Gauged Sigma Models and Magnetic Skyrmions

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Abstract

We define a gauged non-linear sigma model for a 2-sphere valued field and a $su(2)$ connection on an arbitrary Riemann surface whose energy functional reduces to that for critically coupled magnetic skyrmions in the plane for a particular choice of the gauge field. We use the interplay of unitary and holomorphic structures to derive a general solution of the first order Bogomol'nyi equation of the model for any given connection. We illustrate this formula with examples, and point out applications to the study of impurities.

1 Introduction

In the recent paper [1], it was shown that a model for static magnetic skrymions with a particular choice of coupling constants, called critical in [1], can be solved explicitly by viewing it as a gauged non-linear sigma model with a fixed $su(2)$ connection. The purpose of this paper is to define and solve the relevant gauged non-linear sigma model in general geometric terms, and to discuss other applications.

Magnetic skyrmions are topological defects in planar magnetic materials and currently the subject of intensive study, see [2] for a seminal paper in this field and [3] for a review. They are mathematically described as topological solitons in the magnetisation field $\boldsymbol{n}$. The latter is a map from a surface Σ to the 2-sphere S^2. The energy expression for the magnetisation field typically involves a Heisenberg or Dirichlet term (quadratic in derivatives) the crucial Dzyaloshinskii-Moriya or DM term (linear in derivatives) [4, 5] and various potential terms (no derivatives). To be specific, consider a special case of the critical model analysed in [1], defined on $\Sigma = \mathbb{R}^2$ and involving one real constant κ. The energy functional is

$$E[\boldsymbol{n}] = \int_{\mathbb{R}^2} \frac{1}{2}(\nabla \boldsymbol{n})^2 + \kappa \boldsymbol{n} \cdot \nabla \times \boldsymbol{n} + \frac{\kappa^2}{2}(1-n_3)^2 \, \mathrm{d}x_1 \mathrm{d}x_2, \tag{1.1}$$

where n_3 is the third component of the unit vector $\boldsymbol{n}$. In order to analyse such a model, it is useful to clarify the mathematical structures which enter its definition.

In two dimensions, the Dirichlet energy expression only depends on a conformal structure of the domain Σ. However, the potential terms require an integration measure, so that one would expect the full model to depend on a metric structure on Σ. In any case, the energy expression makes use of the (standard) metric on the target 2-sphere.

The critically coupled model in [1] also depends on the metric on the target, but only requires a conformal structure on Σ, which is manifest in the formulation as a gauged sigma model. However, its solution in terms of holomorphic data makes essential use of the complex structure on the target, i.e. of the identification of the 2-sphere with the complex projective line $\mathbb{C}P^1$. This suggests a more general formulation of the model on a Riemann surface Σ and also a more general understanding of its solution in terms of the interplay between the metric and complex structure on the target 2-sphere.

In this paper, we give such a formulation and show that the resulting model can always, at least locally, be solved in terms of an $SL(2,\mathbb{C})$-valued map which relates holomorphic and unitary gauges for a 2-sphere bundle over Σ. A similar interplay between unitary and holomorphic structures is much studied in the context of self-dual connections on 4-manifolds [6, 7], but happily the 2-dimensional version which we study here is much simpler.

The natural generalisation of the critical model studied in [1] involves S^2-bundles over Riemann surfaces, and connections on such bundles which are compatible with holomorphic and unitary structures on them. We outline this setting here, but focus on the case $\Sigma = \mathbb{C}$. Our goal is to explain in what sense the critically coupled magnetic skyrmion model of [1] generalises to a study of complex structures on the total space a S^2-bundle over a Riemann surface, and to exhibit the resulting method of solution as clearly as possible in simple cases.

2 Gauged sigma models on a Riemann surface

2.1 Conventions

The gauged sigma model we want to consider can be defined on any Riemann surface Σ, i.e. on any one-dimensional complex manifold, with our without boundary. We will define the model using invariant notation, but for concrete and explicit expressions we use a local complex coordinate $z = x_1 + ix_2$. We also use the standard notation

$$dz = dx_1 + ix_2, \qquad \partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2). \tag{2.1}$$

The Hodge $\star$ operator on 1-forms is determined by the complex structure; in local coordinates it is

$$\star\, dz = dx_2 - idx_1 = -idz, \qquad \star d\bar{z} = dx_2 + idx_1 = id\bar{z}. \tag{2.2}$$

Note also that, for any two 1-forms α, β on Σ,

$$\star\star\alpha = -\alpha \quad \text{and} \quad \alpha \wedge \star\beta = \beta \wedge \star\alpha. \tag{2.3}$$

Consider now a principal $SU(2)$-bundle P over Σ with a connection, and the associated adjoint bundle as well as the unit 2-sphere bundle $P \times^{\mathrm{Ad}} S^2$ in the adjoint bundle. With think of the fibre S^2 as the round 2-sphere of radius 1 inside the Lie algebra $su(2)$, with $SU(2)$ acting in the adjoint representation. Locally, in an open set $U \subset \Sigma$, a section of $P \times^{\mathrm{Ad}} S^2$ is a map

$$n : U \to S^2 \subset su(2), \tag{2.4}$$

and the connection is given by a $su(2)$-valued 1-form A on U. The exterior covariant derivative of n and the curvature is given by the usual expressions

$$Dn = dn + [A, n], \quad F = dA + \frac{1}{2}[A, A]. \tag{2.5}$$

Gauge transformations are determined locally by functions $u : U \to SU(2)$ and take the form

$$n \to unu^{-1}, \qquad A \to uAu^{-1} + udu^{-1}. \tag{2.6}$$

In local coordinates on Σ, we expand

$$A = A_1 dx_1 + A_2 dx_2 = A_z dz + A_{\bar{z}} d\bar{z}, \tag{2.7}$$

where we defined $A_z = \frac{1}{2}(A_1 - iA_2)$, $A_{\bar z} = \frac{1}{2}(A_1 + iA_2)$, and similarly for the curvature

$$F = F_{12}\, dx_1 \wedge dx_2, \qquad F_{12} = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]. \tag{2.8}$$

We use $su(2)$ generators $t_a = -\frac{i}{2}\tau_a$, $a = 1, 2, 3$, where τ_a are the Pauli matrices, which are normalised so that $[t_1, t_2] = t_3$ + cycl., and also define raising and lowering operators

$$t_+ = t_1 + it_2, \quad t_- = t_1 - it_2. \tag{2.9}$$

These naturally lie in the complexified $su(2)$ Lie algebra, so in $sl(2, \mathbb{C})$. For the inner product of $m, n \in su(2)$ we use a re-scaled trace and write

$$(m, n) = -2\mathrm{tr}(mn), \tag{2.10}$$

as well as $|n|^2 = (n, n)$. Our normalisation is such that $(t_a, t_b) = \delta_{ab}$.

2.2 Energy and variational equations

The energy functional defining the gauged sigma model we want to consider is

$$E[A, n] = \frac{1}{2}\int_\Sigma (Dn, \wedge \star Dn) - \int_\Sigma (F, n), \tag{2.11}$$

or, in local coordinates and in terms of $D_i = \partial_i + [A_i, \cdot]$, $i = 1, 2$,

$$E[A, n] = \int_\Sigma \left(\frac{1}{2}|D_1 n|^2 + \frac{1}{2}|D_2 n|^2 - (F_{12}, n)\right) dx_1 \wedge dx_2. \tag{2.12}$$

Clearly, the covariant Dirichlet term depends on the complex structure (through $\star$), but the curvature term does not and is topological in that sense. The energy is manifestly invariant under gauge transformations (2.6).

As our notation indicates, we think of the energy as a functional of the connection A and the section n. Postponing a discussion of boundary terms we initially assume that the Riemann surface Σ has no boundary. Then variation with respect to the connection gives

$$\delta E = \int_\Sigma (Dn, \wedge \star [\delta A, n]) - (n, (d\delta A + [\delta A, A])) = \int_\Sigma (\delta A, \wedge \star [n, Dn]) - (\delta A, \wedge Dn). \tag{2.13}$$

Setting this to zero for all δA gives the Euler-Lagrange equation

$$\star [n, Dn] = Dn. \tag{2.14}$$

Using (2.2) this can also be written as

$$D_{\bar z} n = i[n, D_{\bar z} n]. \tag{2.15}$$

Since $[n, \cdot]$ is the complex structure in the cotangent space to S^2 at n, this is a holomorphicity condition. We will pursue the consequences of this fact in complex coordinates in Sect. 3.2.

The variation with respect to n, using $\delta n = [\epsilon, n]$ to preserve $|n| = 1$ and neglecting boundary terms gives

$$\delta E = \int_\Sigma (D\delta n, \wedge \star Dn) - (\delta n, F) = -\int_\Sigma (\delta n, (D \star Dn + F)) = -\int_\Sigma (\epsilon, [n, D \star Dn + F]). \tag{2.16}$$

Setting this to zero for all ϵ leads to the variational equation

$$[n, D \star Dn + F] = 0. \tag{2.17}$$

It is not difficult to check that the first order equation (2.14) actually implies the second order equation (2.17). Applying $\star$ to (2.14) and differentiating, we obtain

$$D \star Dn = -D[n, Dn] = -[Dn, Dn] - [n, D^2 n] = -[Dn, Dn] - [n, [F, n]], \tag{2.18}$$

where we suppressed the wedge product in the commutator of Lie algebra-valued 1-forms. Now we use that $[Dn, Dn]$ is in the direction of n to deduce

$$[n, D \star Dn] = -[n, [n[F, n]]] = -[n, F], \tag{2.19}$$

as claimed. The equation (2.14) is therefore the only equation we need to consider. We now show that it can also be interpreted as a Bogomol'nyi equation in this model.

2.3 The Bogomol'nyi equation

The logical dependence of the two variational equations can be understood better by noting that (2.14) can also be obtained via a Bogomol'nyi argument. To show this, we need 't Hooft's identity [8] relating the integrand for the degree,

$$4\pi \deg[n] = \int_\Sigma \frac{1}{2}(n, [dn, dn]), \tag{2.20}$$

to its covariant version:

$$\frac{1}{2}(n, [Dn, Dn]) = \frac{1}{2}(n, [dn, dn]) + (F, n) - d(A, n). \tag{2.21}$$

Now use (2.3) and the cyclical property of the triple product to note

$$((Dn - \star[n, Dn]), \wedge \star (Dn - \star[n, Dn])) = 2(Dn, \wedge \star Dn) - 2(n, [Dn, Dn]). \tag{2.22}$$

This allows us to write the energy as

$$E[A, n] = \frac{1}{4}\int_\Sigma ((Dn - \star[n, Dn]), \wedge \star (Dn - \star[n, Dn])) + \frac{1}{2}\int_\Sigma (n, [Dn, Dn]) - \int_\Sigma (F, n). \tag{2.23}$$

Combining this with the identity (2.21), we deduce

$$E[A, n] = \frac{1}{4}\int_\Sigma ((Dn - \star[n, Dn]), \wedge \star (Dn - \star[n, Dn])) + \frac{1}{2}\int_\Sigma (n, [dn, dn]) - \int_{\partial\Sigma} (A, n), \tag{2.24}$$

with the last term of course vanishing when Σ has no boundary. We conclude that the energy is bounded below by terms which only depend on topology (the degree of n) or on boundary behaviour (if there is a boundary). If both are kept fixed, the energy is minimised iff the first order equation (2.14) holds. The energy of such Bogomol'nyi configurations is

$$E_B[A, n] = \frac{1}{2}\int_\Sigma (n, [dn, dn]) - \int_{\partial\Sigma} (A, n). \tag{2.25}$$

The equation (2.14) is thus seen to be a Bogomol'nyi equation in the general sense of characterising minima of energy functionals subject to topological or boundary conditions [9]. Such equations usually imply the variational equations. This is the case here, too, but in a somewhat unusual way. The Bogomol'nyi equation of the model *is* the variational equation (2.14) with respect to the connection, and implies the second order variational equation with respect to the field n, as we already showed.

The Bogomol'nyi equation (2.14) clearly does not uniquely determine both the connection and the section n. It is easy to write down infinitely many solutions for the connection A for *any* given smooth section n in the form

$$A = an - (p[n, dn] + q \star dn) + r(\star[n, dn] + dn), \tag{2.26}$$

where a is a 1-form and p, q, r are real functions, with r arbitrary but p, q satisfying $p+q=1$. The choice $A=0$ is possible when $\star[n, dn] = dn$, and is then realised with $p = q = \frac{1}{2}$, and $a = 0, r = 0$. This is expected because in that case n satisfies the ungauged Bogomol'nyi equation.

Note that the 1-form $[n, dn]$ is naturally associated to n as the Levi-Civita connection on the plane bundle orthogonal to n inside the trivial bundle $\Sigma \times su(2)$, and that the remaining terms other than an are obtained from the Levi-Civita 1-form by applying the complex structures $\star$ on the domain or $[n, \cdot]$ on the target. Also note that gauge transformations $g = \exp(\alpha n)$ which fix n (so α is a function) determine gauge equivalent solutions for fixed n.

Alternatively, one can use transformations (2.6) to rotate n in a fixed direction (say t_3) in the Lie algebra on some open set $U \in \Sigma$. In this gauge, A is determined by the algebraic condition

$$\star [t_3, [A, t_3]] = [A, t_3]. \tag{2.27}$$

We will return to the physical interpretation of solving for A when n is given in the Conclusion. However, we will now focus on the opposite situation where A is a given background gauge field, and we solve (2.14) for n in this background.

2.4 Boundary terms

So far we have ignored boundary terms which arise in the derivation of the variational equations for the functional (2.11). When Σ does have a boundary $\partial\Sigma$, boundary terms will generally only vanish if we impose a suitable boundary condition. When Σ is an open set - such as $\mathbb{C}$, or the upper half-plane - we need to impose suitable fall-off conditions as we approach 'infinity'. In order to discuss these matters in any detail we would need to fix the surface Σ and the background gauge field A we want to consider. We will not do this here, but make some general observations.

The boundary term which arises in the variation (2.16) of (2.11) with respect to n is

$$\int_{\partial\Sigma} (\delta n, \star Dn). \tag{2.28}$$

If $\partial\Sigma$ is an actual boundary and we impose a Dirichlet boundary condition $n_{|\partial\Sigma} = n_\infty$ this term vanishes because we must require $\delta n = 0$ on the boundary. However, when $\Sigma = \mathbb{C}$, the requirements $\lim_{|z|\to\infty} n(z) = n_\infty$ and $\lim_{|z|\to\infty} \delta n(z) = 0$ are not sufficient to ensure the vanishing or even well-definedness of the integral (2.28). The terms which causes problems here is $(\delta n, [A, n])$ when A does not vanish in the limit $|z| \to \infty$. The situation can be improved by considering the modified energy functional

$$\tilde{E}[A, n] = \int_\Sigma \frac{1}{2}(Dn, \wedge \star Dn) - (F, n) + \int_{\partial\Sigma} (A, n). \tag{2.29}$$

The inclusion of the boundary term means that the modified energy is bounded below by the integral defining the degree, see (2.24). Now variation with respect to n gives

$$\delta\tilde{E} = -\int_\Sigma (\delta n, (D \star Dn + F)) + \int_{\partial\Sigma} (\delta n, (\star Dn + A)). \tag{2.30}$$

With the modified energy, the problematic term involving the connection A vanishes if we fix $n_{|\partial\Sigma} = n_\infty$ and require

$$[\star A_{|\partial\Sigma}, n_\infty] + A_{|\partial\Sigma} = 0. \tag{2.31}$$

This condition is consistent with the Bogomol'nyi equation in the constant gauge (2.27), and appears to be a natural boundary condition to impose in this model.

3 Solving the Bogomol'nyi equation

3.1 Holomorphic versus unitary structures

We will now show how to solve the Bogomol'nyi equation (2.14) for a given connection. The idea is to exploit the interplay between holomorphic and unitary structures on complex vector bundles over Σ, and the special properties of connections which are compatible with both. The underlying theory is covered, for example in [7] and also [6].

We consider only the setting which is relevant for our discussion, so look at holomorphic $\mathbb{C}^2$-bundles over Σ with a unitary structure (a Hermitian inner product on the fibres). Any such vector bundle, denoted E in the following, has an associated projective bundle; this is a holomorphic $\mathbb{C}P^1$-bundle and, with the unitary structure, will be identified with the S^2-bundle $P \times^{\mathrm{Ad}} S^2$ of Sect. 2.1. We use the standard notation of ∂ and $\bar{\partial}$ for the exterior derivative followed by projection onto differential forms of type $(1,0)$ and $(0,1)$ on Σ, so in our local coordinates and applied to functions f,

$$\partial f = \partial_z f dz, \qquad \bar{\partial} f = \partial_{\bar{z}} f d\bar{z}. \tag{3.1}$$

Now consider a connection on the vector bundle E. The associated covariant derivative $D = d + A$ can be split into

$$\partial_A = \partial + A_z dz, \qquad \bar{\partial}_A = \bar{\partial} + A_{\bar{z}} d\bar{z}, \tag{3.2}$$

where $A = A_z dz + A_{\bar{z}} d\bar{z}$ is simply the split (2.7) of the matrix-valued 1-form A into forms of type $(1,0)$ and $(0,1)$. Such a connection is called unitary if it preserves the Hermitian inner product on the fibres, and it is called compatible with the holomorphic structure of E if $\bar{\partial}_A \vec{w} = 0$ for every holomorphic section[1] $\vec{w}$ of E. If a connection is compatible with both structures then, in a unitary gauge (a local choice of an orthonormal basis of the fibre), the gauge potential has to satisfy the anti-Hermiticity condition

$$(A^u_{\bar{z}} d\bar{z})^\dagger = -A^u_z dz. \tag{3.3}$$

On the other hand, in a holomorphic gauge (a local choice of a holomorphic basis of the fibre), we have

$$\bar{\partial}_{A^h} = \bar{\partial}. \tag{3.4}$$

It is now straightforward to check that any connection which is compatible with both the unitary and the holomorphic structure must have curvature of type $(1,1)$. This follows by a short computation which is important for us and which we therefore spell out. The gauge change from the holomorphic to the unitary gauge must be via a locally defined map $g : U \subset \Sigma \to GL(2,\mathbb{C})$ satisfying

$$\bar{\partial}_{A^u} = \bar{\partial} + g\bar{\partial} g^{-1}. \tag{3.5}$$

But then the condition (3.3) implies an explicit formula for the gauge potential in the unitary gauge, valid in the open set $U \subset \Sigma$:

$$A^u = g\bar{\partial} g^{-1} + (g^{-1})^\dagger \partial g^\dagger. \tag{3.6}$$

[1]Sections are holomorphic if they are holomorphic as maps from Σ into the total space of the bundle E.

This shows in particular that if A^u is the gauge potential of an $SU(2)$ connection, then g has determinant 1 and is therefore $SL(2,\mathbb{C})$-valued. This expression for a gauge potential in two dimensions is also frequently used in the physics literature on planar $SU(2)$ Yang-Mills theory, see e.g. [10].

We can transform back, using g^{-1}, from the unitary to the holomorphic gauge to deduce the $(1,0)$ component and hence the entire gauge field in the holomorphic gauge as

$$A^h = g^{-1}(g^{-1})^\dagger \partial g^\dagger g + g^{-1}\partial g = h^{-1}\partial h, \tag{3.7}$$

where we defined $h = g^\dagger g$. The matrix h is manifestly Hermitian and positive definite, and defines Hermitian inner product on the fibre in the holomorphic gauge [7]. The curvature 2-form then comes out as

$$F = \bar{\partial}(h^{-1}\partial h), \tag{3.8}$$

which is manifestly of type $(1,1)$ (and will remain so after gauge transformations), as claimed.

One can also prove the converse result [6, 7]. If a complex vector bundle E over a complex manifold, with a unitary structure, has a connection which is unitary and has curvature of type $(1,1)$ then there is a unique holomorphic structure on E such that the connection has the forms (3.6) and (3.7) in the unitary and holomorphic gauge respectively.

In one complex dimension, any connection has curvature of type $(1,1)$ and it follows that the unitary connections on Σ which we considered in Sect. 2 define a complex structure on the total space of the S^2-bundle $P \times^{\mathrm{Ad}} S^2$ over Σ. Since we are interested in $SU(2)$ connections, they can always locally be expressed in the form (3.6) for $g : U \to SL(2,\mathbb{C})$. This is the result which will put to practical use in the next section. As a final preparation we recall the Iwasawa decomposition of $g \in SL(2,\mathbb{C})$ via

$$g = u\rho, \tag{3.9}$$

with $u \in SU(2)$ and ρ an upper-triangular matrix with unit determinant of the form

$$\rho = \begin{pmatrix} \lambda & c \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \qquad \lambda \in \mathbb{R}^+,\ c \in \mathbb{C}. \tag{3.10}$$

Since the unitary factor u acts as an overall unitary gauge transformation in (3.6) we can express any Hermitian gauge potential on Σ up to $SU(2)$ gauge transformation locally as

$$A = \rho\bar{\partial}\rho^{-1} + (\rho^{-1})^\dagger \partial \rho^\dagger, \tag{3.11}$$

where ρ is a matrix-valued function of the form (3.10).

3.2 Holomorphic structure of the gauged sigma model

In order to apply the theory of the previous section to the gauged sigma model of Sect. 2, we need a little more notation. We write vectors in $\mathbb{C}^2$ as

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{3.12}$$

and use the standard Hermitian product $\langle \vec{v}, \vec{w} \rangle = v_1\bar{w}_1 + v_2\bar{w}_2$ on $\mathbb{C}^2$. The Hopf projection maps the unit sphere S^3 in $\mathbb{C}^2$ to the unit sphere S^2 in the Lie algebra $su(2)$ via

$$\pi : S^3 \subset \mathbb{C}^2 \to S^2 \subset su(2), \quad \vec{w} \mapsto n = Wt_3W^{-1}, \qquad W = \begin{pmatrix} w_1 & -\bar{w}_2 \\ w_2 & \bar{w}_1 \end{pmatrix}, \tag{3.13}$$

or, with $n = n_1 t_1 + n_2 t_2 + n_3 t_3$,

$$n_1 + in_2 = 2w_2\bar{w}_1, \qquad n_3 = |w_1|^2 - |w_2|^2. \tag{3.14}$$

The standard action of $u \in SU(2)$ on $\mathbb{C}^2$,

$$u : \mathbb{C}^2 \to \mathbb{C}^2, \quad \vec{w} \mapsto u\vec{w}, \tag{3.15}$$

induces the adjoint action,

$$u : su(2) \to su(2), \qquad n \mapsto unu^{-1}, \tag{3.16}$$

which preserves the inner product (2.10) in $su(2)$.

We use the conventions of [1] to define a stereographic coordinate $w \in \mathbb{C} \cup \{\infty\}$ for the 2-sphere by projection from the south pole,

$$w = \mathrm{St}(n) = \frac{n_1 + in_2}{1 + n_3}, \tag{3.17}$$

with inverse

$$n_1 + in_2 = \frac{2w}{1 + |w|^2}, \quad n_3 = \frac{1 - |w|^2}{1 + |w|^2}. \tag{3.18}$$

One checks that the Hopf projection (3.13) followed by stereographic projection can now also be written as

$$\mathrm{St} \circ \pi : \vec{w} \mapsto w = \frac{w_2}{w_1}. \tag{3.19}$$

An element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}) \tag{3.20}$$

acts on $\vec{w} \in \mathbb{C}^2$ by ordinary matrix multiplication

$$g : \vec{w} \mapsto g\vec{w}, \tag{3.21}$$

and on our projective coordinate w by fractional linear transformation, which we write as

$$w \mapsto g[w] := \frac{c + dw}{a + bw}. \tag{3.22}$$

For $u \in SU(2) \subset SL(2, \mathbb{C})$, this action agrees with (3.16) when w and n are related via the stereographic map (3.17). However, non-unitary elements in $SL(2, \mathbb{C})$ act as conformal transformations which do not preserve the round metric induced by the embedding $S^2 \subset su(2)$.

To write the gauged sigma model in terms of the stereographic coordinate w, we note that our $sl(2, \mathbb{C})$ Lie algebra generators (2.9) are explicitly

$$t_+ = t_1 + it_2 = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \quad t_- = t_1 - it_2 = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}. \tag{3.23}$$

Writing their action on the projective coordinate w simply as juxtaposition, we have, for general $t \in sl(2, \mathbb{C})$,

$$tw = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \exp(\epsilon t)[w], \quad t \in sl(2, \mathbb{C}), \tag{3.24}$$

and compute

$$t_- w = -i, \quad t_3 w = iw, \quad t_+ w = iw^2. \tag{3.25}$$

Defining the Lie algebra components (as opposed to the 1-form components (2.7)) of the gauge potential A via

$$A = \frac{1}{2}(A_+ t_- + A_- t_+) + A_3 t_3, \tag{3.26}$$

and similarly for the curvature

$$F = \frac{1}{2}(F_+ t_- + F_- t_+) + F_3 t_3, \tag{3.27}$$

we can write the covariant derivative as

$$Dw = dw + Aw = dw - \frac{i}{2}A_+ + iA_3 w + \frac{i}{2}A_- w^2, \tag{3.28}$$

and have the identity

$$(A, n) = \frac{wF_- + \bar{w}F_+ + F_3(1 - |w|^2)}{1 + |w|^2}. \tag{3.29}$$

Using the standard expression for the Dirichlet term in terms of stereographic coordinates [9], the energy (2.11) of the gauged sigma model then takes the form

$$E[A, w] = \int_\Sigma 2\frac{Dw \wedge \star\overline{Dw}}{(1 + |w|^2)^2} - \int_\Sigma \frac{wF_- + \bar{w}F_+ + F_3(1 - |w|^2)}{1 + |w|^2}, \tag{3.30}$$

and the identity (2.21) reads

$$\begin{aligned} 2i\frac{Dw \wedge \overline{Dw}}{(1 + |w|^2)^2} = 2i\frac{dw \wedge \overline{dw}}{(1 + |w|^2)^2} &+ \frac{wF_- + \bar{w}F_+ + F_3(1 - |w|^2)}{1 + |w|^2} \\ &- d\left(\frac{wA_- + \bar{w}A_+ + A_3(1 - |w|^2)}{1 + |w|^2}\right). \end{aligned} \tag{3.31}$$

With

$$(Dw - i\star Dw) \wedge \star\overline{(Dw - i\star Dw)} = 2Dw \wedge \star\overline{Dw} - 2iDw \wedge \overline{Dw}, \tag{3.32}$$

the energy can be therefore written as

$$\begin{aligned} E[A, w] = \int_\Sigma \frac{(Dw - i\star Dw) \wedge \star\overline{(Dw - i\star Dw)}}{(1 + |w|^2)^2} &+ 2i\int_\Sigma \frac{dw \wedge \overline{dw}}{(1 + |w|^2)^2} \\ &- \int_{\partial\Sigma} \frac{wA_- + \bar{w}A_+ + A_3(1 - |w|^2)}{1 + |w|^2}. \end{aligned} \tag{3.33}$$

The second term is $4\pi\times$ the degree of w, and the last term is a boundary term. If both degree and boundary behaviour are kept fixed, minima of the energy are therefore determined by the equation

$$Dw = i\star Dw \Leftrightarrow D_{\bar{z}} w = 0, \tag{3.34}$$

where we used the basic properties (2.2) of the $\star$-operator on 1-forms. This is therefore the Bogomol'nyi equation (2.14) in stereographic coordinates, as can also be checked by explicitly changing coordinates according to (3.18) in (2.14).

A key feature of the equation (3.34), which was not obvious in the formulation (2.14), is its gauge invariance under the larger group of $SL(2, \mathbb{C})$-valued gauge transformation

$$A_{\bar{z}} \mapsto gA_{\bar{z}}g^{-1} + g\partial_{\bar{z}}g^{-1}, \qquad w \mapsto g[w], \tag{3.35}$$

where $g : U \subset \Sigma \to SL(2, \mathbb{C})$, and we used the notation (3.22) for fractional linear transformations.

3.3 A general solution and its applications to magnetic skyrmions

We can apply the insights of Sect. 3.1 to solving the Bogomol'nyi equation (3.34) for a given $su(2)$-connection A on the principal bundle P as follows. We consider the $\mathbb{C}^2$-bundle associated to P via (3.15). By the results of Sect. 3.1, the unitary connection defines a holomorphic structure on this bundle, and hence also on the associated projective $\mathbb{C}P^1$-bundle. Locally, we can go to a holomorphic gauge via an $SL(2,\mathbb{C})$ gauge transformation. In this gauge $\bar{\partial}_A = \bar{\partial}$, so that the Bogomol'nyi equation (3.34) can easily be solved.

Explicitly, this means that, for a given unitary connection A, we need to find a locally defined map

$$g : U \subset \Sigma \to SL(2,\mathbb{C}), \tag{3.36}$$

so that the anti-holomorphic component of A is

$$A_{\bar{z}} = g\partial_{\bar{z}}g^{-1}. \tag{3.37}$$

Then the Bogomol'nyi equation (3.34) becomes simply

$$\partial_{\bar{z}}w + g\partial_{\bar{z}}g^{-1}w = 0. \tag{3.38}$$

Using again our notation (3.22) for the action of g on w by fractional linear transformations, this means that $f = g^{-1}[w]$ is a holomorphic function. Thus we obtain the general solution, valid in some open set $U \subset \Sigma$, as

$$w = g[f], \quad \text{with} \quad f : U \to \mathbb{C}P^1 \quad \text{holomorphic.} \tag{3.39}$$

We illustrate this method by solving equations arising in the study of magnetic skyrmions at critical coupling on $\Sigma = \mathbb{C}$. In this case, both holomorphic and unitary gauges can be chosen globally and therefore we obtain a global solution of the form (3.39).

The key observation used in [1] is that the magnetic skyrmion energy functional at critical coupling can be written as a gauged linear sigma model with a given connection. More precisely, writing the energy functional of [1] (a slight generalisation of the motivating example (1.1) in the Introduction) in our notation as

$$E_S[n] = \int_{\mathbb{R}^2} \left(\frac{1}{2}|\partial_1 n|^2 + \frac{1}{2}|\partial_2 n|^2 + \kappa(n^\alpha, [t_1\partial_1 + t_2\partial_2, n^\alpha]) + \frac{\kappa^2}{2}(1-n_3)^2 \right) \mathrm{d}x_1 \wedge dx_2, \tag{3.40}$$

where κ is a real parameter, α an angular parameter and $n^\alpha = e^{\alpha t_3} n e^{-\alpha t_3}$, one checks that with

$$A_S = -\kappa e^{-\alpha t_3}(t_1 dx_1 + t_2 dx_2)e^{\alpha t_3} = -\frac{1}{2}\kappa \left(e^{i\alpha} d\bar{z}\, t_+ + e^{-i\alpha} dz\, t_- \right), \tag{3.41}$$

the energy expression (2.11) of the gauged sigma model equals the energy expression (3.40) for critically coupled magnetic skyrmions, i.e. we have

$$E[A_S, n] = E_S[n]. \tag{3.42}$$

To apply our method of solution, we note that with

$$\rho = \exp\left(\frac{\kappa}{2} e^{i\alpha} \bar{z} t_+ \right) = \begin{pmatrix} 1 & -\frac{i}{2}\kappa e^{i\alpha}\bar{z} \\ 0 & 1 \end{pmatrix} \tag{3.43}$$

the anti-holomorphic component of the gauge potential (3.41) can be written as

$$(A_S)_{\bar{z}} = \rho\partial_{\bar{z}}\rho^{-1}. \tag{3.44}$$

Thus, the general solution is given in terms of a holomorphic function $f : \mathbb{C} \to \mathbb{C}P^1$ as

$$w = \rho[f] = \frac{f}{1 - \frac{i}{2}\kappa e^{i\alpha}\bar{z}f}, \tag{3.45}$$

or, in terms of $v = 1/w$,

$$v = -\frac{i}{2}\kappa e^{i\alpha}\bar{z} + \frac{1}{f}, \tag{3.46}$$

which, after re-naming $f \to 1/f$, is the general solution found in [1].

The example is particularly simple since the $SL(2, \mathbb{C})$ gauge transformation which links the unitary and the holomorphic gauge is nilpotent. In the context of magnetic skrymions, one would normally expect the energy to be translation invariant in the plane. It is instructive to consider the most general gauge potential which yields such an energy:

$$A = A_1 dx_1 + A_2 dx_2, \tag{3.47}$$

where A_1, A_2 are Lie-algebra valued constants. Then

$$F = [A_1, A_2]\; dx_1 \wedge dx_2, \tag{3.48}$$

and

$$E[A, n)] = \int_{\mathbb{R}^2} \left(\frac{1}{2}|\partial_1 n|^2 + \frac{1}{2}|\partial_2 n|^2 - (n, [A_1\partial_1 + A_2\partial_2, n]) \right.$$
$$\left. + \frac{1}{2}|[A_1, n]|^2 + \frac{1}{2}|[A_2, n]|^2 - (n, [A_1, A_2]) \right) dx_1 \wedge dx_2. \tag{3.49}$$

Now we can solve

$$A_{\bar{z}} = \frac{1}{2}(A_1 - iA_2) = g\partial_{\bar{z}} g^{-1} \tag{3.50}$$

with

$$g = \exp(-\frac{1}{2}(A_1 - iA_2)\bar{z}), \tag{3.51}$$

and obtain the general solution via (3.39).

The explicit form of g as a 2×2 matrix can easily be calculated, but its general form is not particularly illuminating. However, note that $A_{\bar{z}}$ is nilpotent and thus leads to the simple exponentiation (3.43) if and only if

$$|A_1|^2 = |A_2|^2, \qquad (A_1, A_2) = 0, \tag{3.52}$$

i.e. if A_1 and A_2 form an, up to scale, orthonormal basis of $su(2)$. Up to an overall choice of the plane which A_1 and A_2 span, this is also the defining property of (3.41). In fact, one could use a constant $SU(2)$ gauge transformation to bring (3.47) with the orthonormality (3.52) assumed into the form (3.41), and could even use it to set $\alpha = 0$. This shows in particular that changing gauge in the gauged sigma model can make a physical difference in the interpretation as a model of magnetic skyrmions.

4 Conclusion

The non-linear sigma model defined by the energy functional (2.11) provides a framework for systematically studying generalisations of the critically coupled magnetic skyrmion theory considered and solved in [1]. The applications to magnetic skyrmions suggests that one should think of these models as providing a theory for the field n for a given connection A.

In this paper we have shown how to solve the resulting Bogomol'nyi equation (3.34) by exploiting the relation between holomorphic and unitary gauges. This leads to the explicit formula (3.39) for local solutions. In the simple applications considered here we assumed $\Sigma = \mathbb{C}$, in which case we obtain global solutions.

It would certainly be of interest to consider compact Riemann surfaces of any genus, and to study global properties of solutions to (3.39) there. This requires a choice of unitary connection. As we explained, such connections define complex structures on $\mathbb{C}^2$-bundles and hence on associated $\mathbb{C}P^1$-bundles over such Riemann surfaces, which provides a natural geometrical interpretation for any chosen connection.

It would also be of interest to consider Riemann surfaces with boundary, and to clarify the correct boundary terms in that case. We only briefly touched on the relevant issues in Sect. 2.4.

Even though we have assumed a given connection in our discussion, in some applications it may be natural to start with a given configuration n and then choose A so that n solves the Bogomol'nyi equation. We have indicated in equation (2.26) how this can be done, and one could easily repeat this discussion in the projective formulation (3.34) of the Bogomol'nyi equation. One context where this approach could be fruitful is in the study of impurities. As discussed in [11] they can be described by an equation rather similar to (3.34), but with a given impurity configuration instead of a gauge field. Our discussion shows how to construct a connection so that a given impurity solves the Bogomol'nyi equation. Solving this Bogomol'nyi equation then generates other configurations in the presence of the initial impurity.

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