

Why space must be quantised on a different scale to matter

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1 Abstract

The scale of quantum mechanical effects in matter is set by Planck's constant, \hbar . This represents the quantisation scale for material objects. In this article, we give a simple argument why the quantisation scale for space, and hence for gravity, cannot be equal to \hbar . Indeed, assuming a single quantisation scale for both matter and geometry leads to the 'worst prediction in physics', namely, the huge difference between the observed and predicted vacuum energies. Conversely, assuming a different quantum of action for geometry, $\beta \neq \hbar$, allows us to recover the observed density of the Universe. Thus, by measuring its present-day expansion, we may in principle determine, empirically, the scale at which the geometric degrees of freedom must be quantised.

1 Wave-particle duality and \hbar

Classical mechanics is deterministic [1]. If its initial conditions are known, the probability of finding a particle at a given point on its trajectory, at the appropriate time t , is 100%. The corresponding state is described by a delta function, $\delta^3(\mathbf{x} - \mathbf{x}')$, with dimensions of (length)⁻³. This is the probability density of the particle located at $\mathbf{x} = \mathbf{x}'$.

In quantum mechanics (QM), probability amplitudes are fundamental. Position eigenstates, $|\mathbf{x}\rangle$, are the rigged basis vectors of an abstract Hilbert space, where $\langle \mathbf{x} | \mathbf{x}' \rangle = \delta^3(\mathbf{x} - \mathbf{x}')$. These have dimensions of (length)^{-3/2} and more general states may be constructed by the principle of quantum superposition [2]. The resulting wave function, $\psi(\mathbf{x})$, represents the probability amplitude for finding the particle at each point in space, and the corresponding probability density is $|\psi(\mathbf{x})|^2$ [3].

Since $\psi(\mathbf{x})$ can also be decomposed as a superposition of plane waves, $e^{i\mathbf{k}\cdot\mathbf{x}}$, an immediate consequence is the uncertainty principle $\Delta_\psi x^i \Delta_\psi p_j \geq (1/2)\delta^i_j$, where $i, j \in \{1, 2, 3\}$ label orthogonal Cartesian axes. This is a purely mathematical property of ψ that follows from elementary results of functional analysis [4]. In canonical QM, we relate the particle momentum \mathbf{p} to the wave number \mathbf{k} via Planck's constant, following the proposal of de Broglie, $\mathbf{p} = \hbar\mathbf{k}$. It follows that

$$\Delta_\psi x^i \Delta_\psi p_j \geq (\hbar/2)\delta^i_j. \quad (1)$$

This is the familiar Heisenberg uncertainty principle (HUP). We stress that the HUP is a consequence of two distinct physical assumptions:

1. the principle of quantum superposition, and

32 2. the assumption that \hbar determines the *scale* of wave–particle duality.¹

33 Let us also clarify the meaning of the word ‘particle’. We stress that canonical QM
 34 treats all particles as point-like, so that eigenstates with zero position uncertainty may
 35 be realised, at least formally. However, gravitational effects are expected to modify the
 36 HUP by introducing a minimal length, $\Delta x > 0$ [6, 7]. Next, we discuss how this relates to
 37 theoretical predictions of the vacuum energy.

38 2 Minimal length and the vacuum energy

39 In canonical QM, the background space is fixed and classical. Individual points are sharply
 40 defined and the distances between them can be determined with arbitrary precision [8].
 41 By contrast, thought experiments in the quantum gravity regime suggest the existence
 42 of a minimum resolvable length scale of the order of the Planck length, $\Delta x \simeq l_{\text{Pl}}$, where
 43 $l_{\text{Pl}} = \sqrt{\hbar G/c^3} \simeq 10^{-33}$ cm [6]. Below this, the classical concept of length loses meaning,
 44 so that perfectly sharp space-time points cannot exist [7].

45 This motivates us to take l_{Pl} as the UV cut off for vacuum field modes, but doing so
 46 yields the so-called ‘worst prediction in physics’ [9], namely, the prediction of a Planck-
 47 scale vacuum density:

$$\rho_{\text{vac}} \simeq \frac{\hbar}{c} \int_{k_{\text{dS}}}^{k_{\text{Pl}}} \sqrt{k^2 + \left(\frac{mc}{\hbar}\right)^2} d^3k \simeq \rho_{\text{Pl}} = \frac{c^5}{\hbar G^2} \simeq 10^{93} \text{ g} \cdot \text{cm}^{-3}. \quad (2)$$

48 Unfortunately, the observed vacuum density is more than 120 orders of magnitude lower,

$$\rho_{\text{vac}} \simeq \rho_{\Lambda} = \frac{\Lambda c^2}{8\pi G} \simeq 10^{-30} \text{ g} \cdot \text{cm}^{-3}. \quad (3)$$

49 In Eq. (2), the mass scale $m \ll m_{\text{Pl}} = \hbar/(l_{\text{Pl}}c) \simeq 10^{-5}$ g is set by the Standard Model
 50 of particle physics [11] and the limits of integration are $k_{\text{Pl}} = 2\pi/l_{\text{Pl}}$, $k_{\text{dS}} = 2\pi/l_{\text{dS}}$, where
 51 $l_{\text{dS}} = \sqrt{3/\Lambda}$ is the de Sitter length. This is comparable to the present day radius of the
 52 Universe, $r_{\text{U}} \simeq 10^{28}$ cm, which may be expressed in terms of the cosmological constant,
 53 $\Lambda \simeq 10^{-56} \text{ cm}^{-2}$ [10].

54 More detailed calculations alleviate this discrepancy [12], but our naive calculation
 55 highlights the problem of treating l_{Pl} and m_{Pl} as interchangeable cutoffs. We now discuss
 56 an alternative way to obtain a minimum length of order l_{Pl} without generating unfeasibly
 57 high energies.

58 3 Wave–point duality and $\beta \neq \hbar$

59 Clearly, one way to implement a minimum length is to discretise the geometry, as in loop
 60 quantum gravity and related approaches [13]. However, in general, quantisation is *not*
 61 discretisation [14]. The key feature of quantum gravity is that it must allow us to assign a
 62 quantum state to the background, giving rise to geometric superpositions, and, therefore,
 63 superposed gravitational field states [15]. The associated spectrum may be discrete or
 64 continuous, finite or infinite.

¹Note that these assumptions consistent with Poincaré invariance, and, hence, with Galilean invariance in the non-relativistic limit of canonical QM, if and only if $p \propto k$ and $E \propto \omega$ [5]. Ultimately, it is the constant of proportionality in these relations that determines the length and momentum (energy) scales at which quantum effects become important. The ‘quantisation scale’ of any system is, therefore, an action scale, which must be determined empirically. For canonical quantum particles, this scale is $\hbar = 1.05 \times 10^{-34}$ J s.

65 But how to assign a quantum state to space itself? A simple answer is that we must
 66 first assign a quantum state to each *point* in the classical background. Individual points
 67 then map to superpositions of points and the unique classical geometry is mapped to a
 68 superposition of geometries, as required [16]. In effect, we apply our quantisation proce-
 69 dure point-wise, and, in the process, eliminate the concept of a classical point from our
 70 description of physical reality.

71 This can be achieved by first associating a delta function with each coordinate ‘ \mathbf{x} ’. We
 72 then note that $\delta^3(\mathbf{x} - \mathbf{x}')$ is obtained as the zero-width limit of a Gaussian distribution,
 73 $|g(\mathbf{x} - \mathbf{x}')|^2$, with standard deviation $\Delta_g x$. Taking $\Delta_g x > 0$ therefore ‘smears’ sharp
 74 spatial points over volumes of order $\sim (\Delta_g x)^3$, giving rise to a minimum observable length
 75 scale [16]. Motivated by thought experiments [6], we set $\Delta_g x \simeq l_{\text{Pl}}$.

76 Since g may also be expressed as a superposition of plane waves, an immediate con-
 77 sequence is the wave-point uncertainty relation, $\Delta_g x^i \Delta_g k_j \geq (1/2)\delta^i_j$. This is an uncer-
 78 tainty relation for delocalised ‘points’, not point-particles in the classical background of
 79 canonical QM [16]. A key question we must then address is, what is the momentum of
 80 a quantum geometry wave? For matter waves, $\mathbf{p} = \hbar\mathbf{k}$, but we have no *a priori* reason
 81 to believe that space must be quantised on the same scale as material bodies. In fact,
 82 setting $\Delta_g x \simeq l_{\text{Pl}}$ and $\mathbf{p} = \hbar\mathbf{k}$ yields $\Delta_g p \simeq m_{\text{Pl}}c$, giving a vacuum density of order
 83 $\rho_{\text{vac}} \simeq (\Delta_g p)/(\Delta_g x)^3 c \simeq c^5/(\hbar G^2)$. This is essentially the same calculation as that given
 84 in Eq. (2), which results from the same physical assumptions. Hence, we set

$$\Delta_g x^i \Delta_g p_j \geq (\beta/2)\delta^i_j, \quad (4)$$

85 where $\beta \neq \hbar$ is the fundamental quantum of action for geometry.²

86 Smearing each point in the background convolves the canonical probability density
 87 with a Planck-width Gaussian. The resulting total uncertainties are

$$\Delta_\Psi X^i = \sqrt{(\Delta_\psi x^i)^2 + (\Delta_g x^i)^2}, \quad \Delta_\Psi P_j = \sqrt{(\Delta_\psi p_j)^2 + (\Delta_g p_j)^2}, \quad (5)$$

88 for each $i, j \in \{1, 2, 3\}$, where $\Psi := \psi g$ denotes the composite wave function of a particle in
 89 smeared space [16, 19]. Finally, we note that the existence of a finite cosmological horizon
 90 implies a corresponding limit on the particle momentum, which may be satisfied by setting
 91 $\Delta_g p \simeq \hbar\sqrt{\Lambda/3}$. The resulting quantum of action for geometry is

$$\beta \simeq \hbar\sqrt{\frac{\rho_\Lambda}{\rho_{\text{Pl}}}} \simeq \hbar \times 10^{-61}. \quad (6)$$

²In the relativistic regime, the tensor nature of gravitational waves must also be accounted for, but this may be neglected in the non-relativistic limit in which Eq. (4) remains valid [16]. In this model, a function is associated to each spatial point by doubling the degrees of freedom in the classical phase space and the classical point labeled by x is associated with the quantum probability amplitude $g(x - x')$. This is the mathematical representation of a delocalized ‘point’ in the quantum nonlocal geometry. For each x , the additional variable x' may take any value in R^3 . Together, x and x' cover $R^3 \times R^3$, which is interpreted as a superposition of 3D Euclidean spaces [16]. The process of ‘smearing’ points is easiest to visualize in the case of a toy one-dimensional universe. In this case, the original classical geometry is the x -axis and the (x, x') plane on which $g(x - x')$ is defined represents the smeared superposition of geometries. These issues are considered in detail in the refs. [16–19] (see, in particular, see Fig. 1 of ref. [16]), but are not discussed at length in the present article for want of space. Note also that classical points are defined, where necessary, as in standard differential geometry. However, the model considered here is not based on classical points or on the fixed manifolds that form the mathematical basis of classical spacetimes. Instead, we associate each point in the classical background, labelled by x , with a vector in a quantum Hilbert space, $|g_x\rangle$. The associated wave function, $\langle x'|g_x\rangle = g(x - x')$, may be regarded as a Gaussian of width $\sigma_g \simeq l_{\text{Pl}}$. This represents the quantum state of a delocalized ‘point’ in the quantum geometry, but this term is used here in an imprecise sense, only for illustration. (Hence the inverted commas.)

92 The new constant β sets the Fourier transform scale for $g(\mathbf{x} - \mathbf{x}')$, whereas the matter
 93 component $\psi(\mathbf{x})$ transforms at \hbar [16, 19].³ However, this does not violate the existing
 94 no-go theorems for the existence of multiple quantisation constants. These apply only to
 95 species of material particles [20], and still hold in the smeared-space theory, undisturbed
 96 by the quantisation of the background [19].

97 4 The vacuum energy, revisited

98 The introduction of a new quantisation scale for space radically alters our picture of the
 99 vacuum, including our naive estimate of the vacuum energy. This must be consistent with
 100 the generalised uncertainty relations (5). Expanding $\Delta_\Psi X^i$ with $\Delta_g x^i \simeq l_{\text{Pl}}$ gives the
 101 generalised uncertainty principle (GUP) and expanding $\Delta_\Psi P_j$ with $\Delta_g p_j \simeq \hbar\sqrt{\Lambda/3}$ yields
 102 the extended uncertainty principle (EUP), previously considered in the quantum gravity
 103 literature [21, 22].

104 Equations (5) may also be combined with the HUP, which holds independently for
 105 ψ [16, 19], to give two new uncertainty relations of the form $\Delta_\Psi X^i \Delta_\Psi P_j \geq \dots \geq (\hbar +$
 106 $\beta)/2 \cdot \delta^i_j$. The central terms in each relation depend on either $\Delta_\psi x^i$ or $\Delta_\psi p_j$, exclu-
 107 sively. Minimising the product of the generalised uncertainties, $\Delta_\Psi X^i \Delta_\Psi P_j$, we obtain
 108 the following length and momentum scales:

$$\begin{aligned} (\Delta_\psi x)_{\text{opt}} &\simeq l_\Lambda := \sqrt{l_{\text{Pl}} l_{\text{dS}}} \simeq 0.1 \text{ mm}, \\ (\Delta_\psi p)_{\text{opt}} &\simeq m_\Lambda c := \sqrt{m_{\text{Pl}} m_{\text{dS}}} c \simeq 10^{-3} \text{ eV}/c, \end{aligned} \quad (7)$$

109 where $m_{\text{dS}} = \hbar/(l_{\text{dS}} c) \simeq 10^{-66}$ g is the de Sitter mass. This gives a vacuum energy of
 110 order

$$\rho_{\text{vac}} \simeq \frac{3}{4\pi} \frac{(\Delta_\psi p)_{\text{opt}}}{(\Delta_\psi x)_{\text{opt}}^3 c} \simeq \rho_\Lambda = \frac{\Lambda c^2}{8\pi G} \simeq 10^{-30} \text{ g} \cdot \text{cm}^{-3}, \quad (8)$$

111 as required. Taking $k_\Lambda = 2\pi/l_\Lambda$ as the UV cut off in Eq. (2), with $m = m_\Lambda$, also gives
 112 the correct value order of magnitude, $\rho_{\text{vac}} \simeq \rho_\Lambda$ [16].

113 In this model, vacuum modes seek to optimise the generalised uncertainty relations
 114 induced by both \hbar and β , yielding the observed vacuum energy. Any attempt to excite
 115 higher-order modes leads to increased pair-production of neutral dark energy particles,
 116 of mass $m_\Lambda \simeq 10^{-3}$ eV/ c^2 , together with the concomitant expansion of space required to
 117 accommodate them, rather than an increase in energy density [19]. The vacuum energy
 118 remains approximately constant over large distances, but exhibits granularity on scales of
 119 order $l_\Lambda \simeq 0.1$ mm [16, 23, 24]. It is therefore intriguing that tentative evidence for small
 120 oscillations in the gravitational force, with approximately this wavelength, has already
 121 been observed [25, 26].

122 5 Summary

123 This simple analysis shows that, if space-time points are delocalised at the Planck length,
 124 $\Delta x \simeq l_{\text{Pl}}$, the associated momentum uncertainty cannot be of the order of the Planck
 125 momentum, $\Delta p \neq \hbar/\Delta x \simeq m_{\text{Pl}} c$. We are then prompted to ask: is it reasonable to assume
 126 that quantised waves of space-time carry the same quanta of momentum as matter waves

³The term ‘quantum geometry wave’, introduced above Eq. (4), therefore has a precise meaning. It refers to the plane wave components of $\tilde{g}_\beta(\mathbf{p} - \mathbf{p}')$, which is the β -scaled Fourier transform of $g(\mathbf{x} - \mathbf{x}')$.

127 with the same frequency? Though a common assumption, underlying virtually all attempts
128 to quantise gravity that utilise a single action scale, \hbar , we note that it has, *a priori*,
129 no theoretical justification. We have shown that relaxing this stringent requirement by
130 introducing a new quantum of action for geometry, $\beta \neq \hbar$, leads to interesting possibilities,
131 with the potential to open up brand new avenues in quantum gravity research [19]. These
132 include the proposal that the observed vacuum energy, and the present-day accelerated
133 expansion of the universe that it drives, are related to the quantum properties of space-
134 time [17, 18]. In this model, a measurement of the dark energy density constitutes a de
135 facto measurement of the geometry quantisation scale, β , fixing its value to $\beta \simeq \hbar \times 10^{-61}$.

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