Analysis of Sharma-Mittal Holographic Dark Energy Model in Chern-Simons Modified Gravity

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Abstract

In this paper, the Sharma-Mittal Holographic dark energy model has been investigated in the framework of Chern-Simons modified gravity theory taking into account FRW universe. We explored a number of cosmological parameters such as deceleration parameter, equation of state, square of sound speed and density using redshift parameter. The graphical behavior indicated ‘ω’ decreasing approaching to −1 for −1 < z < 0 and increased for z > 0. It advocated that the universe contracts, bounces and expands. The density parameter showed the decreasing behavior and converges to zero asymptotically, gives the condition that universe will collapse ultimately at infinite time. It is also found that the square speed of sound turned to be positive for all parametric values predicted the stability of system.

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1 Introduction

Cosmology is in a disturbance. Its standard model accepted a few years back has as of late been deserted and supplanted by new thoughts. The purpose behind this sensational change is new estimations of the geometry and matter contents of the universe. The new model infers a dynamical age of the universe that obliges the most established known stellar objects but raises the need for a dark energy component to be explained within the current particle physics theories. It is a well-established fact the universe is in the accelerated phase of expansion and there is an exotic quantity named dark energy that is responsible for such type of expansion. A number of efforts have been done to explain the fact but a satisfactory answer is still a dream.

One of the models of modified gravity that attained much attention in recent years is, the four-dimensional Chern-Simons (CS) gravity theory introduced by Jackiw and Pi [1]. This modification built in the addition of Pontryagin density, which suggests the violation of parity symmetry in Einstein-Hilbert action. In four-dimensional space, the Pontryagin density behaves like topological quantity unless the coupling constant $\theta$ is not turned to be constant or promoted to a scalar field.

Nandi with fellows [2] investigated the effects of CS modified gravity on the quantum phase shift of de Broglie waves in neutron interferometry. It turned in a single equation showing a combination of effects coming from Newtonian gravity, inertial forces, Schwarzschild metric, and CS modified gravity. Alexander and Nicolas [3] explained CS modified theory in three different ways such as loop quantum gravity, particle physics, and string theory. They found vacuum approximation and exact solutions of modified theory and discussed the cosmic baryon asymmetry and inflation. They also explained how myriad astrophysical, cosmological probes, solar system and gravitational waves that bound CS modified gravity. Silva and Santos [4] discussed the energy density of the universe which is proportional to the Ricci scalar curvature with CS modified gravity and showed the scale factor is similar to those arise in modified Chaplygin gas.

Cardoso and Gualtieri [5] analyzed stability properties and perturbation formalism of black holes and found no effect of the CS coupling parameter on the polar sector but axial perturbations coupled to CS scalar field. Ahmedov and Aliev [6] showed four-dimensional spacetime with source-free CS modified gravity decouples into Cotton and Einstein constituents and explained how cosmological constant vanishes and the null hypersurface orthogonal
Killing vector of constant length become parallel to the gradient of CS scalar field. Amarilla with collaborators [7] proposed that the null geodesic equation can be integrated with approximation and explains the shadow cast of a black hole. They also discussed the coupling to CS term deforms the shape of black hole shadow and solution of angular momentum.

Chen and Jing [8] explained the geodetic precession and the strong gravitational lensing in the slowly rotating black hole in the framework of dynamical CS modified gravity theory. On the open choice of the external field as a function of angular parameter $\theta$, the Gödel-type solutions are studied and Non-static spherical symmetric metrics are investigated for the external field as a function of radial parameter $r$ in the context of non-dynamical and dynamical CS modified gravity theory [9] respectively. Furtado with his fellows [10] also studied the compatibility of Gödel metric in the context of CS modified gravity. Moon and Myung [11] investigated the stability of a Schwarzschild black hole in $f(R)$ gravity with parity-violating CS term coupled to a dynamical scalar field. They discuss $f(R)$ gravity does not influence the Zerilli equation, while CS coupling influences the Regge-Wheeler equation.

Extreme-Mass-Ratio Inspirals are one of the most promising sources of gravitational waves for space-based detectors like the Laser Interferometer Space Antenna. A parameter estimation study of EMRIs has been conducted in a particular modification of GR described by four-dimensional CS gravitational term [12]. Yagi with collaborators [13] studied the binary and isolated neutron stars in dynamical CS gravity. They found the corrections to post-Keplerian parameters are very similar to be observable today even with data from the double binary pulsar.

Contreras [14] using the minimal geometric deformation method, solve the Einstein field equation in 2+1 dimensional string theory to explore the anisotropic solution of the isotropic sector and decoupler matter. He also analyzed the solution of a regular and non-regular black hole with exotic or non-exotic hair solutions depending upon the behavior of cosmological constant. Eloisa [15] with his collaborators explained the Discrete Cosmology and black hole Lattice models. Using the techniques to construct string theory and observe the dressing of mass due to the interaction between individual black hole along with features of direct observational interest such as the distance-to-redshift relation. Amir and Ali [16]-[17] studied different holographic dark energy models in CS modified gravity taking into account the FRW universe.
A number of entropies have been used to produce modified Holographic dark energy models [18, 19], but there is nothing available in the literature that gives a single example of generalized entropy formalism to construct on the holographic principle. Rényi and Tsallis generalized entropies to develop suitable models for the current universe [20, 21]. These are widely used to study various gravitational and cosmological setups. Jahromi with his fellows [22] studied a new model Sharma-Mittal holographic dark energy (SMHDE) model in which Hubble horizon plays a role of an IR cut-off and there is no mutual interaction between the components of cosmos. In actuality, this model is a generalization of previously stated entropies.

In this paper, the Sharma-Mittal holographic dark energy model is considered to explore the deceleration parameter, energy density, Equation of state and square of sound speed in the context of CS modified gravity theory. This is arranged in the following order. The basic formalism of CS modified gravity is discussed in section 2. A brief introduction of the Sharma-Mittal holographic dark energy model is given in section 3. The deceleration, energy density, and EoS parameters are discussed in section 4, 5 and 6 respectively. Summary and concluding remarks are in the last section.

2 Chern-Simons Modified Gravity

The CS modified gravity is an effective extension of GR that captures leading-order gravitational parity violation. It is also motivated by anomaly cancellation in particle physics and string theory.

\[ S = S_{EH} + S_{\text{mat}} + S_{\Theta} + S_{CS}, \]

(1)

Where, \( \kappa = \frac{1}{16\pi G} \), \( \alpha \) and \( \beta \) are the dimensional coupling constants, \( \nabla_a \), \( g \) and \( R \) are the covariant derivative, determinant of the metric and Ricci scalar respectively. \( ^*RR \) is the Pontryagin density, defined as

\[ ^*RR = ^*R^a_{~b~cd}R^b_{~acd}. \]

(2)

\[
\begin{align*}
S = &\kappa \int_v d^4x \sqrt{-g} \{ \frac{\alpha}{\kappa} C_{ab} - \frac{1}{2\kappa} T_{ab} + R_{ab} - \frac{1}{2} g_{ab} R \} \delta g^{ab} \\
&+ \int_v d^4x \sqrt{-g} \{ \beta \Box \Theta + \frac{\alpha}{4} ^*RR - \beta \frac{dV}{d\Theta} \} \delta \Theta \\
&+ \sum_{CS} + \sum_{EH} + \sum_{\Theta},
\end{align*}
\]

(3)
where \( *R^b_{\, cd} = \frac{1}{2} \varepsilon^{def} R^a_{\, bdf} \) and \( \varepsilon^{def} \) is the 4-dimensional Levi-Civita tensor. Ceremonially, \( *RR \propto R \wedge R \) and Curvature tensor using as a Riemann tensor. \( \Theta \) is the function of spacetime called CS coupling field and not a constant value. If \( \Theta \) consider as a constant then CS modified gravity reduces to GR. \( \nabla_a \Theta \) is an embedding coordinate which helps to the generalization of the standard three dimensional CS theory to four-dimensional spacetime. So, \( \nabla_a \Theta \) and \( \nabla_a \nabla_b \Theta \) used as a deformation parameters in multidimensional space theories. By the variation of action in Eq. (1), the authors found the equation of motion of CS modified gravity and \( \Box \) is the D’Alembertian operator and equal to \( g^{mn} \nabla_m \nabla_n \). The stress-energy tensor into external matter contribution \( T_{mn}^\theta \) and scalar field.

\[
T_{mn}^\theta = \beta \{(\nabla_m \Theta)(\nabla_n \Theta) - g_{mn} V(\Theta) - \frac{1}{2} g_{mn} (\nabla_n \Theta)(\nabla^n \Theta)\} \quad (4)
\]

\( C_{mn} \) is the 4-dimensional C-tensor in Eq. (3) which is the generalization of three-dimensional Cotton-York tensor and quantity is obtained by

\[
C^{mn} = v_c * R^{p(mn)o} + v_c \varepsilon^{opq(m} \nabla_q R^{n)}
\]

where \( v_c \) and \( v_{op} \) are the velocity and covariant acceleration of \( \Theta \) respectively. The modified field equations are

\[
G_{mn} + \frac{\alpha}{\kappa} C_{mn} = \frac{1}{2\kappa} T_{mn}, \quad (5)
\]

\[
g^{mn} \nabla_m \nabla_n \Theta = - \frac{\alpha}{4\pi} *RR \quad (6)
\]

where \( G_{mn} \) is the Einstein tensor and equal to \( R_{mn} - \frac{1}{2} g_{mn} R \). The CS modified Eq. (3) of motion represent two different classes dynamical and non-dynamical frameworks. In the dynamical framework, \( \alpha \) is the variable in the equation but \( \beta \) is equal to zero. Dynamical describes a theory in which the scalar field introduces stress-energy into the modified field equation. In this framework, the forces of vacuum spacetime to possess a certain amount of scalar hair. On the other hand, \( \alpha \) and \( \beta \) are both equal to zero in the non-dynamical framework. In this framework, scalar hairy spacetime is absent. For FRW metric the pontryagin term turned to be zero identically, so the 2nd CS field equation reduced to

\[
g^{mn} \nabla_m \nabla_n \Theta = g^{mn} [\partial_m \partial_n \Theta - \Gamma^o_{\, mn} \partial_o \Theta] = 0
\]

\[
\dot{\Theta} = ca^{-3} \quad (7)
\]
is calculated simultaneously. The 00-component of Energy momentum tensor in case of external scalar field is given as

$$T_{00}^\Theta = \frac{1}{2} \dot{\Theta}^2. \tag{8}$$

Substituting Eq. (7) in Eq. (8)

$$\rho_\Theta = T_{00}^\Theta = \rho_\Theta_0 a^{-6}, \tag{9}$$

where $\rho_\Theta_0 = \frac{1}{2}c^2$.

3 Sharma-Mittal HDE Model

Some system involves statistical mechanics and corresponding thermodynamics at large distances, do not necessarily retain extensivity and additivity properties. Shannon’s entropy used to construct those interactions in which additivity and extensiveness on the basis of all results such as a system includes $W$ states in which $P_i$ probability of $i^{th}$ on $\sum_{i=1}^{W} P_i = 1$ condition. As a rule, these are systems that are better described by a power distribution of probabilities, namely $P_i^{1-\delta}$, where $\delta$ is a real parameter, rather than the usual distribution of $P_i$. It means that other measures of entropy are needed to describe these systems. Sharma and Mittal [23]-[24] introduced a generalized entropy with free parameter $r$ such as

$$S_{SM} = \frac{1}{1 - r} \left[ \left( \sum_{i=1}^{W} P_i^{1-\delta} \right)^{\frac{1-r}{r}} - 1 \right]. \tag{10}$$

In the case of black hole entropy in a quantum gravity loop $\sum_{i=1}^{W} P_i^{1-\delta} = 1 + \frac{\delta A}{4}$ putting in Eq. (10).

$$S_{SM} = \frac{1}{R} \left[ \left( 1 + \frac{\delta A}{4} \right)^{\frac{R}{2}} - 1 \right], \tag{11}$$

where $A$ is the horizon area and $R = 1 - r$. On using of holographic principle, the relation between IR ($L$), UV ($\Lambda$) cut-off and system horizon ($S$) is termed as

$$\Lambda^4 \propto \frac{S}{L^4}. \tag{12}$$
The HDE hypothesis, DE density \((\rho_D)\) corresponding to cut-off and using \(H = \frac{1}{L} = \sqrt{\frac{4\pi}{A}}\), reached at

\[
\rho_D = \frac{3C^2H^4}{8\pi R} \left[ \left(1 + \frac{\delta\pi}{H} \right) \frac{R}{\gamma} - 1 \right],
\]

(13)

where \(\frac{3C^2}{8\pi}\) is the proportionality constant. This is energy density of SMHDE model in which \(C^2\) is the unknown parameter. The original HDE model can be constructed when \(R \to \delta\). For the flat FRW universe, energy density is given by

\[
p_D = - \left( \frac{\dot{\rho}_D}{3H} + \rho_D \right) = - \left( \frac{\dot{\rho}_D}{3H} + \rho_D \right),
\]

(14)

where \(\dot{\rho}_D = \frac{d\rho_D}{dH}\) and \(\dot{\rho}_D = \frac{d\rho_D}{d\alpha}\). The 1st Friedmann equation in CS modified gravity is represented as

\[
H^2 = \frac{8\pi}{3} \left( \rho_m + \rho_D + \rho_\Theta \right),
\]

(15)

where \(\rho_m = \rho_0a^{-3}\) represented the matter energy density. Substituting the relevant values in Eq. (15), one arrived at

\[
H^2 = \frac{8\pi}{3} \left( \rho_0a^{-3} + \rho_\Theta a^{-6} + \frac{3C^2H^4}{8\pi R} \left[ \left(1 + \frac{\delta\pi}{H} \right) \frac{R}{\gamma} - 1 \right] \right).
\]

(16)

For the sake of simplicity, it is defined as \(H = E(z)H_0\), where \(H_0\) is the initial value of Hubble parameter, and in redshift parameter \(\frac{1}{a} = 1 + z\). Let \(\Omega_m = \frac{8\pi}{3H_0^2}\rho_0\), \(\Omega_\Theta = \frac{8\pi}{3H_0^2}\rho_\Theta\) and \(\gamma = \frac{R}{\delta}\), then the Eq. (16) became

\[
E^2(z) = \Omega_\Theta (1 + z)^6 + \Omega_m (1 + z)^3 + \frac{C^2E^4(z)H_0^2}{R} \left[ \left(1 + \frac{\delta\pi}{E^2(z)H_0^2} \right) \gamma - 1 \right].
\]

(17)

The factor \(\frac{C^2H_0^2}{R}\) is constant and its value be elaborated at initial value \(z = 0\) and \(E(0) = 1\), such that

\[
\frac{C^2H_0^2}{R} = \frac{1 - \Omega_m - \Omega_\Theta}{\left(1 + \frac{\delta\pi}{H_0^2} \right) \gamma - 1}.
\]

(18)
For analytic solution of the Eq. (17), suppose that \( \alpha = \frac{C^2 H_0^2}{\delta} \), and it became

\[
E^2(z) = \Omega_0 (1 + z)^6 + \Omega_m (1 + z)^3 + \alpha E^4(z) \left[ \left( 1 + \frac{\delta}{E^2(z) H_0^2} \right)^\gamma - 1 \right].
\]

(19)

4 Deceleration Parameter

The deceleration parameter is the dimensionless value which explains the expansions of the universe slow down due to self-gravity. It is defined as

\[
q = -\frac{\ddot{a}(t) a(t)}{\dot{a}^2(t)},
\]

(20)

where the dots indicate for proper time derivative and negative sign for deceleration. If the universe accelerating then value \( \ddot{a}(t) > 0 \) and \( q > 0 \). The relation between deceleration parameter and Hubble parameter is given as

\[
q = -1 - \frac{\dot{H}}{H^2}.
\]

(21)

Using \( H(z) = H_0 E(z) \) and putting \( \dot{H} = H_0 \frac{d}{dz}(E(z)) \) in Eq. (21), it is obtained as

\[
q = -1 - \frac{H_0 \frac{d}{dz}(E(z))}{H_0^2 E^2(z)}.
\]

(22)

Hence the deceleration parameter in terms of \( z \) is

\[
q = -1 + H_0 \frac{d}{dz}(1 + z) \gamma
\]

\[
- \left[ \frac{3\Omega_m + 6\Omega_0 (1 + z)^3}{2 H_0^2 (1 + z)^2 - 4\alpha ((1 + \delta \pi (1 + z)^2)^\gamma - 1) + 2\alpha \gamma \delta \pi (1 + z)^2 (1 + \delta \pi (1 + z)^2)^{\gamma-1}} \right].
\]

(23)

To study the behavior of deceleration parameter \( q \), a graph is plotted against \( q \) and \( z \) for different values of all parameters used in this relation.
Fig. (1) represents a combined graph of deceleration parameter $q$ versus redshift parameter $z$ at $H_0 = 67(Km/s)Mpc$, $\Omega_m = 0.23$, $\delta = -10$, $\gamma = -40$ and for three values of $\Omega_{\Theta} = 0.2, 0.4, 0.7$. $0 > q > -1$ when $0 > z > -1$, it means the universe is in the decelerating phase and expanding known as de Sitter expansion. $q > 0$ with $z > 0$, it means the universe is in the accelerating phase. The rate of expansion of the universe increase with increasing of $z$. Jahromi [22] with his fellows developed a model in which the universe is in the decelerating phase at $-1 < z < 0.6$ and static at $z = -0.6$ than the universe accelerating at the rate of $q = 0.5$.

By the Eq. (23), $q$ is negative value when $\gamma \leq 3$ with all $\delta$ value. It means that universe is in decelerating phase. For $\gamma > 2$, then $q$ is positive and universe is in accelerating phase. $q$ is equal to -1 when $z$ approaching to -1 and the function undefined at $z = -1$. If the $\delta = 0$ or $\gamma = 0$ then the Eq. (23) reduced to

$$q = -1 + \frac{1}{2}(1 + z)^5(2\Omega_m + 6\Omega_\Theta(1 + z)^3)$$  \hspace{1cm} (24)

By the Eq. (24), universe in accelerating phase at $z < -30$ and $z > 30$. For $-30 \leq z \leq 30$, the universe in decelerating phase.

5 Density Parameter

The density parameter is the ratio of the average density of universe ($\rho$) to critical density ($\rho_c$), mathematically it is given as

$$\Omega_0 = \frac{\rho}{\rho_c}.$$  \hspace{1cm} (25)
The value of \( \Omega_0 \) is approaching 1. If the \( \Omega_0 < 1 \), the universe is open and expanding forever. For \( \Omega_0 > 1 \), then the universe is close and eventually stops its expansion and re-collapse. If \( \Omega_0 = 1 \), then the universe is flat and enough matter to stop expansion, but not enough to collapse it again. The density parameter is the sum of different components including normal baryonic matter (\( \Omega_B \)), DM (\( \Omega_D \)) and DE (\( \Omega_\Lambda \)). Authors using DM density parameter \( \Omega_D(z) = \frac{\rho_D}{3H_0^2} \Rightarrow \rho_D = \Omega_D(z)\frac{3H_0^2}{8\pi} \) in Eq. (13), got

\[
\Omega_D(z)\frac{3H_0^2}{8\pi} = \frac{3C^2H_0^4}{8\pi R}\left[\left(1 + \frac{\delta \pi}{H}\right)^\gamma - 1\right].
\]

(26)

In terms of redshift parameter it became

\[
\Omega_D(z) = \frac{\alpha}{H_0^4(1 + z)^4}\left[\left(1 + \delta \pi(1 + z)\right)^\gamma - 1\right],
\]

(27)

where \( \alpha = \frac{C^2H_0^2}{R_0} \).

To analyze the behavior of the density parameter a graph is plotted. Fig. (2) represent graph of \( \Omega_D \) versus \( z \) \( H_0 = 67(Km/s)\) Mpc, \( \Omega_m = 0.26 \), \( \Omega_\Theta = 0.26 \), \( \delta = 20 \) and three different value of \( \gamma = -10, -20, -40 \). It is obvious that the graph showed decreasing behavior and converges to zero asymptotically. The Density parameter value approach to 0 with increasing \( z \). It predicted that the universe will be collapse at \( z = 0.4 \) [22].
6 Square of Sound Speed

For the model stability, Authors use a square of sound speed \( v_s^2 \) which is defined as

\[
v_s^2 = \frac{dp_D}{d\rho_D} = \frac{d\rho_D}{dE(z)}.
\]  

(28)

The 2nd field equation for flat FRW universe is

\[
H^2 + \frac{2}{3} \dot{H} = -\frac{8\pi}{3} p_D.
\]

(29)

Using Eq. (14) in (29) and simplifying, one arrived at

\[
p_D = \frac{\rho_D - \frac{E(z)}{2} \frac{d\rho_D}{dE(z)}}{\left(\frac{4\pi}{3H_0^2 E(z)}\right) \frac{d\rho_D}{dE(z)} - 1}.
\]

(30)
Differentiating Eq. (30) and Eq. (13) w.r.t $E(z)$ and substituting the corresponding expressions in Eq. (28), one arrived at

\[
v_s^2 = - \frac{H_0^3(1+z)^2}{24\pi (-1+2\pi\gamma\delta(1+\pi\delta(1+z)^2)^{\gamma-1} + 2(1+\pi\delta(1+z)^2)^{\gamma})} \times \frac{1}{(-H_0^3(1+z)^3 - 2-\pi(1+z)^2\alpha\gamma\delta(1+\pi\delta(1+z)^2)^{\gamma-1} + 2(1+\pi\delta(1+z)^2)^{\gamma})^2}
\]

\[
\times \{12\pi(1+z)(H_0^2(1+z)^2 + \pi(1+z)^2\alpha\gamma\delta(1+\pi\delta(1+z)^2)^{\gamma-1} - 2(-1+\pi(1+z)^2\gamma))\} \{-H_0^3\pi^2(1+z)^5\alpha(\gamma-1)\gamma\delta^2(1+\pi\delta(1+z)^2)^{\gamma-2} + 5H_0(-1+H_0^2)\gamma(1+z)^3\alpha\gamma\delta(1+\pi\delta(1+z)^2)^{\gamma-1} - 6\alpha(1+\pi\delta(1+z)^2)^{\gamma-1} - H_0(1+z)\{\pi(1+z)^2\alpha\gamma\delta(1+\pi\delta(1+z)^2)^{\gamma-1} - 2(-1+\pi\delta(1+z)^2\gamma)\}\} + \{3H_0\pi(1+z)^2\alpha\gamma\delta(1+\pi\delta(1+z)^2)^{\gamma-1} - 6H_0(-1+\pi\delta(1+z)^2)^{\gamma-1} - 4\pi(1+z)\alpha\{-2H_0^3\pi^2(1+z)^3(\gamma-1)\gamma\delta^2 \times (1+\pi\delta(1+z)^2)^{\gamma-2} + 5H_0(H_0^2-1)\pi(1+z)^3\gamma\delta(1+\pi\delta(1+z)^2)^{\gamma-1} - 6(-1+\pi\delta(1+z)^2)^{\gamma-1}\}\} \{3\pi(1+z)^2\alpha\gamma\delta(1+\pi\delta(1+z)^2)^{\gamma-1} - 6(-1+\pi\delta(1+z)^2)^{\gamma-1}\} + 8H_0^2\pi(1+z)^4\rho_D\}.
\]

Now plot a graph for square of sound speed $v_s^2$ versus redshift parameter. Authors plotted a graph of $v_s^2$ versus $z$ for the value of $H_0 = 67(Km/s)Mpc$, $\Omega_m = 0.26$, $\Omega_\Theta = 0.26$, $\delta = 20$, $\rho = 0.8$ and $\gamma = -10, -20, -40$ represented in Fig. (3). It is clearly positive function showing decreasing behavior for some $-1 < z < -0.6$ but positive and than it tends to increasing remained positive.

7 Equation of State

The equation of state of ideal fluid is the ration of pressure and energy density, such that

\[
\omega = -\frac{p_D}{\rho_m + \rho_D + \rho_\Theta}.
\]

The equation of state is used to study the effect of different quantities on the universe. For example, it is considered under the effect of the relativistic matter when $\omega = 0$. If $0 < \omega < \frac{1}{3}$, the decelerated phase of the universe is under the effect of the radiation era. For $\omega < -1$, $\omega = -1$ and $\omega < -\frac{1}{3}$
Figure 3: The graph of square of sound speed versus redshift parameter corresponding to the phantom, cosmological constant and quintessence eras respectively. Substituting the value of different parameters in Eq. (32), it is obtained

$$\omega = -\frac{H^2 + \frac{2}{3}\dot{H}}{H^2}. \quad (33)$$

In term of deceleration parameter, the equation of state can be written as

$$\omega = \frac{2}{3} \left( q - \frac{1}{2} \right). \quad (34)$$

Using Eq. (23) in Eq. (34)

$$\omega = -1 + \frac{2}{3}H_0^2(1+z)^7 \times \left[ \frac{3\Omega_m + 6\Omega_\Theta(1+z)^3}{2H_0^2(1+z)^2 - 4\alpha\{(1 + \delta\pi(1 + z)^2)\gamma - 1\} + 2\alpha\gamma\delta\pi(1 + z)^2(1 + \delta\pi(1 + z)^2)\gamma - 1} \right]. \quad (35)$$

To analysis the behavior of equation of state parameter with respect to redshift parameter a graph in being plotted.

In the Fig. (4), authors plotted a graph of $\omega$ versus $z$ for $H_0 = 67 (Km/s) Mpc$, $\Omega_m = 0.23$, $\delta = -40$, $\gamma = 3$ and for three different values of $\Omega_\Theta = 0.2, 0.4, 0.6$. 

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The Eq. (35) showed that the value of $\omega < 0$ if $\Omega_\Theta < 0.25$ and positive for $\Omega_\Theta \geq 0.25$. When the value of $\delta$ is $+ve$ and $\gamma$ is negative then $\omega$ turned $-ve$. The equation of state is positive if $\gamma$ and $\delta$ have same sign. In the absence of $\gamma$ or $\delta$ the equation of state is given as

$$\omega = -1 + 3H_0^4 \left[ 3(1+z)\Omega_m + \frac{6\Omega_\Theta}{(1+z)^4} \right]$$  \hspace{1cm} (36)$$

The dynamical properties of the universe are studied using $\omega - \omega'$ plane. It has thawing region and freezing region. In the thawing region, the EoS parameter ($\omega$) resulted negative and in the evolutionary case, it turned positive while in freezing region the value of EoS parameter remained negative. The term $\omega'$ is a derivative of EoS with respect to $z$ of Eq. (35) and simpli-
fication gives

$$\omega' = \frac{1}{\{2H_0^2(1+z)^2 + 2\pi\alpha\gamma\delta(1 + \pi\delta(1+z)^2)^\gamma - 4\alpha(-1 + (1 + \pi\delta(1+z)^2)^\gamma)\}^2}$$

$$\times \{12H_0^2(1+z)^6 \{5H_0^2(1+z)^2\Omega_m - 2\pi^2\alpha\gamma\delta^2\Omega_m(\gamma - 1)(1+z)^4(1 + \pi\delta(1+z)^2)^\gamma - 2 + 9\pi\alpha\gamma\delta(1+z)^2(1 + \pi\delta(1+z)^2)^\gamma - 14\alpha\Omega(-1 + (1 + \pi\delta(1+z)^2)^\gamma)
+ 24\pi\alpha\delta\Omega_\Theta(1+z)^5(1 + \pi\delta(1+z)^2)^\gamma - 1 - 40\alpha\Omega_\Theta(1+z)^3(-1 + (1 + \pi\delta(1+z)^2)^\gamma)
+ 16H_0^2\Omega_\Theta(1+z)^5 - 4\pi^2\alpha\delta^2(\gamma - 1)\Omega_\Theta(1+z)^7(1 + \pi\delta(1+z)^2)^\gamma - 2\}\}. \quad (37)$$

Figure 5: The graph of derivative of EoS versus redshift parameter

In Fig. (5), the graph has been plotted for $\omega'$ versus $z$ at $H_0 = 67(Km/s)Mpc$, $\Omega_m = 0.23$, $\Omega_\Theta = 0.2, 0.4, 0.6$, $\delta = -40$, and $\gamma = 3$. The term $\omega'$ showed negative behavior on the choice of $\gamma > 0$ and $\delta > 0$. It is worth mentioning that the $\omega'$ decreased on the increasing value of $z$ in negative axis and increased with increasing of $z$ in positive region.

8 Conclusion

In this paper, the Sharma-Mittal Holographic dark energy model has been investigated in the framework of Chern-Simons modified gravity theory taking into account FRW universe, we explored a number of cosmological parameters such as deceleration parameter, equation of state, the square of sound speed and density using redshift parameter. The deceleration parameter $q$ predicted that the universe is in decelerating and expanding phase known as a Sitter expansion for $-1 < z < 0$ and the expansion rate of the universe
is increased at acceleration level for all \( z > 0 \), which is same as a model developed by Johrami [1].

It is observed that the equation of state tuned positive on the choice of the same behavior of \( '\gamma' \) and \( '\delta' \) and it gives the negative results when alternating symbols of \( '\gamma' \) and \( '\delta' \) are chosen. The graphical behavior indicates that the value of \( '\omega' \) decreasing approaching to \(-1\) for \(-1 < z < 0\) and increasing for \( z > 0 \). It advocated that the universe contracts bounces and expands.

The density parameter of the SMHDE model is well defined for all values of \( \gamma \) and \( \delta > 0 \). It is obvious that the density parameter showed the decreasing behavior and converges to zero asymptotically which gives the condition that the universe will collapse at infinite time. The test for square speed of sound also investigated in this work. It is found that the square speed of sound turned to be positive for all parameter values which predict the stability of the system.

References