Reply to the Referee's reports "Curvature effects on phase transitions in chiral magnets"

First of all we thank the Editor for the efficient review process and the Referees for their time as well as constructive criticism aimed at improving our manuscript. Following the suggestions we have improved the paper. All revisions are indicated in the manuscript in blue color.

Sincerely yours,
Authors.

Reply to Referee #1

Comment 1:
In the conclusion, it would be good to go beyond the calculation results and explain in simple terms why the effect of curvature is stronger when competing with Neel type DMI. It would be good also to have some outlook. Indeed, the paper shows that large curvatures are required to get large effects. But when the curvature is too large, the hedgehog structure becomes unstable. Indeed, the κ^2 energy of the hedgehog state should be compared to that of the uniform (in 3D space) magnetization, which is 1/2 in the same units, discarding magnetostatic terms. This leads to $\kappa < 0.707$, close to the value 0.72 often considered in the figures. From this, I reckon that the graphs stop at $\kappa = 0.72$ because, above it, one goes to the uniform magnetization. This would thus be the largest possible curvature.

Answer 1:
The answer on this comment we would like to split into two parts:

- Following Referee's comment we added into the conclusions the explanation why the curvature induced effects is stronger in the case of the Néel type of DMI:

  > page 7 3rd sentence in conclusions
  >
  > The curvature effects are more pronounced for the case of Néel intrinsic DMI because the curvature-induced DMI is usually of the Néel type, thus a direct competition takes place. Note, that for the same reason the Néel skyrmions are more strongly affected by the curvature gradients as compared to the Bloch skyrmions [33] and the DMI-free skyrmions stabilized by curvature are of Néel type [45].

- Indeed, with the increase of the curvature the hedgehog state becomes unstable and the transition to a nonuniform state takes place. This transition is denoted by the blue solid line on the phase diagrams shown in Fig. 3. For small enough DMI, the nonuniform state is the two-domain state ($q = 1$) which can be thought of as an onion state of the tube. With the further curvature increase the two-domain state asymptotically approaches the uniform state with $\boldsymbol{m} = \boldsymbol{m}_0 \perp \hat{\boldsymbol{z}}$. This uniform state has energy 1/2 in the considered units. Note

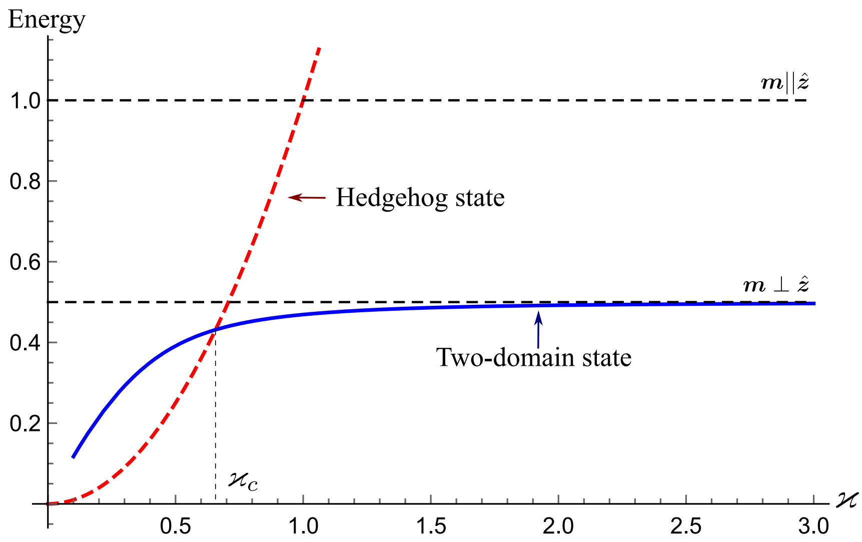

Figure R.1: Energies of different states of the tube for the case $d = 0$.

that the other uniform state with $\boldsymbol{m} = \boldsymbol{m}_0||\hat{\boldsymbol{z}}$ has higher energy equal to 1. The comparison of energies is shown in the Fig. R.1. As one can see there is only one critical curvature value $\varkappa_c \approx 0.657$ (for $d = 0$), which separates the hedgehog and two-domain states. This critical curvature was previously found in Ref. [53]. There are no other critical values. The fact that the phase diagrams in Fig. 3 are limited to $\varkappa \approx 0.72$ is just a necessary contingency (the plots must be terminated somewhere).

Comment 2:

The paper makes a large use of analytical calculations. So the formulas should be carefully checked. I found several mistakes in them, which costed me some time.

(a) in (B.1), line 2, the second term should have $\sin^2(\phi + \psi)$

(b) in (B.1), line 3, the cross product $\nabla\theta \times \epsilon$ should be transformed to a scalar by a dot product with the normal vector n, like in (A.5d)

(c) in (B.2), the first equalities for each line do not hold, as numerical factors are lacking. These play no role for the second equalities, as the right-hand side is zero. But they are important if the reader wants to rederive these formulas. So these factors ($1/2$ in front of $\mathrm{d}E/\mathrm{d}\theta$ and $\mathrm{d}E/\mathrm{d}\phi$, $1/(2\kappa)$ in front of $\mathrm{d}E/\mathrm{d}\psi$) should be restored.

(d) for (B.4), second line, same comment as for (B.1)

(e) for (B.5), same comment as for (B.2)

Answer 2:

We appreciate that the Referee has read our manuscript so carefully and we thank him/her for the spend time. Indeed, in all cases mentioned by the Referee we made misprints. Fortunately, these mistakes are just typos and they do not affect the subsequent calculations. We fixed the misprints in the manuscript.

Comment 3:

The paper uses sometimes the CGS system, sometimes the SI system. This forces to replicate the column of Table I (with a mistake there: 1 mJ/m^2 is equal to 1 erg/cm^2). I suggest to follow the (not so) modern practice, namely to use SI units throughout.

Answer 3:

Following the recommendation of the Referee we proceed to SI units. Note that the supplementals are mainly affected, since we use the dimensionless units for the main text.

Comment 4:

Check the English. Especially for the abstract. The "del operator" mentionned below (A.2) is not a standard term. Why not simply say "gradient" ? The word "whereas" in between (C.2d) and (C.2e) seems to stand for "whether". Ref. [46] should refer to Appendix D.

Answer 4:

We thanks the Referee for his comment. In the DMI energy we have a divergence, curl, and gradient. Therefore we used term "del operator". We also revised text accordingly.

Reply to Referee #2

Comment 1:

Do the authors study the possibility of the solution corresponding to the hedgehog state is a special case of a general solution given by Eq. (5)?

Answer 1:

No, formulas (5) and (6) describe the *nonuniform* multidomain state with $q \geq 1$. The *uniform* hedgehog state with $q = 0$ we consider separately. In principle, formula (5) can be used for the hedgehog state at the limit case $T \to \infty$ ($C \to 0$). However, we believe that such a generalization may confuse the reader.

Comment 2:

From the analysis of Fig. 3, one can state that there are regions in which Neel DMI and Bloch DMI coexist? If yes, it would be useful to include this discussion in the text.

Answer 2:

The DMI strength shown in Fig. 3 is the intrinsic DMI (not curvature induced). And in our MS we do not consider the joint action of intrinsic Bloch and Néel DMI. In order to prevent the possible confusion we denoted it in the caption of Fig. 3:

> page 5 caption of Fig. 3
>
> ... with Bloch and Néel type of intrinsic DMI ...

Comment 3:

There are some parts of the text that are confusing. For instance, the presentation of hedgehog and inhomogeneous solutions are presented without proper separation. This fact can bring some difficulties in the understanding of the results. I recommend the authors to perform a revision in the text to better present their results. For instance, there are some parameters that are not presented immediately after appearing in the equations, as the integration constant C.

Answer 3:

We thanks the referee for his comment. The homogeneous hedgehog state is trivial, therefore we did not consider it in a subsection. Following the Referee's recommendation we highlighted words "homogeneous" and "inhomogeneous" in the main text. We insert the following text before formula (5)

page 5 before Eq. (5)

... normal magnetization component of the inhomogeneous state ...

And we add the description of constant C:

page 5 after Eq. (5)

... C is an integration constant ...

Comment 4:
The text should be revised. There are some problems with English. For instance: "one obtains", "is reads", and others.

Answer 4:
We proofread the text.

Comment 5:
What do the authors mean with the "simultaneous action of DMI and curvature"?

Answer 5:
We reformulate it as follows:

page 6 before formula (7)

... for the case $\varkappa > 0$ and $d^2 > 0$ the DW width decreases as compared to the case $\varkappa = 0$ (planar film) or $d = 0$.

And as follows:

page 7 before formula (11)

... due to non-zero DMI and curvature the width of the well separated Néel DWs is increased. This behavior is opposite to the case of the Bloch DWs.

Comment 6:
I call the attention of the authors for some interesting results regarding curvature effects in nanomagnets with DMI. Some of them were developed by authors of this paper: Phys. Rev. B 102, 014432 (2020); https://doi.org/10.1038/s42005-020-0387-2; Phys. Rev. B 102, 024444 (2020); Nanotechnology 31, 125707 (2020); J. Appl. Phys. 108, 033917 (2010); https://doi.org/10.1038/s41598-019-45553-w; and others.

Answer 6:
We thank the Referee's for his comment. We added the corresponding citations in the introduction with Refs. [33, 34] and Ref. [23] into the discussion about magnetostatic interaction.