The quasilocal degrees of freedom of Yang-Mills theory

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Abstract

Gauge theories possess nonlocal features that, in the presence of boundaries, inevitably lead to subtleties. In the $D+1$ formulation of Yang-Mills theories, we employ a generalized Helmholtz decomposition rooted in the functional geometry of the theory's configuration space to (i) identify the quasilocal radiative and pure-gauge/Coulombic components of the gauge and electric fields, and to (ii) fully characterize the properties of these components upon gluing of regions. The analysis is carried out at the level of the symplectic structure of the theory, i.e. for linear perturbations over arbitrary backgrounds.

Contents

1 Introduction and summary of the results

Physical degrees of freedom in gauge theories cannot be completely localized, since gauge-invariant quantities have a certain degree of nonlocality; the prototypical example being a Wilson line.

Here, we will address the problem of defining *quasilocal* degrees of freedom (quasilocal dof) in electromagnetism and Yang-Mills (YM) theories. By "quasilocal", we specifically mean "confined to a finite and bounded region", with a certain degree of nonlocality allowed *within* the region. To emphasize that specific meaning, we will call such properties *regional.*

In electromagnetism, or any Abelian YM theory, although the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ provides a local gauge-invariant observable, a canonical formulation unveils the underlying nonlocality. The components of $F_{\mu\nu}$ (i.e. the electric and magnetic fields) do not provide gauge-invariant *canonical* coordinates on field space: in 3 space dimensions, $\{E^i(x), B^j(y)\} = \epsilon^{ijk}\partial_k\delta(x, y)$ is not a canonical Poisson bracket and the presence of the derivative on the right-hand-side is the first sign of a nonlocal behavior.

From a canonical perspective, the constraint whose Poisson bracket generates gauge transformations, namely the Gauss constraint, is responsible for the non-local attributes of gauge theories. The Gauss constraint gives an *elliptic* equation which must be satisfied by initial data on a Cauchy surface Σ. In other words, the initial values of the fields cannot be freely specified throughout Σ. Ultimately, this is the source of both the nonlocality and the difficulty of identifying freely specifiable initial data—the "true" dof of the theory. The viewpoint often adopted in the literature is that such nonlocality also prevents the factorizability of gauge-invariant observables and of physical degrees of freedom across regions. [1–3]. In this paper, we will clarify these statements, identifying the different types of non-locality appropriate to different decompositions of the dof.

That is, we will address the definition of YM quasilocal dof in a perturbative setting around a background configuration. We refer to these perturbative dof as "perturbations" or, often, as "modes". These modes are identified with tangent vectors to the YM configuration space over a Cauchy hypersurface Σ. Such tangent vectors are the basic objects required by the study of symplectic geometry, as encoded in the symplectic form Ω.

Our approach seamlessly adapts to the treatment of bounded regions $R \subset \Sigma$ without ever requiring any restriction on the dof: *not even in the form of boundary conditions* ∂R. This feature makes our approach uniquely adaptable to the study of arbitrary fiducial subsystems and boundaries, with *foreseen* applications in e.g. entanglement entropy computations. Nonetheless, restrictive boundary conditions (see e.g. [4]) on the physical content can in principle be incorporated in the formalism by restricting the definition of the configuration space (but we will not analyze this possibility here; see also [5]).

To be clear, in this paper we do *not* impose boundary conditions, and—most importantly—the gauge freedom is never fixed, not even at the boundary. This is a central feature that fundamentally distinguishes our approach from other standard approaches to gauge theories in regions with either finite or asymptotic boundaries (e.g. [6–9]). Since we also restrain from introducing any additional dof at the boundary, our approach is more economical than the edge-mode approach [10–14]—of which it shares, and explains, various important features (to be discussed in due time).

This paper has two parts. Each answers one of two related, but distinct, questions. In the rest of this introduction, we will summarize our result.

1.1 Quasilocal dof

The first question concerns the *organization of the regional dof into conjugate symplectic pairs, on-shell of the Gauss constraint.* In particular, we will focus on a split of the dof into "physical", or "radiative", dof associated to the bulk of a region $R \subset \Sigma$, and more subtle "Coulombic" dof whose importance lies in a quasilocal contribution to Ω associated to the boundary of R.

The radiative/Coulombic split descends from a generalized Helmholtz decomposition applied to the modes of both the electric field and the gauge potential. The decomposition can be derived from the gauge-theoretic and geometric properties of the configuration space intrinsic to each region, and it will be non-locally determined within R. The meaning of this decomposition will be investigated at length in this paper and reviewed at the end of the introduction.

The radiative dof are freely specifiable within R. Conversely, the Coulombic dof prescribe the (nonlocal) coupling of the fields in R to the physics outside of R; they cannot be tuned or affected from within R; their existence is due to the gauge nature of YM and to the ensuing Gauss constraint.

The radiative contribution to Ω is physically intuitive: it features gauge invariant and (quasi-)locally measurable dof that encode the locally tuneable modes of the electric, magnetic, and matter fields. In particular, the radiative mode of the gauge potential is a generalization of the standard notion of a transverse photon: transverse photons are usually defined for infinite plane waves, but our generalization is also appropriate within finite and bounded regions. The conjugate to the radiative mode of the gauge potential is the radiative mode of the electric field, which does not enter the Gauss constraint and is functionally independent from the (value of the) electric flux f through ∂R. Indeed, the modes associated to the flux f appear entirely in the Coulombic contribution to Ω.

This Coulombic contribution appears in the symplectic structure by means of an integration by parts applied to the Gauss constraint, thus the *pure-boundary* nature of the Coulombic contribution to Ω.[1] This boundary contribution features the electric flux f as

[1] As for the matter sector of Ω, it can be itself decomposed into a pure-gauge part that feeds into the Gauss constraint—thus feeding into the Coulombic part of Ω,—and a "radiative" gauge-invariant part.

the conjugate variable to the pure-gauge mode of the gauge-potential A. The flux f and the regional charge density due to the matter sector completely determine the Coulombic component of the electric field in R. The pure gauge-mode of the gauge potential is named $\varpi(\delta A)$; it is a regional nonlocal function of the perturbation δA throughout R. The pure-gauge nature of $\varpi(\delta A)$ makes it non-physical, i.e. non-measurable. Nonetheless, its conjugate quantity, the electric flux f, *is* physical but ultimately encodes dof that are not directly accessible from within R. Indeed, f, as opposed to the regional matter charge density, *precisely summarizes those dof localized beyond the boundary of* R that nonlocally influence the regional physics by means of the Gauss law. We will extensively elaborate on this point in section 3.3 (see also 3.6.2 and 3.7 for the symplectic significance).

These characteristics of the Coulombic pair mathematically substantiate the intuition that gauge dof acquire significance in the (relational) coupling between subsystems [15,16]. The distinguished character of gauge dof at boundaries is made apparent through the physical role of f [15, 16]. (more on this in the second part of the paper).

The characteristics of the Coulombic pair also support the claim that, from the purely regional perspective of a putative quantum theory of the fields supported in R, the electric flux f must be superselected [2, 17, 18]: being pure gauge, its conjugate quantity does not correspond to any physical observable accessible in the quasilocal theory. In the next section, 1.2, we will discuss how this picture is modified if knowledge of radiative fields in *both* R and its complement is assumed.

It is important to emphasize that, *in this first part of the paper, complete ignorance of the field configuration outside of* R *is assumed.* E.g. the only available information about the field configuration beyond ∂R is that imprinted on f. This is important, because in the second part of the paper, we will address the topic of "gluing", and there the knowledge of the fields in R *and in its complementary regions within* Σ will be assumed.

1.2 Gluing the quasilocal dof

Consider a simply connected, boundary-less, Cauchy hypersurface Σ that has been subdivided into two regions,[2] R^+ and R^- such that $\Sigma = R^+ \cup_S R^-$. Each region can individually be treated according to the precepts of the previous section. Now, we are interested in their "gluing". Here, gluing refers to the composition of two regional perturbations supported on R^+ and R^- into a global smooth perturbation supported on Σ. Specifically, we will ask the question: given such a bipartite system Σ, which regional degrees of freedom from R^+ and R^- suffice to reconstruct the global physics? That is, *do the radiative degrees of freedom from both regions* R^+ *and* R^- *suffice to encode all the gauge-invariant modes in* Σ*? The answer will be "yes"*, and an explicit reconstruction formula will be provided. This formula prescribes in particular the unique way to reconstruct the regional Coulombic components of the electric field solely from the *mismatch* of its regional radiative components at the interface S.

This reconstruction formula is remarkable since it shows that, given full knowledge of both regional radiatives, regional modes can be glued into a global description unambiguously, and without reference to any further (boundary, Coulombic) dof. The superfluous status of the Coulombic information in this context is especially surprising because, as described above, from an intrinsically regional perspective the Coulombic information is instead an integral piece of information.

The fact that *no* extra boundary information is needed for gluing YM fields contradicts previous claims from the literature which took such modes as physically necessary to

[2] In general terms, the case where the union of $R^\pm$ has a boundary can also be treated similarly by keeping into account the flux f through $\overline{S} = \partial(R^+ \cup_S R^-)$. However, the case where S and $\overline{S}$ intersect at a "corner" can be subtle. We comment on this below.

perform the gluing of the regions [2, 18–20] (see e.g. section 2.5 in [10] for an example). Nonetheless, various features of our approach parallel formal aspects of those works. This will be discussed in section 3.6. Looser parallels with the BV-BFV formalism [21–23] are also suggestive, see e.g. section 4.2, but will not be discussed in this publication.

As a simple corollary to our gluing formula, we will show that the global radiative sector always contains more energy than the disjoint union of its regional counterparts (the exact converse holding for the Coulombic sector): a conversion, or "promotion", of Coulombic to radiative dof takes place upon gluing. This conversion is reflected in the gluing properties of symplectic potentials: the global symplectic potential differs from the sum of the regional radiative symplectic potentials by an (involved) term that depends solely on the mismatch of the regional radiative field components at the interface S. Such properties can be traced back to the fact that new, global, (and nonlocal) dof are seeded by the mismatch of the regional radiative field components at the interface S, i.e. by information which by its very nature is not part of either regional configuration space.

The uniqueness result for the reconstruction of the global Coulombic modes comes with two physically interesting exceptions. The two exceptions concern: (i) a non-simply-connected Σ, and (ii) reducible background configurations in the presence of charged matter.

We start with the first, which is easily understood: global degrees of freedom supported by the first homology of Σ cannot be regionally detected. That is, if the gluing produces new noncontractible cycles on Σ that were not present in either region, $R^{\pm}$, then the reconstruction formula is ambiguous; technically, this happens because the reconstruction requires the inversion of an operator with a nontrivial kernel (essentially a Laplace-Beltrami operator). Physically, this kernel encodes global Aharonov-Bohm phases associated to the newly produced cycles. The conjugate momenta to these phases are imprinted in the electric flux, f. As a consequence, in the topologically nontrivial case, f cannot be fully reconstructed from the knowledge of the regional radiatives, and can acquire a finite number of topologically sourced components. Whereas most of our study of gluing focuses on the topologically trivial case, $\Sigma \cong \mathbb{R}^D$, we will nonetheless explicitly work out the simplest topologically nontrivial example of two intervals glued into a circle.

To understand the second exception, we need to introduce the notions of stabilizers and charges, which requires a slight detour.

The term "stabilizer" stands for those (infinitesimal) gauge parameters χ that leave the background field configuration invariant,[3] $\delta_\chi A = 0 = \delta_\chi E$. Such χ's are also called *reducibility parameters* or "*stabilizers*" of the conifiguration (see e.g. [24]), and are the YM analogue of Killing vector fields in general relativity. Being reducible—i.e. admitting a nontrivial reducibility parameter—is a gauge-invariant property of a field configuration, and the number of reducibility parameters is imprinted in the geometrical properties of field space. Any configuration can have at most $\dim(G)$ reducibility parameters, with G the charge group of the theory in question.

In non-Abelian YM only certain configurations are reducible while in electromagnetism all configurations admit one and only one reducibility parameter (up to an inconsequential normalization), $\chi_{\rm EM} = const$. To each reducibility parameter is associated a conserved charge which generalizes the total regional electric charge, $Q[\chi] = \int_R \mathrm{Tr}(\chi\rho)$, where ρ is the Lie(G)-valued charge density of the matter fields.[4] Symplectically, these charges generate a symmetry flow in the matter field sector, $\psi \mapsto e^{-t\chi}\psi$, while leaving the YM

[3] Even if stabilizers are defined by action on gauge-dependent configurations, χ's are gauge covariant, and therefore their existence is still a gauge-invariant notion: if A has stabilizer χ, then A^g will have $\mathrm{Ad}_{g^{-1}}\chi$ as a stabilizer.

[4] In the context of enlarged asymptotic symmetry groups [7], the association of charges solely to global reducibility parameters may appear too strict; a resolution of this puzzle was proposed in [5].

fields invariant by construction. In the presence of boundaries, the charges $Q[\chi]$ stand out as the only non-vanishing gauge-flow generators of the radiative sector. This is actually best interpreted as a signature of the physical, i.e. non-gauge, nature of the reducibility parameters [24–27].

The point of this detour is that the existence of such reducibility parameters also gives rise to a new ambiguity in the gluing formula since they are in the kernel of the covariant Laplace operator: each reducibility parameter $\chi^\pm$ in either region $R^\pm$ introduces a 1-parameter ambiguity in the reconstruction of the regional "pure gauge" adjustments for the YM field entering the gluing formula. However, this ambiguity is consequential to gluing *only* in the presence of charged matter. That is, if matter is not present at all, the glued fields will be insensitive to this particular ambiguity, simply because the $\chi^\pm$ leave the YM fields in $R^\pm$ unaffected. If matter is not present at the gluing interface but does not vanish in the bulk of the regions, genuine gluing ambiguities arise.

In other words, as a consequence of the stabilizer ambiguity and of the enabling of regional charges, on symmetric backgrounds with matter there exists a continuum of global configurations that are regionally indistinguishable but globally distinct in a way that is detectable by appropriate global observables, e.g. Wilson lines connecting charge particles across S.

1.3 Configuration space geometry

So far we have summarized the physical content of our results on the characterization of the quasilocal degrees of freedom in YM. However, this summary would be incomplete if we did not spell out the significance of the tools used to arrive at these results. We have mentioned that we make use of a generalized Helmholtz decomposition of the YM modes. It is important to appreciate that this decomposition is not arbitrarily chosen, but has deep roots in the gauge-thereotical and metric properties of the geometry of configuration space [25, 26].

To start with, the fields' gauge orbits fibrate the configuration space $\mathcal{A}$ of YM theory, turning it into a (generalized) infinite dimensional fibre bundle [27–36]. On this bundle one can introduce a connection form $\varpi \in \Lambda^1(\mathcal{A}, \mathrm{Lie}(\mathcal{G}))$, i.e. a gauge covariant object which is a 1-form on $\mathcal{A}$ taking value in the Lie algebra $\mathrm{Lie}(\mathcal{G})$ of the infinite dimensional group of gauge transformations $\mathcal{G} = \{g : \Sigma \to G\}$.

Physically, the role of this connection is to decompose in a gauge-covariant way the tangent space $\mathrm{T}\mathcal{A}$ into "vertical" (or pure-gauge) directions tangent to the gauge orbits, and "horizontal" (or physical) directions transverse to them. In particular, the connection acts on vectors in $\mathrm{T}\mathcal{A}$ as a projector onto their pure-gauge part and, consequently, also extracts the linearly independent horizontal, or physical, or "gauge-fixed", part. Interpreting tangent vectors as field perturbations, we see that the pure-gauge part of a perturbation δA, denoted above as $\varpi(\delta A)$, is precisely given by the contraction of the vector δA with the connection form ϖ. One can thus understand horizontal directions as given by a perturbative gauge-fixing, but one which is fully covariant with respect to choices of the base point and which does not impose any restrictions on the boundary's physical content.

However, as it is well known, fibre bundles do not carry a canonical connection form, which means no canonical horizontal/vertical decomposition of $\mathrm{T}\mathcal{A}$ is given. Therefore, to uniquely select a connection form on $\mathcal{A}$, more structure is needed beyond its fibration by gauge orbits. On the configuration space of YM, this extra structure is provided by the kinematical (super-)metric $\mathbb{G}$ [27–34].

The name of this metric comes from the fact that it features in the Lagrangian: it is implicitly employed in the kinetic term of YM theory when interpreted as the squared norm of the velocity vector $\dot{\mathbb{A}} \in \mathrm{T}\mathcal{A}$ understood as the tangent vector to a certain history

$A(t) \subset \mathcal{A}$. This appearance of $\mathbb{G}$ in the kinetic term of the Lagrangian also means that $\mathbb{G}$ features in the conversion of velocities into canonical momenta and thus enters the expression of the (pre)symplectic potential and (pre)symplectic form.

Now, compatibility between the gauge-theoretic (fibration) and metric structures of $\mathcal{A}$ singles out the unique connection form whose horizontal planes are *orthogonal* to the fibres [27–34]. In [26], we named this connection form the Singer-DeWitt (SdW) connection and generalized its definition to the case where $\mathcal{A}$ is the space of fields on a region with boundaries.

It is the SdW connection that provides us—through the corresponding vertical/horizontal split of $\mathrm{T}\mathcal{A}$—the appropriate generalization of the Helmholtz decomposition to non-Abelian fields and bounded domains; a generalization crucially employed to obtain the results described above. In particular, the SdW connection's origin in the compatibility of the gauge-theoretic and metric structures of $\mathcal{A}$ explains why the associated decomposition is *uniquely* suited to address the interplay between the gauge and the symplectic structures of YM in finite regions.

We emphasize that this abstract construction of a functional connection form on configuration space is more than a curious mathematical superstructure: it is a concrete computational tool. As it often happens, the geometrization of a problem makes it conceptually clear and thus easier to manipulate, and, ultimately, altogether more treatable. In the present case, it is a given that the introduction of ϖ makes the gauge covariance of the generalized Helmholtz decomposition completely manifest and easy to manipulate. But it does more than that: it also allows us to streamline and algebraically treat computations which would otherwise require manipulations of nonlocal expressions, Green functions, and nontrivial boundary conditions.

Lastly, ϖ elucidates [26] the role that certain geometrical features of field-space (e.g. its SdW curvature) implicitly played in interesting previous constructions in gauge theory (e.g. the definition of dressed matter fields and the geometric quantum effective action). As a justification of these relations, we present in appendix D a novel and independent construction of ϖ through the analysis (in the non-Abelian and finite-region setting) of Dirac's notion of the dressed electron field.

1.4 Broad roadmap

We devote the beginning of the paper to a review of the geometry of the field space of YM theory and of the SdW connection, section 2. However, in contrast with our previous work, which was done in the covariant Euclidean context [25, 26, 37], we specialize our discussion to configuration space and the $D+1$ decomposition. Next, in section 3, we will discuss in great detail the symplectic geometry of YM theory and show how the SdW connection is uniquely suited to answer the questions dealt with in this paper. This section contains our first main result: a full characterization of the quasilocal dof of YM theory in the presence of boundaries. The second main topic of this paper, gluing, is instead the subject of section 4. In section 5 we summarize and conclude.

To ease the reading of the paper, we tried to organize the material in such a way that more conceptual discussions of the results are separated from their technical derivations.

2 Field space geometric preliminaries

2.1 Horizontal splittings in configuration space

To start, we introduce notation and recall some basic facts.

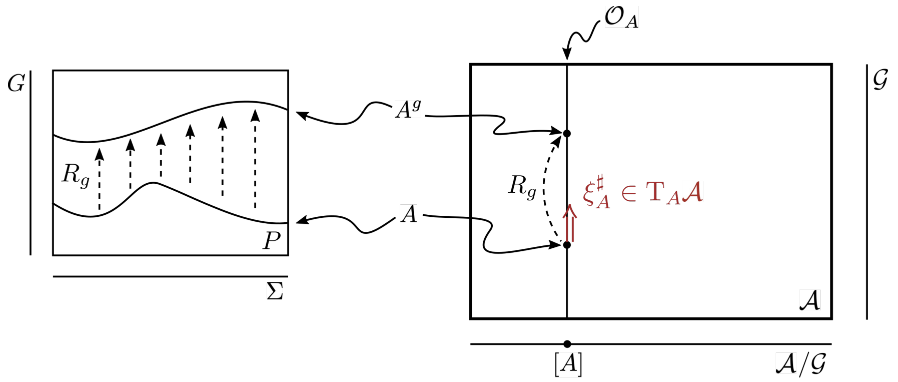

Figure 1: A pictorial representation of the configuration space $\mathcal{A}$ seen as a principal fibre bundle, on the right. We have highlighted a generic configuration A, its (gauge-transformed) image under the action of $R_g : A \mapsto A^g$, and its orbit $\mathcal{O}_A \cong \mathcal{G}$. We have also represented the quotient space of 'gauge-invariant configurations' $\mathcal{A}/\mathcal{G}$. On the left hand side of the picture, we have "zoomed into" a representation of A and A^g as two gauge-related local sections of a connection ω on P, the finite dimensional principal fibre bundle with structural group G over Σ. The principal fibre bundle picture of $\mathcal{A}$ will be partially revisited in section 3.8—see figure 3.

Consider a Lagrangian $D+1$ formulation of YM theory on a globally hyperbolic spacetime M foliated by equal-time Cauchy surfaces $\Sigma_t \cong \Sigma$.

To distinguish issues of global (topological) nature—which we will discard with the exception of section 4.7—from those associated with finite boundaries—which constitute our main focus,—we assume $\Sigma \cong \mathbb{R}^D$. This choice is made for mere convenience and will play no role in the following where our focus will be on compact subregions $R \subset \Sigma$.

Denote the corresponding YM configuration space $\mathcal{A}$ (see figure figure 1). This is the space of Lie-algebra valued one-forms on Σ,[5]

$$A \in \mathcal{A} := \Lambda^1(\Sigma, \mathrm{Lie}(G)). \tag{1}$$

Notice that the component of A in the transverse direction to Σ, A_0, is left out of the description—for now. We will reintroduce it in due time.

The group G is assumed to be compact and semisimple and will be referred to as the *charge group* of the theory. In specific applications, we will have $G = \mathrm{SU}(N)$ in mind. We write, $A = A_i \mathrm{d}x^i = A_i^\alpha \mathrm{d}x^i \tau_\alpha$, where $\{\tau_a\}$ is a basis of generators of $\mathrm{Lie}(G)$.

The space of gauge transformations $\mathcal{G} := \mathcal{C}^\infty_o(\Sigma, G) \ni g$, i.e. the space of smooth G-valued functions on Σ, inherits a group structure from G via pointwise multiplication. Call $\mathcal{G}$ the *gauge group*. The gauge transformation $g : \Sigma \to G$ acts on the gauge potential's configuration A as

$$A_i \mapsto A_i^g = g^{-1} A_i g + g^{-1}\partial_i g. \tag{2}$$

This defines an action of $\mathcal{G}$ on $\mathcal{A}$. This action induces a fiducial principal fibre bundle structure on configuration space, $\pi : \mathcal{A} \to \mathcal{P}$. The orbits of this action, $\mathcal{O}_A$, are called gauge orbits and their space $\mathcal{P} \cong \mathcal{A}/\mathcal{G}$ is the space of physical configurations. This is

[5] Rigorously speaking, dealing with a non-compact Cauchy surface would require us to consider only fields that vanish fast enough at infinity. However, our focus on compact region will make this restriction virtually irrelevant in the following. Therefore, we do not concern ourselves with a precise determination of the fall off rates and hereafter neglect them completely. For an application of our formalism where asymptotic conditions at null infinity are carefully treated, see [5].

the "true", or "reduced", or "gauge invariant", configuration space of the theory, but it is only defined abstractly through an equivalence relation, and is most often inaccessible for practical purposes.

An infinitesimal gauge transformation $\xi \in \mathrm{Lie}(\mathcal{G})$ defines a vector field tangent to the gauge orbits. This is denoted by $\xi^\sharp \in \mathrm{T}\mathcal{A}$, and its value at A is

$$\xi^\sharp_A = \int_\Sigma \mathrm{d}^D x \; (\mathrm{D}_i\xi)^\alpha(x) \frac{\delta}{\delta A_i^\alpha(x)} \in \mathrm{T}_A\mathcal{A}, \tag{3}$$

where $\mathrm{D}_i\xi := \partial_i\xi + [A_i, \xi]$ is the gauge-covariant derivative. The span of the $\xi^\sharp$'s defines the *vertical* subspace of $\mathrm{T}\mathcal{A}$, i.e. $V := \mathrm{Span}(\xi^\sharp) \subset \mathrm{T}\mathcal{A}$. V comprises the "pure gauge directions" in $\mathcal{A}$: *vertical changes are "pure gauge" and carry no physical information.*

The "physical" directions in $\mathrm{T}\mathcal{A}$ are therefore those transverse to V, i.e. the *horizontal directions* $H \subset \mathrm{T}\mathcal{A}$. The decomposition $\mathrm{T}\mathcal{A} = V \oplus H$ is not canonically defined by the fibre bundle structure. The *choice* of any such decomposition that is compatible with the gauge structure of $\mathcal{A}$ is in one-to-one correspondence with the choice of a connection form ϖ on the bundle $\pi : \mathcal{A} \to \mathcal{P}$, and vice-versa.

A connection form ϖ is a (functional) 1-form on $\mathcal{A}$ valued in the Lie algebra of the gauge group,

$$\varpi \in \Lambda^1(\mathcal{A}, \mathrm{Lie}(\mathcal{G})), \tag{4}$$

characterized by the following two properties (see [26,38]):

$$\begin{cases} \mathbb{i}_{\xi^\sharp}\varpi = \xi, \\ \mathbb{L}_{\xi^\sharp}\varpi = [\varpi, \xi] + \mathbb{d}\xi. \end{cases} \tag{5}$$

Hereafter, double-struck symbols refer to geometrical objects and operations in configuration space: $\mathbb{d}$ is the (formal) field-space de Rham differential,[6] $\mathbb{i}$ is the inclusion operator of field-space vectors into field-space forms, and $\mathbb{L}_{\mathbb{X}}$ is the field-space Lie derivative along the vector field $\mathbb{X} \in \mathfrak{X}^1(\mathcal{A})$. Its action on field-space forms is given by Cartan's formula, $\mathbb{L}_{\mathbb{X}} = \mathbb{i}_{\mathbb{X}}\mathbb{d} + \mathbb{d}\mathbb{i}_{\mathbb{X}}$.

The first of the properties (5), the projection property, means that ϖ defines a horizontal complement H to the fixed vertical space V, via

$$H := \ker \varpi. \tag{6}$$

The second of the properties (5) ensures the compatibility of the above definition with the group action of $\mathcal{G}$ on $\mathcal{A}$, i.e. it embodies covariance under gauge transformations. See figure 2 for a pictorial representation.

The term $\mathbb{d}\xi$ on the right hand side of the covariance condition is only present if ξ is chosen differently at different points of $\mathcal{A}$, i.e. if ξ is an infinitesimal *field-dependent* gauge transformation.[7] Geometrically, the field-dependent gauge transformations correspond to generic vertical vector fields on $\mathcal{A}$, generalizing the rigid vertical elements corresponding to the action of the (field-independent) Lie algebra. Therefore, this generalization does not require any extra geometrical information. Nonetheless, it provides a refined diagnostic tool in the presence of boundaries (in their absence, generalizing to field-dependent gauge transformations changes nothing). More on this point in section 3.7.

[6] We prefer this notation to the more common δ, because the latter is often used to indicate vectors as well as forms, hence creating possible confusions.

[7] The form of this equation can be deduced from the standard transformation property, $\mathbb{L}_{\xi^\sharp}\varpi = [\varpi, \xi]$, for ξ's constant throughout $\mathcal{A}$ (i.e. for ξ's that are field independent) and the projection property of ϖ which holds pointwise in field-space. See [26].

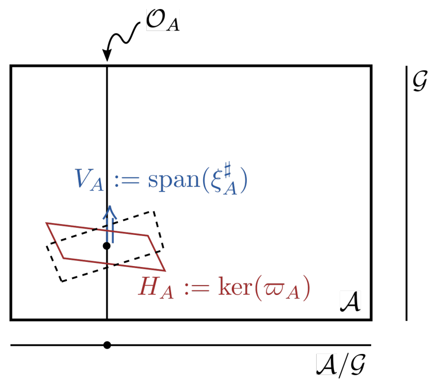

Figure 2: A pictorial representation of the split of $\mathrm{T}_A\mathcal{A}$ into a vertical subspace V_A spanned by $\{\xi^\sharp_A, \xi \in \mathrm{Lie}(\mathcal{G})\}$ and its horizontal complement H_A defined as the kernel at A of a functional connection ϖ. With dotted lines, we represent a different choice of horizontal complement associated to a different choice of ϖ.

Given a connection form characterized by (5), alongside $\mathbb{d}$ we can introduce the horizontal differential, $\mathbb{d}_H$ [25, 26, 37]. Horizontal differentials are by definition transverse to the vertical, pure gauge, directions: $\mathbb{i}_{\xi^\sharp}\mathbb{d}_H A = 0$.

Horizontal differentials can be loosely understood as "ϖ-covariant" differentials on field space. E.g. the horizontal differentials of A_i is

$$\mathbb{d}_H A_i = \mathbb{d}A_i - \mathrm{D}_i\varpi. \tag{7}$$

The covariance property of this horizontal differential under any (possibly field-dependent) gauge transformation is easy to show; it reads

$$\mathbb{L}_{\xi^\sharp}\mathbb{d}_H A_i = [\xi, \mathbb{d}_H A_i]. \tag{8}$$

For details and generalizations to differentials of equivariant field-space forms of general degree, see [37].

We conclude this section by introducing for future reference the curvature $\mathbb{F}$ of a functional connection ϖ [26, 29]:

$$\mathbb{F} := \mathbb{d}_H\varpi \equiv \mathbb{d}\varpi + \tfrac{1}{2}[\varpi \curlywedge \varpi] \in \Lambda^2(\mathcal{A}, \mathrm{Lie}(\mathcal{G})). \tag{9}$$

The curly wedge $\curlywedge$ denotes the wedge product in $\Lambda^\bullet(\mathcal{A})$, where $\bullet$ stands in for arbitrary degrees. By definition, $\mathbb{F}$ is purely horizontal, i.e. $\mathbb{i}_{\xi^\sharp}\mathbb{F} \equiv 0$.

2.2 Horizontality and the electric field

In this section we introduce the electric field into the picture.

In temporal gauge ($A_0 \equiv 0$) the electric field is nothing more than the velocity[8] of A, $E_i = \dot{A}_i$. Now, $\dot{A}_i$ is geometrically best understood as the tangent vector $\dot{\mathbb{A}} \in \mathrm{T}\mathcal{A}$ to a

[8] More generally, $E_i := n^\mu F_{\mu i}$, where n^μ is the unit timelike normal to Σ. The appearance of n^μ forces us to consider the extrinsic geometry of our foliation, i.e. how Σ is embedded in spacetime. Unless stated otherwiese, all our formulae we will hold when Σ belongs to an Eulerian foliation of spacetime, i.e. to a foliation whose lapse is equal to one and whose shift vanishes. In other words, Σ is an equal-time hypersurface in a spacetime with metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{ij}(t,x)\mathrm{d}x^i\mathrm{d}x^j$. The inclusion of nontrivial lapse and shift is in principle straightforward, but makes some formulae more cluttered, and most likely wouldn't add much to our considerations here. However, we point the reader to [5] for a situation where the introduction of a nontrivial shift plays a crucial role in dealing with asymptotic gauge transformations and charges.

curve (history) $A(t) \subset \mathcal{A}$, i.e.

$$\dot{\mathbb{A}}_A := \int_\Sigma \mathrm{d}^D x \, \dot{A}_i^\alpha(x) \frac{\delta}{\delta A_i^\alpha(x)} \in \mathrm{T}_A\mathcal{A}. \tag{10}$$

Irrespectively of temporal gauge, we thus associate E_i to a configuration-space vector

$$\mathbb{E} := \int_\Sigma \mathrm{d}^D x \, E_i^\alpha(x) \frac{\delta}{\delta A_i^\alpha(x)} \in \mathrm{T}\mathcal{A} \tag{11}$$

We will now study the properties of $\mathbb{E}$ in relation to a choice of horizontal/vertical split of $\mathrm{T}\mathcal{A}$. We start by proving a simple lemma that follows from the fact that the field-space scalar coordinate functions E_i must transform covariantly, i.e. $\mathbb{L}_{\xi^\sharp} E_i \equiv \xi^\sharp(E_i) = [E_i, \xi]$. The lemma states that the vector field $\mathbb{E}$ is constant along the flow of $\xi^\sharp$ if ξ is field independent, and otherwise changes along that flow by the vertical vector field[9] $(\mathbb{E}(-\xi))^\sharp$. Here is the proof:

$$\begin{aligned}\mathbb{L}_{\xi^\sharp}\mathbb{E} \equiv [\![\xi^\sharp, \mathbb{E}]\!]_{\mathrm{T}\mathcal{A}} &= \int \Big(\xi^\sharp(E_i) - \mathbb{E}(\partial_i \xi - [A_i, \xi])\Big) \frac{\delta}{\delta A} \\ &= \int \Big([E_i, \xi] - [E_i, \xi] - \mathrm{D}_i \mathbb{E}(\xi)\Big) \frac{\delta}{\delta A} = -(\mathbb{E}(\xi))^\sharp, \end{aligned} \tag{12}$$

with $[\![\cdot, \cdot]\!]_{\mathrm{T}\mathcal{A}}$ the Lie-bracket between configuration-space vectors. We will use this lemma in a moment.

In the last section, we saw that any vector on $\mathrm{T}\mathcal{A}$ is decomposed by ϖ into its horizontal and vertical components. In the case of $\mathbb{E}$, we denote its vertical component as:[10]

$$\varphi := \varpi(\mathbb{E}) \in \mathrm{Lie}(\mathcal{G}). \tag{13}$$

The leftover horizontal component of $\mathbb{E}$ can be written as the horizontal projection of any $\dot{A}$ associated to E_i. That is, in coordinates, if $E_i = \dot{A}_i + \mathrm{D}_i A_0$, then:

$$\mathbb{E} := \dot{\mathbb{A}}^H + \varphi^\sharp, \tag{14}$$

for

$$\dot{\mathbb{A}}^H := \dot{\mathbb{A}} - \varpi^\sharp(\dot{\mathbb{A}}) = \int_\Sigma \mathrm{d}^D x \, \underbrace{\Big(\dot{A}_i^\alpha - \mathrm{D}_i \varpi^\alpha(\dot{\mathbb{A}})\Big)}_{=: \dot{A}_i^H} \frac{\delta}{\delta A_i^\alpha}. \tag{15}$$

These formulae identify $\varphi = \varpi(\dot{\mathbb{A}}) - A_0$.

Now, thanks to the lemma (12) and the defining properties of ϖ given in (5), it is easy to see that $\varphi := \varpi(\mathbb{E})$ transforms in the adjoint representation under a gauge transformation ξ:

$$\delta_\xi \varphi \equiv \mathbb{L}_{\xi^\sharp}\varphi = [\varphi, \xi]. \tag{16}$$

Indeed,

$$\mathbb{L}_{\xi^\sharp}\varphi = \mathbb{L}_{\xi^\sharp}\varpi(\mathbb{E}) = \Big([\varpi, \xi] + \mathbb{d}\xi\Big)(\mathbb{E}) + \varpi([\![\xi^\sharp, \mathbb{E}]\!]) = \Big([\varphi, \xi] + \mathbb{E}(\xi)\Big) - \varpi(\mathbb{E}(\xi)^\sharp) = [\varphi, \xi], \tag{17}$$

It follows from (14) and (12) that $\dot{A}_i^H$ also transforms in the adjoint representation,

$$\delta_\xi \dot{A}_i^H \equiv \mathbb{L}_{\xi^\sharp}\dot{A}_i^H = [\dot{A}_i^H, \xi], \tag{18}$$

[9]Here, and in the following, $\mathbb{X}(f) = \int X_i \frac{\delta f}{\delta A_i}$, for any function f on $\mathcal{A}$ and any a vector field $\mathbb{X}$ in $\mathrm{T}\mathcal{A}$.

[10]Here, and in the following, $\varpi(\mathbb{X}) \equiv (\mathbb{i}_\mathbb{X}\varpi)$ and $\varpi^\sharp(\mathbb{X}) \equiv (\mathbb{i}_\mathbb{X}\varpi)^\sharp$ for any $\mathbb{X} \in \mathrm{T}\mathcal{A}$.

which can also be seen as a consequence of $\varpi(\dot{\mathbb{A}})$ having the same transformation properties of A_0, i.e.

$$\delta_\xi \varpi(\dot{\mathbb{A}}) \equiv \mathbb{L}_{\xi^\sharp}\varpi(\dot{\mathbb{A}}) = \dot{\xi} + [\varpi(\dot{\mathbb{A}}), \xi]. \tag{19}$$

This concludes the derivation of the transformation properties of E and its components.

In our framework, it is actually possible to reverse-engineer the transformation properties of E_i from those of φ and A_i. By using only the defining properties of ϖ and techniques similar to those employed in the proof of (12), it is possible to explicitly prove the transformation property (19), independently of any of the above considerations (see appendix A). Then, from (19) and the postulated covariance of φ, we find that $E_i = \dot{A}^H_i - \mathrm{D}_i\varphi$ transforms covariantly and can be recast in its usual form $E_i = \dot{A}_i + \mathrm{D}_i A_0$ as soon as A_0 is identified with the combination

$$A_0 = -\varphi + \varpi(\dot{\mathbb{A}}). \tag{20}$$

Note that, thanks to the geometrical nature of our constructions, these statements hold for field-dependent gauge transformations ($\mathbb{d}\xi \neq 0$) as well—see appendix A.

Summarizing, we start from the configuration space $\mathcal{A} = \Lambda^1(\Sigma, \mathrm{Lie}(\mathcal{G}))$ and define its tangent bundle $\pi : \Phi \to \mathcal{A}$,

$$\Phi := \mathrm{T}\mathcal{A}, \tag{21}$$

which we coordinitize with (A, E). We identify unphysical directions with vertical directions in $\mathrm{T}\Phi$, which, at the base point (A, E), take the form $(\mathrm{D}\xi, [E, \xi])$.

On $\mathrm{T}\Phi$, vertical vectors are along pure-gauge directions, whereas horizontal vectors with respect to the pulled-back connection form $\pi^*\varpi$ are along physical directions, encoding a variation between neighbouring gauge orbits. Notice that now the identification of pure-gauge with verticality is done in $\mathrm{T}\Phi = \mathrm{T}(\mathrm{T}\mathcal{A})$, i.e. the phase space that is the arena for symplectic geometry.[11]

In other words, Φ also has a fibre bundle structure, and the pullback $\pi^*\varpi$ defines a connection form[12] on it. This pulled-back connection allows us to define the horizontal differential of E:

$$\mathbb{d}_H E = \mathbb{d}E - [E, \pi^*\varpi] \in \Lambda^1(\Phi) \tag{22}$$

However, using ϖ, we can also further decompose $\mathbb{E} = \int E \frac{\delta}{\delta A} \in \Phi$ as an element of the phase space $\Phi = \mathrm{T}\mathcal{A}$ into its horizontal and vertical components, $\dot{\mathbb{A}}^H$ and $\varphi^\sharp$ respectively. Since this decomposition happens in Φ, and not $\mathrm{T}\Phi$, it has to be understood as a generalized Helmholtz decomposition into the gradient-free and pure-gradient components of E, rather than as a gauge versus physical split. Indeed, in electromagnetism, both components of E are equally physical.

Now, this decomposition of E induces a finer decomposition of the *coordinates* of Φ:

$$(A_i, \dot{A}^H_i, \varphi) \in \Phi. \tag{23}$$

[11] The natural arena for the symplectic geometry of finite dimensional systems is the cotangent bundle of configuration space. Here, we will formally work in the tangent bundle of configuration space equipped with a supermetric that allows the translation of velocities (tangent vector) into momenta (cotangent vectors). I.e. we will work with the pullback of the canonical symplectic structure by the supermetric seen as a map from the tangent to the cotangent of configuration space. This treatment is standard in any second-order Lagrangian theory and will be elaborated on in section 3.2.

[12] This statement is true *generically*: i.e.: in an everywhere dense set (according to the standard Inverse Limit Hilbert (ILH) manifold metric structure on field-space [35, 36], see also [39]) on Φ. But it fails at some specific configurations that we will analyze in section 3.8. Until then, these exceptions will be ignored. But on this topic we should also point out that being generic is here a kinematical statement: dynamics might still favor topologically meagre sets.

With this coordinatization of Φ, gauge transformations read

$$\begin{cases} A_i \mapsto g^{-1}A_i g + g^{-1}\partial_i g \\ \dot A_i^H \mapsto g^{-1}\dot A_i^H g \\ \varphi \mapsto g^{-1}\varphi g \end{cases} \qquad \text{or, infintesimally,} \qquad \begin{cases} A_i \mapsto A_i + \mathrm{D}_i\xi \\ \dot A_i^H \mapsto \dot A_i^H + [\dot A_i^H, \xi] \\ \varphi \mapsto \varphi + [\varphi, \xi] \end{cases} \tag{24}$$

The infinitesimal transformations of (24) give us the vertical, pure-gauge, directions of $\mathrm{T}\Phi$ in the new coordinates: i.e. at the base point $(A_i, \dot A_i^H, \varphi)$ they are of the form $(\mathrm{D}_i\xi, [\dot A_i^H, \xi], [\varphi, \xi])$.

Let us emphasize once again that the identification of "vertical" with "unphysical" does *not* apply to the above coordinate-decomposition of $\mathbb{E}$: $\mathbb{E}$ is seen as a point of the phase space Φ and does not constitute a *direction* in $\mathrm{T}\Phi$; only these directions get decomposed into their physical and pure-gauge components by the pulled-back connection $\pi^*\varpi$. In particular, both $\dot A_i^H$ and φ encode physical information. Indeed, it is one of the goals of this first part of the paper to provide a definite physical *significance* to these two components. This goal will be accomplished in the next section, following the introduction of a particular choice of connection form according to which the horizontal/vertical splits are performed.

In the following, we will commit an abuse of notation and denote the pullback connection form on Φ simply by ϖ.

We conclude with one important remark: all the constructions of this section can be performed at the quasilocal level, by formally replacing Σ with any compact subregion R, with $\partial R \neq \emptyset$.

Since our interest lies mostly in bounded regions, we now transition our discussion to focus mostly on such subregions, R. As announced in the introduction, motivated by the study of *fiducial* subregions of Σ, we will assume *no* boundary condition at ∂R, not even in the allowed gauge freedom.

3 Quasilocal degrees of freedom

3.1 The symplectic potential

The Lagrangian $L = K - V$ of $(D+1)$-dimensional YM on $\Sigma \times I$ has kinetic term[13]

$$K = \tfrac{1}{2}\int_\Sigma \mathrm{d}^D x \sqrt{g}\, g^{ij}\,\mathrm{Tr}(E_i E_j). \tag{25}$$

and potential $(F_{ij} = 2\partial_{[i}A_{j]} + [A_i, A_j])$

$$V = \tfrac{1}{4}\int_\Sigma \mathrm{d}^D x \sqrt{g}\, g^{ii'} g^{jj'}\,\mathrm{Tr}(F_{ij}F_{i'j'}). \tag{26}$$

Variation of the YM action $S = \int \mathrm{d}t L$, gives

$$\delta S = \int_I \mathrm{d}t \int_\Sigma \mathrm{d}^D x \sqrt{g}\,\mathrm{Tr}\Big(g^{ij}(\mathrm{D}^k F_{ik} - \mathrm{D}_t E_i)\delta A_j + (\mathrm{D}^i E_i)\delta A_0\Big) + \partial_t \mathrm{Tr}\Big(\vartheta(\delta A)\Big), \tag{27}$$

where we recognize the YM equations of motion, the Gauss law, as well as the evaluation of the (pre)symplectic density on Σ, that is $\vartheta = \sqrt{g}g^{ij}\mathrm{Tr}(E_i \mathbb{d} A_j)$—here understood as a functional 1-form,—on the perturbation δA (here understood as a functional vector).

[13] The trace is normalized so that $\mathrm{Tr}(\tau_\alpha\tau_\beta) = \delta_{\alpha\beta}$ in the fundamental representation W of G.

The symplectic density ϑ encodes the gauge-variant dof of YM theory at each point of Σ. Since we are interested in the quasilocal properties of the YM dof, we will henceforth focus on the integral of ϑ throughout a region $R \subset \Sigma$. This integral gives us $\theta = \int_R \vartheta$, the symplectic-potential 1-form on Φ (the phase-space of YM theory on R):[14]

$$\theta = \int_R \mathrm{d}^D x \sqrt{g}\, g^{ij} \mathrm{Tr}(E_i \mathbb{d} A_j) \in \Lambda^1(\Phi). \tag{28}$$

As any 1-form on Φ, θ can be decomposed into its horizontal and vertical components [25, 26, 37]:

$$\theta = \theta^H + \theta^V \quad \text{with} \quad \begin{cases} \theta^H := \int \sqrt{g}\, g^{ij} \mathrm{Tr}(E_i \mathbb{d}_H A_j) \\ \theta^V := \int \sqrt{g}\, g^{ij} \mathrm{Tr}(E_i \mathrm{D}_j \varpi) \end{cases} \tag{29}$$

Now, it is natural to ask which components of the horizontal/vertical split $E = \dot{A}^H + \mathrm{D}\varphi$ enter θ^H and θ^V respectively.

3.2 Kinetic supermetric and the Singer-DeWitt connection

To clarify this question, we turn to the interpretation of E as a vector field on the configuration space $\mathcal{A}$. This interpretation prompts us to write the kinetic term K as the squared norm of $\mathbb{E}$,

$$K = \tfrac{1}{2} \mathbb{G}(\mathbb{E}, \mathbb{E}), \tag{30}$$

with respect to the kinetic supermetric $\mathbb{G}$ on $\mathcal{A}$ defined by

$$\mathbb{G}(\mathbb{X}, \mathbb{Y}) := \int_R \mathrm{d}^D x \sqrt{g}\, g^{ij} \mathrm{Tr}(\mathbb{X}_i \mathbb{Y}_j) \qquad \forall \mathbb{X}, \mathbb{Y} \in \mathrm{T}\mathcal{A}. \tag{31}$$

With this notation, and interpreting the kinetic supermetric as a map $\mathbb{G} : \Phi = \mathrm{T}\mathcal{A} \to \mathrm{T}^*\mathcal{A}$, the symplectic potential reads[15]

$$\theta = \mathbb{G}(\mathbb{E}) \equiv \mathbb{G}(\mathbb{E}, \mathbb{d}A), \tag{32}$$

with the following expressions for its horizontal and vertical components as a one-form in $\mathrm{T}^*\mathcal{A}$:

$$\theta^H = \mathbb{G}(\mathbb{E}, \mathbb{d}_H A) \quad \text{and} \quad \theta^V = \mathbb{G}(\mathbb{E}, \varpi^\sharp). \tag{33}$$

Hence, using the horizontal/vertical (coordinate-)split of $\mathbb{E}$,

$$\theta^H = \mathbb{G}(\dot{\mathbb{A}}^H + \varphi^\sharp, \mathbb{d}_H A) \quad \text{and} \quad \theta^V = \mathbb{G}(\dot{\mathbb{A}}^H + \varphi^\sharp, \varpi^\sharp). \tag{34}$$

Note that the "horizontal coordinates" are not paired solely with the horizontal one-forms, and mutatis mutandis for the vertical parts. However, fixing the choice of connection form ϖ by demanding its compatibility with the metric structure of $\mathcal{A}$ (i.e. with $\mathbb{G}$) indeed leads to a complete factorization of θ into parts featuring only the horizontal or vertical

[14] There are total derivative ambiguities in the definition of θ from the Lagrangian of the theory. Our choice here is made such that $\mathbb{L}_{\xi^\sharp}\theta = 0$ for field-independent ξ (valid for *invariant* Lagrangian densities). See [26, eq. 6.3] for more on this choice. And also [23] for a more general treatment in the BV-BFV formalism.

[15] The last expression of (32) has been introduced for notational convenience, even if geometrically imprecise. But the meaning is intuitively clear: $\theta(\mathbb{X}) \equiv \mathbb{i}_{\mathbb{X}} \mathbb{G}(\mathbb{E}) \equiv \mathbb{G}(\mathbb{E}, \mathbb{X})$ for any $\mathbb{X} \in \mathrm{T}\mathcal{A}$. We also notice that $\mathbb{i}_{\mathbb{X}}(\mathbb{d}_H A) = \widehat{H}(\mathbb{X})$ and $(\mathbb{i}_{\mathbb{X}}\varpi)^\sharp = \widehat{V}(\mathbb{X})$ where $\widehat{H}$ and $\widehat{V}$ are the horizontal and vertical projections respectively. The notation (32) will be especially convenient when the one-form basis will be restricted to the horizontal space, as in (33).

data respectively. This compatibility amounts to demanding that the horizontal spaces with respect to ϖ are $\mathbb{G}$-*orthogonal* to the gauge fibres of $\mathcal{A}$. This condition completely fixes ϖ and ensures its projection property (the first equation of (5)); its gauge covariance (the second of (5)) is also guaranteed as long as the supermetric is gauge invariant in the sense that[16] $\mathbb{L}_{\xi^\sharp}\mathbb{G} = 0$ [26, 27]. We named the unique connection ensuing from these requirements, the *Singer-DeWitt (SdW) connection* $\varpi_{\rm SdW}$ [26].

In formulae, the SdW connection is defined by the following condition:

$$\mathbb{G}\Big(\mathbb{X} - \varpi^\sharp_{\rm SdW}(\mathbb{X})\,,\,\xi^\sharp\Big) \stackrel{!}{=} 0 \qquad \forall \mathbb{X} \in {\rm T}\mathcal{A},\ \forall \xi \in {\rm Lie}(\mathcal{G}). \tag{35}$$

In a compact region R, $\partial R \neq \emptyset$, it is easy to see[17] that this is equivalent to defining $\varpi_{\rm SdW}$ through the following elliptic boundary value problem [25, 26] ($\cdot_s$ denotes the contraction with the outgoing normal s^i at ∂R)

$$\begin{cases} {\rm D}^2 \varpi_{\rm SdW} = {\rm D}^i \mathbb{d} A_i & \text{in } R, \\ {\rm D}_s \varpi_{\rm SdW} = \mathbb{d} A_s & \text{at } \partial R. \end{cases} \tag{36}$$

In electromagnetism, this boundary value problem is of the Neumann type, while in non-Abelian YM theories it is of a specific Robin type fixed by the background configuration $A \in \mathcal{A}$.

Let us reiterate this point: the above boundary value problem involves boundary conditions for $\varpi_{\rm SdW}$, but *not* for the gauge potential A_i nor for its perturbations. The boundary conditions on $\varpi_{\rm SdW}$ ensure that the connection in a region R—and the corresponding horizontal projections—are uniquely defined; they do not pose any restrictions on the fields nor on the gauge parameters that we allow, neither in R nor at ∂R.

As a consequence of the bulk and boundary properties of $\varpi_{\rm SdW}$, SdW-*horizontal* perturbations, i.e. those in the kernel of $\varpi_{\rm SdW}$, do satisfy specific bulk and boundary properties: they are covariantly divergenceless in the bulk and vanish when contracted with s^i at the boundary.

From this observation, it follows that the SdW-*horizontal* perturbations of the gauge potential, $\delta A = h$, are (covariantly) *transverse* photons (gluons, resp.), with vanishing normal-component to the boundary ∂R:

$$\mathbb{h} = \int h \frac{\delta}{\delta A} \quad \text{is SdW-horizontal iff} \quad \begin{cases} {\rm D}^i h_i = 0 & \text{in } R, \\ h_s = 0 & \text{at } \partial R. \end{cases} \tag{37}$$

As a side note to this formula, we add the following comment: at the level of the space of gauge potential $\mathcal{A}$ (not Φ), the surprising fact about (37) interpreted as a perturbative gauge-fixing *in a bounded region*, is that it encompasses *all* the physical content of perturbations over that region: although the boundary conditions might seem restrictive, a completely general perturbation $\mathbb{X} \in {\rm T}\mathcal{A}$, with *any* other boundary condition can be generated from a $\mathbb{h}$ of (37) with the aid of a unique infinitesimal gauge transformation (up to stabilizers). This is only possible because we do not restrict gauge freedom at the boundary, and it means that the horizontal projection is a complete and viable gauge fixing for the perturbation around $A \in \mathcal{A}$.

[16] A weaker condition is in fact sufficient, see [26].

[17] Asking that $0 = \mathbb{G}\Big(\mathbb{X} - \varpi^\sharp_{\rm SdW}(\mathbb{X})\,,\,\xi^\sharp\Big) = \int \sqrt{g}\, g^{ij} {\rm Tr}\Big(\big(\mathbb{X}_i - {\rm D}_i \varpi(\mathbb{X})\big){\rm D}_j \xi\Big)$ holds for all ξ 's and $\mathbb{X}$ gives condition (36) after an integration by parts.

The curvature of the SdW-connection, obtained from (9), satisfies the following boundary value problem (see [29], and especially[18] [26]):

$$\begin{cases} \mathrm{D}^2\mathbb{F}_{\mathrm{SdW}} = g^{ij}[\mathbb{d}_\perp A_i \stackrel{\wedge}{,} \mathbb{d}_\perp A_j] & \text{in } R, \\ \mathrm{D}_s\mathbb{F}_{\mathrm{SdW}} = 0 & \text{at } \partial R, \end{cases} \tag{38}$$

where here and in the following we will use the sub- (or super-)script $\cdot_\perp$ instead of $\cdot_H$ to denote horizontality with respect to ϖ_{SdW}. Notice that $\mathbb{F}_{\mathrm{SdW}} \equiv 0$ in the Abelian case.

We conclude this section by pointing the reader to an independent argument for the derivation of ϖ_{SdW} that is based on generalizing Dirac's dressing of the electron to the regional (non-Abelian) context. This is discussed in appendix D.

3.3 SdW-horizontality and the Gauss constraint

The SdW horizontal condition (37) applies also to the SdW-horizontal component $\dot{\mathbb{A}}^\perp$ of the electric field vector $\mathbb{E}$,

$$\dot{\mathbb{A}}^\perp := \dot{\mathbb{A}} - \varpi^\sharp_{\mathrm{SdW}}(\dot{\mathbb{A}}) = \int \underbrace{\left(\dot{A} - \mathrm{D}\varpi_{\mathrm{SdW}}(\dot{\mathbb{A}})\right)}_{=:\,\dot{A}^\perp} \frac{\delta}{\delta A}, \tag{39}$$

that is

$$\begin{cases} \mathrm{D}^i \dot{A}^\perp_i = 0 & \text{in } R, \\ A^\perp_s = 0 & \text{at } \partial R. \end{cases} \tag{40}$$

Notice that this means that $\dot{A}^\perp_i$ drops from the Gauss constraint $\mathrm{D}^i E_i \approx 0$.

Indeed, the burden of satisfying the Gauss constraint is fully entrusted to the SdW-vertical component of $\mathbb{E}$, i.e. φ_{SdW}. This is readily seen when writing the boundary value problem defining φ_{SdW}: contracting[19] $\mathbb{E}$ with equation (36), one gets

$$\begin{cases} \mathrm{D}^2\varphi_{\mathrm{SdW}} = \mathrm{D}^i E_i \approx 0 & \text{in } R, \\ \mathrm{D}_s\varphi_{\mathrm{SdW}} = E_s & \text{at } \partial R. \end{cases} \tag{41}$$

We will come back to this equation in a moment, after having introduced charged matter to the picture. This will allow us to include sources in the Gauss constraint.

For definiteness, we consider Dirac fermions valued in the fundamental representation W of the gauge group $G = \mathrm{SU}(N)$,

$$\psi^{B,b} \in \mathcal{C}^\infty(\Sigma, \mathbb{C}^4 \otimes W). \tag{42}$$

These transform under the action of a gauge transformation $g \in \mathcal{G}$ as

$$\psi \mapsto g^{-1}\psi. \tag{43}$$

[18] In [26, eq. 5.6], the differential equation for $\mathbb{F}_{\mathrm{SdW}}$ is found in the context without boundary. To find the boundary condition used in (38), we note that, in [26] to obtain equation 5.6, one uses equations 5.4 and 5.5. The first requires no integration by parts, contrary to the second, which yields an extra boundary term: $\oint \sqrt{h}\,\mathrm{Tr}(\xi s^i \mathrm{D}_i \mathbb{F})$. Hence, from the arbitrariness of ξ at the boundary, we deduce the boundary condition of (38). In [26], the significance of $\mathbb{F}_{\mathrm{SdW}}$ for the non-Abelian theory is extensively discussed in relation to: (*i*) the obstruction to extend the dressing of matter fields à la Dirac (see e.g. [40–43]) to the non-Abelian setting [44]; (*ii*) the Gribov problem [28, 45]; and (*iii*) the Vilkovisky-DeWitt geometric effective action [27, 46–50]. See also appendix D.

[19] Recall, $\mathbb{i}_{\mathbb{E}}\varpi \equiv \varpi(\mathbb{E}) = \varphi$ and $\mathbb{i}_{\mathbb{E}}\mathbb{d}A_i = E_i$.

The fermions ψ carry a Lie(G)-current density

$$J^\mu = (\rho, J^i) \quad \text{with} \quad J^\mu_\alpha = \overline{\psi}\gamma^\mu\tau_\alpha\psi \tag{44}$$

where $(\gamma^\mu)^{B'}{}_B$ are the Dirac matrices,[20] $(\tau_\alpha)^{b'}{}_b$ is an anti-Hermitian generator of G in the fundamental representation W, and $\overline{\psi} = i\psi^\dagger\gamma^0$.

The complete Lagrangian is then given by

$$L = \tfrac{1}{e^2}L_{\text{YM}} + L_{\text{Dirac}} \quad \text{with} \quad L_{\text{Dirac}} = -\int \mathrm{d}^D x\, \sqrt{g}\, \overline{\psi}\gamma^\mu \mathrm{D}_\mu\psi \tag{45}$$

where L_{YM} is the (renamed) Lagrangian defined in section 3.1, and $\mathrm{D}_\mu\psi := \partial_\mu\psi + A_\mu\psi$. The unit charge e will be henceforth set to $e \equiv 1$. From L, one deduces the following equations of motion: $\gamma^\mu\mathrm{D}_\mu\psi = 0$ and its conjugate, $(\mathrm{D}^k F_{ik} - \dot{E}_i) = J_i$ and the Gauss constraint

$$\mathrm{D}^i E_i \approx \rho. \tag{46}$$

Turning back to the equation defining φ_{SdW}, we can finally set it in its definitive form on-shell of the Gauss constraint ($\approx$)

$$\begin{cases} \mathrm{D}^2\varphi_{\text{SdW}} = \mathrm{D}^i E_i \approx \rho & \text{in } R, \\ \mathrm{D}_s\varphi_{\text{SdW}} = f & \text{at } \partial R. \end{cases} \tag{47}$$

where we defined the local electric flux through the boundary

$$f := E_s \quad \text{at } \partial R. \tag{48}$$

Thus, we recognize φ_{SdW} to be the Coulombic component of E in R, i.e. the component of E sourced by the charged matter density. In a bounded region, the source equation would not be enough to fully determine φ_{SdW}. That is why this equation appears here complemented by the information encoded in the electric flux f through ∂R. This can be seen as summarizing the Coulombic, nonlocal, influence of the charges beyond ∂R on the fields in R.[21]

In particular, note that *from within* R it is actually impossible to compute the part of f that is ultimately sourced by the charges contained in R: to actually compute the regional charges' contribution to f, one would require knowledge of the *global* Green's function of the Laplace operator on the whole Σ, a knowledge that is *not* available from within R. Moreover, away from reducible backgrounds[22], it is always possible to add a charged particle in the bulk without altering in any way f at the boundary through the choice of Green's functions with appropriate boundary conditions. In other words, from a regional perspective, f is an integral part of the definition of the Laplace equation which defines the Coulombic potential from the charge density. The flux f and the charge density ρ constitute fully[23] independent data (see also appendix D).

For these reasons, we will refer to φ_{SdW} as the *Coulombic* component of the electric field, and to $\dot{A}^\perp$ as its *radiative component.*

[20] For a metric g_{ij} on Σ, the commutator is $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} = 2\text{diag}(-1, g^{ij})$, i.e. $\gamma^\mu := e^\mu_I\gamma^I$ for γ^I the flat-space Dirac matrices and e^μ_I a local inertial frame, $g_{\mu\nu}e^\mu_I e^\nu_J = \eta_{IJ}$. See footnote 8. We adopt the following conventions for the γ^I: $\gamma^0 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^j = -i\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$ with σ^j the Hermitian Pauli matrices.

[21] In D=3, it is easy to see that $f = E_s$ does not encode any *incoming/outgoing* radiation through ∂R, since E_s does not feature in the normal component through ∂R of the Poynting vector.

[22] Cf. section 3.8.1 and comments on section 1.2 for the meaning of a 'reducible' background.

[23] Even at reducible background configurations, the region's charge content only imposes a small *finite* number of boundary-global constraints on f.

Finally, since from this point onwards all our considerations will concern the SdW connection and related objects, we will hereafter drop the redundant "SdW" subscript:

$$\varpi_{\rm SdW} \rightsquigarrow \varpi, \quad \varphi_{\rm SdW} \rightsquigarrow \varphi, \quad \text{and} \quad \mathbb{F}_{\rm SdW} \rightsquigarrow \mathbb{F}. \tag{49}$$

3.4 SdW decomposition of the symplectic potential

At this point, we are simply left with the task of applying the results obtained so far to the symplectic potential and form, thus unveiling the intimate relationship between the field-space geometry, the Gauss law, and the gauge nature of the YM dof.

To start with, we revisit the construction of section 3.1 to the case with charged matter field. From (45), we find

$$\theta = \theta_{\rm YM} + \theta_{\rm Dirac} \quad \text{with} \quad \theta_{\rm Dirac} = -\int \mathrm{d}^D x\, \sqrt{g}\, \overline{\psi}\gamma^0 \mathbb{d}\psi \in \Lambda^1(\Phi') \tag{50}$$

where Φ' is now the field space with matter

$$\Phi' = \Phi \times \{(\overline{\psi}, \psi)\}. \tag{51}$$

This also has a fibre bundle structure and ϖ can be pulled-back onto it, where it is still a valid connection form.[24] We thus define the SdW-horizontal differentials

$$\mathbb{d}_\perp \psi := \mathbb{d}\psi + \varpi\psi \quad \text{and} \quad \mathbb{d}_\perp \overline{\psi} := \mathbb{d}\overline{\psi} - \overline{\psi}\varpi, \tag{52}$$

with (we recall) the subscript $\cdot_\perp$ evoking "orthogonality" to the vertical directions (with respect to $\mathbb{G}$). In (52), actions in the fundamental representation and its transpose are assumed (in our conventions, $(\tau_\alpha)^{b'}{}_b \in W \otimes W^\dagger$ is antihermitian and therefore the Lie($\mathcal{G}$)-valued connection form satisfies $\varpi^\dagger = -\varpi$). By construction, the horizontal differentials satisfy $\mathbb{L}_{\xi^\sharp}\mathbb{d}_\perp\psi = -\xi\mathbb{d}_\perp\psi$. As discussed in appendix D (see also section 9 of [26]), in a perturbative setting, the SdW-horizontal modes of ψ can be understood in terms of a generalized Dirac dressing of electrons in two ways: to non-Abelian theories and to finitely bounded domains.[25]

We are now in the position to spell out the detailed form of the SdW-horizontal and SdW-vertical components of the symplectic potential θ. However, to emphasize the geometrical aspects of our construction, let us start without matter. Then, specializing the decomposition $\theta_{\rm YM} = \theta^H_{\rm YM} + \theta^V_{\rm YM}$ of (34) to the SdW choice of ϖ, we obtain

$$\theta^\perp_{\rm YM} = \mathbb{G}(\dot{\mathbb{A}}^\perp, \mathbb{d}_\perp A) \quad \text{and} \quad \theta^V_{\rm YM} = \mathbb{G}(\varphi^\sharp, \varpi^\sharp), \tag{53}$$

where we used the metric-compatibility properties of the SdW-connection to set $\mathbb{G}(\dot{\mathbb{A}}^\perp, \varpi^\sharp) \equiv 0 \equiv \mathbb{G}(\varphi^\sharp, \mathbb{d}_\perp A)$—see equation (40).

Including matter, and spelling out the various contributions, on-shell of the Gauss constraint (47), we obtain ($\theta = \theta^\perp + \theta^V$, $\theta^\perp \equiv \theta^\perp_{\rm YM} + \theta^\perp_{\rm Dirac}$)

$$\begin{cases} \theta^\perp_{\rm YM} &= \displaystyle\int_R \mathrm{d}^D x \sqrt{g}\, g^{ij} \mathrm{Tr}(\dot{A}^\perp_i \mathbb{d}_\perp A_j), \\ \theta^\perp_{\rm Dirac} &= -\displaystyle\int_R \mathrm{d}^D x \sqrt{g}\, \overline{\psi}\gamma^0 \mathbb{d}_\perp \psi, \\ \theta^V &\approx \displaystyle\oint_{\partial R} \mathrm{d}^{D-1} x \sqrt{h}\, \mathrm{Tr}(f\,\varpi), \end{cases} \tag{54}$$

[24] In [26] we named this fact the "corotation principle", meaning that a gauge transformation must act on every field at the same time and can therefore be detected by assessing just the transformation of A (which is arguably the most "sensitive" to the action of gauge transformations). See [26, Section 5].

[25] Cf. also footnote 18 for the role of the SdW-curvature $\mathbb{F}_{\rm SdW}$ in obstructing a non-perturbative definition of the dressed quark as opposed to the dressed electron.

where $h = \iota^*_{\partial R} g$ is the induced metric on ∂R. Note that the vertical contribution to $\theta_{\rm Dirac}$ is $\theta^V_{\rm Dirac} = \int \sqrt{g}\,{\rm Tr}(\rho\varpi)$, which therefore just plays into the Gauss constraint and the associated integration by parts.

3.5 SdW decomposition of the symplectic form

We now turn our attention to the symplectic form and its interplay with the SdW-decomposition.

The symplectic form is defined as

$$\Omega = \mathbb{d}\theta \in \Lambda^2(\Phi). \tag{55}$$

In [26] it was shown that generally, and therefore a fortiori in relation to the SdW connection, the horizontal part $\theta^\perp$ of θ gives

$$\Omega^\perp := \mathbb{d}_\perp \theta^\perp = \mathbb{d}\theta^\perp, \tag{56}$$

and that $\Omega^\perp$ differs from Ω by the term $\Omega^\partial = \mathbb{d}\theta^V$ which, on-shell of the Gauss constraint has support on the boundary *only* (i.e. is "pure boundary"):

$$\Omega = \Omega^\perp + \Omega^\partial. \tag{57}$$

As proved in appendix B, Ω is given by ($\Omega^\perp \equiv \Omega^\perp_{\rm Dirac} + \Omega^\perp_{\rm YM} \equiv \mathbb{d}\theta^\perp_{\rm Dirac} + \mathbb{d}\theta^\perp_{\rm YM}$)

$$\begin{cases} \Omega^\perp_{\rm Dirac} = -\displaystyle\int \sqrt{g}\left(\mathbb{d}_\perp\overline{\psi} \curlywedge \gamma^0 \mathbb{d}_\perp \psi \;-\; {\rm Tr}(\rho\,\mathbb{F})\right) \\ \Omega^\perp_{\rm YM} \;\;= \displaystyle\int \sqrt{g}\, g^{ij}{\rm Tr}\Big(\mathbb{d}_\perp \dot{A}^\perp_i \curlywedge \mathbb{d}_\perp A_j\Big) \\ \Omega^\partial \quad\;\; \approx \displaystyle\oint \sqrt{h}\,{\rm Tr}\Big(\mathbb{d}_\perp f \curlywedge \varpi + f\,\mathbb{F}\Big) \end{cases} \tag{58}$$

where the SdW curvature $\mathbb{F}$ was defined in (9) and (38), and the horizontal differentials $\mathbb{d}_\perp A$ and $\mathbb{d}_\perp\psi$ were established in (7) and (52). Instead, the horizontal differential of $\mathbb{d}_\perp \dot{A}^\perp_i$ is equal to $\mathbb{d}\dot{A}^\perp_i - [\dot{A}^\perp_i, \varpi]$ as follows from (18) (cf. (22)).

Observe that the SdW curvature $\mathbb{F}$ is a functional on the horizontal radiative modes $\mathbb{d}_\perp A$ (see (38)). This prevents $\Omega^\perp_{\rm Dirac} = \mathbb{d}\theta^\perp_{\rm Dirac}$ and $\Omega^\partial = \mathbb{d}\theta^V$ to be a symplectic forms of only the matter and Coulombic modes, respectively.[26] The role of the terms containing the SdW curvature is to ensure that $\Omega^\perp_{\rm Dirac}$, $\Omega^\perp_{\rm YM}$, and Ω^∂ are all $\mathbb{d}$-closed and $\mathbb{d}$-exact, i.e. that they descend from a symplectic potential.[27]

Nonetheless, it is convenient to separate the matter, radiative, and Coulombic contributions more neatly; thus we introduce a second decomposition:

$$\Omega \approx \Xi_{\rm Dirac} + \Xi_{\rm rad} + \Xi_{\rm Coul}, \tag{59a}$$

where

$$\begin{cases} \Xi_{\rm Dirac} := -\displaystyle\int \sqrt{g}\left(\mathbb{d}_\perp\overline{\psi} \curlywedge \gamma^0 \mathbb{d}_\perp \psi\right), \\ \Xi_{\rm rad} \;\;:= \displaystyle\int \sqrt{g}\, g^{ij}{\rm Tr}\Big(\mathbb{d}_\perp \dot{A}^\perp_i \curlywedge \mathbb{d}_\perp A_j - \rho\,\mathbb{F}\Big) + \oint \sqrt{h}\,{\rm Tr}\Big(f\,\mathbb{F}\Big), \\ \Xi_{\rm Coul} \;:= \displaystyle\oint \sqrt{h}\Big(\mathbb{d}_\perp f \curlywedge \varpi\Big), \end{cases} \tag{59b}$$

[26] However, recall that $\mathbb{F}_{\rm Abelian} \equiv 0$ in the Abelian case.

[27] On Φ, an n-form $\Theta^{(n)} \in \Lambda^n(\Phi)$ is closed if and only if it is exact, $\Theta^{(n)} = \mathbb{d}\Theta^{(n-1)}$.

As a bonus, the purely radiative component $\Xi_{\rm rad}$ can be rewritten in the following more suggestive (and also manifestly gauge-invariant) form

$$\Xi_{\rm rad} \approx \int \sqrt{g}\, g^{ij} {\rm Tr}\Big((\tfrac{\rm D}{{\rm d}t} \mathbb{d}_\perp A_i) \curlywedge \mathbb{d}_\perp A_j \Big). \tag{60}$$

where $\frac{\rm D}{{\rm d}t}$ is the gauge covariant time derivative. To get to this last formula, $\partial_t g_{ij} = 0$ in R was assumed—see appendix C for a proof.

A quick translation into common notation

To conclude, we provide a quick bridge to a more common notation (e.g. [51] or [52]). Let $\mathbb{X}_{1,2} = \int (X_i^\alpha)_{1,2} \frac{\delta}{\delta A_i^\alpha}$ be two tangent vectors on configuration space. In interpreting them as two infinitesimal variations, we denote their components with the more common notation $(X_i^\alpha)_{1,2} \equiv \delta_{1,2} A_i^\alpha$. Then, the SdW decomposition of $\delta_{1,2}A$ is given by the generalized Helmholtz decomposition

$$\delta_{1,2} A_i = (h_{1,2})_i + {\rm D}_i \eta_{1,2} \tag{61}$$

where ${\rm D}^i (h_{1,2})_i = 0$ in R and $(h_{1,2})_s = 0$ at ∂R (see (37)) is the horizontal part of $\delta_{1,2}A$ and $\lambda_{1,2}$ its vertical part.

On-shell of the Gauss constraint and in vacuum, to obtain a complete basis of variations in Φ (rather than just $\mathcal{A}$), we define the field space vectors

$$\delta_{1,2} := (\delta_{1,2}A, \delta_{1,2}E) = (h_{1,2}, \eta_{1,2}, \tfrac{\rm d}{{\rm d}t} h_{1,2}, \delta_{1,2} f) \tag{62}$$

where we traded the variation of the radiative part of the electric field, i.e. $\delta \dot{A}_i^\perp$, for $\frac{\rm d}{{\rm d}t} h$ (see (177) in appendix C). Then,

$$\begin{cases} \Xi_{\rm rad}(\delta_1, \delta_2) = \displaystyle\int \sqrt{g}\, g^{ij} {\rm Tr}\Big(\Big(\tfrac{\rm D}{{\rm d}t} (h_1)_i \Big) (h_2)_j - (1 \leftrightarrow 2) \Big) \\ \Xi_{\rm Coul}(\delta_1, \delta_2) = \displaystyle\oint \sqrt{h}\, {\rm Tr}\Big(\delta_1 f \, \lambda_2 - (1 \leftrightarrow 2) \Big). \end{cases} \tag{63}$$

Similarly, ${\rm D}^2 \mathbb{F}(\delta_1, \delta_2) = 2 g^{ij} [(h_1)_i, (h_2)_j]$.

3.6 SdW decomposition of the symplectic structure: discussion

3.6.1 Horizontal symplectic potential

In section 3.4, we have decomposed the symplectic potential of YM theory with matter into its horizontal and vertical parts, i.e. $\theta = \theta^\perp + \theta^V$ respectively (54). Upon contraction with a generic vector $\delta = (\delta A, \delta E, \delta\psi)$ in ${\rm T}\Phi'$, the SdW-horizontal component of the symplectic potential,

$$\theta^\perp(\delta) = \int_R \sqrt{g} \Big(g^{ij} {\rm Tr}(\dot{A}_i^\perp \delta_\perp A_j) - \overline{\psi} \gamma^0 \delta_\perp \psi \Big), \tag{64}$$

involves: the (covariantly) transverse gluon perturbations $h \equiv \delta_\perp A = \delta A - {\rm D}\varpi(\delta A)$ (37), the radiative component of the electric field $\dot{A}^i_\perp$ (40), and the (infinitesimal non-Abelian finite-region) analogue $\delta_\perp \psi = \delta\psi + \varpi(\delta A)\psi$ of the Dirac-dressed electron (eq. 52 and appendix D). These degrees of freedom are all gauge-invariant, quasilocal in R, and can be affected only through operations within R. In other words, the SdW horizontal contribution $\theta^\perp$ has all the *regional* localization properties one would expect from a standard field theory with no gauge symmetry—even if, *within that region*, the radiative dof are non-local.

3.6.2 Vertical symplectic potential

On the other hand, contracted with δ, the SdW-vertical component

$$\theta^V(\delta) \approx \oint_{\partial R} \sqrt{h}\mathrm{Tr}(f\,\varpi(\delta A)) \tag{65}$$

involves (on-shell of the Gauss constraint) only the electric flux f and the pure-gauge part $\varpi(\delta A)$ of the perturbation δA (and no matter field). Let us emphasize some important characteristics of $\varpi(\delta A)$ and f.

First, the pure-gauge part $\varpi(\delta A)$ is a *nonlocal* functional of δA *throughout* R. Only when the perturbation is pure gauge, $\delta A = \mathrm{D}\xi$, does this contribution become local, entering θ^V only through the local value of ξ at the boundary (this happens by construction, see the first of (5)).

Second, as we have already stressed in section 3.3 (see in particular text below equation (48)), the electric flux f summarizes the information *from beyond* ∂R which non-locally influences the physics within R through the elliptic Gauss constraint. Here, this is confirmed in the symplectic treatment: f enters the only contribution to θ that is not directly influenced by the region's interior.

The vertical part θ^V and its properties are unique to gauge theory: they are a manifestation of the local Gauss laws, which are in turn seen as the distinguished signature of "gauge" [53]. It is then satisfactory to see that these properties fit, and mathematically translate, the expectation that the raison d'être of "gauge" hinges on the coupling of subsystems [15, 16].

We notice some analogies between the SdW vertical part θ^V and the constructions of [10]. In particular, our ϖ seems to occupy the place there taken by their 'extra' boundary dof, aka "edge modes". The fact that edge modes are introduced as gauge-compensating fields, or gauge reference frames, adds to the analogy (see [37]). However, it is important to notice that here ϖ is not a new boundary dof, but the pure-gauge part of δA from which it is nonlocally constructed. And crucially, in the SdW-decomposition of θ, the bulk (radiative) and boundary (Coulombic) components of the electric field are *functionally independent from each other*. See also section 3.7.

3.6.3 Symplectic form

The structure of the symplectic forms $\Omega = \mathbb{d}\theta$, unravelled in section 3.5, for the most part parallels the one we just described, modulo one subtlety. That is the decomposition of Ω into radiative and Coulombic components (59) is more subtle than that of θ because of the role played by the SdW curvature $\mathbb{F}$ (9). In particular, this means that the purely radiative and Coulombic components of Ω, i.e. Ξ_{rad} and Ξ_{Coul}, are *not* given by $\mathbb{d}\theta^\perp_{\mathrm{YM}}$ and $\mathbb{d}\theta^V$; in fact, they do not descend from a potential at all. (This discussion does not apply to the Abelian case, where $\mathbb{F}_{\mathrm{Abelian}} \equiv 0$; cf. footnote 18).

3.6.4 Flux superselection

However, there is one fact that is most clearly emphasized by the structure of Ω: i.e. that the symplectic form does not have a *purely* vertical component. Indeed, the Coulombic component Ξ_{Coul} is of mixed horizontal/vertical nature,

$$\Xi_{\mathrm{Coul}} = \oint \sqrt{h}\,\mathrm{Tr}\Big(\mathbb{d}_\perp f \curlywedge \varpi\Big). \tag{66}$$

This of course reflects the fact that the physical modes of f are conjugate to purely gauge modes of A that are by definition non-physical. Therefore, in a putative quasilocal quantization intrinsic to the region R, the observable associated to f should be superselected.

This fact, already recognized on the lattice[28] and important for the computation of entanglement entropy in a gauge theory [2, 18, 19], is here supported and made more precise by a continuum analysis of the quasilocal symplectic geometry.

This symplectic pairing also suggests that working within a superselection sector requires us to fix the gauge freedom at the boundary ∂R (recall the discussion just below equation (65)). This remark will be relevant for the discussion of symplectic generators and charges in the next sections.

3.7 Gauge invariance and the symplectic potential

In this section we provide some comments on the gauge invariance properties of the YM symplectic potential. To start, let us define

$$H[\xi] := \mathbb{i}_{\xi^\sharp}\theta = \int_R \sqrt{g}\,\mathrm{Tr}\big(E^i \mathrm{D}_i\xi + \rho\xi\big). \tag{67}$$

Then, the gauge transformation properties of θ, easily obtained from (50) and its components' behavior under gauge transformations, are encoded in the equation

$$\mathbb{L}_{\xi^\sharp}\theta = \int \sqrt{g}\,\mathrm{Tr}\big(E^i\mathrm{D}_i \mathbb{d}\xi + \rho\mathbb{d}\xi\big) = H[\mathbb{d}\xi]. \tag{68}$$

Using Cartan's formula on its left-hand side, i.e. $\mathbb{L}_{\xi^\sharp}\theta = \mathbb{i}_{\xi^\sharp}\mathbb{d}\theta + \mathbb{d}\mathbb{i}_{\xi^\sharp}\theta$, we recognize:

$$\mathbb{i}_{\xi^\sharp}\Omega = -\mathbb{d}(H[\xi]) + H[\mathbb{d}\xi], \tag{69}$$

which provides a measure of the obstruction for the definition of a Hamiltonian generator for the gauge transformation ξ.

We readily see that for field-independent gauge transformations, $\mathbb{d}\xi = 0$, the functional $H[\xi]$ is a valid Hamiltonian generator.

Moreover, in the absence of boundaries, it turns out that the field-dependence of ξ does not create any issue. Indeed, although the flow equation does not strictly hold, in that case $H[\mathbb{d}\xi]$ vanishes on-shell of the Gauss constraint and thus does not "interfere" with the gauge structure. This situation can be translated into the standard Hamiltonian formalism as follows: for a field-dependent gauge-parameter—i.e. a smearing $\lambda^\alpha = \lambda^\alpha(p, q)$ of first class constraints H_α—we obtain

$$\{H_\alpha[\lambda^\alpha], F\} = \{H_\alpha, F\}[\lambda^\alpha] + H_\alpha[\{\lambda^\alpha, F\}] \approx \{H_\alpha, F\}[\lambda^\alpha], \tag{70}$$

i.e. the constraints still effect a homomorphism from the gauge group of transformations into canonical transformations in phase space.

However, the situation is different in the presence of boundaries [10, 12–14, 26]. To relate $H[\mathbb{d}\xi]$ and the Gauss constraint, an integration by parts is needed and thus, if $\partial R \neq \emptyset$ one gets the following obstruction for ξ to admit a Hamiltonian generator:

$$H[\mathbb{d}\xi] \approx \oint_{\partial R} \mathrm{Tr}\big(f\mathbb{d}\xi\big). \tag{71}$$

In the presence of boundaries $\partial R \neq \emptyset$, field dependent gauge transformations do *not* admit a Hamiltonian generator within R even on-shell of the Gauss constraint. Of course, this obstruction is related to another puzzling fact, which does not require field-dependence,

[28] It is however interesting to note that in [2, 54] the possibility to superselect a magnetic flux was made concrete both for Abelian and (2+1 dimensional) non-Abelian gauge theories (on the lattice).

namely, the fact that in the presence of boundaries, the Hamiltonian generator of a gauge transformation does not necessarily vanish on-shell:

$$H[\xi] \approx \oint_{\partial R} \mathrm{Tr}\big(f\xi\big). \tag{72}$$

One route to resolving these issues is to consider f as superselected (cf. the discussion at the end of the previous section) and their conjugate gauge transformations at the boundary as fixed (cf. section 3.6.2), thus setting $\xi_{|S} = 0$ and $H[\xi] = 0$. But this creates issues if one wants to recover conserved charges, leading to several distinct proposals to allow only a selected class of gauge transformations at the boundary, taken to represent 'physical dof' (see e.g. [6–8]). The constructions of [10] can be seen as a way of maintaining the arbitrary gauge transformations at the boundary while recovering non-trivial charges.[29]

Here, we rely instead on yet a different, more minimal, geometric perspective on this question. Prompted by the observation that the Hamiltonian nature of the gauge flow is obstructed by field-dependent ξ, we introduced a connection form in the treatment of symplectic geometry in bounded regions [5,16,25,26,37] and thus reframed the symplectic analysis of YM in the presence of boundaries in terms of an SdW-horizontal/vertical decomposition—as detailed in the previous sections.

In light of our analysis, the fact that the obstruction to the Hamiltonian nature of the gauge flow is controlled by the electric flux f acquires a precise physical meaning: f is the observable at the boundary of R that summarizes the exterior information needed to reconstruct the physics inside R. The point is that f holds this role without being conjugated to—and therefore actable upon by—any *physical* quantity intrinsic to R (or ∂R).

In this perspective, the unique gauge-invariant, or physical, charge *intrinsic to the region*, is given not by $H[\xi] = \mathbb{i}_{\xi^\sharp}\theta$, but by $\mathbb{i}_{\xi^\sharp}\theta^\perp$. In other words, without treating gauge transformations at the boundary any differently—i.e. without fixing them or endowing them with physical significance—we can define horizontal charges. These can be loosely understood as "perturbatively gauge-fixed" charges. It has often been forcefully argued that on a Cauchy surface gauge-fixed/gauge-invariant objects are the only ones which *can* carry physical meaning [53]. Here we are, in a way, extending this criterion to quantities intrinsic to the region.

In the absence of boundaries, and on-shell of the Gauss constraint, θ^V vanishes and there is no difference between θ and $\theta^\perp$. In their presence, however, θ^V precisely encodes the Coulombic obstruction to the Hamiltonian nature of the gauge flow (71)

$$\mathbb{L}_{\xi^\sharp}\theta^V = H[\mathbb{d}\xi] \approx \oint \sqrt{h}\,\mathrm{Tr}\big(f\mathbb{d}\xi\big), \tag{73}$$

whereas $\theta^\perp$ maintains its gauge invariance properties

$$\mathbb{L}_{\xi^\sharp}\theta^\perp = \mathbb{i}_{\xi^\sharp}\Omega^\perp + \mathbb{d}\mathbb{i}_{\xi^\sharp}\theta^\perp \equiv 0. \tag{74}$$

From the Hamiltonian perspective, this equation is somewhat trivial, since both $\mathbb{i}_{\xi^\sharp}\theta^\perp$ and $\mathbb{i}_{\xi^\sharp}\Omega^\perp$ vanish identically and independently.

Of course, if we are truly to entrust regional physical status to those charges for which $\mathbb{i}_{\xi^\sharp}\theta^\perp \neq 0$ it would seem that we are forced to conclude *no* such regionally intrinsic charges exist. However, as we will see, these regional horizontal charges will only vanish *almost*-everywhere on $\Phi' \ni \{(A, E, \psi, \overline{\psi})\}$; they are sometimes nontrivial. The time has come to investigate where the connection-form properties of ϖ fail. This is the goal for the next section.

[29] It should be noted that they recover an infinite group of charges, and that it is not clear in which manner such charges obey non-trivial conservation laws. See e.g. [55–57] for attempts in this direction.

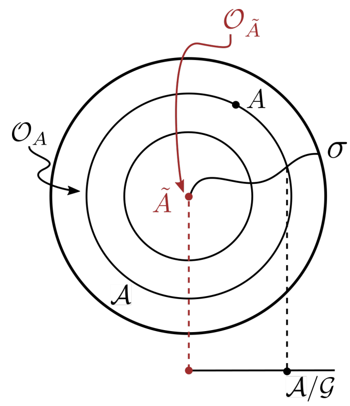

Figure 3: In this representation $\mathcal{A}$ is the page's plane and the orbits are given by concentric circles; σ is a section in $\mathcal{A}$. The field A is generic, and has a generic orbit, $\mathcal{O}_A$. The field $\tilde{A}$ has a nontrivial stabilizer group (i.e. it has non-trivial reducibility parameters), and its orbit $\mathcal{O}_{\tilde{A}}$ is of a different dimension than $\mathcal{O}_A$. The projection of $\tilde{A}$ on $\mathcal{A}/\mathcal{G}$ therefore sits at a qualitatively different point than that of A (a lower-dimensional stratum of $\mathcal{A}/\mathcal{G}$).

3.8 Charges

3.8.1 Reducible configurations

Consider a configuration $\tilde{A}_i$ such that there exists a gauge transformation χ with the following property:

$$\delta_\chi \tilde{A}_i = \tilde{\mathrm{D}}_i \chi = 0. \tag{75}$$

Then, $\tilde{A}_i$ is said "reducible" and χ is called a "reducibility parameter" or "stabilizer". The stabilizers χ depend on the global properties of $\tilde{A}_i$ and constitute a finite dimensional vector space (possibly of dimension zero).

In non-Abelian theories reducible configurations form a meager set,[30] in the same way as those spacetime metrics which admit non-trivial Killing vector fields are "extremely rare" (i.e. form a meager set).[31] In this respect, electromagnetism is an exception, since all its configurations have as their reducibility parameter the constant gauge transformation, $\chi_{\mathrm{EM}} = const.$

Since $(\mathbb{i}_{\chi^\sharp} \mathbb{d} A_i)_{|\tilde{A}} = \delta_\chi \tilde{A}_i = 0$, it follows from the definition (3) that at these configurations of $\mathcal{A}$, $\chi^\sharp{}_{|\tilde{A}} \in \mathrm{T}_{\tilde{A}}\mathcal{A}$ vanishes, thus establishing the degeneracy of the gauge orbit $\mathcal{O}_{\tilde{A}} \subset \mathcal{A}$. See figure 3.

Notably, on simply connected manifolds, the reducibility parameters χ constitute the *only*[32] kernel of the SdW boundary value problem featured in the defining equations of ϖ_{SdW} (36), φ_{SdW} (47), and $\mathbb{F}_{\mathrm{SdW}}$ (38). As we will discuss, this fact has far reaching consequences for the nature of charges in YM.

[30] A *meager* set is one whose complement is an everywhere dense set: roughly, an arbitrarily small perturbation takes one out of a meager set. Here, reducible configurations form a meager set according to the standard field-space metric topology on $\mathcal{A}$ (the Inverse-Limit-Hilbert topology [35, 36], see also [39]). cf. footnote 12.

[31] Because of the existence of these configurations, $\mathcal{A}$ is not quite a bona fide fibre bundle, and its base manifold is in fact a stratified manifold, figure 3 (see [26] for a more thorough discussion of the geometry involved and further references on the topic).

[32] This is easy to see: demanding that ξ is in the kernel of the SdW boundary value problem, i.e. that $\mathrm{D}^2\xi = 0$ and $\mathrm{D}_s\xi_{|\partial R} = 0$, implies that $0 = \int \sqrt{g}\,\mathrm{Tr}(\xi \mathrm{D}^2 \xi) - \oint \sqrt{h}\,\mathrm{Tr}(\xi \mathrm{D}_s \xi) = -\int \sqrt{g}\, g^{ij}\,\mathrm{Tr}(\mathrm{D}_i\xi \mathrm{D}_j \xi)$. From which $\mathrm{D}\xi = 0$.

So far we have discussed the situation in the YM configuration space $\mathcal{A}$, but what happens in $\Phi' = \Phi \times \{(\psi, \overline{\psi})\}$, that is when we add the electric and matter fields to the picture?

3.8.2 Electromagnetism

Let us start with electromagnetism. In EM, although all configurations $A \in \mathcal{A}_{\rm EM}$ are reducible with respect to the constant gauge transformations $\chi_{\rm EM} = const$ (and so are all $(A, E) \in {\rm T}\mathcal{A} = \Phi_{\rm EM}$), none of the matter field configurations for which $\psi \neq 0$ is:

$$\delta_{\chi_{\rm EM}}\psi = -\chi_{\rm EM}\psi \neq 0. \tag{76}$$

Therefore, $\chi^\sharp_{\rm EM}$ as a vector field on $\Phi'_{\rm EM}$ reads[33]

$$\chi^\sharp_{\rm EM} = \int (-\chi_{\rm EM}\psi)^B(x)\frac{\delta}{\delta\psi^B(x)} \in {\rm T}\Phi'_{\rm EM}. \tag{77}$$

Hence, in $\Phi'_{\rm EM}$, the vector $\chi^\sharp_{\rm EM}$ does not vanish and is nonetheless in the kernel of (the pullback of) $\varpi_{\rm EM}$:

$$\varpi_{\rm EM}(\chi^\sharp_{\rm EM}) = 0. \tag{78}$$

Thus, we see that we have geometrically isolated the only "gauge transformations" of EM that have a physical significance in R, i.e. those that are associated to the total electric charge contained in R. At the light of these findings, it is particularly instructive to revisit the gauge invariance of $\theta^\perp_{\rm EM}$ (74). With this purpose we define,

$$Q_{\rm EM}[\chi_{\rm EM}] := \theta^\perp_{\rm EM}(\chi^\sharp_{\rm EM}) = \int_R \sqrt{g}\,(\chi_{\rm EM}\rho) \neq 0 \tag{79}$$

and thus obtain, through Cartan's formula, the *nontrivial* flow equation

$$0 = \mathbb{L}_{\chi_{\rm EM}}\theta^\perp_{\rm EM} = \mathbb{d}Q_{\rm EM}[\chi^\sharp_{\rm EM}] + \mathbb{i}_{\chi^\sharp_{\rm EM}}\Omega^\perp_{\rm EM}. \tag{80}$$

Furthermore, the fact that the SdW boundary value problem has a kernel in EM, implies the (integrated) Gauss law—usually expressed for $\chi_{\rm EM} = 1$:

$$Q_{\rm EM}[\chi_{\rm EM}] \approx \int \sqrt{g}\,(\chi_{\rm EM}{\rm D}^2\varphi) = \oint \sqrt{h}\,(\chi_{\rm EM}f). \tag{81}$$

Finally, we notice that, if $\chi_{\rm EM}$ is not only a constant in space but also in time, $Q_{\rm EM}[\chi_{\rm EM}]$ is a quantity satisfying a balance equation (electric charge conservation).

That such charges are physical, and are thus distinguished from gauge transformations is thus not postulated, but derived. After all, the transformations corresponding to $\chi_{\rm EM}$ are entirely generated by the $\theta^\perp$ components, and are thus *not* generated by the Gauss constraint (which is entirely in θ^V).

3.8.3 Yang-Mills theory

We now want to generalize these considerations to the non-Abelian theory. The construction is mathematically completely analogous, albeit more subtle. Physically, however, these subtleties have fundamental consequences.

To start with, as we have already said, not all configurations of the gauge potential $A \in \mathcal{A}$ are reducible. Only at reducible configurations $\tilde{A} \in \mathcal{A}$ there are $\chi \in {\rm Lie}(\mathcal{G})$ for

[33] Here, B is a spinorial index in $\mathbb{C}^4$, e.g. the Dirac gamma matrices γ^μ have components $(\gamma^\mu)^{B'}{}_B$.

which ${\chi^\sharp}_{|\tilde A}$ vanishes. Extending the above construction to include the electric field, we move our attention to $(\tilde A, E) \in \Phi$. Then, the question arises whether $\chi^\sharp$ also vanishes on Φ. For this to happen χ has to also stabilize E, i.e. $\delta_\chi E = [E, \chi]$ also has to vanish. However, this condition does not automatically follow from the reducibility of $\tilde A$:

$$\delta_\chi \tilde A = \tilde{\mathrm{D}}\chi = 0 \quad \not\Rightarrow \quad \delta_\chi E = [E, \chi] = 0 \text{ at } (\tilde A, E) \in \Phi. \tag{82}$$

Heuristically, one can see the extra requirement as demanding that stabilizer not only stabilize the instantaneous A, but also (the physical component of) its velocity.

Configurations $(\tilde A, E) = (\tilde A, \dot A^\perp, \varphi)$ which are reducible in A but not in $\dot A^\perp$, that is $\tilde D\chi = 0$ but $[\chi, \dot A^\perp] \neq 0$, provide an exception to equation[34] (74). In fact, at such configurations one has $\mathbb{L}_{\chi^\sharp}\theta^\perp = \int g^{ij}\mathrm{Tr}([\dot A_i^\perp, \chi]\delta_\perp A_j) \neq 0$. That is, while generically $\mathbb{i}_{\xi^\sharp}\mathbb{d}_\perp \dot A_\perp = 0$, for a stabilizer χ of A but not of $\dot A_\perp$ we have $\mathbb{i}_{\chi^\sharp}\mathbb{d}_\perp \dot A_\perp = [\dot A^\perp, \chi]$. We will henceforth assume our stabilizers are not of this form.[35]

Thus, we focus on those configurations $(\tilde A, \tilde{\dot A}^\perp, \varphi) \in \Phi$ at which there exists a $\chi \in \mathrm{Lie}(\mathcal G)$ such that

$$\delta_\chi \tilde A = \tilde{\mathrm{D}}\chi = 0 \quad \text{and} \quad \delta_\chi \tilde{\dot A}^\perp = [\tilde{\dot A}^\perp, \chi] = 0. \tag{83}$$

At these configurations of the YM field, and in the presence of matter, one has

$$\varpi(\chi^\sharp) = 0. \tag{84}$$

even though

$$\chi^\sharp = \int (-\chi\psi)\frac{\delta}{\delta\psi} \in \mathrm{T}\Phi' \tag{85}$$

is nonvanishing—which means that at $(\tilde A, \tilde{\dot A}^\perp, \varphi, \psi) \in \Phi'$ the projection property $\varpi^\sharp(\xi^\sharp) = \xi^\sharp$ fails for $\xi = \chi$. Then, the quantity[36]

$$Q[\chi] := \theta^\perp(\chi^\sharp) = \int_R \sqrt{g}\,\mathrm{Tr}(\chi\rho) \neq 0 \tag{86}$$

is the analogue of the EM charge built above and nontrivially satisfies

$$0 = \mathbb{L}_\chi \theta^\perp = \mathbb{d}Q[\chi] + \mathbb{i}_{\chi^\sharp}\Omega^\perp. \tag{87}$$

Indeed, the two terms on the right-hand side of (87) are both nonvanishing and $Q[\chi]$ satisfies an (integrated) Gauss law:

$$Q[\chi] \approx \int \sqrt{g}\,\mathrm{Tr}(\chi \mathrm{D}^2\varphi) = \oint \sqrt{h}\,\mathrm{Tr}(\chi f). \tag{88}$$

It is crucial to appreciate that this relation between the total charge and the integrated flux holds *only at reducible configurations* $\tilde A$. This means that at a generic (non-Abelian)

[34] This occurs because at such points of $\Phi = \mathrm{T}\mathcal{A}$, the pull-back of ϖ from $\mathcal A$ to Φ does not constitute a well-posed connection-form in Φ. In the absence of matter, $\xi^\sharp = (\mathrm{D}\xi, [E, \xi]) \in \mathrm{T}_{(A,E)}\Phi$. The connection ϖ is only sensitive to the first term. For generic ξ we have as usual $\varpi(\xi^\sharp) = \xi$ and $\varpi^\sharp(\xi^\sharp) = (\mathrm{D}\xi, [E, \xi]) = \xi^\sharp$. But if the specific χ considered above exist, then $\varpi^\sharp(\chi^\sharp) = (0, 0) \neq \chi^\sharp = (0, [E, \chi])$ since $\tilde{\mathrm{D}}\chi = 0$.

[35] The issue discussed in footnote 34 suggests that an improvement of ϖ is necessary so that it satisfies the projection property $\varpi^\sharp(\xi^\sharp) = \xi^\sharp$ everywhere on Φ, and not just on $\mathcal A$. We will not address this problem here.

[36] This charge can be zero, if $\psi = 0$ or if $\chi\psi = 0$. The latter condition is not attainable for $G = \mathrm{SU}(2)$, but it is for larger $N > 2$. This situation was analyzed in [26] through the lens of the Higgs mechanism for condensates. To simplify the discussion, we will hereafter ignore this possibility.

configuration it is possible to add charged matter to the bulk *without* altering the boundary flux at all.[37]

This construction provides the minimal requirement for the definition of non-trivial *horizontal* charges within R. Their time-evolution and conservation properties are however not as clear. Here, we abstain from performing a detailed analysis involving the equations of motion, and rather limit ourselves to the following scenario.

We consider a configuration of Φ' whose time evolution allows an extension of χ to a *time* neighbourhood N of R that satisfies $(\tilde{\mathrm{D}}_0\chi, \tilde{\mathrm{D}}_i\chi) = (0,0)$.[38] Then, the quantity $Q[\chi]$ is conserved in the sense that it satisfies a balance law in terms of the matter current [24,27]:

$$0 = \int_N \sqrt{g}\,\nabla_\mu \mathrm{Tr}(\chi J^\mu) = \Delta Q[\chi] + \int \mathrm{d}t\, F_{\partial R}[\chi], \tag{89}$$

where we introduced the fluxes $F_{\partial R}[\chi] = \oint \sqrt{h}\,\mathrm{Tr}(\chi J_s)$ through ∂R, and used that $\tilde{\mathrm{D}}_\mu\chi = 0$ as well as the equation of motion $\tilde{\mathrm{D}}_\mu J^\mu = 0$. Notice that all integrands are gauge invariant quantities constructed geometrically from the properties of $\tilde{A}$ in N. Indeed, the existence and properties of a χ such that $\tilde{\mathrm{D}}_\mu\chi = 0$ are gauge-invariant features of the configuration history $\tilde{A}(t)$.

Notice that the condition $\tilde{\mathrm{D}}_\mu\chi = 0$ implies that $A = \tilde{A}$ and $E = \tilde{E}$ are reducible, and therefore—via the Gauss constraint—that $\rho = \tilde{\rho}$ commutes with χ:

$$\tilde{\mathrm{D}}_\mu\chi = 0 \quad \Rightarrow \quad \tilde{\mathrm{D}}_i\chi = 0, \quad [\tilde{E}_i, \chi] = 0 \quad \text{and} \quad [\tilde{\rho}, \chi] = 0. \tag{90}$$

The last equation might raise the suspicion that the charge $Q[\chi]$ is bound to vanish, but this suspicion is unfounded. In particular, although a χ which is covariantly conserved, $\tilde{\mathrm{D}}_\mu\chi = 0$, is also a reducibility parameter of Φ, it does not need to be a reducibility parameter of the field space with matter, Φ', as well. To prove this statement for $G = \mathrm{SU}(N)$ it is enough to verify it when $G = \mathrm{SU}(2)$: in ρ the global U(1) phase of ψ plays no role, but it is precisely this phase that enters the definition of $Q[\chi]$, cf. footnote 36 (the U(1) EM case provides an even more basic example of this mechanism).

We conclude this section by noticing that the balance equation expressed in (89) is akin to the conservation of Komar charges for Killing vector fields in general relativity; similarly, the impossibility of identifying a meaningful non-Abelian charge density over generic, i.e. nonreducible, configurations parallels, in general relativity, the difficulties in identifying conserved stress-energy charges away from backgrounds with Killing symmetries [58]. Finally, within the present framework, the construction of asymptotic YM charges at null infinity that are more akin to the Bondi, rather than Komar, charges was carried out in [5].

4 Gluing

4.1 SdW gluing

4.1.1 Mathematical statement of the problem and of its solution

In this and the following sections, we will analyze how the SdW split (for non-Abelian fields, in the presence of boundaries) behaves with regards to the composition, or gluing,

[37]See section 3.6.2. There we explain how f summarizes the physics beyond R, and the fact that, at generic, i.e. non-reducible, YM configurations, f is a datum independent of the local matter charge density in R. In this regard, see also the initial discussion of appendix D.

[38]It is possible that such χ's are uniquely fixed by demanding that they conserve both $\tilde{A}$ and $\tilde{\dot{A}}^\perp$ (and then evolving these solutions in time).

of regions. Specifically, we will here state, and in the next section prove, the theorem at the root of all our physical results on the composition of both the electric field E and of the perturbations $\mathbb{X} \in \mathrm{T}\mathcal{A}$ of the gauge potential A.

For simplicity, we consider a closed, simply connected manifold Σ, with $\partial\Sigma = \emptyset$, which is split into two regions, $R^\pm$, such that $R^+ \cap R^- = S = \partial R^\pm$:

$$\Sigma = R^+ \cup_S R^-. \tag{91}$$

To formally encode the separation of regions, we introduce $\Theta_\pm$ as the characteristic functions of the regions $R^\pm$. Denoting s_i the outgoing co-normal at S with respect to the region R^+, one has

$$\partial_i \Theta_\pm = \mp s_i \delta_S \tag{92}$$

where δ_S is a $(D-1)$-dimensional delta function supported on the interface S.

A generic Lie-algebra-valued vector in $\mathrm{T}\mathcal{A}$ can be written as $\mathbb{Y} = \int_\Sigma Y \frac{\delta}{\delta A} \in \mathrm{T}\mathcal{A}$. It is useful to introduce the following notation for the regional decomposition of $\mathbb{Y}$ supported on Σ:

$$\mathbb{Y} = \mathbb{Y}^+ \oplus \mathbb{Y}^-, \tag{93}$$

where $Y = Y^+\Theta_+ + Y^-\Theta_-$ and $\mathbb{Y}^\pm = \int_{R^\pm} Y^\pm \frac{\delta}{\delta A}$. Notice that smoothness of $\mathbb{Y}$ implies all derivatives of Y^+ match those of Y^- at S.

We then introduce the SdW decompositions of $\mathbb{Y}$, both in Σ and in $R^\pm$. That is, we introduce the SdW connection defined through (36) for the whole manifold and also for its bounded subregions. For the entire manifold Σ, no boundary condition is needed, since $\partial\Sigma = \emptyset$.

Let us start from Σ, and denote the SdW decomposition of $\mathbb{Y}$ as $\mathbb{Y} = \mathbb{H} + \Lambda^\sharp$, where $\Lambda = \varpi(\mathbb{Y})$ and where $\mathbb{H}$ denotes the SdW-horizontal part of $\mathbb{Y}$, i.e. $\varpi(\mathbb{H}) = 0$ by definition. In components,

$$Y = H + \mathrm{D}\Lambda, \tag{94}$$

with $\mathrm{D}^i H_i = 0$ in Σ.

We now turn to the SdW-decomposition in $R^\pm$. We henceforth denote the regional SdW connections $\varpi_\pm$ (i.e. the SdW connections defined through (5) over $R^\pm$); and the regional decompositions of $\mathbb{Y}^\pm$ become

$$Y^\pm = h^\pm + \mathrm{D}\lambda^\pm, \tag{95}$$

where $\varpi_\pm(\mathbb{Y}^\pm) = \lambda^\pm$ and $\varpi_\pm(\mathbb{h}^\pm) = 0$, i.e.

$$\begin{cases} \mathrm{D}^i h_i^\pm = 0 & \text{in } R^\pm, \\ s^i h_i^\pm = 0 & \text{at } S. \end{cases} \tag{96}$$

Importantly, $h^\pm$ and $\lambda^\pm$ are *not* the restrictions of H and Λ to $R^\pm$: the operations of regional restriction and SdW-projection do not commute. Indeed, in general one has

$$H = (h^+ + \mathrm{D}\xi^+)\Theta_+ + (h^- + \mathrm{D}\xi^-)\Theta_- \tag{97}$$

where $\xi^\pm = \lambda^\pm - \Lambda\Theta_\pm$.

These equations govern the relationship between the global and the regional SdW splits. Here, however, we are interested in the following *inverse* problem: given solely the regional information $h^\pm$ and $\lambda^\pm$—i.e. without any a priori knowledge of H and Λ—is it possible to uniquely reconstruct the global H and Λ (see figure 4)? And furthermore: under which conditions can we uniquely determine the Lie-algebra valued functions $\xi^\pm$ solely from the knowledge of the $h^\pm$?

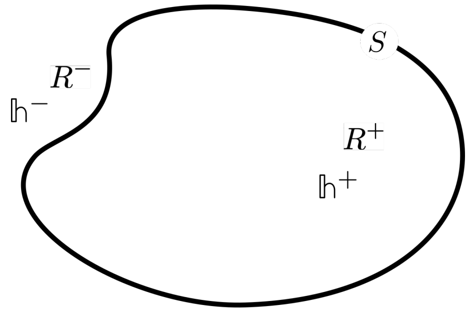

Figure 4: The two subregions of Σ, i.e. $\Sigma^{\pm}$, with the respective horizontal perturbations $\mathbb{h}^{\pm}$ on each side, along with the separating surface S.

In the following we will provide an explicit reconstruction formula for the $\xi^{\pm}$, and hence for H, once the $h^{\pm}$ are given. This will be achieved under the assumption that the smooth, global, H exists. From this assumption we will deduce a constructive proof of its uniqueness (up to possible stabilizer and topological ambiguities). This is the content of this section. In a second moment we will come back to analyze the continuity condition also from a more constructive perspective. This will be done in section 4.2.

Let us chart a roadmap, consisting of three steps, for finding the $\xi^{\pm}$:

1. First, from the smoothness of H at S, we will deduce restrictions on the difference $h_+ - h_-$ at S. Combined with the horizontality of the regional $h^{\pm}$, the requirement of smoothness gives us conditions on the longitudinal and transverse derivatives of $(\xi^+ - \xi^-)$ at the boundary. In particular, the transverse condition states the equality of the derivatives normal to the boundary $\mathrm{D}_s\xi^+{}_{|S} = \mathrm{D}_s\xi^-{}_{|S}$, while the longitudinal condition allows us to solve for the difference $(\xi^+ - \xi^-)_{|S}$ in terms of the interface mismatch $(h^+ - h^-)_{|S}$, which is parallel to the boundary due to (96).

2. Second, imposing the horizontality of the global H, which is now guaranteed to be smooth at S, provides us with one extra condition on the bulk part of the $\xi^{\pm}$'s, stating that the $\xi^{\pm}$ must be (covariantly) harmonic in their own regional domains.

3. Finally, we show that the information above suffices to uniquely and fully reconstruct the $\xi^{\pm}$ in terms of h^+ and h^-: indeed, we will manage to recast the system of equations above into (i) two homogeneous elliptic boundary-value problems, one per region, with generalized Neumann conditions of the type we already encountered in the study of ϖ, and (ii) another equation that fixes the generalized-Neumann boundary condition in terms of $(\xi^+ - \xi^-)_{|S}$ and thus, thanks to step 1, in terms of $(h^+ - h^-)_{|S}$. The equation of (ii) involves the inversion of an elliptic operator intrinsic to the boundary, as well as a combination of 'generalized Dirichlet-to-Neumann operators' $\mathcal{R}_{\pm}$ attached to the boundary (but associated to each region). The details on the nature of these standard operators are postponed to the next section.

The final result is provided by the unique solution to the following boundary value problem

$$\begin{cases} \mathrm{D}^2\xi^{\pm} = 0 & \text{in } R^{\pm}, \\ \mathrm{D}_s\xi^{\pm} = \Pi & \text{at } S, \end{cases} \tag{98}$$

with $\Pi \in \Lambda^0(S, \mathrm{Lie}(G))$ given by

$$\Pi = \left(\mathcal{R}_+^{-1} + \mathcal{R}_-^{-1}\right)^{-1} \left({}^S\mathrm{D}^2\right)^{-1} \; {}^S\mathrm{D}^a \iota_S^*(h^+ - h^-)_a. \tag{99}$$

Here ${}^S\mathrm{D}_a := (\iota_S^*\mathrm{D})_a$ is the covariant derivative intrinsic to S and $h_{ab} = (\iota_S^* g)_{ab}$ is the induced metric there. Similarly ${}^S\mathrm{D}^2 = h^{ab}\,{}^S\mathrm{D}_a\,{}^S\mathrm{D}_b$ is the covariant Laplace operator on S. The next section is devoted to the proof (and explanation) of the above formula.

4.1.2 Proof

For the first step, from the assumed continuity of H across S (97), we get that

$$(h_i^+ - h_i^-)_{|S} = -\mathrm{D}_i(\xi^+ - \xi^-)_{|S}. \tag{100}$$

Observe that a priori this equation not only imposes a series of conditions on our unknown $\xi^\pm$, but also demands the interface mismatch of our variables $h_i^\pm$ to be of a pure-gradient form. This condition is restrictive and will be analyzed in more detail in the following sections. For now, we take it for granted and focus on the consequences of this equation on $\xi^\pm$.

We start by decomposing this equation into its transverse and longitudinal components with respect to S. Now, since the component of $h^\pm$ transverse to S vanishes because of regional horizontality (96), contracting (100) with s^i we obtain that the normal derivatives of $\xi^\pm$ at S must match

$$\mathrm{D}_s(\xi^+ - \xi^-)_{|S} = 0. \tag{101}$$

Therefore, taking the boundary divergence of the pullback of (100) to S (i.e. effectively contracting with ${}^S\mathrm{D}^a$) we find that $(\xi^+ - \xi^-)_{|S}$ is solely determined by the mismatch of the two horizontals at the boundary (recall that $\partial S = \emptyset$):

$$(\xi^+ - \xi^-)_{|S} = -\left({}^S\mathrm{D}^2\right)^{-1}\,{}^S\mathrm{D}^a \iota_S^*(h^+ - h^-)_a, \tag{102}$$

This concludes step 1 of the proof outlined above.

Now we move to the step 2: assuming that the global region Σ has no boundaries, by smearing the global horizontality condition, $\mathrm{D}^i H_i = 0$, with H given by (97), we obtain:

$$\int_\Sigma \mathrm{Tr}\Big[\sigma\,\mathrm{D}^i\Big((h_i^+ + \mathrm{D}_i\xi^+)\Theta_+ + (h_i^- + \mathrm{D}_i\xi^-)\Theta_-\Big)\Big] = 0 \tag{103}$$

for any $\sigma \in C^\infty(\Sigma, \mathrm{Lie}(G))$. Now, thanks to the identity $\partial_i\Theta_\pm = \mp s_i\delta_S$ (92) and to the regional horizontality conditions $\mathrm{D}^i h_i^\pm = 0 = s^i h^\pm_{i\,|S}$ (96), we get:

$$\int_{R^+} \mathrm{Tr}\Big[\sigma\,\mathrm{D}^2\xi^+\Big] + \int_{R^-} \mathrm{Tr}\Big[\sigma\,\mathrm{D}^2\xi^-\Big] - \oint_S \mathrm{Tr}\Big[\sigma\, s^i\mathrm{D}_i\left(\xi^+ - \xi^-\right)\Big] = 0, \tag{104}$$

where the last term above already vanishes due to (101). From the arbitrariness of σ, we obtain the last, bulk, condition mentioned in step 2 of the outline above. We thus deduce that the $\xi^\pm$ must satisfy the following elliptic boundary value problem

$$\begin{cases} \mathrm{D}^2\xi^\pm = 0 & \text{in } R^\pm \\ s^i\mathrm{D}_i\left(\xi^+ - \xi^-\right) = 0 & \text{at } S \\ (\xi^+ - \xi^-) = -\left({}^S\mathrm{D}^2\right)^{-1}\,{}^S\mathrm{D}^a\iota_S^*(h^+ - h^-)_a & \text{on } S \end{cases} \tag{105}$$

This concludes step 2. Now we must use the appropriate PDE tools to show that this boundary value problem determines $\xi^\pm$ in terms of the regional horizontal perturbations $\mathbb{h}^\pm$.

For step 3, we proceed as follows: start by setting

$$\Pi := s^i(\mathrm{D}_i\xi^\pm)_{|S}, \tag{106}$$

from the second equation in (105). Note that in possession of Π, we can determine $\xi^\pm$ by solving the boundary value problem given by (106) and the first equation of (105). In this way, given Π, $\xi^\pm$ are uniquely determined up to stabilizers, i.e. up to elements $\chi^\pm \in C^\infty(\Sigma, \mathrm{Lie}(G))$ such that $\mathrm{D}_i\chi^\pm = 0$ which are nontrivial only at reducible configurations. In the topologically simple case that we have analyzed so far, this is the only ambiguity present in the determination of $\xi^\pm$. We postpone the discussion of reducible configurations and of nontrivial topologies until sections 4.3.2 and 4.7, respectively.

Now, to determine Π, we introduce generalized *Dirichlet-to-Neumann operators* (see e.g. [59] and references therein), $\mathcal{R}_\pm$. In each region, such operators map Dirichlet conditions for a (gauge-covariantly) harmonic function to the corresponding (gauge-covariant) Neumann conditions. In brief, for a given bounded region, $\mathcal{R}$ functions as follows: a given harmonic function with Dirichlet conditions—these conditions are the input of $\mathcal{R}$—will possess a certain normal derivative at the boundary; i.e. will induce certain Neumann conditions there—these conditions are the output of $\mathcal{R}$. But let us be more explicit.

In general, for a manifold with boundary S and outgoing normal s^i, we define the Dirichlet-to-Neumann operator $\mathcal{R}$ by

$$\mathcal{R}(u) := s^i\mathrm{D}_i(\zeta_u)_{|S} \tag{107}$$

where ζ_u is the unique (gauge-covariantly) harmonic Lie-algebra-valued function defined by the elliptic Dirichlet boundary value problem: $\mathrm{D}^2\zeta_u = 0$ with $(\zeta_u)_{|S} = u$. Notice that the subscript u encodes the *Dirichlet* boundary condition employed. Using superscripts to denote (gauge-covariant) Neumann boundary conditions, we would have by definition $\zeta^{\mathcal{R}(u)} \equiv \zeta_u$. Moreover, since the corresponding Neumann problems also have unique solutions of stabilizers), $\mathcal{R}$ is invertible, i.e. $\zeta^\Pi \equiv \zeta_{\mathcal{R}^{-1}(\Pi)}$. We can thus define $\mathcal{R}_\pm$ associated to $R^\pm$ with boundaries $\partial R^\pm = S$, and their inverses $\mathcal{R}_\pm^{-1}$.

Now, from (106) and the fact that ξ is itself (gauge-covariantly) harmonic from the first equation of (105), we have

$$\xi^\pm = {}^{(\pm)}\zeta^{\pm\Pi} \equiv {}^{(\pm)}\zeta_{\mathcal{R}_\pm^{-1}(\pm\Pi)} \tag{108}$$

where the back-superscript $(\pm)$ indicate whether the respective covariantly harmonic functions are defined over R^+ or R^-, respectively. The challenge now is to use the last equation of (105) to fix Π uniquely. Once this is done, (108) contains all the information we sought for the gluing.

Notice that there is a $\pm$ sign in the argument of $\mathcal{R}_\pm^{-1}$ in (108). This sign is due to the fact that, at S, $s^i\mathrm{D}_i\xi^+ = s^i\mathrm{D}_i\xi^-$ but s^i is the outgoing normal on one side and the ingoing normal on the other, so the conditions $s^i\mathrm{D}_i\xi^\pm = \Pi$ fix opposite Neumann conditions on the two sides. By the linearity of $\mathcal{R}$ we have

$$\mathcal{R}_\pm^{-1}(\pm\Pi) = \pm\mathcal{R}_\pm^{-1}(\Pi). \tag{109}$$

Hence, since by defintion $(\zeta_u)_{|S} = u$, together with (108) and (109) we have

$$(\xi^+ - \xi^-)_{|S} = \mathcal{R}_+^{-1}(\Pi) - \mathcal{R}_-^{-1}(-\Pi) = \Big(\mathcal{R}_+^{-1} + \mathcal{R}_-^{-1}\Big)(\Pi). \tag{110}$$

This gives us a relation between the (gauge-covariant) Neumann boundary condition Π and the difference of the Dirichlet boundary conditions $\xi_{\pm|S}$.

This relation finally allows us to provide a formula that fixes Π in terms of the boundary discrepancy of the regional horizontals $(h^+ - h^-)_{|S}$. That is, we insert (110) into the last of the equations (105) to obtain:

$$\Big(\mathcal{R}_+^{-1} + \mathcal{R}_-^{-1}\Big)(\Pi) = \left({}^S\mathrm{D}^2\right)^{-1}\ {}^S\mathrm{D}^a\iota_S^*(h^+ - h^-)_a. \tag{111}$$

This is the equation that Π has to satisfy. Since its solution is unique (as we will discuss in a moment), it also fixes $\xi^\pm$ uniquely through (108), thus subsuming the entire set of equations (105). This concludes step three.

For the uniqueness statement for Π to be meaningful, it is important to check that the operator $(\mathcal{R}_+^{-1} + \mathcal{R}_-^{-1})$ is invertible. That this is the case follows from $\mathcal{R}_\pm$ being positive self-adjoint operators, and from the relative sign appearing on the left-hand-side of (111)—a consequence of the sign in (109).

To show that the generalized Dirichlet-to-Neumann operators $\mathcal{R}_\pm$ are self-adjoint and have positive spectrum we proceed as follows. Consider again $\zeta_u \neq 0$ to be the unique solution to the problem $\mathrm{D}^2\zeta_u = 0$ in the bulk and $(\zeta_u)_{|S} = u$ at the boundary. Then, for any Lie-algebra valued functions u, v on the boundary, one has

$$\begin{aligned}\int_{\Sigma^+}\sqrt{g}\,g^{ij}\mathrm{Tr}(\mathrm{D}_i\zeta_u\mathrm{D}_j\zeta_v) &= -\int_{\Sigma^+}\sqrt{g}\,\mathrm{Tr}(\zeta_u\mathrm{D}^2\zeta_v) + \oint_{\partial\Sigma}\sqrt{h}\,s^i\mathrm{Tr}(\zeta_u\mathrm{D}_i\zeta_v)\\ &= \oint_S\sqrt{h}\,\mathrm{Tr}(u\mathcal{R}_+(v)) = \oint_S\sqrt{h}\,\mathrm{Tr}(\mathcal{R}_+(u)v). \qquad (112)\end{aligned}$$

Notice that the first step in (112) follows from an integration by parts and properties of the commutator under the trace.[39] The last line of (112) proves the self-adjointness of $\mathcal{R}_+$ with respect to the natural inner product $\langle u,v\rangle_S = \oint_S\sqrt{h}\,\mathrm{Tr}(uv)$, while setting $u = v$ in (112), gives positivity:

$$\oint_S\sqrt{h}\,\mathrm{Tr}(u\mathcal{R}_+(u)) \geq 0. \qquad (113)$$

At irreducible configurations, the equality holds if and only if $\zeta_u = 0$ and therefore if and only if $u = 0$. Similar manipulations lead to the analogous conclusion for $\mathcal{R}_-$. This concludes our proof.

4.2 A dimensional tower of compatibility conditions on $h^\pm$

Recall that, whereas the normal component of the continuity condition for H (100) is a condition on the $(\xi^+ - \xi^-)_{|S}$ only, its parallel component to S not only encodes a relation between $(\xi^+ - \xi^-)_{|S}$ and $(h^+ - h^-)_{|S}$, but also requires $(h^+ - h^-)_{|S}$ to be a pure gradient parallel to S. This is a necessary and sufficient condition on $h^\pm$ for there to exist a smooth global horizontal field H corresponding to their composition.

In this section we will discuss a more constructive procedure to understand this condition on $(h^+ - h^-)_{|S}$. This procedure can be iteratively applied to the "boundaries of the boundaries", opening a door to the discussion of the more general gluing schemes involving corners.

In a gauge theory, the space of the pullbacks to S of the fields in $\mathcal{A}$ defines a new "boundary configuration space", ${}^S\!\mathcal{A}$ which is isomorphic to the space of gauge fields intrinsic to S:

$${}^S\!A := \iota_S^* A \in {}^S\!\mathcal{A}. \qquad (114)$$

Moreover, the induced metric on S defines a supermetric ${}^S\mathbb{G}$ on ${}^S\!\mathcal{A}$. From this, one can define an SdW connection ϖ_S on ${}^S\!\mathcal{A}$ and hence, via pullback, on ${}^S\Phi = \mathrm{T}({}^S\!\mathcal{A})$. Now, thanks to the second of the equations (96), i.e. $s^i\mathbb{h}_i = 0$, the difference between two

[39] The following identity is valid for any smearing $\sigma \in C^\infty(\Sigma, \mathrm{Lie}(G))$:

$$\mathrm{Tr}\Big(-\sigma\,\partial^i\mathrm{D}_i\zeta + g^{ij}[A_i,\sigma]\mathrm{D}_j\zeta\Big) = \mathrm{Tr}\Big(-\sigma\,\partial^i\mathrm{D}_i\zeta - g^{ij}\sigma[A_i,\mathrm{D}_j\zeta]\Big) = \mathrm{Tr}\Big(-\sigma\,\mathrm{D}^2\zeta\Big).$$

generic[40] horizontal perturbations $\mathbb{h}^\pm$ defines, without any loss of information, a vector field intrinsic to the boundary:

$$ {}^S\mathbb{Y} := \oint_S \iota_S^*(h^+ - h^-)\frac{\delta}{\delta({}^S A)} \in \mathrm{T}_{({}^S A)}({}^S\mathcal{A}). \tag{115} $$

This vector field can be decomposed via ϖ_S into its horizontal and vertical parts *within* ${}^S\mathcal{A}$:

$$ {}^S\mathbb{Y} = {}^S\mathbb{H} + \left({}^S\xi\right)^{\sharp_S}, \tag{116} $$

where the $\cdot^{\sharp_S}$ operation is the S-intrinsic analog of $\cdot^\sharp$. Given equations (115) and (116), then it becomes clear that the parallel component of the continuity condition for a fiducial boundary (100), is equivalent to demanding that ${}^S\mathbb{Y}$ has no horizontal component, i.e. ${}^S\mathbb{H} = 0$. Of course, in this case, the ${}^S\xi$ of (116) is identified with the $(\xi^- - \xi^+)$ of (100).

From these observations we conclude that *the parallel continuity condition is satisfied if and only if* ${}^S\mathbb{Y}$ *is purely vertical, that is if and only if* ${}^S\mathbb{Y} = \varpi_S^{\sharp_S}\left({}^S\mathbb{Y}\right)$. In such a case, this last equation is only a more formal way to write (100), with $(\xi^+ - \xi^-)_{|S} = -\varpi_S({}^S\mathbb{Y})$ being a rewriting of (102).

We conclude this section by observing that the parallel continuity condition bears an interesting possibility. Note that if S itself had corners, i.e. if it was subdivided into regions $S^\pm$ sharing a boundary, we could have repeated the same treatment for two possible horizontal differences, $({}^S h)^+ - ({}^S h)^-$, themselves arising from the difference of horizontals in a manifold of one higher dimension, as expressed in (116). This chain of descent to the boundaries of boundaries might become useful in discussions of more complex gluing patterns involving corners; a necessary extension for building general manifolds from fundamental building blocks. We conclude this section by noticing that this chain of descent is reminiscent of the BV-BFV formalism [21–23], but we will leave an investigation of these matters to future work.

4.3 Gluing of the gauge potential

4.3.1 Unique gluing of the gauge potential

We are now ready to apply the above results to the gluing of the perturbations of the gauge potential A. We include matter in the next section, and apply the construction to the elecric field in section 4.4.

Therefore, we consider

$$ \mathbb{X} = \int X\frac{\delta}{\delta A} \in \mathrm{T}_A\mathcal{A}, \tag{117} $$

and decompose it, and its regional restrictions, into their SdW-horizontal and -vertical components

$$ X = H + \mathrm{D}\Lambda \quad \text{and} \quad X^\pm = h^\pm + \mathrm{D}\lambda^\pm. \tag{118} $$

Physically, whereas Λ and $\lambda^\pm$ encode the "pure gauge" components of X in Σ and $R^\pm$ respectively, H and $h^\pm$ encode their physical components. Therefore, the gluing question can be rephrased as the following: given only the regional gauge invariant perturbations $h^\pm$, is the global gauge invariant perturbation[41] H *uniquely* reconstructed? And if yes, under which conditions is this possible?

[40] I.e. that do *not* have to necessarily satisfy the continuity condition (100).

[41] Notice that the theorem involves the perturbations of A (elements of $\mathrm{T}\mathcal{A}$) over a globally smooth, fixed, background configuration A.

The theorem of the previous sections states that—whenever possible—*the reconstruction of a continuous H from $h^\pm$ is indeed unique*, and no additional information is needed to perform the gluing.

In particular, the theorem provides an explicit formula (99) for the reconstruction of the gauge transformations $\xi^\pm$ that relate the regional and global horizontals according to

$$H = (h^+ + \mathrm{D}\xi^\pm)\Theta_+ + (h^- + \mathrm{D}\xi^-)\Theta_-, \tag{119}$$

where the $\xi^\pm$ were fully determined in (111) and (108), i.e. by a covariant Laplace equations with boundary conditions determined in terms of the mismatch $\iota_S^*(h^+ - h^-)$.

However, the derivation assumed the mismatch $\iota_S^*(h^+ - h^-)$ to be a pure (gauge-covariant) gradient intrinsic to S. As explained in the previous section, whether this is the case can be checked by considering an SdW connection ϖ_S intrinsic to S, and verifying whether $\iota_S^*(h^+ - h^-)$ is purely vertical with respect to ϖ_S. If this mismatch is not purely boundary-vertical, then there is a physical discontinuity in the magnetic flux across S, i.e. in F_{ab} (a, b are tangential indices over S).[42] It is interesting to observe that such a discontinuity is *not* the consequence of a distributional surface current density on S, which would rather contribute a discontinuity in $s^i F_{ia}$. Rather, it is the consequence of a distributional surface density of magnetic monopole charges.

However, postulating the configuration space of Yang-Mills theory to be fundamentally given by the space of smooth (or at once-differentiable) connections $\mathcal{A}$, we are implicitly excluding this possibility from the onset: the algebraic validity of the Bianchi identities $\mathrm{D}F[A] = 0$ excludes the existence of magnetic monopoles—and thus guarantees that the perturbations $h^\pm$ are always glueable.[43]

4.3.2 Gluing of the gauge potential, with matter: ambiguities, reducibility parameters, and charges

In the presence of matter, gluing is more subtle. Let us first introduce some notation. Let $\mathbb{h}^\pm = \mathbb{h}^\pm_A + \mathbb{h}^\pm_\psi$ and $\mathbb{H} = \mathbb{H}_A + \mathbb{H}_\psi$, be horizontals, which decompose according to

$$\begin{cases} H_A = (h^+_A + \mathrm{D}\xi^+)\Theta_+ + (h^-_A + \mathrm{D}\xi^-)\Theta_- \\ \\ H_\psi = (h^+_\psi - \xi^+\psi)\Theta_+ + (h^-_\psi + \xi^-\psi)\Theta_- \end{cases}. \tag{120}$$

and e.g.

$$\mathbb{H} = \mathbb{H}_A \oplus \mathbb{H}_\psi = \int H_A \frac{\delta}{\delta A} + \int H_\psi \frac{\delta}{\delta \psi}. \tag{121}$$

As above, we are here implicitly using the SdW connection to assess horizontality. It is important to note that the matter horizontal components $\mathbb{h}^\pm_\psi$ are then, in a sense, parasitic on the gauge-field: they are just the matter perturbations corrected by the

[42] More precisely, in a neighbourhood of S, the relation between the curvature and the perturbation is: $F_{ab}(A + X) - F_{ab}(A) = [F_{ab}(A), \Lambda] + {}^S\mathrm{D}_{[b}H_{a]} + \mathcal{O}(X^2)$, where the first term on the right-hand side is an inconsequential perturbation in the gauge (vertical) direction and the second is the physical perturbation. Thus, only if $(h^+ - h^-)_a = \mathrm{D}_a\Xi$ does ${}^S\mathrm{D}_{[b}(h^+ - h^-)_{a]} = [F_{ab}, \Xi]$ feed into the gauge ambiguity; otherwise, a physical discontinuity in the parallel curvature will emerge.

[43] Notice that, the discontinuity in the components $s^i F_{ia}$ of the magnetic field at S induced by the presence of surface currents is more subtle from a gluing perspective since it does not necessarily stem from a discontinuity of h_i (it could also be due to a discontinuity in its normal derivative). Given any vector field u in a neighbourhood of S that is tangent to S, and recalling that $h_s = 0$ by the horizontality condition, one has that the perturbation of $F^\pm_{su} \equiv s^i u^j F^\pm_{ij}$ is given by $s^i u^j \mathrm{D}_i h^\pm_j = \mathrm{D}_s h^\pm_u - h^\pm_j (\pounds_u s)^j$.

vertical displacement provided by the gauge sector. Namely, for a fermion field in the fundamental representation of $\mathcal{G}$ [26],

$$h_\psi = X_\psi - \varpi(\mathbb{X}_A)X_\psi. \tag{122}$$

where $\mathbb{X}_\psi$ and $\mathbb{X}_A$ denote arbitrary (not necessarily horizontal) matter and gauge-potential perturbations respectively. In other words, the $\mathbb{h}^\pm_\psi$ do not satisfy horizontality conditions of their own. In appendix D we provide an interpretation of this in terms of Dirac dressings.

Then, we see that $\mathbb{H}$ (and $\mathbb{h}^\pm$) is horizontal (regionally horizontal, respectively) if and only if $\mathbb{H}_A$ ($\mathbb{h}^\pm_A$, respectively) is. This means in particular that the above procedure aimed at the determination of $\xi^\pm$ is completely insensitive to the presence of matter, and can be applied in the same way. The hypothesis of continuity of the original global perturbation $\mathbb{X} = \mathbb{X}_A + \mathbb{X}_\psi$ ensures that the same $\xi^\pm$ needed to glue one field will work for the other as well.

Now, all previous results on gluing go through seamlessly, unless either one of the *regional* configurations of the gauge potential, i.e. $A^\pm = A_{|R^\pm}$, is reducible. If, say, $A^+ = \tilde{A}^+$ is reducible, then the resulting ambiguity in the reconstruction of ξ^+ will have no effect on the reconstruction of the global horizontal gauge potential H_A, but it *will* generically render the reconstruction of the horizontal matter field H_ψ ambiguous.[44] We will now spell this out.

Let us suppose, for definiteness, that only the regional configuration $A^+ = \tilde{A}^+$ is reducible by a single reducibility parameter, i.e. χ^+ such that $\tilde{\mathrm{D}}\chi^+ = 0$, while A^- is not reducible. (Generalizations are straightforward.) This means that we have a continuous 1-parameter family of solutions for ξ^+ that we write, by choosing an origin ξ^+_o and introducing the parameter r (depending on the charge group), as

$$\xi^+_r := \xi^+_o + r\chi^+ \qquad r \in \mathbb{R} \text{ or } \mathbb{C}. \tag{123}$$

Then, two distinct possibilities are given: either ψ vanishes at S or it does not. The second case allows us to glue the two perturbations together if and only if we can find an r such that

$$\xi^+_r \psi_{|S} = \xi^- \psi_{|S}. \tag{124}$$

With the continuity hypothesis for the original global field perturbation $\mathbb{X} = \mathbb{X}_A + \mathbb{X}_\psi$, this equation would then fix the global ambiguity, but for generic values of $\psi_{|S}$ no solution exists.[45] If no solution exists, it means that the two perturbations are not glueable, i.e. they do not descend from a global smooth perturbation.

Conversely, in the first case, which is realized if ψ^+ vanishes at S, gluing of the two perturbations $\mathbb{h}^\pm$ is possible for any r but will give rise to *physically distinct global perturbations.*

To see this, we observe that the 1-parameter family of global horizontal perturbations

[44] This is always the case in QED, where we can always add constants $c^\pm$ to the reconstructed $\xi^\pm$ and where a constant phase shift will affect the Dirac fermions, unless they vanish. In a non-Abelian theory, the zoology is more complicated, and will depend on the gauge group as well as the type of matter fields (fundamental, adjoint, etc).

[45] These compatibility requirements between χ^+ and ψ^+ could be further formalized in terms of the kernel of the Higgs functional connection introduced in [26]. However, the presence of distributional charged matter at S—as manifested over e.g. an idealized conducting plate—generally blocks the possibility of a smooth gluing *of the electric field*, E, discussed in the following section.

corresponding to ξ_r^+, i.e.[46] $\mathbb{H}^r = \mathbb{H}^r_A \oplus \mathbb{H}^r_\psi$, is given by

$$H^r_A \equiv H^o_A \qquad \text{and} \qquad H^r_\psi = H^o_\psi - r\chi^+\psi\,\Theta_+. \tag{125}$$

Note that each of the $\mathbb{H}^r$ for different r is horizontal—hence physical, according to our identification.

Now, two possible situations are given: either χ^+ stabilizes ψ^+ throughout R^+, or it does not.

If ψ^+ is also stabilized (for this we need $G = \mathrm{SU}(N \geq 3)$ if $\psi \neq 0$ in R^+, see [26, Sec.7]), then uniqueness of the reconstructed global radiative mode is untouched: even if the regional gauge transformations $\xi^\pm$ are ambiguous, $\mathbb{H}$ of (120) will not be since in this case $\mathbb{H}^r \equiv \mathbb{H}^o$. The generally quite restrictive condition of χ^+ stabilizing ψ^+ trivially applies if matter is absent from R^+, in which case $\mathbb{H} = \mathbb{H}_A$ is clearly unaffected by χ^+ such that $\tilde{\mathrm{D}}\chi^+ = 0$.

If ψ^+ is *not* stabilized by χ^+, on the other hand, the resulting $\mathbb{H}^r$ are indeed *distinct* from one another. This setup formalizes 't Hooft's beam splitter thought experiment [60], and can be used to provide a concrete example for the considerations of Wallace and Greaves, characterizing "symmetries with direct empirical significance" (aka DES) [61].

To prove that the ensuing states are regionally indistinguishable but globally distinct, let us consider the following simplified scenario: $R^\pm$ contain one charged particle each, located at $x^\pm \in R^\pm$ and A^+ admits a reducibility parameter χ^+. Denoting the particle's spinorial configurations[47] by $|\psi^\pm\rangle$, we consider then the following global *gauge-invariant* Wilson-line observable (with obvious notation):

$$W = \langle\psi^-|\mathrm{Pexp}\left(\int_\gamma A\right)|\psi^+\rangle, \tag{126}$$

where γ is some path connecting across S the positions $x^\pm \in R^\pm$ of the charged particles $|\psi^\pm\rangle$. Now, if we "unglue" the two regions, perform the (infinitesimal) gauge transformation χ^+ in R^+ and glue back (which as we saw above is a seamless operation), we will find that: whereas $\mathrm{Pexp}\left(\int_\gamma A\right)$ and $|\psi^-\rangle$ have not changed at all, $|\psi^+\rangle$ has changed by the (infinitesimal) amount $\delta_{\chi^+}|\psi^+\rangle = \chi^+|\psi^+\rangle$; in turn this means that the global observable W is able to distinguish the two global states, since generically

$$\delta_{\chi^+}W = \langle\psi^-|\mathrm{Pexp}\left(\int_\gamma A\right)\chi^+|\psi^+\rangle \neq 0. \tag{127}$$

Of course, this construction is strictly related to the ability of defining a charge for the $|\psi^+\rangle$ on the reducible background A^+, and confirms that *(regional) stabilizers must be attributed a different status than generic gauge transformation*, as discussed in section 3.8 (see in particular the last paragraph of 3.8.2).

4.4 Gluing of the electric field

We now turn our attention to the gluing of the electric field E. Let us first stress a somewhat trivial point: in the study of the purely regional properties of the YM fields

[46] We are using the following notation:

$$\mathbb{H} = \int H_A \frac{\delta}{\delta A} + H_\psi \frac{\delta}{\delta \psi} \equiv \mathbb{H}_A \oplus \mathbb{H}_\psi.$$

[47] The bra-ket notation is employed to ease the writing of the following formulae, it does not refer to any quantum treatment.

performed in the first part of this paper, we were not allowed to access the physics beyond a given region, say R^+. In this context, we argued that such physics was "summarized" by the electric flux f at S. However, now that we are considering the gluing of regions, we *do* have access to the physics beyond R^+: it is simply the physics within R^-. This trivial observation allows us to shift viewpoints and ask questions that were not "allowed" from a purely regional perspective. This shift will shed further light on the quasi-local properties of the YM fields.

We start by recalling the representation of the electric field as a configuration-space vector $\mathbb{E}$:

$$\mathbb{E} = \int E \frac{\delta}{\delta A} \in \mathrm{T}_A \mathcal{A} \subset \Phi, \tag{128}$$

and (the components) of the global and regional SdW decompositions of $\mathbb{E}$

$$E = \dot{A}_\perp + \mathrm{D}\varphi \quad \text{and} \quad E^\pm = \dot{A}^\pm_\perp + \mathrm{D}\varphi^\pm. \tag{129}$$

We emphasize that, as was the case with $h^\pm$ and $\lambda^\pm$ in relation to H and Λ in (96), $\varphi^\pm$ is *not* the regional restriction[48] of φ to $R^\pm$, and similarly $\dot{A}^\pm_\perp$ is *not* the regional restriction of $\dot{A}_\perp$ to $R^\pm$; instead,

$$\begin{cases} \varphi = (\varphi^+ - \eta^+)\Theta_\pm + (\varphi^- - \eta^-)\Theta_- \\ \dot{A}_\perp = (\dot{A}^+_\perp + \mathrm{D}\eta^+)\Theta_+ + (\dot{A}^-_\perp + \mathrm{D}\eta^-)\Theta_- \end{cases} \tag{130}$$

where, according to the theorem of section 4.1, the $\eta^\pm$ are fully determined by the mismatch of $(\dot{A}^+_\perp - \dot{A}^-_\perp)_{|S}$; the appropriate behaviour of φ merely follows. Notice also that the electric flux f through S corresponds precisely to $\Pi = f$ of (99) in our main gluing theorem in section 4.1.

In the case of the electric field, we do not interpret the SdW vertical component φ of $\mathbb{E}$ as a pure-gauge quantity, but as a Coulombic component of the electric field, while $\dot{A}_\perp$ is what we called its radiative component.

Therefore, equation (130) states that the Coulombic/radiative split of the electric field depends on the choice of region in which the split is performed. In light of section 3.6.2, this should not come as a surprise. It is nonetheless worth revisiting those considerations in the context of gluing.

In this context, as we stressed at the beginning of this section, we do have access to both R^+ *and* to its complement R^-. This simple observation crucially affects the physical interpretation and properties that we ascribe to both the flux f through the interface S—from the regional perspective f was interpreted as summarizing the physics beyond e.g. R^+—and to the (regional) Coulombic components of E in general.

To ascertain more specifically how the interpretation of f and the Coulombic components are affected, it is particularly instructive to consider first the case without matter. For simplicity we will also assume that the simply connected global region $\Sigma = R^+ \cup_S R^-$ has no boundary, i.e. $\partial\Sigma = \emptyset$. It then follows that $\varphi \equiv 0$, and therefore, according to (130) $\varphi^\pm = \eta^\pm$. Since $\eta^\pm$ are entirely functions of $\dot{A}^\pm_{\perp|S}$, it follows that all components of the global electric field are determined solely by its regional radiatives. Indeed, for a globally radiative electric field (i.e. no global boundary and no charges), $E = \dot{A}_\perp$ and $E_{|R^\pm} = \dot{A}^\pm_\perp - \mathrm{D}\eta^\pm$ with $\eta^\pm$ functionals of $(\dot{A}^+_\perp - \dot{A}^-_\perp)_{|S}$ only.[49] Thus, in this case, once *both* regional radiatives are known, even the *regional* Coulombic components are completely

[48] Having run out of symbols, we could not use the same capitalized vs. lower case variables to indicate that relationship.

[49] Again, as already stressed, it is important to note that regional restriction and horizontal projection do not commute, thus e.g.: $\dot{A}^\pm_\perp \neq \dot{A}_{\perp|R^\pm}$.

determined. Indeed, knowledge of a reconstruction of E includes any knowledge of f, which is thus no longer an independent degree of freedom once the radiative modes are accessible in *both* regions.

Thus, in this case—when the larger (glued) region R has no boundary,—the *regional radiative modes encode the entire physical information of the system.*

The addition of charged matter does not change this conclusion: the matter fields merely join the radiative modes in determining φ and therefore in fixing the full physical content of the system.

In sum, once the radiative modes are accessible in both regions, the role of the flux f at S—i.e. to regionally fix $\varphi^\pm$—is taken over by $(\dot{A}^+_\perp - \dot{A}^-_\perp)_{|S}$. Thus f—which is often claimed to embody the "new boundary degrees of freedom" [18] or their momenta [10]—also constitutes a piece of redundant information for the final result of the gluing: it only shows up when encoding one subregion's ignorance of the other, i.e. when we do not have access to both radiatives, $\dot{A}^\pm_\perp$.

Explicitly, playing the role of Π in the theorem of section 4.1, the flux is given by

$$\Big(\mathcal{R}_+^{-1} + \mathcal{R}_-^{-1}\Big)(f) = \big({}^S\mathrm{D}^2\big)^{-1}\ {}^S\mathrm{D}^a \iota_S^*(\dot{A}^+_\perp - \dot{A}^-_\perp)_a. \tag{131}$$

This conclusion is only challenged in the presence of nontrivial cohomological 1-cycles in the Cauchy surface, a point exemplified in section 4.7.

Concerning the analogues of the continuity conditions explored in section 4.2 for the gauge potential, we observe that on-shell the electric field is continuous across S if and only if there is no *distributional* charge density there. Such a charge density would create a discontinuity in the fluxes $E^\pm_{s}{}_{|S} \equiv f^\pm$. No analogous physical discontinuity can be found in the components of the electric field parallel to S. Moreover, if there is no charge density and therefore E is continuous, the difference $(\dot{A}^+_\perp - \dot{A}^-_\perp)_{|S}$ is the same as the difference $(\mathrm{D}_i\varphi^+ - \mathrm{D}_i\varphi^-)_{|S}$. Since the latter is always of the pure-gradient form, the radiative parts of a continuous electric field (on-shell) satisfy the analogue of (100) and are therefore guaranteed to be glueable.

4.5 On the energy of radiative and Coulombic modes

The radiative/Coulombic split of E satisfies a monotonocity property, which roughly states that *in a composite region $\Sigma = R^+ \cup_S R^-$, a larger portion of the energy is attributed to the radiative part of the electric field than it is in the disjoint union of R^+ and R^-; the converse holds for its Coulombic part.* This section is devoted to establishing and interpreting this result.

Let us start by writing the energy $\mathcal{H}$ contained in Σ. We decompose this energy into its electric (kinetic) and magnetic (potential) parts,

$$\mathcal{H} = \mathcal{E} + \mathcal{B} = \int_\Sigma \sqrt{g}\,\mathrm{Tr}(E^i E_i) + \int_\Sigma \sqrt{g}\,\tfrac{1}{2}\mathrm{Tr}(F^{ij}F_{ij}) \tag{132}$$

Since F is fully determined by the background value of A (which undergoes no SdW splitting), we will henceforth focus on the electric contribution. This can be written more abstractly as

$$\mathcal{E} = ||\mathbb{E}||^2 = ||\mathbb{E}||^2_+ + ||\mathbb{E}||^2_-, \tag{133}$$

with $||\cdot||$ and $||\cdot||_\pm$ the SdW-norms on Σ and $R^\pm$ respectively.[50]

[50] E.g. $||\mathbb{E}||^2_+ = \mathbb{G}_{R^+}(\mathbb{E},\mathbb{E}) = \int_{R^+}\sqrt{g}\,g^{ij}\mathrm{Tr}(E_iE_j)$.

Consider now the radiative/Coulombic decomposition of E, and recall that it corresponds to a horizontal/vertical orthogonal decomposition with respect to the SdW supermetric. Then,

$$||\mathbb{E}||^2 = ||\dot{\mathbb{A}}_\perp||^2 + ||\varphi^\sharp||^2, \tag{134}$$

and similarly on $R^\pm$. Applying the same decomposition to the second gluing formula of (130) gives

$$\begin{aligned}||\dot{\mathbb{A}}_\perp||^2 &= ||\dot{\mathbb{A}}^+_\perp + (\eta^+)^\sharp||^2_+ + ||\dot{\mathbb{A}}^-_\perp + (\eta^-)^\sharp||^2_- \\ &= ||\dot{\mathbb{A}}^+_\perp||^2_+ + ||\dot{\mathbb{A}}^-_\perp||^2_- + ||(\eta^+)^\sharp||^2_+ + ||(\eta^-)^\sharp||^2_- \\ &\geq ||\dot{\mathbb{A}}^+_\perp||^2_+ + ||\dot{\mathbb{A}}^-_\perp||^2_- . \end{aligned} \tag{135}$$

From the additivity of $\mathcal{E}$ (133), the gluing formula (130) and the equation above, it follows that the total Coulombic contribution correspondingly decreases by the same amount:[51]

$$\begin{aligned}||\varphi^\sharp||^2 &= ||(\varphi^+)^\sharp||^2_+ + ||(\varphi^-)^\sharp||^2_- - ||(\eta^+)^\sharp||^2_+ - ||(\eta^-)^\sharp||^2_- \\ &\leq ||(\varphi^+)^\sharp||^2_+ + ||(\varphi^-)^\sharp||^2_- . \end{aligned} \tag{136}$$

We have thus proved (and qualified) our statement above.

So, if to the radiative part of E we ascribe the kinetic energy of the radiative modes, the following question arises: which new radiative field strengths are included in Σ that are not present in the disjoint union of R^+ and R^-?

The answer lies at the interface S: the regional Coulombic and vertical adjustments, $\eta^\pm$ and $\xi^\pm$, respectively, from the global perspective are additions to the radiative sector of Σ with respect to the radiative sectors of $R^\pm$. Although supported on the whole regions $R^\pm$ respectively, these new components, are completely determined by the mismatch at S of the two regional radiative modes, $\dot{A}^\pm_{\perp|S}$ (or $h^\pm{}_{|S}$, resp). In other words, the new global radiative field strength that emerges on Σ upon gluing $R^\pm$ is entirely determined by the standard regional radiative modes at the boundary. However, it is important to stress that these new global contributions are not encoded in either region, since they depend on the mismatch at S of the two regional components. *Thus, in this precise sense, we can claim that there is an additional component to the global radiative field strength: it results from the gluing and arises from the relation of the two subsystems at their common boundary.*

4.6 Gluing of the symplectic potentials

It is now straightforward to study the gluing of the symplectic potential. As above, we focus on the situation where a D-dimensional simply connected hypersurface without boundary $\partial\Sigma = \emptyset$ is split into two regions $R^\pm$ sewn at $S = \partial R^\pm$, i.e. $\Sigma = R^+ \cup_S R^-$.

In this case, from (54), the total symplectic potential reads

$$\theta = \int_\Sigma \sqrt{g}\left\{g^{ij}\mathrm{Tr}\Big(E_i \mathbb{d} A_j\Big) - \overline{\psi}\gamma^0 \mathbb{d}\psi\right\} \approx \int_\Sigma \sqrt{g}\left\{g^{ij}\mathrm{Tr}\Big(\dot{A}^\perp_i \mathbb{d}_\perp A_j\Big) - \overline{\psi}\gamma^0 \mathbb{d}_\perp\psi\right\} = \theta^\perp, \tag{137}$$

where $\theta \approx \theta^\perp$ since $\partial\Sigma = \emptyset$.

[51] This follows from the comparison of the following two expressions

$$\begin{cases}||\mathbb{E}||^2 &= ||\dot{\mathbb{A}}_\perp - \varphi^\sharp||^2 = ||\dot{\mathbb{A}}_\perp||^2 + ||\varphi^\sharp||^2 = ||\dot{\mathbb{A}}^+_\perp||^2_+ + ||\dot{\mathbb{A}}^-_\perp||^2_- + ||(\eta^+)^\sharp||^2_+ + ||(\eta^-)^\sharp||^2_- + ||\varphi^\sharp||^2 \\ ||\mathbb{E}||^2 &= ||\mathbb{E}||^2_+ + ||\mathbb{E}||^2_- = ||\dot{\mathbb{A}}^+_\perp - (\varphi^+)^\sharp||^2_+ + ||\dot{\mathbb{A}}^-_\perp - (\varphi^-)^\sharp||^2_+ = ||\dot{\mathbb{A}}^+_\perp||^2_+ + ||(\varphi^+)^\sharp||^2_+ + ||\dot{\mathbb{A}}^-_\perp||^2_- + ||(\varphi^-)^\sharp||^2_+ . \end{cases}$$

Now, θ can also be decomposed into $\theta = \theta^+ + \theta^-$ simply by factorizing the integration domain in the first expression above,

$$\theta^\pm = \int_{R^\pm} \sqrt{g} \left\{ g^{ij} \mathrm{Tr}\Big(E_i \mathbb{d} A_j \Big) - \overline{\psi}\gamma^0 \mathbb{d}\psi \right\}. \tag{138}$$

Each of these regional contributions can be written in the SdW decomposition following (54):

$$\theta^\pm \approx \int_{R^\pm} \sqrt{g} \left\{ g^{ij} \mathrm{Tr}\Big(\dot{A}_i^{\perp(\pm)} \mathbb{d}_{\perp(\pm)} A_j \Big) - \overline{\psi}\gamma^0 \mathbb{d}_{\perp(\pm)}\psi \right\} \pm \oint_S \sqrt{h}\, \mathrm{Tr}\Big(f \varpi_\pm \Big). \tag{139}$$

where $\perp(\pm)$ denotes that the SdW decomposition intrinsic to $R^\pm$ has been respectively used, and the sign of the last term depends on the fact that, in $f = s^i E_{i|S}$, the normal s^i to S is outgoing for R^+ and ingoing for R^-. Thus, we find

$$\theta \approx \theta^{\perp(+)} + \theta^{\perp(-)} + \oint_S \sqrt{h}\, \mathrm{Tr}\Big(f (\varpi_+ - \varpi_-) \Big). \tag{140}$$

The results of section 4.1, and in particular equation (102), can be applied[52] to $\varpi_\pm$ to obtain

$$(\varpi_+ - \varpi_-)_{|S} = -({}^S\mathrm{D}^2)^{-1} {}^S\mathrm{D}^a \iota_S^* (\mathbb{d}_{\perp(+)} A - \mathbb{d}_{\perp(-)} A)_a \equiv -\frac{{}^S\mathrm{D}[\mathbb{d}_\perp A]_S^\pm}{{}^S\mathrm{D}^2}. \tag{141}$$

Here, we have introduced a new short-hand symbol for the interface difference of a given quantity $\bullet$, namely $[\bullet]_S^\pm$. For more compact notation, we have also schematically denoted the inverse operator by a fraction $({}^S\mathrm{D}^2)^{-1}(\bullet) := \frac{\bullet}{({}^S\mathrm{D}^2)}$. Similarly, we recall[53] (131)

$$\Big(\mathcal{R}_+^{-1} + \mathcal{R}_-^{-1} \Big)(f) = -({}^S\mathrm{D}^2)^{-1} {}^S\mathrm{D}^a \iota_S^* (\dot{A}^{\perp(+)} - \dot{A}^{\perp(-)})_a \equiv -\frac{{}^S\mathrm{D}[\dot{A}^\perp]_S^\pm}{{}^S\mathrm{D}^2}. \tag{142}$$

Hence, combining these results, and remembering that $\mathcal{R}_\pm$ is self-adjoint, we find the following gluing formula for the composition of the symplectic potential:

$$\theta \overset{\partial\Sigma=\emptyset}{\approx} \theta^\perp = \theta^{\perp(+)} + \theta^{\perp(-)} + \oint_S \sqrt{h}\, \mathrm{Tr} \left(\frac{{}^S\mathrm{D}[\dot{A}^\perp]_S^\pm}{{}^S\mathrm{D}^2} \Big(\mathcal{R}_+^{-1} + \mathcal{R}_-^{-1} \Big)^{-1} \frac{{}^S\mathrm{D}[\mathbb{d}_\perp A]_S^\pm}{{}^S\mathrm{D}^2} \right). \tag{143}$$

This formula emphasizes the role of the "new" radiative degrees of freedom that emerge upon gluing as discussed in section 4.5: the *mismatch* of the horizontal/radiative modes at the interface S plays—from the global perspective—the role of a new horizontal/radiative dof which is not present in *either* region, but is instead—from a strictly regional perspective—summarized in the vertical part of $\theta^\pm$ (cf. (54)).

In sum, as the symplectic treatment confirms, the often-heard "gauge-turned-physical at the boundary" narrative is superseded by one in which certain global radiative modes can only be revealed upon gluing. This is because, due to their nonlocal nature, certain global radiative modes are not accessible as regional radiative modes within either region taken alone, but are rather encoded in the *mismatch* between the two regional radiative modes at the common interface S. Consistently, upon gluing of these new radiative modes, we are no longer required to superselect the electric flux f trough S (which was conjugate to the pure-gauge part of the gauge potential). Indeed, f is reconstructed from the mismatch of the *electric* radiative modes (see section 3.6.4 for the discussion on f's superselection and (131) for its reconstruction).

[52]This is entirely compatible with the standard definition of ϖ, which can be seen by noticing that given a vector $\mathbb{Y} \in \mathrm{T}_A\mathcal{A}$: $\mathbb{i}_\mathbb{Y} \mathbb{d}_\perp A_i = H_i$, $\mathbb{i}_\mathbb{Y}\varpi = \Lambda$, and similarly $\mathbb{i}_\mathbb{Y}\mathbb{d}_{\perp(\pm)} A_i = h_i^\pm$, $\mathbb{i}_\mathbb{Y}\varpi_\pm = \lambda^\pm$.

[53]The notation used for (131) has been here (slightly) adapted to fit with the notation used in the rest of this section. We apologize with the reader for the inconvenience.

4.7 Example: 1-dimensional gluing and the emergence of topological modes

In this final section we work out a simple example, implementing the gluing of 1-dimensional intervals. Two cases are given: two closed intervals are glued into a larger interval, and one interval is glued on itself to form a circle. This second case falls outside the simply-connected setup we adopted for the rest of the paper. Nonetheless, this case allows us to easily discuss, without introducing a host of new technologies, the emergence of new global (or "topological") degrees of freedom associated to the non trivial cohomology of the circle.

4.7.1 Gluing into an interval

Let us start by considering two closed intervals $I^+ = [0,1]$ and $I^- = [-1,0]$, that we shall glue together to form a new closed interval $I = [-1,1]$. We shall see that, since on the interval the gauge potential must be pure gauge, the regional horizontal perturbations must vanish—a fact consistently encoded by our gluing formula. Although somewhat trivial, this example helps us set the stage for the gluing into a circle.

We first characterize the 1-dimensional gauge fields and their horizontal perturbations. One dimensional gauge fields are always locally pure gauge,

$$A^\pm = g_\pm^{-1} \mathrm{d} g_\pm \tag{144}$$

for $g_+(x) = \mathrm{Pexp} \int_0^x A$ on I^+ and similarly on I^-, where we choose g_- such that $g_-(0) = \mathbb{1}$ too ($x = 0$ is where the gluing takes place). Since in one dimension $s^i h_{i|S} = 0$ implies $h_{|S} = 0$, SdW-horizontal perturbations $\mathbb{h}^\pm$ in $I^\pm$, according to (96) must satisfy the equations

$$\mathrm{D}^\pm h^\pm = 0 \qquad \text{and} \qquad h^\pm{}_{|\partial I^\pm} = 0, \tag{145}$$

which can be rewritten in terms of $\tilde{h}^\pm := g_\pm h^\pm g_\pm^{-1}$ as $\partial \tilde{h}^\pm = 0$ and $\tilde{h}^\pm{}_{|\partial I^\pm} = 0$. Now, these equations can be solved to give $\tilde{h}^\pm = 0$ and hence

$$\mathbb{h}^\pm = 0. \tag{146}$$

This is solely an immediate consequence of the pure gauge character of all 1-dimensional configurations, and therefore all perturbations over topologically trivial regions must be purely vertical.

Applying these results on the horizontal/vertical decomposition of fields on the interval to the electric field, we deduce that on the interval all electric fields are purely Coulombic. As per section 4.4, without any knowledge of regions outside of the interval $I^+ \equiv R^+$, this is entirely characterized by the charge content of the interval and by f at its boundary S. The latter encodes our ignorance of the outside of the region.

Let us now analyze the gluing. Again, the global horizontal vector is denoted by

$$H = (h^+ + \mathrm{D}\xi^+)\Theta_+ + (h^- + \mathrm{D}\xi^-)\Theta_- = \mathrm{D}\xi^+\Theta_+ + \mathrm{D}\xi^-\Theta_- \tag{147}$$

as in (97). The relevant equations for gluing arise as in (105), with a couple of new features: (i) there is no analogue to the last equation of (105), since h_i has only one component that is transverse to the zero-dimensional gluing surface S; and (ii) we have to add one equation per global boundary of the interval $I = [-1,1]$, since the total horizontal vector has now (two) endpoint boundaries, $\partial\Sigma \equiv \partial I = \{-1\} \cup \{+1\} \neq \emptyset$.

Thus,

$$\begin{cases} \mathrm{D}^2\xi^\pm = 0 & \text{in } I^\pm \\ \mathrm{D}\left(\xi^+ - \xi^-\right) = 0 & \text{at } \partial I^+ \cap \partial I^- = \{0\} \\ \mathrm{D}\xi^\pm = 0 & \text{at } \partial I = \{\pm 1\} \end{cases} \tag{148}$$

Now, again, by defining $\tilde{\xi}^\pm := g_\pm \xi^\pm g_\pm^{-1}$, we can turn the covariant derivatives into ordinary ones. This allows us to readily solve these equations. In fact, the bulk equations (the first of (148)) tell us that

$$\tilde{\xi}^\pm = \pm\tilde{\Pi}^\pm x + \tilde{\chi}^\pm, \tag{149}$$

where $\tilde{\chi}^\pm$ are constant functions valued in Lie(G) corresponding to two arbitrary reducibility parameters of the vanishing configuration $\tilde{A}^\pm = 0$. This is a concrete example of the discussion in the previous section.

Now, the second equation of (148) sets $\tilde{\Pi}^+ = -\tilde{\Pi}^-$, and the third one sets them equal to zero. Since the $\tilde{\chi}_\pm$ don't affect the value of the regional horizontal fields, we hence conclude that in this case the unique solution to the gluing problem at hand is $\xi^\pm = 0$ which readily leads to $\mathbb{H} = 0$, consistently with the general regional result (146). This concludes the gluing of two intervals $I^\pm$ into a larger one $I = [-1, 1]$.

4.7.2 Gluing into a circle

We now move on to the second case, where one interval, $I = [-\pi, \pi] \ni \phi$, has its ends glued to form a unit circle. To keep the two cases notationally distinct, we have denoted an element of the circle by ϕ, as opposed to x of the interval in the previous case. This case requires a little more care.

The idea is to split I into two intervals which overlap around $\phi = 0$, e.g. on the interval $U_\epsilon := (-\epsilon, \epsilon)$. Thus we consider $I^- = [-\pi, \epsilon)$ and $I^+ = (-\epsilon, \pi]$, so that we can glue at $\phi = \pm\pi$ according to the procedures of the above section, while matching the overlap of charts around $\phi = 0$ to close the interval into a circle.

This allows us to separate the problem of gluing from the problem of covering the circle. The latter is accomplished by overlapping open charts, with transition functions which appropriately match the gauge configuration.

Let us start by analyzing the background configuration $A^\pm$ on $I^\pm$. We assume, as in the previous sections, that the configurations $A^\pm$ join smoothly at $\phi = \pm\pi$.[54]

As above, $A^\pm$ are pure gauge, i.e. $A^\pm = g_\pm^{-1}\mathrm{d}g_\pm$ with $g_+(\pi) = g_-(-\pi)$. On the other hand, on U_ϵ, the configurations $A^\pm$ do not have to be equal; they need only be related by the action of a gauge transformation κ, the transition function. Since we are in 1-dimension, this does not constitute a restriction; one simply has $\kappa = g_-^{-1}g_+$.

Now, we move on to consider the horizontal perturbations. We shall find that the relevant horizontality equations for $\mathbb{h}^\pm$ involve boundary conditions only at $\phi = \pm\pi$, and the one for $\mathbb{H}$ does not involve boundary conditions at all. In particular no boundary conditions are imposed at the open-extrema of the intervals $I^\pm$. This is not because the intervals are open, but rather because there are no boundaries from the perspective of the global $\mathbb{H}$. But let us be more detailed.

We start from the observation that on the overlap region U_ϵ, generic perturbations $\mathbb{X}^\pm$ must be gauge related through $X^+ = \mathrm{Ad}_\kappa X^-$. This means that, using the appropriate partitions of unity over S^1, there is no difficulty, nor ambiguity, in the patching of the SdW inner products over I^+ and I^-: we obtain an inner product over S^1 between two

[54]It would be more appropriate to introduce new coordinates to glue at a specific value of the coordinate. We persist in this slightly sloppier, but more compact, language, merely flagging the possibility for confusion.

perturbations $\mathbb{X}^\pm$ and $\mathbb{Y}^\pm$ that satisfy the overlap condition we have just described. Recalling that SdW-horizontality is the requirement of being orthogonal to any purely vertical vector with respect to the SdW supermetric, we see that the horizontality condition for $\mathbb{H}$ does *not* involve boundary conditions at the non-glued boundaries of $I^\pm$, i.e. at $\phi = \pm\epsilon$. Of course, this was an expected result from the closed nature of the manifold on which $\mathbb{H}$ resides.

Focusing now on horizontal perturbations, it is easy to see that this discussion doesn't change the fact that $\mathbb{h}^\pm = 0$, since the manifold on which they reside still has boundaries at $\phi = \pm\pi$. Note moreover that $\mathbb{h}^\pm = 0$ implies that their matching on U_ϵ is automatic. However, this discussion leads us to a horizontality condition for $\mathbb{H}$ that is distinct from the one found for the gluing into an interval (148). Indeed, in the present case, we find

$$\begin{cases} \mathrm{D}^2\xi^\pm = 0 & \text{in } I^\pm \\ \mathrm{D}\left(\xi^+ - \xi^-\right) = 0 & \text{at } \phi = \pm\pi \end{cases} \tag{150}$$

with *no* extra conditions at $\phi = \pm\epsilon$. Hence, it is readily clear that the solutions for $\xi^\pm$ are here much less restricted than they were in the closed interval case considered above: in this case we find that

$$\xi^\pm = g_\pm^{-1}(\tilde{\Pi}\phi + \tilde{\chi}^\pm)g_\pm. \tag{151}$$

with the same, possibly non-vanishing, $\tilde{\Pi}$ for both the $\pm$ choices. From this we obtain,

$$H = g_\pm^{-1}\tilde{\Pi}g_\pm. \tag{152}$$

As for the background, matching the perturbed configurations in U_ϵ comes at no cost (since $\mathbb{h}_\pm = 0$).

In summary, we see that the gluing procedure has no unique solution in this case, as a consequence of the absence of a second "outer" boundary for the interval (which is glued into a circle). The second outer boundary is instead replaced by the chart matching.[55] We thus obtain a one-parameter family of solutions parametrized by an element $\tilde{\Pi} \in \mathrm{Lie}(G)$. This element constitutes the perturbation of the Wilson-loop observable around the circle (Aharonov-Bohm phase), which is precisely the unique physical degree of freedom present there. The existence of this new topological mode is of course related to the non-contractibility ($\pi_1(S_1) = \mathbb{Z} \neq 0$) of the circle. Also note that in our formalism this topological mode arises automatically and originates solely from the interplay between gauge and topology.

Similarly, application of these results to the gluing of the electric field on the circle leads to the following analogous result: $\eta^\pm$—i.e. the homologue of $\xi^\pm$ above—is now entrusted with the encoding of the global *radiative* mode of the electric field on the circle, even if the latter is locally of a pure Coulombic form. Then the analogue of $\tilde{\Pi}$ in equation (151) for $\eta^\pm$ is not free, but fixed by the electric flux $f = E_{s|S}$. Therefore, only in the topologically non-trivial case, would f come into its own. That is, only to the extent that f encodes possible global, i.e. topological, radiative modes which do not correspond to regional radiative modes. This consideration only partly endorses the attribution of "new

[55] The decoupling of chart transitioning and horizontal gluing can be made into a more general feature. For instance, had we wished to cut up the circle into three segments, we would divide the interval $[0, 2\pi]$ into three sets, $I_1 = [0, 2\pi/3], I_2 = [2\pi/3, 4\pi/3], I_3 = [4\pi/3, 2\pi]$, with $\mathbb{h}_i \in I_i$. Then we can cover the circle with three charts $U_{1,2,3}$, given in larger, but largely overlapping, domains: $D_1 = [0, 4\pi/3], D_2 = [\pi/3, 2\pi], D_3 = [4\pi/3, \pi/3]$. Then $\mathbb{h}_1$ and $\mathbb{h}_2$ glue entirely within the U_1 chart domain D_1; $\mathbb{h}_2$ and $\mathbb{h}_3$ similarly glue in D_2; and $\mathbb{h}_3, \mathbb{h}_1$ glue in D_3. In this way, one decouples the chart matching from the horizontal gluing; we can cyclically glue all $\mathbb{h}_i$'s first and find the appropriate chart transition later, independently. In that case, it is the cyclicity of the equations that yields one less condition. This type of concatenating construction can be extended to higher dimensional manifolds.

edge mode degrees of freedom" to boundaries [10, 18]. Namely, it grants such status only to those, *finitely many* degrees of freedom which encode information about a (global!) nontrivial first cohomology.[56]

5 Conclusions

5.1 Summary

Previous work on the functional connection form ϖ focused on three main themes [5, 16, 25, 26, 37]. Namely: (*i*) that it provides a regional decomposition of field perturbations into physical and gauge, (*ii*) that it identifies gauge-invariant, regional charges without the need for gauge-fixing, and (*iii*) that it provides a general and unifying notion of dressing (see appendix D).

After reviewing the basic technology of the framework, this paper advanced our previous work in two main directions: (*1*) we introduced and developed in sections 2 and 3 the connection form formalism within the $D+1$ formulation of the field dynamics—as opposed to the Euclidean covariant formulation of [26],—with particular attention to the properties of the symplectic potential; and (*2*) for the first time, we thoroughly studied the problem of gluing and derived, in section 4, the explicit formulae dictating its behaviour, for both the the gauge potential and the electric field. Finally, in appendix D, we proposed a new argument for the relevance of ϖ to dressings, through a generalization of Dirac's prescription to finite regions and in the non-Abelian setting.

5.1.1 The SdW-connection and symplectic geometry in configuration space

In treating the $D+1$ properties of the framework, we revived and expanded on the observations of [30–32, 34, 35] to argue that the choice of the Singer-DeWitt (SdW) connection is naturally inherited from the Lagrangian of the theories in question. Most importantly, and unlike all previous work, we have also encompassed boundaries in our analysis. In appendix D we complemented this result, by showing that the choice of the SdW connection can also be uniquely fixed by consistency with the Dirac dressing condition for non-Abelian fields in finite regions.

But, as far as choices of connection go, both the $D+1$ and the Euclidean picture have downsides.[57] In the Euclidean picture, everything is manifestly covariant, and the downside is the arbitrariness in the choice of a functional connection where none is canonically given.[58] Conversely, in the $D+1$ case, the downside is the manifest breaking of Lorentz symmetry: the canonical SdW connection we construct is anchored on a specific choice of foliation. This tie between gauge and Lorentz symmetry is vaguely reminiscent of [8, 17, 62]. However, in the finite-region setting, we notice that the choice of corners as the waist of causal diamonds of "finite observers" explicitly breaks Lorentz covariance. Whether a restriction of covariance (now understood as a refoliation invariance) to the causal diamond anchored at ∂R still applies thus seems a natural question to ask, but further investigations in this direction are left to future work.

[56] Of course, this distinction and the ensuing identification of finitely many topological modes cannot be performed at the regional level.

[57] The covariant and Lorentzian setting gives rise to hyperbolic equations for ϖ whose interpretation is less clear than when it is built from elliptic equations, as it is in the Euclidean covariant setting and in the D+1 framework.

[58] Having said that, the covariant SdW connection is *somewhat* natural also in the Euclidean context: it descends from a field-space supermetric that is uniquely fixed by ultralocality. See e.g. [27–29, 33, 36].

In the $D+1$ framework, we started our discussion of the SdW functional connection ϖ with its introduction on the space of gauge potentials, $A \in \mathcal{A}$; we then argued that ϖ has a natural pull-back to phase space $(A, E) \in \Phi \cong \mathrm{T}\mathcal{A}$. We then showed that on both spaces, $\mathcal{A}$ and Φ, the role of the functional connection is to split physical and pure gauge directions of the respective tangent bundles, interpreted as the spaces of linear perturbations of the gauge potential A and of the electric field E. Moreover, within Φ, we employed the SdW connection to find particularly useful coordinates on phase space, by splitting the electric field in terms of its "Coulombic" and "radiative" components.

Since both the symplectic potential and the SdW connection are derived from the Lagrangian of the theory, the symplectic potential inherits this radiative/Coulombic split: the radiative components of E are conjugated to horizontal components of the gauge potential, and the Coulombic components of E are conjugated to the vertical—i.e. pure gauge—components of the gauge potential. We thus split the symplectic potential into a *radiative* and a *Coulombic* component. Conversely, such a split of the symplectic potential uniquely defines the the SdW connection.

(Often, for notational convenience, we will collectively refer to "horizontal" and "radiative" quantities as "radiative", and similarly join "pure gauge" and "Coulombic" under "Coulombic". When a finer distinction is necessary, we will provide it.)

The radiative components of the symplectic potential represent its fully (perturbatively) gauge-fixed (gauge-invariant) content.[59] For a region without boundary, the full symplectic potential is gauge-invariant and purely radiative. For a bounded region R, the radiative contribution is still gauge invariant and is entirely determined by the intrinsic content of R, requiring no additional boundary information.

Due to the Gauss constraint, the Coulombic components of the symplectic potential are entirely encoded at the boundary, where the normal electric flux f is conjugated to the pure gauge part of the gauge potential (evaluated at the boundary). These symplectic conjugacy relations prompt an important question: what does it mean for the boundary-flux f, which contains a host of gauge-invariant information, to be conjugate to a pure-gauge contribution?

To answer this question, we turn to the Laplace boundary value problem (47) defining the Coulombic component of the electric field through the Gauss constraint in R. This equation has two independent inputs: the charge density within R (bulk) and the electric flux f through $S = \partial R$ (boundary). These two inputs are fully independent[60] from the point of view of the boundary value problem: and whereas the charge density encodes the matter dof in R, f encodes those dof which influence a region R and yet are independent of the contents of the region itself: i.e. f is the *influence from the outside*.

To avoid misunderstandings, a clarification is in order. It is true that, physically, a change in the distribution of the charge within R, keeping everything unchanged in the outer region, does modify f. However, this reasoning is unwarranted from a purely regional perspective: it requires us to know that everything stays unchanged in the outer region that we have no access to. In fact, at a closer analysis, the situation is even more serious than this, since—from a purely regional perspective—we cannot tell apart the

[59] Here, it is important to notice that "perturbatively" means that the gauge-fixing section in the bundle is only defined locally in Φ, i.e. "at the level of the tangent spaces": the non-vanishing SdW curvature of ϖ prevents, in the non-Abelian case, the integrability of the horizontal distribution into a globally defined gauge fixation Moreover, if one wants to interpret ϖ as yielding a perturbative gauge-fixing, one must keep in mind that such a gauge-fixing must retain full gauge-covariance with respect to the base-point of the perturbation (i.e. the perturbed configuration). This is morally similar to the BRST enhancement of a gauge fixing by a new global symmetry. [26, 28, 29].

[60] More precisely, they are independent modulo exceptions that arise at reducible configurations only. These exceptions are however limited to certain global modes of f on S determined by its smearing against the regional reducibility parameter χ pulled-back onto S.

contributions to f from the charges inside and the ones outside of R. Indeed, to compute these contributions to f, we would need to know the Green functions of the Laplace operator on the whole Σ, thus requiring global topological and geometrical knowledge about Σ. In contrast, the boundary value problems that we use to define the regional Coulombic part of the electric field treats the regional charge destribution and the electric flux as completely independent and uncorrelated inputs.[61] This should clarify why, and in which sense, we state that—from a purely regional perspective—f is independent of ρ and summarizes the influence on R coming from the outside. Consistently, *both* $\varpi(\delta A)$ and f are *functionally independent* from the radiative modes, including the charged matter content of the region.

With this clarification, we can now make sense of the fact that the physically-meaningful f is conjugated to the pure-gauge mode of A. In fact, whereas f can be measured from within one region, mathematically it should be considered as a complementary datum that *defines* the physics within R (cf. also [4]) and cannot be independently controlled from within R. This matches the fact that its conjugate variable is pure-gauge and therefore does *not* correspond to any physical observable, since this means that, in a putative quasilocal quantum theory, f commutes with all regional observables and is therefore superselected: each value of f defines a separate and independent "branch" of physics within R.

This picture of the electric flux is corroborated by our treatment of gluing, to which we now turn.

5.1.2 The gluing of radiative modes and its caveats

In section 4, we considered the problem of "gluing" regional radiative modes into a global radiative mode. Gluing radiative modes requires an adjustment of each regional radiative mode, so that the joined field is smooth and itself radiative. Since the adjustment cannot affect the regional radiative description, the regional adjustments must be along the pure-gauge/Coulombic direction. The main question is whether these pure-gauge/Coulombic adjustments exist at all, and if they exist whether they are ambiguous or rather uniquely defined by the regions' radiative content.

In this paper we proved that for fiducial boundaries, regional, or quasi-local, radiative modes contain the necessary and sufficient information for gluing into a global, smooth, and purely radiative mode. Indeed, assuming the continuity of the resulting global mode, the appropriate regional adjustments used for gluing are unique; they are given by a formula exploiting the properties of the Dirichlet-to-Neumann operator (but again, no particular boundary conditions on the underlying fields is ever imposed). The pure-gauge/Coulombic adjustments turn out to be explicit functionals of the mismatch between the original regional radiative at the common boundary.

This uniqueness result comes with two, physically significant, caveats.

The first caveat arises if the process of gluing generates new cohomological 1-cycles that were not present in either subregion. In such cases, the gluing formula still formally applies but—due to the presence of kernels in the Laplacian—its result is not unique. Here we must distinguish the effects for the gauge potentials and for the electric field: (a) On a non-simply connected manifold, the global horizontal mode of the gauge potentials is not fully fixed by the regional horizontal modes, due to the emergence of global Aharonov-Bohm phases that leave no imprint in the *regional* physics. (b) The situation is slightly different for the gluing of the electric field, since the flux f across the gluing interface contains relevant information to fix the ambiguity. Although we did not analyze this situation

[61] Cf. the previous footnote.

in general detail, we worked out an explicit example in 1+1 dimensions that showcases these topological effects without introducing other complications. We will return to the topological ambiguity at the end of section 5.1.3.

The second caveat is an ambiguity which arises from the presence of stabilizers of the fields, i.e. it arises when the underlying background fields are invariant under a finite subgroup of gauge transformations, also called the *reducibility parameters* "stabilzers", of the *reducible* field configurations. In the presence of matter, the stabilizers give rise to a well-defined notion of (conserved) charges [24, 27]—see also section 3.8—to which this ambiguity is strictly related. Indeed, this ambiguity is irrelevant in the absence of matter, and is independent of the topology; we studied it in section 4.3.2.

This relation between conserved charges and the ambiguities of gluing—both occurring in the joint presence of matter and of stabilizers—elicit a new take on the following conceptual question, much discussed in the literature on the foundations of gauge theory: do gauge symmetries possess "direct empirical significance"? Symmetries with direct empirical significance [60, 61] are transformations of the universe which are *not globally gauge*, but which, when restricted to certain subregions, are *regional* gauge transformations. Our results link this notion of "direct empirical significance" to the conservation of charges—usually considered an indirect empirical significance of gauge. That is because gluing employs regional gauge transformations as adjustments, and yet, when the two adjustments are ambiguous in ways that do not match at the boundary, they can produce different global physical states—as we showed in section 4.3.2.

5.1.3 Gluing and the role of the electric flux

Interesting features of this work emerge when one considers the gluing properties of the radiative and Coulombic components of the electric field, a topic investigated in section 4.4.

In the gluing process of a bipartite system, we consider a complete set of data, $(\delta_\perp A, \varpi(\delta A), \dot{A}_\perp, f, \psi, \overline{\psi})$, for each region. Of these, there are precisely two components which are superfluous for gluing and reconstructing the global physics: the pure gauge part of δA in each region (i.e. $\varpi(\delta A)$), and the electric flux at the interface, f. Therefore, in possession of the radiative components of *both* regions, we can fully recover the entire physical content of the theory: there is no need for f when we have all the radiative components.

As previously emphasized, from a strictly regional perspective, f is a "given function", a naturally superselected quantity when viewed from within the region alone. The results on gluing are compatible with this view of f: indeed f turns out to be entirely redundant when (radiative) information from the *both* regions becomes available. Moreover, f also turns out to be entirely fixed as a function of the regional radiative modes—and thus no longer needs to be superselected from the global perspective.

Even if the radiative components are intrinsic and accessible from within each region, their nonlocal nature confers them holistic properties: generically, the global radiative field is not the direct sum of the regional radiative fields. A possible mismatch of the regional radiative field components at the interface between the regions will require a Coulombic readjustment on each region. The physical nature of the Coulombic readjustments bears more interesting consequences than the vertical readjustments of the gauge potential, which are pure-gauge.

For example, it turns out that the energy carried by the glued radiative electric field is *always larger* than the sum of the energies carried by the regional radiative electric field within each region before gluing. The exact opposite applies for the energy carried by the Coulombic component of the electric field. Indeed, as we showed in section 4.5, energy is transferred upon gluing from the Coulombic to the radiative sector of the electric field (since the readjustment of A is pure-gauge, the amount of energy contained in the magnetic field is not affected). From this we conclude that the act of gluing partly converts regional Coulombic dof of E into global radiative dof of E. The conversion arises due to the existence of global radiative modes which cannot be encoded in *either* region: indeed, the conversion is a function of the mismatch of the values of the regional radiative modes at the interface *between* the two regions.

We had to consider both the readjustments of the gauge potential and of the electric field in order to describe the regional decomposition of the symplectic potential, as we saw in section 4.6. In complete parallel to the findings for the field components and their energies, as described above, the global symplectic potential decomposes into regional radiative symplectic potentials plus an interface term, which depends functionally on the mismatch of the regional radiative fields there.

However, as briefly discussed in section 5.1.2, the normal electric flux is irrelevant for recovering global physics only when the topology of the underlying manifold Σ is trivial. For a non-simply connected manifold, the boundary flux of the electric field incorporates a finite number of "independent degrees of freedom". Namely, it is in these cases that the regional information consisting of radiative modes and charge content fails to determine the full regional and global information about the field: since global degrees of freedom supported by the cohomology of Σ exist, the horizontal regional information must be supplemented by the normal electric flux f and its conjugated variables—the Aharonov-Bohm phase around the nontrivial cycles. These quantities cannot be retrieved at the regional level.

In section 4.7, we studied these matters in the simple case of one-dimensional manifolds taken in two different topologies: the line and the circle. For the line, no radiative modes survive: as expected, all configurations are pure gauge. However, upon gluing two segments into a circle, we explicitly found that the gluing procedure is ambiguous, i.e. its solution is non-unique. For what concerns the electric field, this ambiguity is fixed by considering the boundary Coulombic information (i.e. the flux f), which is thus "converted" into a global radiative mode of the electric field; for what concerns the modes of the gauge potential, on the other hand, the ambiguity cannot be resolved, since the Aharonov-Bohm phase is necessarily left undetermined by the regional information.

5.2 Outlook

We conclude this article by mentioning a couple of physically relevant questions that we expect our quasilocal framework will address and clarify.

Superselection Sectors and the Asymptotic Limit We start by discussing the superselection of the electric flux f that, in the asymptotic limit $\partial R \to \infty$, has been argued to have highly nontrivial and somewhat puzzling consequences such as the spontaneous breaking of Lorentz symmetry [8, 17, 62, 63]. In this context, we expect the following remark to be of interest. Following the considerations of section 3.6 (also summarized in 5.1.1), the superselection of f is found to hold for all quasilocal boundaries on the basis of an assumption of *ignorance* of the physics outside of R: indeed, as we argued at length, from our quasilocal perspective f summarizes the influence on R of the dof not present in R; an influence enacted through the Gauss law. That is, in our case f is superselected, has influence in the bulk of R, but—encoding information from beyond R—it is *not* used to deduce properties of matter within R. This should be contrasted with the considerations of [8,17,62,63]. There, f and its superselection are used to deduce properties of matter *within* R. This deduction is essentially based on an assumption of *knowledge* of the physics beyond $\partial R \to \infty$: i.e. that there can be nothing "beyond infinity".[62] Such marked difference therefore tempts us to speculate that the conclusions of [8, 62, 63] might have to be revisited at the light of the tension between a mathematical construction, i.e. the limit $\partial R \to \infty$, and the physical assumptions on which it is based, i.e. the introduction of infinity as a way to describe in an idealized manner an isolated system *within* the universe—rather than the universe as a whole. In this regard, it would be crucial to properly understand the role boundary (and fall-off) conditions for the (asymptotic) fields play in our arguments (e.g. [4, 9]). This work begun in [5], where null-infinity was analyzed, but we leave a more detailed analysis of these ideas to future work.

Entanglement Entropy Another question that we expect our formalism can help clarify concerns the nonstandard properties of entanglement entropy of gauge systems [64]. In gauge theories, the entanglement entropy turns out to quantify not only the standard, "distillable", (quantum and classical) correlations between local excitations, but also a more exotic "edge" (or "contact") component. The latter component is classical, and descends from the probability distribution for finding the super-selected flux f in a certain configuration [1, 2, 18–20]. Given our understanding of the interplay between gauge, fiducial interfaces,[63] and gauge symmetry, it is clear that the present formalism will shed light on the interpretation and computation of the edge component to the entanglement entropy.

In this regard, we notice that our formalism is well-suited not only for a broad generalization of the ideas of [65] on the computation and interpretation of the contact term of the (3d Abelian) Yang-Mills theory, but also for inscribing them in a larger and firmer theoretical landscape.

In [65], the computation of the contact term is set up in terms of the comparison between a globally gauge-fixed path integral and its regional counterparts. The main ingredient of this the computation is the Forman-BFK formula for the factorization of (zeta-regularized, Faddeev-Popov) functional determinants of Laplacians [59, 66, 67] (the relevance of this ingredient to calculations of black-hole entropy was already identified[64] by Carlip [69]). This formula features precisely the Abelian analogue of the operator $(\mathcal{R}_+^{-1} + \mathcal{R}_-^{-1})$ that is central to our gluing formula. Indeed, interpreting horizontal modes as corresponding to the perturbatively gauge fixed ones, our gluing formula gives a precise non-degenerate[65] Jacobian for the transformation of the global radiatives to the regional radiatives, whose determinant yields the relevant factor in the factorization of the path integrals.

Corners and Gluing So far we have considered only gluing patterns in which two regions are glued along their *whole* boundaries. More generally, one should consider cases in which the gluing happens on portions of the boundaries bounded by corner surfaces, and the boundary of those, and so on. In particular, these more general gluing patterns

[62] Of course, in these works the asymptotic superselection of f is not assumed, but derived. Roughly, its derivation is based on the observation that the observable associated to f at *infinity* is spacelike separated from, and hence commutes with, *all* the local operators of the theory (since they must have a finite support): hence, commuting with all other operators, f at infinity is superselected. Notice how the idealization of "infinity" is given a central role in these arguments.

[63] Fiducial interfaces—i.e. interfaces at which no boundary condition is imposed—are crucial to the generic definition of entanglement entropy, but for gauge theories they are not easily implementable in previous set-ups (see e.g. the "brick wall" of [18, 20]).

[64] See also [68] for an even earlier application of Forman's results to the gluing, or "sewing", of string amplitudes.

[65] However, subtleties are expected to arise for non-simply-connected manifolds and at reducible background configurations.

are necessary to build arbitrary manifolds from topologically trivial building blocks. This is therefore an important topic that deserves deeper study. In section 4.2, we noticed that the continuity condition parallel to the interface S takes the form of a verticality condition *in the space of boundary fields* for the difference of the *pullbacks* of the regional horizontals on S. In this scenario, it seems that a chain of descent could apply for horizontal/vertical decomposition at boundaries of boundaries, etc. with analogies to the nested structures featured in the BV-BFV formalism (when interfaces of multiple codimensions are considered) [21–23].

Further Directions We conclude by mentioning a few other topics that we leave to future work: a complete study of the descent chain and of the gluing into topologically nontrivial domains; the generalization of this work to general relativity and diffeomorphism symmetry (see [4,12,14,37,57]), as well as to other types of gauge theories such as Chern-Simons and BF theories.

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A Time dependent gauge transformations

(Note: all manipulations in this section are independent of the choice $\varpi = \varpi_{\rm SdW}$ and hold for any connection form on Φ.)

Given an evolution flow $\frac{\rm d}{{\rm d}t}$ in configuration space, the function $\dot{A}_i$ is understood to be the *total* derivative of A_i (understood as a coordinate function on $\mathcal{A}$) along this flow:

$$\dot{A}_i := \frac{\rm d}{{\rm d}t} A_i. \tag{153}$$

This function, constitutes the components of the configuration-space "velocity" field

$$\dot{\mathbb{A}} = \int \sqrt{g} \left(\frac{\rm d}{{\rm d}t} A_i^\alpha \right)(x) \frac{\delta}{\delta A_i^\alpha(x)} \in {\rm T}\mathcal{A} \tag{154}$$

where (hereafter, $\dot{\mathbb{A}}(\xi) \equiv \mathbb{i}_{\dot{\mathbb{A}}} \mathbb{d}\xi$ etc.)

$$\dot{\xi} := \partial_t \xi + \dot{\mathbb{A}}(\xi) \tag{155}$$

With this, it is immediate to compute

$$\delta_\xi \dot{A}_i := \mathbb{L}_{\xi^\sharp} \frac{\mathrm{d}}{\mathrm{d}t} A_i = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{D}_i \xi) = \partial_i \dot{\xi} + [\dot{A}_i, \xi] + [A_i, \dot{\xi}] = \mathrm{D}_i \dot{\xi} + [\dot{A}_i, \xi]. \tag{156}$$

Notice that this the derivative along $\xi^\sharp$ of the function $\dot{A}_i$, and *not* the component of[66] $\mathbb{L}_{\xi^\sharp} \dot{\mathbb{A}} \equiv [\![\xi^\sharp, \dot{\mathbb{A}}]\!]$, which instead is equal to

$$[\![\xi^\sharp, \dot{\mathbb{A}}]\!] = \int \sqrt{g} \left(\xi^\sharp(\dot{A}_i^\alpha) - \dot{\mathbb{A}}(\mathrm{D}_i\xi)^\alpha \right)(x) \frac{\delta}{\delta A_i^\alpha(x)} = (\dot{\xi})^\sharp - \dot{\mathbb{A}}(\xi)^\sharp, \tag{157}$$

where we used (156) for $\xi^\sharp(\dot{A}_i) = \delta_\xi \dot{A}_i$, as well as

$$\dot{\mathbb{A}}(\mathrm{D}_i\xi) = \dot{\mathbb{A}}(\partial_i \xi + [A_i, \xi]) = [\dot{A}_i, \xi] + \mathrm{D}_i \dot{\mathbb{A}}(\xi). \tag{158}$$

Hence, from the defining properties (5) of ϖ—i.e. $\varpi(\xi^\sharp) = \xi$ and $\mathbb{L}_{\xi^\sharp}\varpi = [\varpi, \xi] + \mathbb{d}\xi$,— as well as the above identities, it follows that

$$\begin{aligned} \mathbb{L}_{\xi^\sharp}(\varpi(\dot{\mathbb{A}})) &= (\mathbb{L}_{\xi^\sharp}\varpi)(\dot{\mathbb{A}}) + \varpi([\![\xi^\sharp, \dot{\mathbb{A}}]\!]) \\ &= [\varpi(\dot{\mathbb{A}}), \xi] + \dot{\mathbb{A}}(\xi) + \varpi((\dot{\xi})^\sharp - \dot{\mathbb{A}}(\xi)^\sharp) \\ &= [\varpi(\dot{\mathbb{A}}), \xi] + \dot{\xi}. \end{aligned} \tag{159}$$

Therefore, we showed that $\varpi(\dot{\mathbb{A}})$ *has the same transformation properties of* A_0.

Now, combing this equation with the definition (20) of A_0 and the covariance of φ (required by construction),

$$A_0 := -\varphi + \varpi(\dot{\mathbb{A}}) \qquad \text{and} \qquad \mathbb{L}_{\xi^\sharp}\varphi = [\varphi, \xi], \tag{160}$$

it readily follows that

$$\delta_\xi A_0 \equiv \mathbb{L}_{\xi^\sharp} A_0 = -\mathbb{L}_{\xi^\sharp}\varphi + \mathbb{L}_{\xi^\sharp}\varpi(\dot{\mathbb{A}}) = [-\varphi + \varpi(\dot{\mathbb{A}}), \xi] + \dot{\xi} = [A_0, \xi] + \dot{\xi} = \mathrm{D}_0\xi. \tag{161}$$

Of course this formula implies the covariance of $E_i = \dot{A}_i - \mathrm{D}_i A_0$,

$$\begin{aligned} \delta_\xi E_i = \mathbb{L}_{\xi^\sharp} E_i &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{D}_i\xi) - \Big([\delta_\xi A_i, A_0] + \mathrm{D}_i([A_0, \xi] + \dot{\xi}) \Big) \\ &= \Big(\mathrm{D}_i\dot{\xi} + [\dot{A}_i, \xi] \Big) - \Big([\mathrm{D}_i\xi, A_0] + \mathrm{D}_i[A_0, \xi] + \mathrm{D}_i\dot{\xi} \Big) \\ &= [\dot{A}_i, \xi] - [\mathrm{D}_i A_0, \xi] = [E_i, \xi], \end{aligned} \tag{162}$$

whereas a completely analogous computation, obtained by replacing A_0 with $\varpi(\dot{\mathbb{A}})$ shows the covariance of $\dot{A}^\perp$:

$$\delta_\xi \dot{A}_i^\perp \equiv \mathbb{L}_{\xi^\sharp} \dot{A}_i^\perp = [\dot{A}^\perp, \xi] \tag{163}$$

To avoid confusions, we re-iterate that $\delta_\xi E_i = \mathbb{L}_{\xi^\sharp} E_i$ is (very!) different from $\mathbb{L}_{\xi^\sharp}\mathbb{E} = [\![\xi^\sharp, \mathbb{E}]\!]$. Indeed, a short computation shows that $[\![\xi^\sharp, \mathbb{E}]\!] = -\mathbb{E}(\xi)^\sharp$, where $\mathbb{E}(\xi)$ stands for the functional derivative of ξ along $\mathbb{E}$. Both $\mathbb{L}_{\xi^\sharp} E_i$ and $[\![\xi^\sharp, \mathbb{E}]\!]$ are Lie-derivative along $\xi^\sharp$, but the first is the Lie derivative of a function on Φ, while the second is the Lie derivative of a vector field on $\mathrm{T}\Phi$.

We conclude this appendix by noticing that equation (161) shows the announced result that A_0 has automatically the correct transformation properties once we assume that φ transforms in the adjoint representation: the burden of instilling A_0 with its typical non-homogenous transformation property under time- (and field-)dependent gauge transforamtions is fully carried by the term built out of ϖ.

[66] Here, $[\![\cdot, \cdot]\!]$ denotes the Lie bracket between vector fields in $\mathrm{T}\mathcal{A}$ (or $\mathrm{T}\Phi$, depending on the context).

B Computation of the symplectic form

In this appendix we prove the expression for the symplectic form given in (58).

First, we notice the following identities,

$$\mathbb{d}_\perp \mathbb{d}_\perp A_i = -\mathrm{D}\mathbb{F}_i \quad \text{and} \quad \mathbb{d}_\perp \mathbb{d}_\perp \psi = \mathbb{F}\psi, \tag{164}$$

which follow from the general definition of the horizontal differential for horizontal and equivariant field-space forms. That is, for any p-form $\lambda \in \Lambda^p(\Sigma, W)$ valued in a representation (W, ρ) of $\mathcal{G}$ such that $\mathbb{i}_{\xi^\sharp}\lambda = 0$ and $\mathbb{L}_{\xi^\sharp}\lambda = \rho(\xi)\lambda$ (see [37]), we define

$$\mathbb{d}_\perp \lambda := \mathbb{d}\lambda - \rho(\varpi)\lambda, \tag{165}$$

which is a horizontal and equivariant field-space $(p+1)$-form (see [37]).

Second, we notice that the identity ($\cdot_\bullet$ stands for $\cdot_{\text{YM}}$ or $\cdot_{\text{Dirac}}$)

$$\mathbb{d}_\perp \theta^\perp_\bullet = \mathbb{d}\theta^\perp_\bullet, \tag{166}$$

follows from the above definition and the $\theta^\perp_{\text{YM}}$ and $\theta^\perp_{\text{Dirac}}$ being gauge invariant, i.e. $\mathbb{L}_{\xi^\sharp}\theta^\perp_\bullet = 0$. Hence,

$$\Omega_\bullet = \mathbb{d}\theta_\bullet = \mathbb{d}(\theta^\perp_\bullet + \theta^V_\bullet) = \mathbb{d}_\perp \theta^\perp_\bullet + \mathbb{d}\theta^V_\bullet = \Omega^\perp_\bullet + \Omega^{\text{rest}}_\bullet. \tag{167}$$

Now, we are ready to compute $\Omega_\bullet$. We start from the horizontal contributions. The YM one gives

$$\Omega^\perp_{\text{YM}} = \mathbb{d}_\perp \theta^\perp_{\text{YM}} = \int \sqrt{g}\, g^{ij}\,\mathrm{Tr}(\dot{A}^\perp_i \mathbb{d}_\perp A_j) = \int \sqrt{g}\, g^{ij}\,\mathrm{Tr}(\mathbb{d}_\perp \dot{A}^\perp_i \curlywedge \mathbb{d}_\perp A_j), \tag{168}$$

where we used (164) and the fact that $\mathbb{G}(\dot{\mathbb{A}}^\perp, \mathbb{F}^\sharp) \equiv 0$. Similarly, the matter one gives

$$\Omega^\perp_{\text{Dirac}} = \mathbb{d}_\perp \theta^\perp_{\text{Dirac}} = -\mathbb{d}_\perp \int \sqrt{g}\,(\overline{\psi}\gamma^0 \mathbb{d}_\perp \psi) = -\int \sqrt{g}\,(\mathbb{d}_\perp \overline{\psi} \curlywedge \gamma^0 \mathbb{d}_\perp \psi + \mathrm{Tr}(\rho\,\mathbb{F})). \tag{169}$$

We then focus on the differential of the vertical contributions to the symplectic potential. Once again, we start from the YM one, which reads

$$\begin{aligned}\Omega^{\text{rest}}_{\text{YM}} &= \mathbb{d}\int \sqrt{g}\, g^{ij}\,\mathrm{Tr}(\mathrm{D}_i\varphi \mathrm{D}_j\varpi) = -\mathbb{d}\int \sqrt{g}\,\mathrm{Tr}(\varpi \mathrm{D}^2\varphi) + \mathbb{d}\oint \sqrt{h}\,\mathrm{Tr}(f\,\varpi) \\ &= -\int \sqrt{g}\,\mathbb{d}\mathrm{Tr}(\mathrm{D}^2\varphi\,\varpi) + \oint \sqrt{h}\,\mathrm{Tr}(\mathbb{d}_\perp f\,\varpi + f\,\mathbb{F}).\end{aligned} \tag{170}$$

where we used the defining relations $\mathbb{d}f = \mathbb{d}_\perp f + [f, \varpi]$ and $\mathbb{d}\varpi = \mathbb{F} - \frac{1}{2}[\varpi \stackrel{\curlywedge}{,} \varpi]$. The Dirac part gives on the other hand:

$$\Omega^{\text{rest}}_{\text{Dirac}} = \int \sqrt{g}\,(\overline{\psi}\gamma^0\varpi\psi) = \mathbb{d}\int \sqrt{g}\,\mathbb{d}\mathrm{Tr}(\rho\varpi). \tag{171}$$

Combining all the terms, we obtain:

$$\begin{aligned}\Omega = &\underbrace{\int \sqrt{g}\, g^{ij}\,\mathrm{Tr}\Big(\mathbb{d}_\perp \dot{A}^\perp_i \curlywedge \mathbb{d}_\perp A_j\Big)}_{=\Omega^\perp_{\text{YM}}} + \underbrace{\int \sqrt{g}\,\Big(-\mathbb{d}_\perp\overline{\psi} \curlywedge \gamma^0 \mathbb{d}_\perp\psi - \mathrm{Tr}(\rho\,\mathbb{F})\Big)}_{=\Omega^\perp_{\text{Dirac}}} + \\ &+ \underbrace{\int \sqrt{g}\,\mathbb{d}\mathrm{Tr}\Big(\underbrace{(\rho - \mathrm{D}^2\varphi)}_{\approx 0}\varpi\Big) + \oint \sqrt{h}\,\mathrm{Tr}(\mathbb{d}_\perp f\,\varpi + f\,\mathbb{F})}_{=\Omega^\partial}.\end{aligned} \tag{172}$$

C Proof of equation (60)

(Note: manipulations in this section *do* depend of the choice $\varpi = \varpi_{\text{SdW}}$, unless stated otherwise.)

In this appendix we prove (60), in its more complete form:

$$\Xi_{\text{rad}} = \int \sqrt{g}\,\text{Tr}\Big(g^{ij}(\tfrac{\text{d}}{\text{d}t}\mathbb{d}_\perp A_i + [A_0, \mathbb{d}_\perp A_i]) \curlywedge \mathbb{d}_\perp A_j + (\text{D}^2\varphi - \rho)\mathbb{F}\Big), \tag{173}$$

under the hypothesis that $\partial_t g_{ij} = 0$ in R.

We start by the definition (59),

$$\Xi_{\text{rad}} := \int \sqrt{g}\,\text{Tr}\Big(g^{ij}\mathbb{d}_\perp \dot{A}_i^\perp \curlywedge \mathbb{d}_\perp A_j - \rho\,\mathbb{F}\Big) + \oint \sqrt{h}\,\text{Tr}\Big(f\,\mathbb{F}\Big), \tag{174}$$

and observe that, on-shell of the (source-free) Gauss constraint (47), the last two terms can written as

$$\begin{aligned}\int \sqrt{g}\text{Tr}(-\rho\,\mathbb{F}) + \oint \sqrt{h}\,\text{Tr}(f\,\mathbb{F}) &= \int \sqrt{g}\,\text{Tr}\Big((\text{D}^2\varphi - \rho)\mathbb{F} + \text{D}^i\varphi\text{D}_i\mathbb{F}\Big)\\ &= \int \sqrt{g}\,\text{Tr}\Big((\text{D}^2\varphi - \rho)\mathbb{F} - \varphi g^{ij}[\mathbb{d}_\perp A_i \stackrel{\curlywedge}{,} \mathbb{d}_\perp A_j]\Big),\end{aligned} \tag{175}$$

where in the first step we used the defining equation $\text{D}_s\varphi = f$ (47) as well as the divergence theorem; the second step follows from an integration by parts and the properties (38) of the SdW-curvature $\mathbb{F}$.

Hence, (59) ca be put into the form

$$\Xi_{\text{rad}} = \int \sqrt{g}\,\text{Tr}\Big(g^{ij}(\mathbb{d}_\perp \dot{A}_i^\perp - [\varphi, \mathbb{d}_\perp A_i]) \curlywedge \mathbb{d}_\perp A_j + (\text{D}^2\varphi - \rho)\mathbb{F}\Big), \tag{176}$$

Next, we prove that

$$\mathbb{d}_\perp(\dot{A}_i^\perp) = \tfrac{\text{d}}{\text{d}t}\mathbb{d}_\perp A + [\varpi(\dot{\mathbb{A}}), \mathbb{d}_\perp A] + \text{D}\mathbb{i}_{\dot{\mathbb{A}}}\mathbb{F}. \tag{177}$$

As the proof shows, this formulae holds for *any* connection form satisfying (5) provided $\partial_t\varpi = 0$ (which in turn follows from $\partial_t g_{ij} = 0$ in R). This goes as follows:

$$\begin{aligned}\tfrac{\text{d}}{\text{d}t}\mathbb{d}_\perp A_i &= \tfrac{\text{d}}{\text{d}t}(\mathbb{d}A_i - \text{D}_i\varpi) = \mathbb{d}(\dot{A}_i) - \text{D}_i\tfrac{\text{d}}{\text{d}t}\varpi - [\dot{A}_i, \varpi]\\ &= \mathbb{d}\dot{A}_i - \text{D}_i\mathbb{L}_{\dot{\mathbb{A}}}\varpi - [\dot{A}_i, \varpi]\\ &= \mathbb{d}(\dot{A}_i^\perp + \text{D}_i\varpi(\dot{\mathbb{A}})) - \text{D}_i\mathbb{d}\varpi(\dot{\mathbb{A}}) - \text{D}_i\mathbb{i}_{\dot{\mathbb{A}}}\mathbb{d}\varpi - [\dot{A}_i, \varpi]\\ &= \mathbb{d}(\dot{A}_i^\perp) + [\mathbb{d}A_i, \varpi(\dot{\mathbb{A}})] - \text{D}_i\Big(\mathbb{i}_{\dot{\mathbb{A}}}\mathbb{F} - [\varpi(\dot{\mathbb{A}}), \varpi]\Big) - [\dot{A}_i, \varpi]\\ &= \mathbb{d}(\dot{A}_i^\perp) + [\mathbb{d}_\perp A_i, \varpi(\dot{\mathbb{A}})] - [\dot{A}_i^\perp, \varpi] - \text{D}_i\mathbb{i}_{\dot{\mathbb{A}}}\mathbb{F}\\ &= \mathbb{d}_\perp(\dot{A}_i^\perp) + [\mathbb{d}_\perp A_i, \varpi(\dot{\mathbb{A}})] - \text{D}_i\mathbb{i}_{\dot{\mathbb{A}}}\mathbb{F}.\end{aligned} \tag{178}$$

In the first line, we used the definition of $\mathbb{d}_\perp A_i = \mathbb{d}A_i - \text{D}_i\varpi$ and the Leibniz rule; in the second, we used the identity $\mathbb{L}_{\dot{\mathbb{A}}}\varpi \equiv \frac{\text{d}}{\text{d}t}\varpi$ which follows from the fact that ϖ only depends on t though its functional dependence on A; in the third, we used the definition of $\dot{A}_i^\perp = \dot{A}_i - \text{D}_i\varpi(\dot{\mathbb{A}})$ and Cartan's formula $\mathbb{L}_{\mathbb{X}} = \mathbb{d}\mathbb{i}_{\mathbb{X}} + \mathbb{i}_{\mathbb{X}}\mathbb{d}$; in the fourth, we used the Leibniz rule for the functional differential as well as the definition of the functional curvature $\mathbb{F} = \mathbb{d}\varpi + \frac{1}{2}[\varpi \stackrel{\curlywedge}{,} \varpi]$; in the fifth, we distributed D_i and recollected terms according the definitions of $\mathbb{d}_\perp A_i$ and $\dot{A}_i^\perp$; finally, in the sixth line, we simply recognized (cf. (18) or (163) of appendix A)

$$\mathbb{d}_\perp(\dot{A}_i^\perp) = \mathbb{d}\dot{A}_i^\perp - [\dot{A}_i^\perp, \varpi]. \tag{179}$$

Combining equation (177) and (176) with the relation $A_0 = -\varphi + \varpi(\dot{\mathbb{A}})$ (equation (20)) gives the sought result, equation (173). The term involving $\mathrm{D}\mathbb{i}_{\dot{\mathbb{A}}}\mathbb{F}$ cancels because it involves the integration (i.e. the $\mathbb{G}$-inner product) of a pure gauge-covariant gradient (i.e. of a purely-vertical object) against the SdW-horizontal differential $\mathbb{d}_\perp A_i$.

Note that (177) can be used to show that

$$\Omega^\perp_{\mathrm{YM}} = \int \sqrt{g}\, g^{ij}\mathrm{Tr}\Big((\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{d}_\perp A_i) \curlywedge \mathbb{d}_\perp A_j + [\varpi(\dot{\mathbb{A}}), \mathbb{d}_\perp A_i]\Big). \tag{180}$$

This expression is also manifestly gauge-invariant, even under time-dependent gauge transformations (see (19)).

D The SdW connection from the Dirac dressing

In this appendix, we revisit Dirac's construction for the dressing of the electron [40], and provide considerations about its generalizability (or lack thereof) to the non-Abelian case. This discussion also offers an *independent* route to the introduction of the SdW connection ϖ.

Dirac's dressing construction can be motivated by the need to define a *physical* electron field that is meant to create not only the "bare" electron but its Coulombic electric field as well, so that the Gauss law is automatically satisfied. With the purport of being physical, this dressed field is expected to be, and indeed is, gauge invariant. However, the dressed field describes a charged electron, and therefore also carries electric charge. This means the electric charge Poisson-generates a global shift in the phase of the dressed electron field, which might seem in contrast with the posited gauge invariance of the dressed field. However, as we saw in section 3.8, these requirements are not mathematically in conflict [70]: constant "gauge transformations" associated to the electric charge do have a different (geometric) status in Φ_{EM} with respect to generic (local) gauge transformations (see last paragraph of section 3.8.2).

Starting from these ideas, we will now revisit Dirac's construction. We will work in finite regions and in the non-Abelian setting, for generality.

Denoting the dressed field with a hat, $\widehat{\psi}$, the classical condition corresponding to the demand that the corresponding quantum field creates an electron at x together with its electrostatic field is

$$\{E^\beta_j(y), \widehat{\psi}(x)\} = -\big(\mathrm{D}_j G_{\alpha,x}\big)^\beta(y)\tau_\alpha\widehat{\psi}(x) \tag{181}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket and[67] $G_{\alpha,x} \in \Lambda^0(R, \mathrm{Lie}(\mathcal{G}))$ is the $\mathrm{Lie}(\mathcal{G})$-valued Green's function defined by (here, $\delta_x(y) = \delta^D(x,y)$ is Dirac's delta distribution)

$$\begin{cases} \mathrm{D}^2 G_{\alpha,x}(y) = \tau_\alpha\delta_x(y) & \text{in } R,\\ \mathrm{D}_s G_{\alpha,x}(y) = 0 & \text{at } \partial R.\end{cases} \tag{182}$$

Although at this level any other choice of boundary conditions would have worked, the (covariant) Neumann boundary condition is chosen here for future convenience. Physically,

[67] Here (α, x) are seen as labels of the Green's function, not indices in $\mathrm{Lie}(\mathcal{G})$. Note that $\{\tau_\alpha\delta_x\}_{(\alpha,x)}$ provides a basis of $\mathrm{Lie}(\mathcal{G})$, since a generic $\xi \in \mathrm{Lie}(\mathcal{G})$ can be written $\xi(x) = \int_R \mathrm{d}^D y\, \xi^\alpha(y)\tau_\alpha\delta_x(y)$.

this choice corresponds to the demand that the dressed particle created at x does *not* contribute to the flux f at ∂R. A posteriori, with the knowledge acquired from the construction of the SdW connection, it is possible to see that these boundary conditions are moreover the only ones that make $\widehat{\psi}$ gauge invariant with respect to gauge transformations whose support is not limited to the interior of R, extending also to its boundary ∂R.

As familiar from EM, at reducible configurations $\tilde{A}$, the above choice of boundary conditions is inconsistent and should be amended, e.g. by demanding that it creates a constant flux compatible with the integrated Gauss law.[68] However, notice that, at (generic!) non-Abelian configurations that are *not* reducible, there is *no integrated Gauss law* that the Green's function should respect. Indeed, the extension of the Gauss law, $\int \rho = \int \partial^i E_i = \oint f$, to the non-Abelian context would be

$$\int \mathrm{Tr}(\tau_\alpha \rho) = \int \mathrm{Tr}(\tau_\alpha \mathrm{D}^i E_i) = -\int \mathrm{Tr}(E^i \mathrm{D}_i \tau_\alpha) + \oint \mathrm{Tr}(\tau_\alpha f), \tag{183}$$

but the bulk term on the rightmost side vanishes only if A is reducible and τ_α is replaced by a reducibility parameter. Therefore, at a generic configuration of a non-Abelian YM theory, there is no (integrated) Gauss law relating total charges and (integrated) electric fluxes.[69]

With definition (182) at hand, using the following non-Abelian extension of Green's theorem (e.g. [71])

$$\int_R \sqrt{g}\, \mathrm{Tr}(\phi \mathrm{D}^2 \psi - \psi \mathrm{D}^2 \phi) = \oint_{\partial R} \sqrt{h}\, \mathrm{Tr}(\phi \mathrm{D}_s \psi - \psi \mathrm{D}_s \phi) \quad \forall \phi, \psi \in \Lambda^0(R, \mathrm{Lie}(\mathcal{G})), \tag{184}$$

one can choose $\psi = G_{\alpha,x}$ and $\phi = \varphi$, to obtain the Coulombic component of the electric field in terms of the charge density ρ and the flux f:

$$\begin{aligned} \varphi_\alpha(x) &= \int_R \sqrt{g(y)}\, \mathrm{Tr}\Big(G_{\alpha,x}(y) \mathrm{D}^2 \varphi(y)\Big) - \oint_{\partial R} \sqrt{h(y)}\, \mathrm{Tr}\Big(G_{\alpha,x}(y) \mathrm{D}_s \varphi(y)\Big) \\ &\approx \int_R \sqrt{g(y)}\, \mathrm{Tr}\Big(G_{\alpha,x}(y) \rho(y)\Big) - \oint_{\partial R} \sqrt{h(y)}\, \mathrm{Tr}\Big(G_{\alpha,x}(y) f(y)\Big). \end{aligned} \tag{185}$$

At reducible configurations, this formula must again be amended by the addition of constant offsets due to the modified boundary condition. The addition of such offsets is possible thanks to the freedom of redefining $G_{\alpha,x}$ by some combination of the reducibility parameters, since they lie in the kernel of (182).

Going back to the definition of the dressed matter field, and working formally, we consider the ansatz

$$\widehat{\psi}(x) \equiv e^{F[A](x)} \psi(x) \quad \text{with} \quad F[A] \in \mathrm{Lie}(\mathcal{G}), \tag{186}$$

from which, via substitution into the requirement (181), we get the condition:

$$\frac{1}{\sqrt{g}} \frac{\delta}{\delta A} F = \{E, F\} = -\mathrm{D}G, \tag{187}$$

[68] In EM one possible natural boundary condition is $\partial_s G = 1/\mathrm{Vol}(\partial R)$ with $\mathrm{Vol}(\partial R)$ the volume of the region's boundary ∂R, whereas in YM at a $\tilde{A}$ with a single reducibility parameter χ, the following y-constant boundary condition plays a similar role $\mathrm{D}_s G_{\alpha,x} = \mathrm{Tr}(\chi(x)\tau_\alpha)/||\chi||^2_{\partial R}$.

[69] The issue is formally the same as the difficulties present in defining quasi-local conserved quantities in general relativity. Also, notice that bringing the gauge field contribution on the left-hand side of (183) to make the ensuing "integrated Gauss law" satisfied identically is a trick with no bearing on the dressing problem and the definition of the Green's functions for the matter field. Cf. the discussion in section 3.8.

or, in full detail,

$$\frac{g_{ji}(y)}{\sqrt{g(y)}}\frac{\delta}{\delta A_i^\beta(y)}F^\alpha[A](x) = \{E_j^\beta(y), F^\alpha(x)\} = -\big(\mathrm{D}_j G_{\alpha,x}\big)^\beta(y), \tag{188}$$

where we used $\Omega(\frac{\delta}{\delta A_i^\beta(y)}) = -\sqrt{g(y)}\, g^{ij}(y)\delta E_j^\beta(y)$ to individuate the Hamiltonian vector field associated to $E_j^\beta(y)$ (notice that this expression is valid also in the presence of boundaries).

Equation (181) can then be formally solved through a line integral in configuration space $\mathcal{A}$, that we denote $\iint^A$:

$$F[A]^\alpha(x) = -\iint^A \int_R \mathrm{d}^D y\sqrt{g(y)}\, g^{ij}\sum_\beta \left(\mathrm{D}_i G_{\alpha,x}\right)^\beta(y)\, \mathbb{d}A_j^\beta(y) \tag{189}$$

Using a more compact notation, this can be written

$$F[A](x) = -\iint^A \int_R \sqrt{g}\,\mathrm{Tr}\Big(\mathrm{D}^i G_{\alpha,x}\, \mathbb{d}A_i^\beta\Big)\tau_\alpha. \tag{190}$$

Integrating by parts, one obtains

$$F[A](x) = -\iint^A \left(-\int_R \sqrt{g}\,\mathrm{Tr}\Big(G_{x,\alpha}\mathrm{D}^j\mathbb{d}A_j\Big)\tau_\alpha + \oint_{\partial R}\sqrt{h}\,\mathrm{Tr}\Big(G_{x,\alpha}\mathbb{d}A_s\Big)\tau_\alpha\right) \tag{191}$$

Now, to be able to use Green's theorem and simplify this expression, it is natural to introduce

$$\varpi \in \Lambda^1(R, \mathrm{Lie}(\mathcal{G})) \tag{192}$$

defined by (5)

$$\begin{cases} \mathrm{D}^2\varpi = \mathrm{D}^i\mathbb{d}A_i & \text{in } R, \\ \mathrm{D}_s\varpi = \mathbb{d}A_s & \text{at } \partial R. \end{cases} \tag{193}$$

Hence, using this definition and Green's theorem (184) for $\psi = G_{\alpha,x}$ and $\phi = \varpi$, we obtain the formal solution

$$F[A](x) = \iint^A \varpi(x) \tag{194}$$

and $\widehat{\psi} = \exp\left(\iint^A \varpi\right)\psi$ is the formal general solution to the demands imposed by Dirac's dressing.

This construction provides an independent motivation for the introduction of the SdW connection form ϖ—even though at this level, its connection-form properties (5), and in particular its covariance property, are *not* manifest.

But with hindsight knowledge of the connection-form nature of ϖ, we introduce the following gauge-covariant expression (i.e. even under field-*dependent* gauge transformations) involving a *field-space Wilson-line*: we call this the dressing factor (see [26, Sec. 9] for details and crucial subtleties regarding its gauge-covariance):[70]

$$\widehat{\psi}(x) = \underbrace{\mathbb{P}\mathbb{exp}\left(\iint^A \varpi(x)\right)}_{=:\text{dressing factor } e^F}\psi(x). \tag{195}$$

[70]This is a 1-dimensional integral along a curve embedded in an infinite dimensional space. It is the latter property that the doublestruck-face of the symbol $\mathbb{P}\mathbb{exp}\iint$ is meant to emphasize. Cf. equation (126), where a Wilson line in space—rather than in configuration space—is considered.

In EM, the SdW connection is Abelian and flat. Therefore, both the path ordering and the choice of path in $\mathcal{A}$ are inessential. In particular, one can choose the trivial configuration $A^\star = 0$ as a starting point for the field-space line integral so that the resulting expression (seemingly) depends only on the final configuration A. Indeed, using the fact that in EM the Green's function G does not depend on A, we can perform the integral explicitly and readily find—cf. (189):

$$\widehat{\psi}(x) := e^{i\int_R \sqrt{g(y)}\,G(x,y)\partial^i A_i(y)}\psi(x). \tag{196}$$

This provides a generalization to finite and bounded regions of the Dirac dressing, which in $\mathbb{R}^{D=3}$ reads (see [40]):

$$\widehat{\psi}(x) := e^{i\int \frac{\mathrm{d}^3 y}{4\pi}\frac{\partial^i A_i(y)}{|x-y|}}\psi(x). \tag{197}$$

In the non-Abelian setting, we first proposed the expression (195) (without reference to the derivation presented here) in [26]. There, this formula was framed in relation to the work on dressings by Lavelle and McMullan [41–44], and also to the Gribov-Zwanziger framework [72, 73] (see [74] for a review and relation to confinement), and, finally, to the geometric approach to the quantum effective action by Vilkovisky and De-Witt [27, 46–50]. In particular, in [26], we studied in detail the properties and limitations of (195) and we related the limitations to certain obstructions appearing in the previous works [27, 44, 46–48, 72, 73]. More specifically, we showed that: the obstructions found previously come from the curvature of the connection form, which induces a path-dependence ambiguity in the dressing; that this ambiguity can be fixed in a neighbourhood of a given reference configuration $A^\star$ (using field-space geodesics with respect to the so-called Vilkovisky connection [27, 47, 48]); and that, nonetheless, all expressions will still depend on the (gauge-dependent) choice of the reference configuration $A^\star \in \mathcal{A}$ [49, 50]. Finally, note that global existence and uniqueness of the Vilkovisky geodesics from $A^\star = 0$ to a generic A, a question related to the non-perturbative existence and uniqueness of the dressing factor, is expected to fail in view of the Gribov problem. We restrain from dissecting these topics here, and refer to [26, Sec. 9] for a thorough discussion.

Instead, we limit ourselves to the following observations: although the notion of a full-blown nonperturbative dressing is not viable in YM due to the involved geometry of $\mathcal{A}$, an infinitesimal version thereof is precisely provided by the SdW horizontal differential. Indeed, formally, the total differential of the (gauge invariant) dressed matter field, $\mathbb{d}\widehat{\psi}$ is directly related to the SdW horizontal differential of the bare matter field, modulo the dressing factor:

$$\mathbb{d}\widehat{\psi} = e^F(\mathbb{d}\psi + \varpi\psi) = e^F \mathbb{d}_\perp \psi. \tag{198}$$

Since the SdW-horizontal differential has a natural place in any Abelian as well as non-Abelian YM theory, it follows that, in this sense, *the SdW horizontal differential constitutes the closest analogue to the Dirac dressing that generalizes to the non-Abelian YM theory.* In particular, the discussion of symmetry charges of section 3.8 shows that the dressed fields (or better, their differentials) do carry charges despite being fully gauge invariant (resp. horizontal covariant) objects.

Although uncharged, the photon can also be made gauge invariant by dressing it with the same dressing factor. Not surprisingly this gives rise to the transverse photon. In the non-Abelian setting a dressed, covariantly-transverse, gluon can be defined with the same caveats as for the dressed quark and with a completely analogous relation to the horizontal differentials.

We conclude by directing the reader to [75, 76] for a more algebraic take on dressings and their consequences e.g. for the interpretation of spontaneous symmetry breaking (aka the Higgs mechanism).

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