Summary of changes in the revised Manuscript:

“The authors provided a simple model to capture the essential physics of two strongly interacting flat bands for the cases that the bands have identical (spin-1/2 QH) or opposite (Z2) Chern number. By studying the model numerically with a cylinder geometry, it was found that the gap to the first excited states remains finite for the Z2 case and it is significantly shrunk for the spin-1/2 QH case. Although the obtained results are rather obvious, the used numerical approaches would be useful information for future studies of quantum Hall system. Furthermore, the discussion on the nature of the excited states, especially a connection to 1D Ising-type spin physics, is interesting. Therefore, the contents of this manuscript seem to be enough for the publication in SciPost.”

Our reply: *We thank the referee for recommending our Manuscript for publication in SciPost.*

“However, I think that the discussion about numerical results is insufficient and ambiguous descriptions are seen in the presentation. Hence, I cannot recommend the publication in the present form. Then, I would like to request the improvement of the manuscript with addressing the following questions/suggestions.”

Our reply: *We thank the referee for pointing out possible improvements for our Manuscript. We implemented the suggestions provided by the referee carefully and answered their questions in the following.*

“- Is the number of LLL orbitals counted like $N=LxLy/(2\pi)$? If yes, how can one take N=4,6,8, …, as shown in FIG.3 and FIG.4? Namely, what is the relation between N and Lx?”

Our reply: *N is the number of orbitals in the lowest Landau level, which is proportional to the size of the 1D discrete chain after dimensional reduction technique is applied to the 2D problem in continuum. Thus N could be any positive integer number. The size of the 2D system in the x-axis, namely Lx, can be calculated from $N=LxLy/(2\pi)$ for a given Ly. We briefly noted this in page 4 in Section III.*

“- What is the definition of gap? Is it $\Delta E=E1-E0$? Also, does the single-particle gap provide the same result in the Z2 case or not? It would be a useful information for DMRG calculation.”

Our reply: *The gap is defined as the energy difference between the many body ground and first excited states. We note this in the captions of Figs. 3 and 5 as well as the main text in Section IV. In our revised Manuscript, we study the comparison between the energies of the first excited states and the lowest-lying spin-flip state which is a product state. For small Ly circumference values, the first excited state is indeed a product state where the spin-flip excitation is observed at the edge of the 1D chain. As Ly increases, an entangled state at the edge of the 1D chain becomes energetically more favorable compared to the edge spin-flip product state. Fig. 4 in our revised Manuscript shows this change in the nature of the first excited states with respect to Ly for orbital numbers 6 to 8 performed with exact diagonalization. This observation is intuitive and can be explained via the localization of the Wannier wavefunctions. For a fixed magnetic length l_B, as Ly increases, the distance between the centers of the wavefunctions decreases, thus giving rise to more significant overlaps between nearest neighbor Wannier wavefunctions. As a result of this decrease in localization, the first excited states become extended at the edge of the 1D chain. In Section IV, we discuss this observation and give an example of an extended first excited state based on ED data.*

- It is unclear how the parameters ls, lB are chosen in FIG.3 and FIG.4. For example, the used choice of lB=1 corresponds to B=1. Does it fulfill the condition that the gap between Landau levels is much larger than the interaction strength?

Our reply: *l_B, the magnetic length is set to 1 for all computations in the Manuscript. This choice sets the scale for the rest of the parameters in the problem. One can always scale the interaction strength accordingly to fulfill the condition that higher Landau levels are ignored thanks to exact flatbands of Landau levels. Scaling the interaction strength, and thus the Hamiltonian, since the interaction term is the only term in the Hamiltonian, only scales the absolute energies of the states. In the revised Manuscript, we plotted the gap of the Z2 Hamiltonian for different l_s, the range of Coulomb interaction (Fig. 10 in Supplement D). We find that the gap shrinks with increasing ls, and consequently the gap crashes for the Z2 system with screening length ls=3 l_B, resulting in a ground state with charge accumulation at the edges and center. However the gap is still visible beyond ls=1l_B, as seen in data for ls=1.5. We conclude that while a stable gap -- between the fully polarized ground states and the edgelike excited states -- is persistent for different interaction ranges, the nature of the ground state completely changes after some interaction range to a more fluidlike state with charge accumulation at the edges from a uniform one particle per site incompressible state. Nonetheless, the fully polarized states remain eigenstates for ls=3l_B, albeit at a higher energy. For these reasons, it is likely the change in the nature of the ground state due to increasing the screening length is a consequence of the open boundary conditions. iDMRG or a tunable confining potential, the latter which could be tuned to increase the energy of the edge states, are potential ways to regulate this change. Investigation of whether there is indeed a critical screening length between two different phases, and how it responds to a confining potential, is beyond the scope of our paper. Thus we leave this as a future study.*

- In FIG.3, why the gap is larger for larger Ly, even though the gap between Landau levels typically depends only on B?

Our reply: *We thank the referee for this important question. The calculations produce an additional factor of Ly as given in Supplement A due to normalization of the wavefunctions. We replotted our data with the rescaling performed correctly. Figs. 3 shows the updated gap values for Z2 Hamiltonian for both contact and screened Coulomb interactions as a function of total orbital number N. In both cases, we observe that the gap approaches the same value for different Ly as N increases. The results for spin half Hamiltonian are also updated with the correct scaling, which can be seen in Figs. 5. Here the main conclusion does not change: the gap decreases as N increases for all Ly with a similar trend.*

“-Related to the above question, the data only for two circumferences may be not enough to confirm the saturation of the gap in the thermodynamic limit. If possible, it would be better to add one more data for another Ly even if it is smaller than Ly=15”

Our reply: *As suggested by the referee, we computed another two sets of Ly value, which is Ly=10 and Ly=8, for Z2 Hamiltonian with screened Coulomb interaction. All our data exhibits the same trend, suggesting a gap in thermodynamic limit. For spin half Hamiltonian we computed another three sets of Ly values. In addition to Ly=20 and Ly=15, now we also have Ly=8,10 and 12. From the consistent behavior observed from these five data sets, the gap significantly shrinks as the orbital number increases in the spin half QH system.*

“- In FIG.3, one may naively think that the gap for the short-range contact-like interaction is larger than that for the screened Coulomb interaction but the result is opposite. Is the limit ls→0 of Vsc quantitatively connected to Vδ?”

Our reply: *We thank the referee for this interesting question. Indeed the limit ls goes to zero of Vsc is quantitatively connected to Vdelta. We derived the relation between them in Supplement D under two assumptions, see Eq. 25. The two are equal to each other up to a coefficient which explains why the gap for contactlike interaction is smaller than the gap for screened Coulomb interaction with ls=1l_B. It should be understood from Fig 10 (supplement of updated Manuscript) that the energy cost of a “spin-flip” at the edge grows with shrinking screening length, and therefore the gap is largest in the limit ls→0.*

“- Is the caption of FIG.4 correct? What does "plotted as $(\Delta E)^{-1}$ mean?

-The horizontal axis of FIG.4 should start from 1/N=0”

Our reply: *We fixed these typos.*

If possible, it would be better to discuss the gap in the thermodynamic limit.

Our reply: *We extended the discussion of our results for both Hamiltonians in Section IV and discussed the presence or absence of a gap in the thermodynamic limit. The related discussion for Z2 and spin-half QH Hamiltonians are at pages 6 and 7, respectively.*

“- Which parameter value is used in FIG.5 and FIG.6?”

Our reply: *Fig. 6a and b in revised Manuscript are plotted for Z2 Hamiltonian with contact-like interaction. In the revised manuscript, we noted this both in the caption and the main text. Additionally in the revised manuscript, we looked for inverse participation ratio greater than 0.99, to ensure that we are counting only the product states.*

“- In FIG.5 and FIG. 6 the data for Ly=2 seems to deviate from the systematic size dependence. Is a periodic boundary indeed applied in the y-direction for Ly=2 namely, is the hopping taken twice?”

Our reply: *Periodic boundary conditions are taken around the cylinder. As discussed in the main text and previous replies, the localization of the LLL’s is controlled by the ratio Ly/l_B. Since we fix l_B=1, scaling the cylinder circumference has the effect of controlling the localization. In the limit of an infinitely thin cylinder, we find that the LLL’s are significantly separated, and consequently the entire spectrum (determined by using ED) becomes composed of product states, reminiscent of the Ising Hamiltonian with no external fields (classical limit). As Ly increases, so does the overlap of the LLL wavefunctions, and consequently states composed of entangled orbitals appear around Ly \sim 2\pi. This behavior was discussed in Section IV in detail for the first excited state, which is a product state until Ly\sim 2\pi, and then is replaced by an entangled state for larger Ly (Fig. 4). The product state which is the first excited state for Ly<2\pi, still remains an eigenstate independent of Ly, albeit at a higher energy. Thus the total number of product excited states grows with N regardless of Ly (Fig 6a) but their fraction compared to the number of entangled states shrinks with larger Ly (Fig 6b). Due to this reason, we do not expect that Ly=2 data should have a similar trend with L=8 data. For Ly=2, the increase in the product states as a function of N is faster compared to Ly=8, however the decrease in its r value (ratio of product states to the dimension of Hilbert space) is significantly slower than Ly=8. We noted this observation at page 8, Section V.*

- In FIG.7, it would be helpful for readers to put numbers in the color bar to estimate how strongly a spin is localized.

Our reply: *We have made the requested change. Please see the updated figure.*

- The accuracy of DMRG should be written, even if the discarded weight is negligible.

Our reply: *We have included the limits on the bond dimension and the cutoff for DMRG of Z2 and spin-half in the captions of Fig 3 and 5, respectively.*

Requested changes

1. The parameters and quantities should be clearly defined.
2. The reason why the used parameters were chosen should be clearly written.
2. More discussion about the interpretation of numerical results, especially for FIG.3 and FIG.4, should be added.
3. The presentation (including the caption) of some figures should be improved.

Our reply: *We defined and motivated all parameters as clearly as possible in the revised Manuscript. We expanded the DMRG results section (Sec. IV), in addition to updating our Figures that plot the gap from ground state to the first excited states in Z2 Hamiltonian with the correct Ly scaling. As the referee suggested, we discussed the gap in the thermodynamic limit for both models. We improved the presentation of our Manuscript and the captions of the figures as much as possible. We fixed typos and updated Supplement sections A and B; additionally added new Supplement sections C and D. We improved our understanding of the excited states in Z2 Hamiltonian. We thank the referee for their helpful comments which helped us to improve our Manuscript significantly.*