{\blacksquare})^a (S^{\mu}_{\blacksquare})^a (S^{\mu}_{\blacksquare(i)})^a \\
&\qquad + \frac{1}{2!}\sum_{a,b}\sum_{\blacksquare,\blacksquare'} \frac{1}{N} \sum_{i,j,\mu,\nu} i\eta_{\mu,\blacksquare,a} i\eta_{\nu,\blacksquare',b} (J^i_{\blacksquare})^a (J^j_{\blacksquare'})^b (S^{\mu}_{\blacksquare})^a (S^{\nu}_{\blacksquare'})^b (S^{\mu}_{\blacksquare(i)})^a (S^{\nu}_{\blacksquare'(j)})^b + \dots \Bigg\rangle_{J^i, S^{\mu}} \\
&= \frac{1}{2!}\sum_{a,b}\sum_{\blacksquare,\blacksquare'} \frac{1}{N} \sum_{i,j,\mu,\nu} i\eta_{\mu,\blacksquare,a} i\eta_{\nu,\blacksquare',b} \delta_{\blacksquare,\blacksquare'} \delta_{ij} \delta_{\mu,\nu} Q_{ab,\blacksquare} q_{ab,\blacksquare} q_{ab,\blacksquare(i)} \\
&+ \frac{1}{4!} \sum_{a,b,c,d} \sum_{\blacksquare_1,\blacksquare_2,\blacksquare_3,\blacksquare_4} \frac{1}{N^2} \sum_{i,j,k,l,\mu_1,\mu_2,\mu_3,\mu_4} i\eta_{\mu_1,\blacksquare_1,a} i\eta_{\mu_2,\blacksquare_2,b} i\eta_{\mu_4,\blacksquare_3,c} i\eta_{\mu_4,\blacksquare_4,d} \delta_{\blacksquare_1,\blacksquare_2} \delta_{\blacksquare_1,\blacksquare_3} \delta_{\blacksquare_1,\blacksquare_4} \delta_{ij} \delta_{ik} \delta_{il} \\
&\times \delta_{\mu_1,\mu_2} \delta_{\mu_1,\mu_3} \delta_{\mu_1,\mu_4} \Big[ \langle (J^i_{\blacksquare})^a (J^i_{\blacksquare})^b (J^i_{\blacksquare})^c (J^i_{\blacksquare})^d \rangle_{J^i} \langle (S^{\mu}_{\blacksquare})^a (S^{\mu}_{\blacksquare})^b (S^{\mu}_{\blacksquare})^c (S^{\mu}_{\blacksquare})^d \rangle_{S^{\mu}} \langle (S^{\mu}_{\blacksquare(i)})^a (S^{\mu}_{\blacksquare(i)})^b (S^{\mu}_{\blacksquare(i)})^c (S^{\mu}_{\blacksquare(i)})^d \rangle_{S^{\mu}} \\
&- Q_{ab,\blacksquare} Q_{cd,\blacksquare} q_{ab,\blacksquare} q_{cd,\blacksquare} q_{ab,\blacksquare(i)} q_{cd,\blacksquare(i)} - Q_{ac,\blacksquare} Q_{bd,\blacksquare} q_{ac,\blacksquare} q_{bd,\blacksquare} q_{ac,\blacksquare(i)} q_{bd,\blacksquare(i)} - Q_{ad,\blacksquare} Q_{bc,\blacksquare} q_{ad,\blacksquare} q_{bc,\blacksquare} q_{ad,\blacksquare(i)} q_{bc,\blacksquare(i)} \Big] + \dots \\
&= \frac{1}{2} \sum_{a,b} \sum_{\blacksquare} \sum_{\mu} i\eta_{\mu,\blacksquare,a} i\eta_{\mu,\blacksquare,b} q_{ab,\blacksquare} Q_{ab,\blacksquare} \frac{1}{N} \sum_{i=1}^{N} q_{ab,\blacksquare(i)} \qquad (1)
\end{aligned}
$$

In the last equation we assumed $N \gg 1$ by which only the 2nd order term in the cumulant survives. For instance the 4th order term is smaller than the 2nd order term by a factor $O(1/N)$.