Hawking flux of 4D Schwarzschild blackhole with supertransition correction to second-order

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Abstract

Former part of this article is the proceeding for my talk [1] on [2], which is a report on the issue in the title of this article. Later part is the detailed description of [2].

Contents

1 Motivation for my analyzing Hawking flux

We will start with a 4D spacetime, then impose some falloff condition to its spatially infinite region[1], which means the spacetime asymptotes to the flat spacetime according to that. Then, we will consider the diffeomorphism at its spatially infinite region in the range of the falloff condition (at this time, gauge conditions to eliminate local diffeomorphism ambiguities are also preserved). Transformation of the spacetime by this diffeomorphism is called asymptotic (or BMS) symmetry [3, 4]. I list several facts known for this.

- Diffeomorphism of the asymptotic symmetry is an infinite dimensional group, which contains the Poincarè group as its subgroup.
- There are so-called supertranslation and superrotation in the asymptotic symmetry, which respectively contains the global translation and Lorentz transformation.
- Diffeomorphism of the asymptotic symmetry maps a configuration of an asymptotically flat spacetime as a solution to an other physically different asymptotically flat spacetime as a solution in the range of the falloff condition.
- Asymptotically flat spacetime is infinitely degenerated in the range of asymptotic symmetry, and the symmetry of theory at that asymptotic region is not Poincarè symmetry but the one associated with supertranslation and superrotation.
- Infinite conserved charges for the asymptotic symmetry can be defined. Of course these effects are not zero, therefore the asymptotic symmetry is some kinds of spontaneous broken symmetry[2]. (Brown-Henneaux charge [5] is one of asymptotic symmetry breakings, though not in 4D flat spacetime.)
- Although asymptotic symmetry is not global, one can consider the field like NG boson field for supertranslation.
- Supertranslated spacetimes are normal [6], therefore considering asymptotic symmetry is meaningful realistically.

In [2], I obtain the Schwarzschild blackhole spacetime with supertranslation correction to the second-order, which I sketch as

$$\begin{aligned} ds^2 \,=\, & -(1-2m/r+\cdots+\mathcal{O}(\varepsilon^3))\,dt_s^2+((1-2m/r)^{-1}+\cdots+\mathcal{O}(\varepsilon^3))\,dr_s^2 \\ & +(r^2+\cdots+\mathcal{O}(\varepsilon^3))\,d\theta_s^2+(r^2\sin^2\theta+\cdots+\mathcal{O}(\varepsilon^3))\,d\phi_s^2 \\ & +2((\cdots)\varepsilon+\cdots+\mathcal{O}(\varepsilon^3))\,dr_s d\theta_s, \end{aligned} \tag{1}$$

where ε mean the order of supertranslation correction and the full expressions of metrices are given $j_{\mu\nu}^{(2)}$ in (46a)-(50b). We can obtain the position of the horizon from this is

$$r_{h,4D}=2m-\frac{15m\sin^2(2\theta)}{8\pi}\varepsilon^2+O(\varepsilon^3). \tag{2}$$

The correction of supertranslation enters from the second-order, which is the motivation for our analysis to the second-order. Here, as this is not constant, there may be a concern

[1] Talk slide is in the homepage [1]. I describe the former part of this article by first person.

[2] One may ask what's the configuration before the SSB. there look no studies on this until now and I cannot say any sure things about this. This SSB would be different kind of our familiar SSB. In P.41 in [7], some comment is written. Phenomena in the familiar SSB may not be always exist, and there may be no configuration before the SSB of the asymptotic symmetry, though I am not sure.

for the zeroth law of the blackhole thermodynamics. It is no problem since the Hawking temperature is constant in the range of our analysis's order, ε^2, as shown below.

Since the position of the horizon is displaced for supertranslation correction, it is interesting to check how the Hawking temperature is. Its result is

$$T_H = 1/8\pi m + O(\varepsilon^3), \tag{3}$$

which is no difference from just the Schwarzschild (reason is written in Sec.B.2).

Since Hawking temperature can be calculated from Hawking flux, if the Hawking temperature were preserved the Hawking flux would be expected to be preserved. However I have considered that supertranslation corrections may be involved in the Hawking flux but would be canceled out in the Hawking temperature. This is one of my motivations for my computing the Hawking flux in [2].

There is another motivation, which is that as a result of involving the supertranslation corrections as in (1), it becomes obscure whether field theories can reduce to free 2D or not in the near-horizon. Originally it should be so for the strongly gravitational force at the horizon, and if not, it would be physically abnormal. Although it can be shown in Sec.B.5 that the scalar theory can reduce to free 2D whether it is possible or not is unclear before trying (I comment on the key for the feasibility of this in the last of Sec.B.5).

2 What's supertranslation and its NG boson fields

We start with an expression of general 4D spacetime by the Bondi coordinates (u, r, Θ^A) ($u = t - r$ and Θ^A are the spherical coordinates $(z, \bar{z})$ on the S^2) as

$$ds^2 = -U du^2 - e^{2\beta} du dr + g_{AB}\Big(d\Theta^A + \frac{1}{2}U^A du\Big)\Big(d\Theta^B + \frac{1}{2}U^B du\Big), \tag{4}$$

where the Bondi gauge to fix the local diffeomorphisms, $g_{rr} = g_{rA} = 0$, is imposed. Then, supposing that the spacetime will asymptote to the flat spacetime, let us consider to describe the neighborhood of $\mathcal{I}^+$. At this time we need to impose a falloff condition to the metrices, however there is no systematic ways to determine the falloff condition. Conversely, various falloff conditions can be considered. Typically, it is chosen so that physical solutions can exist and unphysical solutions do not exist.

As an expansion of (4) to r^{-1}, the following one is conventionally adopted [3,4]:

$$\begin{aligned} ds^2 =& - du^2 - 2dudr + 2r^2\gamma_{z\bar{z}} dz d\bar{z} \\ &+ 2m_B/r\, du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + D^z C_{zz} du dz \\ &+ \frac{1}{r}\Big(\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}(C_{zz}C^{zz})\Big) du dz + \text{c.c.} + O\left(r^{-2}\right), \end{aligned} \tag{5}$$

where D_z is the covariant derivative with respect to $\gamma_{\bar{z}z}$. It is usual that the structure to r^{-1} is important. C_{zz}, $C_{\bar{z}\bar{z}}$, m_B and N_z are functions of $(u, z, \bar{z})$ but not of r, and

- m_B is the Bondi mass aspect. $\int_{S^2} dz d\bar{z} m_B$ gives the Bondi mass, which can be ADM mass in the cases of blackhole spacetimes.
- N_z is the angular momentum aspect. $\int_{S^2} dz d\bar{z} N_z V^z$ gives the total angular momentum, which is ADM angular momentum in the blackhole spacetimes.
- C_{zz}, $C_{\bar{z}\bar{z}}$ play the role of potential for gravitational wave (akin to vector potential for electromagnetic field), and N_{zz} is the Bondi news given as $\partial_u C_{zz}$ ($N_{\bar{z}\bar{z}}$ is likewise).

The falloffs of the metrices in (5) are given as follows:

$$\begin{aligned} g_{uu} = & -1+\mathcal{O}(r^{-1}), \quad g_{ur}=-1+\mathcal{O}(r^{-2}), \quad g_{uz}=\mathcal{O}(1), \\ g_{zz}= & \mathcal{O}(r), \quad g_{z\bar{z}}=r^2\gamma_{z\bar{z}}+\mathcal{O}(1), \quad g_{rr}=g_{rz}=0. \end{aligned} \tag{6}$$

Let us turn to the supertranslations. These generation are given as

$$\begin{aligned} \mathcal{L}_\zeta g_{ur} &= -\partial_u\zeta^u+\mathcal{O}(r^{-1}), \\ \mathcal{L}_\zeta g_{zr} &= r^2\gamma_{z\bar{z}}\partial_r\zeta^{\bar{z}}-\partial_z\zeta^u+\mathcal{O}(r^{-1}), \\ \mathcal{L}_\zeta g_{z\bar{z}} &= r\gamma_{z\bar{z}}(2\zeta^r+rD_z\zeta^z+rD_{\bar{z}}\zeta^{\bar{z}})+\mathcal{O}(1), \\ \mathcal{L}_\zeta g_{uu} &= -2\partial_u\zeta^u-2\partial_u\zeta^r+\mathcal{O}(r^{-1}). \end{aligned} \tag{7}$$

$$\begin{aligned} \mathcal{L}_f m_B &= f\partial_u m_B+\frac{1}{4}(N^{zz}D_z^2 f+2D_zN^{zz}D_zf+c.c.), \\ \mathcal{L}_f N_{zz} &= f\partial_u N_{zz}, \\ \mathcal{L}_f C_{zz} &= f\partial_u C_{zz}-2D_z^2 f, \end{aligned} \tag{8}$$

where the vector field for the Lie derivative above is given as

$$\xi = f\partial_u+\frac{1}{r}(D^zf\partial_z+D^{\bar{z}}f\partial_{\bar{z}})+D^zD_zf\partial_r. \tag{9}$$

f is arbitrary function of $(z,\bar{z})$, and normally spherical harmonics are taken.

The NG bosons associated with the asymptotic symmetry breaking is given as

$$\mathcal{L}_f C(z,\bar{z}) = f(z,\bar{z}), \tag{10}$$

C is the NG boson, which is infinite as the asymptotic symmetry is infinite dimensional.

3 Fun in the asymptotic symmetry

First of all, what 4D Mankowski spacetime has not been an unique vacuum but infinitely degenerated would be a surprisingly interesting fact. Surely this had been already found in 1962 [3,4], however it is in just the last decade that hep-th has recognized this problem [7]. As interesting directions from the study of the asymptotic symmetry, the following ones could be taken: **1)** gravitational memory effect **2)** links with/between soft theorems and memory effects, **3)** holography, and **4)** information paradox.

1) is the variation in the relativistic position of two objects near the future null infinity $\mathcal{I}^+$ for the passing of the gravitational wave, which could be measured by the formalism of the asymptotic symmetry.

Consider the gravitational wave is turned on at $u=u_i$ and off at $u=u_f$, and two objects near $\mathcal{I}^+$ are exposed it during the time interval $\Delta u = u_f-u_i$. The Bondi news tensor and the energy momentum tensors are zero at any time except for the time getting the gravitational wave. Then, one can evaluate the displaced amount as

$$\Delta s^{\bar{z}} = \frac{\gamma^{z\bar{z}}}{2r}\Delta C_{zz}s^z, \tag{11}$$

where $\Delta s^A = s^A|_{u=u_f}-s^A|_{u=u_i}$ ($(s,\,\bar{s})$ means the relativistic position of the two objects), and $\Delta C_{AB}=C_{AB}|_{u=u_f}-C_{AB}|_{u=u_i}$.

Hence, the passage of gravitational wave will arise the displacement by the order r^{-1}. To observe this would be highly harder than observing the gravitational wave. Now it is planed at LIGO [8] or via a pulsar timing array [9, 10].

Other types of memory effect are also considered: spin memory effect [11], color memory effect [12], and electromagnetic memory effect [13–15]. Observing the soft graviton may be also planed, however it is so silent that it is not be caught in our current detection.

Regarding **2)**, the equation of the soft theorem can be obtained as the Ward identity with regard to the asymptotic symmetry ([16, 17] and [18, 19] for gauge theories and gravity, respectively). Therefore,

$$\text{asymptotic symmetry} \longleftrightarrow \text{soft theorem} \tag{12}$$

Next, the DC shift (equation given in P.91 of [7]) and "the effect of attaching one soft-graviton line to an arbitrary Feynman diagrams" can be identical each other via Fourier transformation (with adjustment of some notations's convention). From this fact, it is considered that the gravitational wave from blackholes and the soft particles from the elementary particles's collisions will show the similar behavior at the long distance in the observation [7]. Thus, as the phenomena showing similar behavior [20],

$$\text{soft theorem} \longleftrightarrow \text{memory effect.} \tag{13}$$

Lastly, the gravitational wave at the long distance can be considered as fine transformations of the asymptotic symmetry. In this sense,

$$\text{memory effect} \longleftrightarrow \text{asymptotic symmetry.} \tag{14}$$

It is very interesting that different theories and phenomena can get related. Same relations can be obtained in the gauge field theories [7]. Therefore, gravitational and gauge theories would be universal in the IR-region. I list a few of the many achievements obtained from (12)-(14) in what follows.

- Each of the subgroups $\mathrm{BMS}^{\pm}$ is the symmetry separately acting in gravitational scatterings, but the full group $\mathrm{BMS}^{+} \times \mathrm{BMS}^{-}$ is not symmetry [21]. However, by the relation with soft theorem, it turns out that a certain antipodal subgroup of $\mathrm{BMS}^{+} \times \mathrm{BMS}^{-}$ is an exact asymptotic symmetry in gravitational scatterings [18].
- By the fact the asymptotically flat spacetimes are infinitely degenerated, the so-called angular momentum problem in general relativity can be resolved [7].
- Without the relation with the soft theorem, one cannot show superrotation is the symmetry or not for some singularities in these charges [22, 23].

3) means the attempt to find the 2D CFT on the celestial sphere, $\mathcal{CS}^2$, with the correlators which can reproduce the 4D Minkowski $\mathcal{S}$-matrix, based on the fact that 4D Lorentz symmetry $SL(2,\mathbb{C})$ acts as the global conformal symmetry on a two-sphere [24–27].

Regarding **4)**, an initial configuration to form a star or blackhole finally leads to some deformed spacetime by supertranslation [6], the phase space of which is infinite dimensional and characterized by the infinite conserved charges [21]. Hence, we can expect that the information of the initial configuration could be preserved in the configuration of the spacetime, which may be the key to the information paradox.

4 Our 2D effective action with supertranslation correction

From here, I would like to talk on my study. What I want to do first is to obtain the Schwarzschild blackhole metric with the supertranslation correction to the second order in the Schwarzschild blackhole coordinates. For this I will start with the Schwarzschild blackhole spacetime given in the isotropic coordinates:

$$ds^2 = -\frac{(1-m/2\rho_s)^2}{(1+m/2\rho_s)^2}dt_s^2 + (1+m/2\rho_s)^4(d\rho_s^2 + \rho_s^2 d\Omega_s^2), \tag{15}$$

where a flat three-dimensional space part, $d\rho_s^2 + \rho_s^2 d\Omega_s^2$, is convenient to involve the supertranslation correction according to [6].

Then, writing as $d\rho_s^2 + \rho_s^2 d\Omega_s^2 = dx_s^2 + dy_s^2 + dz_s^2$ and $\rho_s^2 = x_s^2 + y_s^2 + z_s^2$, I involve the supertranslation correction according to [6]:

$$x_s = (\rho - C)\sin\theta\cos\phi + \sin\phi\csc\theta\,\partial_\phi C - \cos\theta\cos\phi\,\partial_\theta C, \tag{16a}$$
$$y_s = (\rho - C)\sin\theta\sin\phi - \cos\phi\csc\theta\,\partial_\phi C - \cos\theta\sin\phi\,\partial_\theta C, \tag{16b}$$
$$z_s = (\rho - C)\cos\theta + \cos\theta\cos\phi\,\partial_\theta C, \tag{16c}$$

where the function C is the NG boson field for supertranslation, which I will take as

$$C = m\varepsilon\, Y_2^0(\theta,\phi) = \frac{m\varepsilon}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1). \tag{17}$$

- ε is dimensionless, which I attach to measure the order of supertranslations in our analysis. m is that in (15), which I involve to have C have the same dimension with ρ (now, $G/c^2 = 1$). The correction of ε appears from the second-order (see (2)) in the position of the horizon, which is our motivation for the analysis to ε^2-order.
- Why Y_2^0 is considered is that this mode is expected to be dominant in the process forming a soft-hairy blackhole (e.g. [28]). I have also performed the analysis with Y_1^0 just in case. Although I have not performed the calculation to the end, it has been seemed to be essentially same with what will present in the following.

Involving (16) into the isotoropic coordinates (15) to ε^2-order, I will rewrite it into the Schwarzschild coordinates (for detail, see Sec.A), and finally obtain like (1).

Then, with these 4D metrcies, I consider a complex scalar field theory as

$$S = \int d^4x\sqrt{-g}\, g^{MN}\partial_M\phi^*\partial_N\phi. \tag{18}$$

Writing $\phi(t,r,\theta,\phi) = \phi_{lm}(t,r)Y_m^l(\theta,\phi)$, and taking near-horizon limit by writing $r = r_{h,4D} + \Delta r$, I can get the 4D near-horizon action as

$$\begin{aligned} S = -\sum_{l,m}\sum_{k,n}\int dtdr(2m)^2\Big\{ & \\ \phi_{lm}^*\Big(&-\frac{2m}{r-2m}\Lambda_{lm,\,kn} - \frac{15m^2\varepsilon^2}{4\pi(r-2m)^2}\int d\Omega\sin^2(2\theta)(Y_l^m)^*Y_k^n\Big)\partial_t\partial_t\phi_{kn} \\ +\phi_{lm}^*\partial_r\Big(&\frac{r-2m}{2m}\Lambda_{lm,\,kn} - \frac{15m^2\varepsilon^2}{16\pi}\int d\Omega\sin^2(2\theta)(Y_l^m)^*Y_k^n\Big)\partial_r\phi_{kn}\Big\} + O(\varepsilon^3), \end{aligned} \tag{19a}$$

$$\Lambda_{lm,\,kn} \equiv \int d\Omega \Big\{ 1 + \frac{3}{2}\sqrt{\frac{5}{\pi}}\,(1+3\cos(2\theta))\,\varepsilon + \frac{45}{2\pi}\Big(\frac{\sin(2\theta)}{4} - \cos^2(\theta) + 3\cos^2(\theta)\cos(2\theta)\Big)\varepsilon^2 \Big\}\,(Y_l^m)^* Y_k^n. \tag{19b}$$

Integrating out $(\theta,\,\phi)$, I obtain 2D near-horizon effective action as (Sec.B.4 and B.5)

$$S_{\text{2D eff}} = \sum_{l=0}^{l_{max}} \sum_{|m|=0}^{l} \int d^2x\,\Phi_{lm}\big((g_{\text{eff}})^{tt}_{lm}\,\partial_t\varphi^*_{lm}\partial_t\varphi_{lm} + (g_{\text{eff}})^{rr}_{lm}\,\partial_r\varphi^*_{lm}\partial_r\varphi_{lm}\big), \tag{20}$$

$$\Phi_{lm} = (2(m_{\text{eff}})_{lm,\,lm})^2, \tag{21a}$$

$$(g_{\text{eff}})^{tt}_{lm} = -1/(g_{\text{eff}})^{rr}_{lm} = -\frac{2(m_{\text{eff}})_{kn,\,lm}}{r - 2(m_{\text{eff}})_{kn,\,lm}} + O(\varepsilon^3), \tag{21b}$$

$$(m_{\text{eff}})_{kn,\,lm} = m + \frac{15m}{8\pi r}\mathcal{I}^C_{kn,\,lm}\varepsilon^2 + O(\varepsilon^3). \tag{21c}$$

(For r-dependence in $(m_{\text{eff}})_{kn,\,lm}$, see (82)). Whether the 2D near-horizon effective action can be obtained or not is non-trivial before trying as mentioned in Sec.1, to check which is one of the motivations in this study.

5 Result of Hawking flux with supertranslation correction

I obtain Hawking flux by anomaly cancellation method [53, 54], in which reducing to 2D is crucial, because analysis is performed with the 2D anomaly. For details, see Sec.D.

Anomaly cancellation will focus on the fact: **1)** At the classical level, there is no outgoing flux in the near-horizon region for the strong gravitational effect, **2)** however, at the quantum level, outgoing flux will arise by the quantum tunneling. Hence, the outgoing flux exists in the near-horizon region finally. At this time, if one takes in the analysis as

$$\text{amount of flux from tunneling} = \text{amount of lack of flux at the classical level}, \tag{22}$$

the amount of the flux by the quantum tunneling can be identified as the Hawking flux.

The amount of the outgoing flux is represented by the integral constant obtained from the formulas of the 2D anomaly:

$$\nabla_\mu T^\mu{}_{\nu,\,lm} = -\frac{\partial_\nu \Phi_{lm}}{\sqrt{-(g_{\text{eff}})_{lm}}}\frac{\delta S_{\text{2D}}}{\delta_L \Phi_{lm}} + \text{both/either } \mathscr{A}^{\pm}_{\nu,\,lm}, \tag{23}$$

which can be fixed by the condition that the system is symmetric, which is at the point where the variation of the action vanishes:

$$(\delta S_{\text{2D}})_{lm} = -\int d^2x\sqrt{-(g_{\text{eff}})_{lm}}\,\eta^\nu \nabla_{\mu,\,lm} T^\mu{}_{\nu,\,lm}. \tag{24}$$

The Hawking flux I have obtained has been (127), $\pi T_H^2/12$, which is the same with just a Schwarzschild. The reason of this is written in Sec.D.2.

6 Conclusion

Although the position of the horizon has been displaced and whether the near-horizon field theories can reduce to free 2D has been non-trivial, the Hawking temperature and flux have been obtained without changes. This result, no changes, has been already clear when the near-horizon metrices are obtained, however whether theories can reduce to free 2D has not been so before trying.

The value of the Hawking flux would be always (127) as long as the function C is **1)** to the second-order, and **2)** independent of ϕ. The reason for this is as follows.

First, if the correction is to ε^2 but C is some one other than (17) independent of ϕ,

- highly complicated terms of θ will be newly involved into the each coefficient of $\varepsilon^{1,2}$ in (19). At this time, the feasibility of the integrate out for $(\theta,\,\phi)$ is the problem, however it would be no problem by using the following formula and (78):

$$Y_{l_1}^{m_1} Y_{l_2}^{m_2} = \sum_{L,M} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}} \langle l_1\,0\,l_2\,0|L\,0\rangle\langle l_1 m_1\,l_2 m_2|LM\rangle Y_L^M, \tag{25}$$

- coefficients of $\varepsilon^{1,2}$ in (21c) will get highly involved concerning θ, however the structure of (21b) as the function f would be no changed (see the last of Sec.D.2), since C depends only on θ and ϕ by definition.

Therefore, if the two conditions above are satisfied, one could always get the 2D near-horizon effective action with the f same with (20) as the structure. On the other hand,

- if the correction of ε were involved more than 3rd-order, the feasibility of the analysis to get the free 2D theory as in Sec.B.5 gets unclear. See the last line in Sec.B.5. Namely, if the same behavior with (91) were not held, the analysis to get the free 2D theory would be impossible.
- If ϕ-dependence were mixed in the C, the formulas (25) might get unavailable, and I could not get the 2D action like (20).

It is considered from our result, no changes, that Hawking temperature and flux may be the conserved quantities under the asymptotic symmetry. Lastly I'd like to write down personally interesting talks (not general sense): [29]- [50].

Acknowledgment

Upon writing this article, many discussions with Prof.Feng-Li Lin, the collaborator in [2], were very helpful. Using this opportunity, I would like to offer my thanks to Prof.Hai-Qing Zhang who could financially support the conference.

Appendix

A Metrices with supertranslation correction

This appendix is the detailed description of [2], and in this section, we obtain the metrices for a 4D Schwarzschild blackhole spacetime with supertranslations to the second-order.

A.1 Introduction of supertranslation

We start with the following coordinate system for a 4D Schwarzschild blackhole spacetime:

$$ds^2 = -(1-2m/r_s)dt_s^2 + (1-2m/r_s)^{-1}dr_s^2 + r_s^2 d\Omega_s^2. \tag{26}$$

We refer to this type of coordinate system as the "Schwarzschild coordinates". In order to involve the supertranslations, we rewrite (26) into (15), where $r_s = \rho_s\,(1+m/2\rho_s)^2$. We refer to this type of expression as "isotoropic coordinates".

Note that in this relation, two ρ_s correspond to one r_s as

$$\rho_s = \big(-m + r_s \pm \sqrt{-2mr_s + r_s^2}\big)/2. \tag{27}$$

See Fig.1. We can see **1)** positions of horizon in isotropic and Schwarzschild coordinates correspond each other, **2)** isotropic coordinates do not cover the inside of the horizon.

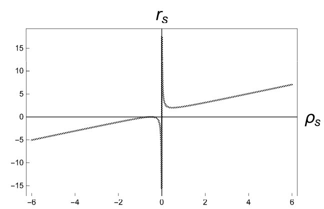

Figure 1: Plot for $r_s = \rho_s\,(1+m/2\rho_s)^2$ for $m=1$.

We denote the supertranslated isotropic coordinates as (t,ρ,θ,ϕ). These and (t_s, x_s, y_s, z_s) are related like (16), where

$$d\rho_s^2 + \rho_s^2 d\Omega_s^2 \text{ in (15)} = dx_s^2 + dy_s^2 + dz_s^2, \quad t_s \text{ in (15)} = t. \tag{28}$$

We take C we consider as (17). Description here overlaps with the one under (17) (but one comment; if we employ Y_1^0 as C, r.h.s. of (28) results in just $d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2$, namely no supertranslation corrections, which get involved from (31), supertranslated ρ.)

A.2 Isotropic coordinates with supertranslations

We now write (15) in terms of (t,ρ,θ,ϕ). For the parts in (15), we can write as

$$\rho_s^2 = x_s^2 + y_s^2 + z_s^2, \tag{29a}$$
$$(1+m/2\rho_s)^4(d\rho_s^2 + \rho_s^2 d\Omega_s^2) = g_{\rho\rho}d\rho^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \tag{29b}$$

We can evaluate $d\rho_s^2 + d\rho_s^2 d\Omega_s^2$ as (28), with which we obtain g_{MN} from now. We write

g_{MN} for the metrices of the supertranslated isotropic coordinates (30a)
j_{MN} for the metrices of the supertranslated Schwarzschild coordinates (30b)

in what follows, where M, N in g_{MN} and j_{MN} refer to (t,ρ,θ,ϕ) and (t,r,θ,ϕ).

We can obtain ρ_s by calculating (31) using (16) to ε^2-order as

$$\rho_s = \rho - \frac{1}{8}\sqrt{\frac{5}{\pi}}\varepsilon m(3\cos(2\theta)+1) + \frac{45\varepsilon^2 m^2 \sin^2(2\theta)}{32\pi\rho} + O(\varepsilon^3). \tag{31}$$

With this we can obtain g_{MN} as

$$\begin{aligned} g_{tt} = &-\frac{(m-2\rho)^2}{(m+2\rho)^2} - \sqrt{\frac{5}{\pi}}\varepsilon m^2\frac{(m-2\rho)}{(m+2\rho)^3}(3\cos(2\theta)+1) - \frac{5\varepsilon^2 m^3}{8\pi\rho(m+2\rho)^4}(22m\rho - 9m^2 \\ &+ 14\rho^2 + 9\cos(4\theta)(m^2+2m\rho-6\rho^2) + 24\rho\cos(2\theta)(m-\rho)) + O(\varepsilon^3), \end{aligned} \tag{32a}$$

$$g_{\rho\rho}=\frac{(m+2\rho)^4}{16\rho^4}+\sqrt{\frac{5}{\pi}}\varepsilon m^2\frac{(m+2\rho)^3}{32\rho^5}(3\cos(2\theta)+1)+\frac{5\varepsilon^2m^3(m+2\rho)^2}{1024\pi\rho^6}(19m-28\rho$$
$$+12\cos(2\theta)(5m+4\rho)+27\cos(4\theta)(3m+4\rho))+O(\varepsilon^3), \tag{32b}$$
$$g_{\theta\theta}=\frac{(m+2\rho)^4}{16\rho^2}+\sqrt{\frac{5}{\pi}}\varepsilon m(m+2\rho)^3\frac{3\cos(2\theta)(5m+6\rho)+m-2\rho}{64\rho^3}+\frac{5\varepsilon^2m^2(m+2\rho)^2}{2048\pi\rho^4}($$
$$12\cos(2\theta)(39m^2+76m\rho+20\rho^2)+9\cos(4\theta)(27m^2+44m\rho+4\rho^2)$$
$$+249m^2+484m\rho+236\rho^2)+O(\varepsilon^3), \tag{32c}$$
$$g_{\phi\phi}=\frac{(m+2\rho)^4\sin^2(\theta)}{16\rho^2}+\sqrt{\frac{5}{\pi}}\varepsilon m\frac{(m+2\rho)^3}{64\rho^3}\sin^2(\theta)(\cos(2\theta)(9m+6\rho)+7m+10\rho)$$
$$+\frac{5\varepsilon^2m^2\sin^2(\theta)(m+2\rho)^2}{2048\pi\rho^4}(249m^2+484m\rho+236\rho^2+12\cos(2\theta)(39m^2+76m\rho$$
$$+20\rho^2)+9\cos(4\theta)(27m^2+44m\rho+4\rho^2))+O(\varepsilon^3). \tag{32d}$$

A.3 Rewriting from Schwarzschild to isotropic coordinates

Since we have obtained the metrices in the supertranslated isometric coordinates (t,ρ,θ,ϕ), we will rewrite these to the following Schwarzschild coordinates:

$$ds^2=-(1-2\mu/r)dt^2+(1-2\mu/r)^{-1}dr^2+j_{\theta\theta}d\phi^2+j_{\phi\phi}d\phi^2. \tag{33}$$

Then, we will find the mass part μ cannot remain constant (if analysis is to ε^1-order, it can be constant). Therefore, we treat μ as $\mu(\rho)$. In what follows, we obtain **1)** a relation between r and ρ, and **2)** $\mu(\rho)$ as the solution, by solving the following relations:

- $$-(1-2\mu(\rho)/r)=g_{tt}, \tag{34a}$$
- $$\frac{1}{1-2\mu(\rho)/r}\Big(\frac{dr}{d\rho}\Big)^2=g_{\rho\rho}. \tag{34b}$$

The argument in $\mu(\rho)$ should be ρ. If we express $\mu(\rho)$ in terms of r, (38) is plugged in.

We can obtain the r satisfying (34a) to ε^2-order as

$$r=\frac{\mu(\rho)(m+2\rho)^2}{4m\rho}+\frac{\varepsilon}{32\rho^2}\sqrt{\frac{5}{\pi}}(3\cos(2\theta)+1)\mu(\rho)(m^2-4\rho^2)$$
$$+\frac{5\varepsilon^2m\mu(\rho)}{512\pi\rho^3}(9\cos(4\theta)(3m^2-8\rho^2)+12m^2\cos(2\theta)-7m^2+72\rho^2)+O(\varepsilon^3). \tag{35}$$

Let us obtain the $\mu(\rho)$. For this, look (34b), then plugging (35) into the r, solve it for $\mu(\rho)$ order by order to ε^2-order. As a result we can obtain

$$\mu(\rho)=m+\frac{c_1\rho\varepsilon}{(m+2\rho)^2}+\frac{\varepsilon^2}{64\pi\rho^2(m+2\rho)^3}\{-24\sqrt{5\pi}c_1m\rho^2\cos(2\theta)(m-2\rho)$$
$$+8\rho^2(8\pi c_2\rho(m+2\rho)-\sqrt{5\pi}c_1m(m-2\rho))$$
$$+30m^3\sin^2(2\theta)(m+2\rho)(m^2-12\rho^2)\}+O(\varepsilon^3), \tag{36}$$

where we took the integral constant at ε^0-order so that ε^0-order becomes m. $c_{1,2}$ are the integral constants at $\varepsilon^{1,2}$-orders (these can depend on (θ,ϕ) and determined at (42)).

Now we have obtained the relation "$r=\cdots$" as in (35), with which, rewriting the Schwarzschild to the isotropic coordinates to ε^2-order is possible:

$$-(1-2\mu(\rho)/r)dt^2+(1-2\mu(\rho)/r)^{-1}dr^2+j_{\theta\theta}d\theta^2+j_{\phi\phi}d\phi^2$$
$$\to g_{tt}dt^2+g_{\rho\rho}d\rho^2+\Big(j_{\theta\theta}+\frac{1}{1-\frac{2\mu(\rho)}{r}}\Big(\frac{\partial r}{\partial\theta}\Big)^2\Big)d\theta^2+\frac{2}{1-\frac{2\mu(\rho)}{r}}\frac{\partial r}{\partial\rho}\frac{\partial r}{\partial\theta}d\rho d\theta+j_{\phi\phi}d\phi^2, \tag{37}$$

However what is needed is rewriting from the isotropic to Schwarzschild coordinates.

A.4 Rewriting from isotropic to Schwarzschild coordinates

We will obtain the relation between ρ and r in the form "$\rho = \cdots$" to ε^2-order to become possible to rewrite (37) in the opposite direction. For this, there are two ways: to solve **1)** (34b) or **2)** (35). As a result of our try, if we solve to ε^1-order, we can get the same ρ from either of them (we checked this sameness numerically). However, if we try to obtain to ε^2-order, we can obtain only from 2) (for some technical reason of mathematica).

Writing what we did, plugging $\mu(\rho)$ in (36) into the μ in (35), then expanding it to ε^2-order, we can obtain ρ order by order. As a result, four solutions are obtained. At this time, the ε^0-order in the two of these do not agree with (27), while those of the rest two can agree with (27). Therefore, we employ the latter two, which are

$$\begin{aligned}\rho^{(1,2)}(r) = {}& \frac{1}{2}(r-m\mp\sqrt{r(r-2m)})+\frac{\varepsilon}{8\pi m r(2m-r)}\{\pi c_1 r(r-2m)\pm\pi c_1 m\sqrt{r(r-2m)}\\ &\mp\pi c_1 r\sqrt{r(r-2m)}-\sqrt{5\pi}m^2 r(3\cos(2\theta)+1)(r-2m)\}-\frac{\varepsilon^2}{128\pi m r^2(r-2m)^2}[\\ &16\pi c_2(r^4\mp\sqrt{r^7(r-2m)})-120m^4 r(4r\pm\sqrt{r(r-2m)})\\ &+60m^3(16r^3\pm 9\sqrt{r^5(r-2m)})-8m^2\{(60\sqrt{r^7(r-2m)}\mp 8\pi c_2 r^2)+r^4\pm 75\\ &+4\pi c_2 r\sqrt{r(r-2m)}\}+m\{120r^5\pm 120\sqrt{r^9(r-2m)}\mp 2\pi {c_1}^2\sqrt{r(r-2m)}\\ &-64\pi c_2 r^3\pm 48\pi c_2\sqrt{r^5(r-2m)}\}+60mr(2m-r)\cos(4\theta)\{\\ &m^2(4r\pm\sqrt{r(r-2m)})+2(r^3\pm\sqrt{r^5(r-2m)})-2mr(3r\pm 2\sqrt{r(r-2m)})\}\\ &]+O(\varepsilon^3),\end{aligned}\tag{38}$$

where the 1 and 2 in the $\rho^{(1,2)}(r)$ correspond as

$$(1,2)\to(+,-)\text{ of }\pm\text{ and }(-,+)\text{ of }\mp.\tag{39}$$

Let us determine which $\rho^{(1,2)}(r)$ we employ and determine $c_{1,2}$. For this, plugging $\rho^{(1,2)}$ in (38) into $\mu(\rho)$ in (36), write it in terms of r to ε^2-order as

$$\begin{aligned}\mu^{(1,2)}\big(\rho^{(1,2)}(r)\big) = {}& m+\frac{c_1\varepsilon}{4r}+\frac{\varepsilon^2}{8\pi m r(\sqrt{r(r-2m)}\mp r)^3(\sqrt{r(r-2m)}\pm m\mp r)^2}[\\ &\pi\{-4c_2 m^4 r\sqrt{r(r-2m)}+48c_2 m^3\sqrt{r^5(r-2m)}-{c_1}^2 m^3\sqrt{r(r-2m)}\\ &-80c_2 m^2\sqrt{r^7(r-2m)}+12{c_1}^2 m^2 r\sqrt{r(r-2m)}+32c_2 m\sqrt{r^9(r-2m)}\\ &+r(\pm 7m^3\mp 28m^2 r\pm 28mr^2\mp 8r^3)(4c_2 mr+{c_1}^2)+8{c_1}^2\sqrt{r^7(r-2m)}\\ &-20{c_1}^2 m\sqrt{r^5(r-2m)}\}-30m^4 r\sin^2(2\theta)\{m^2(\sqrt{r(r-2m)}\mp 7r)\\ &\pm 3mr(5r\mp 3\sqrt{r(r-2m)})+6(\mp r^3+\sqrt{r^5(r-2m)})\}]+O(\varepsilon^3).\end{aligned}\tag{40}$$

Behavior of these at the distant region is given as

$$\begin{aligned}\mu^{(1)}\big(\rho^{(1)}(r)\big) = {}& m+\varepsilon\Big(\frac{c_1}{4r}+O\left(r^{-2}\right)\Big)+\varepsilon^2\Big(\frac{15r^2\sin^2(2\theta)}{2\pi m}-\frac{45\sin^2(2\theta)r}{2\pi}+\frac{45m\sin^2(2\theta)}{4\pi}\\ &+\frac{15m^2\sin^2(2\theta)/\pi+c_2}{4r}+O\left(r^{-2}\right)\Big)+O(\varepsilon^3),\end{aligned}\tag{41a}$$

$$\begin{aligned}\mu^{(2)}\big(\rho^{(2)}(r)\big) = {}& m+\varepsilon\Big(\frac{c_1}{4r}+O\left(r^{-6}\right)\Big)+\frac{\varepsilon^2}{4}\Big(\frac{c_2}{r}+\frac{2{c_1}^2-45m^4\sin^2(2\theta)/\pi}{8mr^2}-\frac{45m^4\sin^2(2\theta)}{8\pi r^3}\\ &-\frac{105m^5\sin^2(2\theta)}{16\pi r^4}-\frac{135m^6\sin^2(2\theta)}{16\pi r^5}+O\left(r^{-6}\right)\Big)+O\left(\varepsilon^3\right).\end{aligned}\tag{41b}$$

It can be seen from the above we should discard $\mu^{(1)}$ by the reason: It is always diverged at the distant region irrelevantly of $c_{1,2}$ for the terms, $\frac{15r^2\sin^2(2\theta)}{2\pi m}-\frac{45r\sin^2(2\theta)}{2\pi}$. Thus, it is enough only with $\mu^{(2)}$ in the following, but we proceed with both just in case.

Now we determine $c_{1,2}$. Since these are integral constants, we are allowed to take these arbitrarily. However in this study, by the reason written in what follows we will take as

$$c_{1,2} = 0. \tag{42}$$

Looking $\mu^{(2)}(\rho^{(2)}(r))$, we can find that it diverges at $m = 0$ unless c_1 is zero for the term $\frac{2c_1{}^2}{32mr^2}$ at its ε^2-order. Hence we take c_1 to 0.

As for our logic for c_2, **1)** consider starting with just a flat spacetime patched by Schwarzschild coordinates with the zero mass, therefore $\mu^{(2)}$ at the starting stage is zero. **2)** Suppose changing it to the isotropic coordinates, involve the supretranslations. Then, back the coordinates to the Schwartzshild. At this time, the expanded $\mu^{(2)}$ is given by (41b). **3)** At this time, the mass should be zero, therefore $\mu^{(2)}$ should be zero. However, if c_2 is not zero, we can see $\mu^{(2)}$ is not zero for the terms $\frac{c_1}{4r}$ at the ε^1-order and $\frac{c_2}{4r}$ at the ε^2-order. **4)** As c_1 has been taken to zero in the above, we take c_2 to zero.

Above, we have considered in terms of the supertanslation toward the flat spacetime and based on the consideration that mass in the spacetime should not be changed by the supertanslation. The same issue is taken up in Sec.24.2 in [55]. There, again mass is not introduced, though C_{zz} and $C_{\bar{z}\bar{z}}$ are introduced.

With (42), (38) and (40) are fixed as

$$\begin{aligned}
\rho^{(1,2)} =& \frac{1}{2}(\mp\sqrt{r(r-2m)} - m + r) + \frac{1}{8}\sqrt{\frac{5}{\pi}}\varepsilon m(3\cos(2\theta)+1) - \frac{15\varepsilon^2\sin^2(2\theta)}{16\pi r^2(r-2m)^2}\{ \\
& -10mr^4 \mp 8m\sqrt{r^7(r-2m)} + 2(r^5 \pm \sqrt{r^9(r-2m)}) - 2m^3 r(4r \pm \sqrt{r(r-2m)}) \\
& + m^2(16r^3 \pm 9\sqrt{r^5(r-2m)})\} + O(\varepsilon^3),
\end{aligned} \tag{43}$$

$$\mu^{(1,2)}(\rho^{(1,2)}(r)) = m + \frac{15\varepsilon^2 m^3\sin^2(2\theta)(m^2 - 3(\sqrt{r(r-2m)} \pm m \mp r)^2)}{8\pi(r \mp \sqrt{r(r-2m)})^2(\sqrt{r(r-2m)} \pm m \mp r)^2} + O(\varepsilon^3). \tag{44}$$

Using these we can rewrite the isotropic to the Schwarzschild coordinates as

$$\begin{aligned}
& g_{tt}dt^2 + g_{\rho\rho}d\rho^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \\
\to & -\Big(1 - \frac{2\mu(\rho)}{r}\Big)dt^2 + \Big(1 - \frac{2\mu(\rho)}{r}\Big)^{-1}dr^2 + \Big(g_{\theta\theta} + g_{\rho\rho}\Big(\frac{\partial\rho}{\partial\theta}\Big)^2\Big)d\theta^2 \\
& + 2g_{\rho\rho}\frac{\partial\rho}{\partial r}\frac{\partial\rho}{\partial\theta}d\rho d\theta + g_{\phi\phi}d\phi^2 \equiv j_{tt}dt^2 + j_{rr}dr^2 + j_{\theta\theta}d\theta^2 + 2j_{r\theta}drd\theta + j_{\phi\phi}d\phi^2,
\end{aligned} \tag{45}$$

where $\rho = \rho(r,\theta)$ and g_{MN} are in (32). We give the expressions of j_{MN} in the next.

A.5 Metrices with correction to ε^2-order in 4D Schwarzschild coord.

We give the expression of j_{MN} in (45) in the case of (42).

- $$j_{tt}^{(1,2)} = -\Big(1 - \frac{2m}{r}\Big) + \frac{15\varepsilon^2\sin^2(2\theta)(m^5 - 3m^3(\sqrt{r(r-2m)} \pm m \mp r)^2)}{4\pi r(r \mp \sqrt{r(r-2m)})^2(\sqrt{r(r-2m)} \pm m \mp r)^2} + O(\varepsilon^3), \tag{46a}$$
- $$\begin{aligned} j_{rr}^{(1,2)} =& \Big(1 - \frac{2m}{r}\Big)^{-1} + \frac{15\varepsilon^2\sin^2(2\theta)}{4\pi m(r-2m)^2}\{m^3 + 3m^2 r - 6mr^2 \pm 2(\sqrt{r^5(r-2m)} \pm r^3) \\ & \mp 4mr\sqrt{r(r-2m)}\} + O(\varepsilon^3), \end{aligned} \tag{47a}$$

- $$j_{\theta\theta}^{(1)} = r^2 - \frac{3\sqrt{\frac{5}{\pi}}\varepsilon m\cos(2\theta)(r-\sqrt{r(r-2m)})^4}{2(\sqrt{r(r-2m)}+m-r)^3} + \frac{15\varepsilon^2 m^2}{8\pi r(r-2m)^2(\sqrt{r(r-2m)}+m-r)^3}$$
$$\times \{r^2(r-2m)^2(\cos(4\theta)(m^2+6mr-9r^2)-(m^2-18mr+15r^2))$$
$$+12m^3\sqrt{r^5(r-2m)}(\cos(4\theta)-1)+m^2\sqrt{r^7(r-2m)}(24\cos(4\theta)+72)$$
$$-m\sqrt{r^9(r-2m)}(33\cos(4\theta)+63)+\sqrt{r^{11}(r-2m)}(9\cos(4\theta)+15)\}+O(\varepsilon^3), \tag{48a}$$

$$j_{\theta\theta}^{(2)} = r^2 + \frac{3\sqrt{\frac{5}{\pi}}\varepsilon m\cos(2\theta)(\sqrt{r(r-2m)}+r)^4}{2(\sqrt{r(r-2m)}-m+r)^3}$$
$$+\frac{15\varepsilon^2 m^2}{8\pi r(r-2m)^2\{m^4-4m^3(\sqrt{r(r-2m)}+4r)+20m^2 r(\sqrt{r(r-2m)}+2r)-24m\sqrt{r^5(r-2m)}-32mr^3+8(\sqrt{r^7(r-2m)}+r^4)\}}[$$
$$-4m^5r^2+8m^4(2\sqrt{r^5(r-2m)}+13r^3)-5m^3(32\sqrt{r^7(r-2m)}+73r^4)$$
$$+m^2(268\sqrt{r^9(r-2m)}+409r^5)+\cos(4\theta)\{4m^5r^2-8m^4(2\sqrt{r^5(r-2m)}+r^3)$$
$$+m^3(-32\sqrt{r^7(r-2m)}-115r^4)+m^2(116\sqrt{r^9(r-2m)}+191r^5)$$
$$-84m\sqrt{r^{11}(r-2m)}-102mr^6+18(\sqrt{r^{13}(r-2m)}+r^7)\}$$
$$+30(\sqrt{r^{13}(r-2m)}+r^7)-6m(26\sqrt{r^{11}(r-2m)}+31r^6)]+O(\varepsilon^3), \tag{48b}$$

- $$j_{r\theta}^{(1,2)} = -\frac{3\sqrt{\frac{5}{\pi}}\varepsilon m\sin(2\theta)(r\mp\sqrt{r(r-2m)})^4}{8\sqrt{r(r-2m)}(\sqrt{r(r-2m)}\pm m\mp r)^3} \pm \frac{15\varepsilon^2 m^4 r\sin(4\theta)}{4\pi(2m-r)(\sqrt{r(r-2m)}\pm m\mp r)^3}$$
$$+O(\varepsilon^3), \tag{49a}$$

- $$j_{\phi\phi}^{(1)} = r^2\sin^2(\theta) - \frac{3\sqrt{\frac{5}{\pi}}\varepsilon m\sin^2(2\theta)(r-\sqrt{r(r-2m)})^4}{8(\sqrt{r(r-2m)}+m-r)^3}$$
$$+\frac{15\varepsilon^2 m^2\sin^2(2\theta)}{8\pi r(r-2m)^2(\sqrt{r(r-2m)}+m-r)^3}\{12m^3\cos(2\theta)\sqrt{r^5(r-2m)}$$
$$-12m^3\sqrt{r^5(r-2m)}+24m^2\sqrt{r^7(r-2m)}+r^2\cos(2\theta)(r-2m)^2(m^2-3r^2)$$
$$-r^2(r-2m)^2(m^2-6mr+3r^2)+3\cos(2\theta)\sqrt{r^{11}(r-2m)}+3\sqrt{r^{11}(r-2m)}$$
$$-9m\cos(2\theta)\sqrt{r^9(r-2m)}-15m\sqrt{r^9(r-2m)}\}+O(\varepsilon^3), \tag{50a}$$

$$j_{\phi\phi}^{(2)} = r^2\sin^2(\theta) + \frac{3\sqrt{\frac{5}{\pi}}\varepsilon m\sin^2(2\theta)(\sqrt{r(r-2m)}+r)^4}{8(\sqrt{r(r-2m)}-m+r)^3}$$
$$+\frac{15\varepsilon^2 m^2\sin^2(2\theta)}{4\pi r^2(r-2m)^2(\sqrt{r(r-2m)}-m+r)^4}[-4m^5r^3+8m^4(2\sqrt{r^7(r-2m)}+7r^4)$$
$$-m^3(64\sqrt{r^9(r-2m)}+125r^5)+m^2(76\sqrt{r^{11}(r-2m)}+109r^6)$$
$$+\cos(2\theta)\{4m^5r^3-16m^4(\sqrt{r^7(r-2m)}+2r^4)+m^3(16\sqrt{r^9(r-2m)}+5r^5)$$
$$+m^2(20\sqrt{r^{11}(r-2m)}+41r^6)+6(\sqrt{r^{15}(r-2m)}+r^8)$$
$$-6m(4\sqrt{r^{13}(r-2m)}+5r^7)\}+6(\sqrt{r^{15}(r-2m)}+r^8)$$
$$-6m(6\sqrt{r^{13}(r-2m)}+7r^7)]+O(\varepsilon^3). \tag{50b}$$

1) numbers in the superscripts mean those j_{MN} are associated with which of $\rho^{(1,2)}$ in (43) with (39). (Origin of $j_{MN}^{(1,2)}$ is (38), then it turns out above (42) that $\mu^{(1)}$ is unphysical and $\mu^{(2)}$ is physical. $j^{(2)}_{MN}$ is associated with $\mu^{(2)}$.) **2)** Killing vector in the system above is $\xi^M = (1,0,0,1)$ as well. With either $j^{(1,2)}_{MN}$, **3)** Einstein eq. is satisfied to ε^2-order. Also from either $j^{(1,2)}_{MN}$, **4)** (32) can be obtained using following one, (35) with (36) and (42):

$$r = \frac{m^2}{4\rho} + m + \rho + \frac{\sqrt{\frac{5}{\pi}}m(m^2-4\rho^2)(3\cos(2\theta)+1)}{32\rho^2}\varepsilon - \frac{5m^4(1-12\cos(2\theta)-21\cos(4\theta))}{512\pi\rho^3}\varepsilon^2 + O(\varepsilon^3). \tag{51}$$

A.6 Comment on $c_{1,2}$

Positions of the horizon in supertranslated isotropic and Schwarzschild coordinates are

$$\rho_{h,4D} = \frac{m}{2} + \frac{1}{8}\sqrt{\frac{5}{\pi}}m\varepsilon(3\cos(2\theta)+1) - \frac{45}{16\pi}\varepsilon^2 m\sin^2(2\theta) + O(\varepsilon^3), \tag{52}$$

$$r_{h,4D} = 2m - \frac{15}{8\pi}\varepsilon^2 m\sin^2(2\theta) + O(\varepsilon^3), \tag{53}$$

where $r_{h,4D}$ gets to this above regardless of $j^{(1,2)}_{MN}$. Then it turns out $\rho_{h,4D}$ can be transformed to $r_{h,4D}$ through (35) ($\mu(\rho)$ is replaced by (36)), but $r_{h,4D}$ is transformed to

$$\frac{m}{2} + \sqrt{\frac{5}{\pi}}m\varepsilon\Big(\frac{3\cos(2\theta)+1}{8} - \frac{1}{16}\sqrt{6(-1+\cos(4\theta))}\Big) - \frac{45\varepsilon^2 m\sin^2(2\theta)}{16\pi} + O(\varepsilon^3). \tag{54}$$

through (43).

Since j_{MN} can be transformed to g_{MN} by (43), r-coordinate corresponds to ρ-coordinate by (43). However, as mentioned above (42), there is freedom for how to take $c_{1,2}$. Moreover, as the problem of how the coordinates are patched toward a spacetime, there is ambiguity up to $c_{1,2}$ for the mapping of each point in r-coordinate to each point in ρ-coordinate through (43), and vice verse through (35).

Actually, the position of horizon in the Schwarzschild coordinate is obtained if one proceeds calculation with unfixed $c_{1,2}$ as

$$2m + \frac{c_1\varepsilon}{4m} + \varepsilon^2\Big(\frac{c_2}{4m} - \frac{15m\sin^2(2\theta)}{8\pi}\Big) + O(\varepsilon^3). \tag{55}$$

Here, the position of horizon in the Schwarzschild coordinate obtained from $\rho_{h,4D}$ through (35) (this (35) is given with unfixed $c_{1,2}$) is (55). Therefore, the position of horizon in the isotropic coordinate is always mapped to that in the Schwarzschild coordinate.

Toward (55), if we take as $c_1 = 0$ and $c_2 = \frac{15m^2\sin^2(2\theta)}{2\pi}$, $\rho_{h,4D}$ can be obtained through (43) (this (43) is given with these $c_{1,2}$). However, the $r_{h,4D}$ at that time is $2m + O(\varepsilon^3)$.

One may consider to determine $c_{1,2}$ based on agreement of the positions of horizon. However these should be zero for the reason under (42), therefore the positions of horizon determine to those obtained from $c_{1,2} = 0$, (52) and (53).

B 2D effective near-horizon action

We have obtained the metrices with the supertranslation correction to the second-order in the Schwarzschild coordinates. In this section, obtaining the near-horizon expression of these, we consider the scalar field theory. Then, expanding the field by the spherical harmonics, we integrate out its (θ, ϕ). We will finally obtain 2D effective near-horizon action. The scalar field theory we consider is

$$S_{\text{scalar}} = \frac{1}{2}\int d^4x\,\sqrt{-g}\,j^{MN}\partial_M\phi^*\partial_N\phi, \tag{56}$$

where $M, N = t, r, \theta, \phi$, and j^{MN} mean $j^{(2)MN}$. We do not include the mass and interaction terms, since these are ignorable in the near-horizon [57][3].

B.1 Near-horizon metrices

To obtain the near-horizon expression of (56), we first obtain the 4D metrices $j^{(2)}_{MN}$ in Sec.A.5 in the near-horizon. For this, we replace r in those $j^{(2)}_{MN}$ with $r_{h,4D} + \Delta r$ ($r_{h,4D}$ is given in (53))[4], then expand around $\Delta r = 0$. Writing these as t_{MN},

$$\begin{aligned}
t_{tt} =& \Big(-\frac{\Delta r}{2m} + \frac{\Delta r^2}{4m^2} + O\left(\Delta r^3\right)\Big) + \varepsilon^2\frac{15\sin^2(2\theta)}{4\pi}\Big\{\frac{3\Delta r}{4m} - \frac{\sqrt{2}\Delta r^{3/2}}{m^{3/2}} + \frac{3\Delta r^2}{4m^2} \\
&+\frac{\Delta r^{5/2}}{2\sqrt{2}m^{5/2}} + O\left(\Delta r^3\right)\Big\} + O(\varepsilon^3),
\end{aligned} \tag{57}$$

$$\begin{aligned}
t_{rr} =& \Big(\frac{2m}{\Delta r} + 1 + O\left(\Delta r^3\right)\Big) + \varepsilon^2\frac{15\sin^2(2\theta)}{\pi}\Big\{\frac{3m}{4\Delta r} - \frac{\sqrt{2m}}{\sqrt{\Delta r}} + \frac{6}{4} - \frac{3\sqrt{\Delta r}}{2\sqrt{2m}} + \frac{2\Delta r}{4m} \\
&-\frac{3\Delta r^{3/2}}{16\sqrt{2}m^{3/2}} + O\left(\Delta r^{5/2}\right)\Big\} + O(\varepsilon^3),
\end{aligned} \tag{58}$$

$$\begin{aligned}
t_{\theta\theta} =& \Big(4m^2 + 4\Delta r m + \Delta r^2 + O\left(\Delta r^3\right)\Big) + \varepsilon 24\sqrt{\frac{5}{\pi}}\cos(2\theta)\Big\{m^2 - m^{3/2}\sqrt{2}\sqrt{\Delta r} \\
&+2m\Delta r - \frac{5}{4}\sqrt{2m}\Delta r^{3/2} + \frac{5}{4}\Delta r^2 + O\left(\Delta r^{5/2}\right)\Big\} + \frac{90\varepsilon^2}{\pi}\Big\{2m^2\cos^2(2\theta) \\
&-4\sqrt{2}m^{3/2}\cos^2(2\theta)\sqrt{\Delta r} + \frac{m(41\cos(4\theta)+39)\Delta r}{8} - \frac{\sqrt{2m}}{3}(14\cos(4\theta)+13)\Delta r^{3/2} \\
&+\frac{5(21\cos(4\theta)+19)\Delta r^2}{16} + O\left(\Delta r^{5/2}\right)\Big\} + O\left(\varepsilon^3\right),
\end{aligned} \tag{59}$$

$$\begin{aligned}
t_{\phi\phi} =& \sin^2(\theta)\left(4m^2 + 4\Delta r m + \Delta r^2 + O\left(\Delta r^3\right)\right) + 3\varepsilon\sqrt{\frac{5}{\pi}}\sin^2(2\theta)\Big\{2m^2 - 2\sqrt{2}m^{3/2}\sqrt{\Delta r} \\
&+4m\Delta r - 5\sqrt{m}\Delta r^{3/2} + \frac{3}{2}\Delta r^2 + O\left(\Delta r^{5/2}\right)\Big\} + \varepsilon^2\frac{45\sin^2(2\theta)}{\pi}\Big\{m^2\cos(2\theta) \\
&-2\sqrt{2}m^{3/2}\cos(2\theta)\sqrt{\Delta r} + \frac{m}{4}(21\cos(2\theta)-1)\Delta r + \frac{5(11\cos(2\theta)-1)\Delta r^2}{8} \\
&+\frac{675}{2}\sqrt{2m}(2-29\cos(2\theta))\Delta r^{3/2} + O\left(\Delta r^{5/2}\right)\Big\} + O(\varepsilon^3),
\end{aligned} \tag{60}$$

[3] It is considered that theories effectively become 2D free massless in the near-horizon at the classical level as the particles effectively fall freely and these longitudinal motions get dominant.

[4] We give the Jacobian and partial derivatives when we change from r to Δr. We denote the old and new coordinates as (r, θ) to (Δr, $\tilde{\theta}$). The relations between these are $\Delta r = r - r_h(\theta)$ and $\tilde{\theta} = \theta$. Then, $drd\theta = \begin{vmatrix} \frac{\partial r}{\partial(\Delta r)} & \frac{\partial r}{\partial\tilde{\theta}} \\ \frac{\partial\theta}{\partial(\Delta r)} & \frac{\partial\theta}{\partial\tilde{\theta}} \end{vmatrix} d(\Delta r)d\tilde{\theta} = d(\Delta r)d\tilde{\theta}$. Further, $\frac{\partial}{\partial r} = \frac{\partial(\Delta r)}{\partial r}\frac{\partial}{\partial(\Delta r)} + \frac{\partial\tilde{\theta}}{\partial r}\frac{\partial}{\partial\tilde{\theta}} = \frac{\partial}{\partial(\Delta r)}$ and $\frac{\partial}{\partial\theta} = \frac{\partial(\Delta r)}{\partial\theta}\frac{\partial}{\partial(\Delta r)} + \frac{\partial\tilde{\theta}}{\partial\theta}\frac{\partial}{\partial\tilde{\theta}} = -\frac{\partial r_h(\theta)}{\partial\theta}\frac{\partial}{\partial(\Delta r)} + \frac{\partial}{\partial\tilde{\theta}}$.

$$t_{r\theta} = -3\varepsilon\sqrt{\frac{5}{\pi}}\sin(2\theta)\Big\{\frac{m^{3/2}\sqrt{2}}{\sqrt{\Delta r}} - 2m + \frac{7}{2}\sqrt{\frac{m}{2}}\sqrt{\Delta r} - 6\Delta r + \frac{27\Delta r^{3/2}}{16\sqrt{2m}} - \frac{\Delta r^2}{2m}$$
$$+O\left(\Delta r^{5/2}\right)\Big\} + \varepsilon^2\frac{15\sin(4\theta)}{\pi}\frac{m^2}{2\Delta r} - \frac{3m^{3/2}}{\sqrt{2}\sqrt{\Delta r}} + \frac{19m}{4} - \frac{41\sqrt{m}\sqrt{\Delta r}}{4\sqrt{2}}$$
$$+\frac{33\Delta r}{4} - \frac{329\Delta r^{3/2}}{32\sqrt{2m}} + \frac{5\Delta r^2}{m} + O\left(\Delta r^{5/2}\right)\Big\} + O(\varepsilon^3). \tag{61}$$

Contravariant metrices toward these are obtained as

$$t^{tt} = \Big(-\frac{2m}{\Delta r} - 1 - \frac{\Delta r}{2m} + O\left(\Delta r^2\right)\Big) - \varepsilon^2\frac{45\sin^2(2\theta)}{2\pi}\Big(\frac{m}{2\Delta r} - \frac{2\sqrt{2m}}{3\sqrt{\Delta r}} + 1 - \frac{\sqrt{\Delta r}}{\sqrt{2m}}$$
$$+\frac{7\Delta r}{8m} + O\left(\Delta r^{3/2}\right)\Big) + O(\varepsilon^3), \tag{62}$$

$$t^{rr} = \Big(\frac{\Delta r}{2m} + O\left(\Delta r^2\right)\Big) + \varepsilon^2\frac{15\sin^2(2\theta)}{\pi}\Big(\frac{3\Delta r}{16m} - \frac{\Delta r^{3/2}}{\sqrt{2}m^{3/2}} + O\left(\Delta r^2\right)\Big) + O\left(\varepsilon^3\right), \tag{63}$$

$$t^{\theta\theta} = \Big(\frac{1}{4m^2} - \frac{\Delta r}{4m^3} + O\left(\Delta r^2\right)\Big) + 3\varepsilon\sqrt{\frac{5}{\pi}}\cos(2\theta)\Big(-\frac{1}{2m^2} + \frac{\sqrt{\Delta r}}{\sqrt{2}m^{5/2}} + O\left(\Delta r^{3/2}\right)\Big)$$
$$+\varepsilon^2\frac{45}{8\pi}\Big(\frac{11\cos(4\theta)+13}{4m^2} - \frac{(11\cos(4\theta)+13)\sqrt{\Delta r}}{\sqrt{2}m^{5/2}} + \frac{(65\cos(4\theta)+79)\Delta r}{8m^3}$$
$$+O\left(\Delta r^{3/2}\right)\Big) + O(\varepsilon^3), \tag{64}$$

$$t^{r\theta} = \varepsilon\frac{3}{4}\sqrt{\frac{5}{\pi}}\sin(2\theta)\Big(\frac{\sqrt{\Delta r}}{\sqrt{2}m^{3/2}} - \frac{\Delta r}{m^2} + O\left(\Delta r^{3/2}\right)\Big)$$
$$-\varepsilon^2\frac{15\sin(4\theta)}{16\pi}\Big(\frac{1}{m} + \frac{6\sqrt{\Delta r}}{\sqrt{2}m^{3/2}} - \frac{16\Delta r}{m^2} + O\left(\Delta r^{3/2}\right)\Big) + O(\varepsilon^3), \tag{65}$$

$$t^{\phi\phi} = \frac{\csc^2\theta}{4}\Big(\frac{1}{m^2} - \frac{\Delta r}{m^3} + O\left(\Delta r^2\right)\Big) + 3\varepsilon\sqrt{\frac{5}{2\pi}}\cot^2\theta\Big(-\frac{1}{\sqrt{2}m^2} + \frac{\sqrt{\Delta r}}{m^{5/2}} + O\left(\Delta r^{3/2}\right)\Big)$$
$$+\varepsilon^2\frac{45\cot^2(\theta)}{\pi}\Big(\frac{\cos(2\theta)+2}{4m^2} - \frac{(\cos(2\theta)+2)\sqrt{\Delta r}}{\sqrt{2}m^{5/2}} + \frac{(11\cos(2\theta)+25)\Delta r}{16m^3}$$
$$+O\left(\Delta r^{3/2}\right)\Big) + O(\varepsilon^3). \tag{66}$$

We can check these are the inverse each other in the range of ε^2. (Leading of these are the same with just a Schwarzschild, which is the technical reason for our result, no change.)

B.2 Hawking temperature in the original 4D

We have given the Killing vector in (A.5), and obtained the position of the horizon in (53) and the near-horizon metrices. With these and the formula: $\kappa^2 = -\frac{1}{2}D^M\xi^N D_M\xi_N$, the Hawking temperature in the original 4D spacetime can be obtained as

$$T_H = 1/8\pi m + O(\varepsilon^3). \tag{67}$$

This is the same with the one in just the Schwarzschild. We can understand this as follows.

Generally, $T_H = \frac{1}{4\pi}\left|\partial_r f(r)\big|_{r=r_h}\right|$ for $ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + \cdots$, where $f(r)|_{r=r_h} = 0$ (these $f(r)$ and r_h are irrelevant with this study). However, our $f(r)$ behaves same with just the Schwarzschild at $\Delta r = 0$ ($r = r_{h,4D}$) as in (57).

Our Hawking temperature might have been expected to depend on the angular directions, which breaks the zeroth law of blackhole thermodynamics. However, we could expect from the result above that it would be always out of the analysis's order.

B.3 Near-horizon action

Having obtained 4D near-horizon metrices, let us obtain near-horizon action. For this, we write down (56) term by term, then express each line order by order as to ε^2-order as

$$\begin{aligned}
\mathcal{L} \text{ of } (56) = &-\phi^*\partial_t\big\{\big(\sqrt{-t}^{(0)}+\sqrt{-t}^{(1)}+\sqrt{-t}^{(2)}\big)\big(t^{tt(0)}+t^{tt(2)}\big)\partial_t\big\}\phi &\text{(68a)}\\
&-\phi^*\partial_r\big\{\big(\sqrt{-t}^{(0)}+\sqrt{-t}^{(1)}+\sqrt{-t}^{(2)}\big)\big(t^{rr(0)}+t^{rr(2)}\big)\partial_r\big\}\phi &\text{(68b)}\\
&-\phi^*\partial_r\big\{\big(\sqrt{-t}^{(0)}+\sqrt{-t}^{(1)}+\sqrt{-t}^{(2)}\big)\big(t^{r\theta(1)}+t^{r\theta(2)}\big)\partial_\theta\big\}\phi &\text{(68c)}\\
&-\phi^*\partial_\theta\big\{\big(\sqrt{-t}^{(0)}+\sqrt{-t}^{(1)}+\sqrt{-t}^{(2)}\big)\big(t^{\theta\theta(0)}+t^{\theta\theta(1)}+t^{\theta\theta(2)}\big)\partial_\theta\big\}\phi &\text{(68d)}\\
&-\phi^*\partial_\theta\big\{\big(\sqrt{-t}^{(0)}+\sqrt{-t}^{(1)}+\sqrt{-t}^{(2)}\big)\big(t^{r\theta(1)}+t^{r\theta(2)}\big)\partial_r\big\}\phi &\text{(68e)}\\
&-\phi^*\partial_\phi\big\{\big(\sqrt{-t}^{(0)}+\sqrt{-t}^{(1)}+\sqrt{-t}^{(2)}\big)\big(t^{\phi\phi(0)}+t^{\phi\phi(1)}+t^{\phi\phi(2)}\big)\partial_\phi\big\}\phi, &\text{(68f)}
\end{aligned}$$

where the numbers in the superscripts mean the part of that quantity at that order when that quantity is expanded with regard to ε [5].

We write the order behavior of the ingredients in (68a)-(68f) based on (62)-(66) as

- $\sqrt{-t}^{(0)}+\sqrt{-t}^{(1)}+\sqrt{-t}^{(2)} \sim (1+\Delta r)+(1+\sqrt{\Delta r})\varepsilon+(1+\sqrt{\Delta r})\varepsilon^2$,
- $t^{tt(0)}+t^{tt(2)} \sim (\Delta r)^{-1}+\varepsilon^2(\Delta r)^{-1}$,
- $t^{rr(0)}+t^{rr(2)} \sim \Delta r+\Delta r\varepsilon^2$,
- $t^{r\theta(1)}+t^{r\theta(2)} \sim \sqrt{\Delta r}\varepsilon+(1+\sqrt{\Delta r})\varepsilon^2$,
- $t^{\theta\theta(0)}+t^{\theta\theta(1)}+t^{\theta\theta(2)} \sim (1+\Delta r)+(1+\sqrt{\Delta r})\varepsilon+(1+\sqrt{\Delta r})\varepsilon^2$,
- $t^{\phi\phi(0)}+t^{\phi\phi(1)}+t^{\phi\phi(2)} \sim (1+\Delta r)+(1+\sqrt{\Delta r})\varepsilon+(1+\sqrt{\Delta r})\varepsilon^2$.

With these, we can get the order behavior of the each line (68a)-(68f) as

$$\begin{aligned}
\text{(68a)} \sim &\Big(\frac{1}{\Delta r}+1+O\left(\Delta r^2\right)\Big)+\varepsilon\Big(\frac{1}{\Delta r}+\frac{1}{\sqrt{\Delta r}}+O\left(\Delta r^{3/2}\right)\Big)\\
&+\varepsilon^2\Big(\frac{2}{\Delta r}+\frac{1}{\sqrt{\Delta r}}+1+O\left(\Delta r^{3/2}\right)\Big)+O(\varepsilon^3),\\
\text{(68b)} \sim &\Big(1+2\Delta r+O\left(\Delta r^2\right)\Big)+\varepsilon\Big(1+\frac{3\sqrt{\Delta r}}{2}+O\left(\Delta r^{3/2}\right)\Big)\\
&+\varepsilon^2\Big(2+\frac{3\sqrt{\Delta r}}{2}+2\Delta r+O\left(\Delta r^{3/2}\right)\Big)+O(\varepsilon^3),\\
\text{(68c)} \sim &\,\varepsilon\Big(\frac{1}{2\sqrt{\Delta r}}+\frac{3\sqrt{\Delta r}}{2}+O\left(\Delta r^{3/2}\right)\Big)+\varepsilon^2\Big(\frac{1}{\sqrt{\Delta r}}+2+\frac{3\sqrt{\Delta r}}{2}+O\left(\Delta r^{3/2}\right)\Big)+O(\varepsilon^3),\\
\text{(68d)} \sim &\big(1+2\Delta r+O(\Delta r^2)\big)+\varepsilon\big(2+2\sqrt{\Delta r}+2\Delta r+O(\Delta r^{3/2})\big)\\
&+\varepsilon^2\big(3+4\sqrt{\Delta r}+3\Delta r+O(\Delta r^{3/2})\big)+O(\varepsilon^3),\\
\text{(68e)} \sim &\,\varepsilon\big(\sqrt{\Delta r}+O(\Delta r^{3/2})\big)+\varepsilon^2\big(1+2\sqrt{\Delta r}+2\Delta r+O(\Delta r^{3/2})\big)+O(\varepsilon^3),\\
\text{(68f)} \sim &\big(1+2\Delta r+O(\Delta r^2)\big)+\varepsilon\big(2+2\sqrt{\Delta r}+2\Delta r+O(\Delta r^{3/2})\big)\\
&+\varepsilon^2\big(3+4\sqrt{\Delta r}+3\Delta r+O(\Delta r^{3/2})\big)+O(\varepsilon^3).
\end{aligned}$$

We find (68a) is dominant and others are vanish or ignorable compared with (68a) at $\Delta r \to 0$. Therefore, from the viewpoint of which parts remain at $\Delta r \to 0$, we may

[5] E.g., $\sqrt{-t}^{(0)}+\sqrt{-t}^{(1)}+\sqrt{-t}^{(2)}$ means the first three terms writing as $\sqrt{-t}=(\cdots)+(\cdots)\varepsilon+(\cdots)\varepsilon^2+\cdots$.

remain only (68a). However, since the $(t,\, r)$-space is crucial in the analysis of Hawking temperature and flux, the parts on and tangling with the $(t,\, r)$-space are indispensable in our analysis. (Also note that (68b) makes the action at $\Delta r \to 0$ finite.) Therefore, remaining (68b), (68c) and (68e) in addition to (68a), we consider the following action:

$$
\begin{aligned}
(68) &= \sqrt{-t}(t^{tt}\partial_t\phi^*\partial_t\phi - t^{rr}\partial_r\phi^*\partial_r\phi - t^{r\theta}\partial_r\phi^*\partial_\theta\phi - t^{\theta r}\partial_\theta\phi^*\partial_r\phi) + \cdots \\
&= \sqrt{-t}\big(t^{tt}\partial_t\phi^*\partial_t\phi + t^{rr}\big(\partial_r\phi^* + \frac{t^{\theta r}}{t^{rr}}\partial_\theta\phi^*\big)\big(\partial_r\phi + \frac{t^{\theta r}}{t^{rr}}\partial_\theta\phi\big) - \frac{(t^{\theta r})^2}{t^{rr}}\partial_\theta\phi^*\partial_\theta\phi\big), \quad (69)
\end{aligned}
$$

as the near-horizon action, where "$\cdots$" is (68d), (68f) (the terms vanishing at $\Delta r \to 0$ and irrelevant for the $(t,\, r)$-space) and terms under ε^3-order.

Let us look $(t^{\theta r})^2/t^{rr}$ and $t^{\theta r}/t^{rr}$ in (69). Using (62)-(66), we can write these as

- $$
\begin{aligned}
\frac{t^{\theta r}}{t^{rr}} &= \varepsilon\frac{3}{2}\sqrt{\frac{5}{\pi}}\sin(2\theta)\Big(\frac{1}{\sqrt{2m}\sqrt{\Delta r}} - \frac{1}{m} + O\left(\sqrt{\Delta r}\right)\Big) \\
&\quad - \varepsilon^2\frac{15}{8\pi}\sin(4\theta)\Big(\frac{1}{\Delta r} + \frac{6}{\sqrt{2m}\sqrt{\Delta r}} + O\left(\Delta r^0\right)\Big) + O(\varepsilon^3), \qquad (70a)
\end{aligned}
$$
- $$
\frac{(t^{\theta r})^2}{t^{rr}} = \varepsilon^2\frac{45\sin^2(2\theta)}{16\pi}\Big(\frac{1}{m^2} - \frac{4\sqrt{\Delta r}}{\sqrt{2}m^{5/2}} + O\left(\Delta r^1\right)\Big) + O(\varepsilon^3). \qquad (70b)
$$

Thus, $(t^{\theta r})^2/t^{rr}$ is ignorable in the limit $\Delta r \to 0$, but $t^{\theta r}/t^{rr}$ is not. However $t^{\theta r}/t^{rr}$ is ignorable finally in the analysis of Hawking temperature and flux for the following facts:

- We can regard $\frac{t^{\theta r}}{t^{rr}}\partial_\theta\phi$ as the r-component of U(1) gauge field in the sense that we can evaluate the anomalies and currents associated with it using the way to evaluate those for U(1) gauge field. The point here is that it is composed of the t-independent r-component only, therefore we can see by looking at (4) in [54] the gauge anomalies do not arise from $\frac{t^{\theta r}}{t^{rr}}\partial_\theta\phi$. Hence, $\frac{t^{\theta r}}{t^{rr}}\partial_\theta\phi$ is irrelevant of this study.

- Next, as for the gravitational anomalies, since J^μ is zero according to (4) in [54], the second term in r.h.s. in (16) in [54] is some constants. The first term in the (16) will be also zero, since our gauge field is composed of only t-independent r-component.

Hence, since $t^{\theta r}/t^{rr}$ has nothing to do with gauge and gravitational anomalies, we can ignore it in our analysis and are allowed to write the near-horizon action we consider as

$$
S_{\rm nh} = \frac{1}{2}\int dx^4\sqrt{-t}(t^{tt}\partial_t\phi^*\partial_t\phi + t^{rr}\partial_r\phi^*\partial_r\phi). \qquad (71)
$$

Let us obtain the concrete expression for (71). For this we write as

$$
\begin{aligned}
\mathcal{L} \text{ of } (71) = &- \phi^*\{(\sqrt{-t})^{(0)}t^{tt(0)}\}\partial_t\partial_t\phi - \phi^*\partial_r\{(\sqrt{-t})^{(0)}t^{rr(0)}\partial_r\phi\} &(72a)\\
&- \phi^*\{(\sqrt{-t})^{(0)}t^{tt(2)} + ((\sqrt{-t})^{(1)} + (\sqrt{-t})^{(2)})g^{tt(0)}\}\partial_t\partial_t\phi &(72b)\\
&- \phi^*\partial_r\{((\sqrt{-t})^{(0)}t^{tt(2)} + ((\sqrt{-t})^{(1)} + (\sqrt{-t})^{(2)})t^{tt(0)})\partial_r\phi\}, &(72c)
\end{aligned}
$$

where the meaning of the numbers in the superscripts are the same with (68).

We list the ingredients needed to calculate (72) as

- $(\sqrt{-t})^{(0)} = 4m\sin(\theta)(\Delta r + m),$ (73a)
- $$\sum_{i=1,2}(\sqrt{-t})^{(i)} = \frac{90\varepsilon^2m^2\sin(\theta)\cos^2(\theta)(3\cos(2\theta)-1)}{\pi} + 6\sqrt{\frac{5}{\pi}}\varepsilon m^2\sin(\theta)(3\cos(2\theta)+1), \qquad (73b)$$
- $t^{tt(0)} = -2m/\Delta r, \quad t^{rr(0)} = \Delta r/2m,$ (73c)
- $t^{tt(2)} = -45\varepsilon^2 m\sin^2(2\theta)/4\pi\Delta r, \quad t^{rr(2)} = 45\Delta r\varepsilon^2\sin^2(2\theta)/16\pi m.$ (73d)

Using these, we can obtain the concrete expression of (71) to ε^2-order as

$$\begin{aligned}(71) &= -\int d^4x\,(2m)^2\sin(\theta)\Big\{1+\frac{3}{2}\sqrt{\frac{5}{\pi}}\,(1+3\cos(2\theta))\,\varepsilon+\frac{45}{2\pi}\Big(\frac{\sin(2\theta)}{4}-\cos^2(\theta)\\ &\quad +3\cos^2(\theta)\cos(2\theta)\Big)\varepsilon^2\Big\}\phi^*(t^{tt}\partial_t\partial_t+\partial_r(t^{rr}\partial_r))\phi \qquad (74)\\ (t^{tt},t^{rr}) &= \Big(-\frac{2m}{\Delta r},\frac{\Delta r}{2m}\Big)\\ &= \Big(-\frac{2m}{r-2m}-\frac{15\varepsilon^2m^2\sin^2(2\theta)}{4\pi(r-2m)^2},\ \frac{r-2m}{2m}-\frac{15\varepsilon^2\sin^2(2\theta)}{16\pi}\Big). \qquad (75)\end{aligned}$$

Note that the mass parts in the denominator of $-\frac{2m}{\Delta r}$ and numerator of $\frac{\Delta r}{2m}$ do not agree each other. We fix this point in (81b) by defining the effective mass (81d).

B.4 Integrate out of $(\theta,\,\phi)$

We will obtain the 2D effective near-horizon action by integrating out $(\theta,\,\phi)$ of (74). For this, we first expand ϕ by the spherical harmonics as

$$\phi(t,r,\theta,\phi)=\sum_{l,m}\phi_{lm}(t,r)Y_l^m(\theta,\phi). \qquad (76)$$

Then defining the following $\Lambda_{lm,\,kn}$ ($d\Omega=d\theta d\phi\sin\theta$), we can write (74) as

$$\begin{aligned}\Lambda_{lm,\,kn} &= \int d\Omega\Big\{1+\frac{3}{2}\sqrt{\frac{5}{\pi}}\,(1+3\cos(2\theta))\,\varepsilon\\ &\quad +\frac{45}{2\pi}\Big(\frac{\sin(2\theta)}{4}-\cos^2(\theta)+3\cos^2(\theta)\cos(2\theta)\Big)\varepsilon^2\Big\}\,(Y_l^m)^*Y_k^n, \qquad (77a)\\ (74) &= -\sum_{l,m}\sum_{k,n}\int dtdr(2m)^2\Big\{\\ &\quad \phi_{lm}^*\Big(-\frac{2m}{r-2m}\Lambda_{lm,\,kn}-\frac{15m^2\varepsilon^2}{4\pi(r-2m)^2}\int d\Omega\sin^2(2\theta)(Y_l^m)^*Y_k^n\Big)\partial_t\partial_t\phi_{kn}\\ &\quad +\phi_{lm}^*\partial_r\Big(\frac{r-2m}{2m}\Lambda_{lm,\,kn}-\frac{15m^2\varepsilon^2}{16\pi}\int d\Omega\sin^2(2\theta)(Y_l^m)^*Y_k^n\Big)\partial_r\phi_{kn}\Big\}. \qquad (77b)\end{aligned}$$

We can evaluate all kinds of the $(\theta,\,\phi)$-integrals in (77b) (totally four) using (78) as[6]

- $$\int d\Omega\cos 2\theta\,(Y_k^n)^*Y_l^m$$
$$\begin{aligned}&= -\frac{4m^2-1}{4l^2+4l-3}\delta_{kl}\delta_{nm}+\frac{2(-1)^{2m}\sqrt{\frac{((l+1)^2-m^2)((l+2)^2-m^2)}{(2l+1)(2l+5)}}}{2l+3}\delta_{k-2,l}\delta_{nm}\\ &\equiv \mathcal{I}_{lm}^{A_0}\,\delta_{kl}\delta_{nm}+\mathcal{I}_{lm}^{A_2}\,\delta_{k-2,l}\delta_{nm}\equiv\mathcal{I}_{lm,\,kn}^{A}, \qquad (79a)\end{aligned}$$

[6]Necessary formulas for the calculations (79a)-(79d):

$$\begin{aligned}&(Y_l^m)^*=(-)^mY_l^{-m},\quad \int d\Omega\,(Y_{l_1}^{m_1})^*Y_{l_2}^{m_2}=\delta_{l_1l_2}\delta_{m_1m_2},\quad \int d\Omega\,\sin 2\theta(Y_{l_1}^{m_1})^*Y_{l_2}^{m_2}=0\quad\text{and}\\ &\int d\Omega\,(Y_{l_1}^{m_1})^*(Y_{l_2}^{m_2})^*Y_L^M=\sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}}\langle l_1\,0\,l_2\,0|L\,0\rangle\langle l_1\,m_1\;l_2\,m_2|L\,M\rangle, \qquad (78)\end{aligned}$$

where $\langle l_1m_1l_2m_2|LM\rangle$ mean Clebsch-Gordan coefficients [56]. We can obtain (79a)-(79d) using (78) by rewriting these integrands into the form of the 3 products of spherical harmonics. To be concrete, express $\cos(2\theta)$, $\cos^2\theta$, $\sin^2(2\theta)$ and $\cos^2\theta\cos(2\theta)$ by Y_0^0, Y_2^0 and Y_4^0 (e.g. $\cos(2\theta)=\frac{8}{3}\sqrt{\frac{\pi}{5}}Y_2^0-\frac{2\sqrt{\pi}}{3}Y_0^0$).

- $$\int d\Omega \cos^2\theta\,(Y_k^n)^* Y_l^m$$

$$= \frac{2l^2+2l-2m^2-1}{4(l+1)^2-4(l+1)-3}\delta_{kl}\delta_{nm} + \frac{(-1)^{2m}\sqrt{\frac{((l+1)^2-m^2)((l+2)^2-m^2)}{(2l+1)(2l+5)}}}{2l+3}\delta_{k-2,l}\delta_{nm}$$
$$\equiv \mathcal{I}^{B_0}_{lm}\,\delta_{kl}\delta_{nm} + \mathcal{I}^{B_2}_{lm}\,\delta_{k-2,l}\delta_{nm} \equiv \mathcal{I}^{B}_{lm,\,kn}, \tag{79b}$$

- $$\int d\Omega \sin^2(2\theta)\,(Y_k^n)^* Y_l^m$$

$$= \frac{8(-1)^{2m}(l(l+1)(l^2+l-5)+2l(l+1)m^2-3m^4+3)}{(2l-3)(2l-1)(2l+3)(2l+5)}\delta_{kl}\delta_{nm}$$
$$+ 4(-1)^{2m-2l}\sqrt{\frac{((l+1)^2-m^2)((l+2)^2-m^2)}{4l^2+12l+5}}$$
$$\times \frac{(-1)^{2l}(2l-1)(2l+7)-4(-1)^{2m}(l(l+3)-7m^2)}{7(2l-1)(2l+3)(2l+7)}\delta_{k-2,l}\delta_{nm}$$
$$- 4(-1)^{2m}\sqrt{\frac{((l+1)^2-m^2)((l+2)^2-m^2)((l+3)^2-m^2)((l+4)^2-m^2)}{(2l+1)(2l+3)^2(2l+5)^2(2l+7)^2(2l+9)}}\delta_{k-4,l}\delta_{nm}$$
$$\equiv \mathcal{I}^{C_0}_{lm}\,\delta_{kl}\delta_{nm} + \mathcal{I}^{C_2}_{lm}\,\delta_{nm}\delta_{k-2,l} + \mathcal{I}^{C_4}_{lm}\,\delta_{nm}\delta_{k-4,l} \equiv \mathcal{I}^{C}_{lm,\,kn}, \tag{79c}$$

- $$\int d\Omega \cos 2\theta \cos^2\theta\,(Y_k^n)^* Y_l^m$$

$$= \frac{(-1)^{2m}(2(-8l(l+1)m^2+l(l+1)(2l(l+1)-7)+6m^4)+30m^2+3)}{(2l-3)(2l-1)(2l+3)(2l+5)}\delta_{kl}\delta_{nm}$$
$$+ (-1)^{2m-2l}\sqrt{\frac{((l+1)^2-m^2)((l+2)^2-m^2)}{4l^2+12l+5}}$$
$$\times \frac{8(-1)^{2m}(l(l+3)-7m^2)+5(-1)^{2l}(2l-1)(2l+7)}{7(2l-1)(2l+3)(2l+7)}\delta_{nm}\delta_{k-2,l}$$
$$+ 2(-1)^{2m}\sqrt{\frac{((l+1)^2-m^2)((l+2)^2-m^2)((l+3)^2-m^2)((l+4)^2-m^2)}{(2l+1)(2l+3)^2(2l+5)^2(2l+7)^2(2l+9)}}\delta_{nm}\delta_{k-4,l}$$
$$\equiv \mathcal{I}^{D_0}_{lm}\,\delta_{kl}\delta_{nm} + \mathcal{I}^{D_2}_{lm}\,\delta_{nm}\delta_{k-2,l} + \mathcal{I}^{D_4}_{lm}\,\delta_{nm}\delta_{k-4,l} \equiv \mathcal{I}^{D}_{lm,\,kn}. \tag{79d}$$

Next problem is it is not diagonalized with regard to k and l. This reflects the shape of the horizon of our 4D blackhole is not a sphere. Actually, it depends on $(\theta,\,\phi)$ as in (53) (for zeroth law of blackhole thermodynamics, see Sec.B.2). In the next subsection, we diagonalize these by redefining fields, which corresponds to rearrange appropriate bases.

B.5 2D effective near-horizon metrices

Using (79a)-(79d), we can write (77b) as

$$\sum_{kn}\sum_{lm}\int dtdr\,(2m)^2\Big\{ \quad \phi^*_{kn}\Big(-\frac{2m}{r-2m}\Lambda_{lm,\,kn} - \frac{15m^2\varepsilon^2}{4\pi(r-2m)^2}\mathcal{I}^{C}_{kn,\,lm}\Big)\partial_t\partial_t\phi_{lm}$$
$$+ \phi^*_{kn}\partial_r\Big(\frac{r-2m}{2m}\Lambda_{lm,\,kn} - \frac{15m^2\varepsilon^2}{16\pi}\mathcal{I}^{C}_{kn,\,lm}\Big)\partial_r\phi_{lm}\Big\}$$
$$= \sum_{kn}\sum_{lm}\int dtdr\,(2m)^2\Lambda_{kn,\,lm}((g_{\text{eff}})^{tt}_{kn,\,lm}\partial_t\phi^*_{kn}\partial_t\phi_{lm} + (g_{\text{eff}})^{rr}_{kn,\,lm}\partial_r\phi^*_{kn}\partial_r\phi_{lm}), \tag{80}$$

where $(g_{\rm eff})^{tt}_{kn,\,lm}$ and $(g_{\rm eff})^{rr}_{kn,\,lm}$ are the 2D effective near-horizon metrices given as

$$\begin{aligned}
\Lambda_{kn,\,lm} =& \left(1+\frac{3}{2}\sqrt{\frac{5}{\pi}}\left(1+3\mathcal{I}^{A_0}_{lm}\right)\varepsilon+\frac{45}{2\pi}\left(-\mathcal{I}^{B_0}_{lm}+3\mathcal{I}^{D_0}_{lm}\right)\varepsilon^2\right)\delta_{kl}\delta_{mn} \\
&+\frac{9}{2}\left(\sqrt{\frac{5}{\pi}}\mathcal{I}^{A_2}_{lm}\varepsilon+\frac{5}{\pi}\left(-\mathcal{I}^{B_2}_{lm}+3\mathcal{I}^{D_2}_{lm}\right)\varepsilon^2\right)\delta_{k-2,l}\delta_{mn}+\frac{135}{2\pi}\mathcal{I}^{D_4}_{lm}\varepsilon^2\delta_{k-4,l}\delta_{mn} \\
\equiv& \Lambda^{(0)}_{lm}\delta_{kl}\delta_{mn}+\Lambda^{(2)}_{lm}\delta_{k-2,l}\delta_{mn}+\Lambda^{(4)}_{lm}\delta_{k-4,l}\delta_{mn}, \qquad (81a)
\end{aligned}$$

$$(g_{\rm eff})^{tt}_{kn,lm} \equiv -\frac{2m}{r-2m-\frac{\mathcal{I}^C_{kn,\,lm}}{\Lambda_{kn,\,lm}}\frac{15m\varepsilon^2}{8\pi r}} = -\frac{2(m_{\rm eff})_{kn,\,lm}}{r-2(m_{\rm eff})_{kn,\,lm}}+O(\varepsilon^3), \qquad (81b)$$

$$(g_{\rm eff})^{rr}_{kn,lm} \equiv ((g_{\rm eff})^{tt}_{kn,lm})^{-1}, \qquad (81c)$$

$$(m_{\rm eff})_{kn,\,lm} \equiv m+\frac{15m}{8\pi r}\mathcal{I}^C_{kn,\,lm}\varepsilon^2+O(\varepsilon^3) \equiv m+\frac{\bar{\Delta}_{kn,\,lm}}{r}\varepsilon^2, \qquad (81d)$$

where **1)** $\frac{\varepsilon^2}{\Lambda_{kn,\,lm}}=1+O(\varepsilon^3)$ in (81d), **2)** we defined $\Lambda^{(0,2,4)}_{lm}$, $\bar{\Delta}_{kn,\,lm}$ and $(m_{\rm eff})_{kn,\,lm}$, and **3)** $(m_{\rm eff})_{kn,\,lm}$ get depended on r, which may be concerned. However metrices before the near-horizon limit in SecA.5 satisfy Einstein equation, and

$$(g_{\rm eff})^{tt}_{kn,lm} = -1+\frac{r}{2m}-\frac{15\,\mathcal{I}^C_{kn,\,lm}}{16\,m\pi}\varepsilon^2+\frac{225\,(\mathcal{I}^C_{kn,\,lm})^2}{128\,m\pi^2 r}\varepsilon^4+O\left(\varepsilon^5\right). \qquad (82)$$

Therefore, r-dependence is out of the analysis's order. (Hawking temperature and flux are obtained without any problems later.)

We perform the summation with regard to k and n. Then, the indeces k and n in all the $(g_{\rm eff})^{tt}_{kn,\,lm}$, $(m_{\rm eff})_{kn,\,lm}$ and $\bar{\Delta}_{kn,\,lm}$ become l and m for the delta-functions in (81a). Therefore, to shorten the expressions of equations, we in what follows denote these as

$$(g_{\rm eff})^{tt}_{lm,\,lm} \to (g_{\rm eff})^{tt}_{lm}, \quad (m_{\rm eff})_{lm,\,lm} \to (m_{\rm eff})_{lm}, \quad \bar{\Delta}_{lm,\,lm} \to \bar{\Delta}_{lm}. \qquad (83)$$

In what follows, tt- and rr-parts are basically same. We check rr-part only at the checkpoint.

In (80), we consider to change the front factor $(2m)^2$ to $(2(m_{\rm eff})_{lm})^2$. For this we evaluate $\frac{(2m)^2}{(2(m_{\rm eff})_{lm})^2}\Lambda_{kn,\,lm}$. With

$$\frac{(2m)^2}{(2(m_{\rm eff})_{lm})^2} = 1-\frac{2\bar{\Delta}_{lm}}{m\,r}\varepsilon^2+O(\varepsilon^3), \qquad (84)$$

and $\Lambda_{kn,\,lm}$ given in (81a), we can calculate in ε^2-order as

- $$\frac{(2m)^2}{(2(m_{\rm eff})_{lm})^2}\Lambda^{(0)}_{lm} = \Lambda^{(0)}_{lm}-\frac{2\bar{\Delta}_{lm}}{m\,r}\varepsilon^2+O(\varepsilon^3) \equiv \Theta^{(0)}_{lm}, \qquad (85a)$$
- $$\frac{(2m)^2}{(2(m_{\rm eff})_{lm})^2}\Lambda^{(2)}_{lm} = \Lambda^{(2)}_{lm}+O(\varepsilon^3), \qquad (85b)$$
- $$\frac{(2m)^2}{(2(m_{\rm eff})_{lm})^2}\Lambda^{(4)}_{lm} = \Lambda^{(4)}_{lm}+O(\varepsilon^3). \qquad (85c)$$

Therefore, we can write the tt-part in (80) as

$$\begin{aligned}
(80) \;=&\; \sum_{l=0}^{l_{max}-4}\sum_{m=-l}^{l}\int d^2x\,(2(m_{\rm eff})_{lm})^2(\mathcal{L}^{(0)}_{lm}+\mathcal{L}^{(2)}_{lm}+\mathcal{L}^{(4)}_{lm}) \qquad (86a)\\
&+\sum_{l=l_{max}-3}^{l_{max}-2}\sum_{m=-l}^{l}\int d^2x\,(2(m_{\rm eff})_{lm})^2(\mathcal{L}^{(0)}_{lm}+\mathcal{L}^{(2)}_{lm}) \qquad (86b)
\end{aligned}$$

$$+\sum_{l=l_{max}-1}^{l_{max}}\sum_{m=-l}^{l}\int d^2x\,(2(m_{\rm eff})_{lm})^2\mathcal{L}^{(0)}_{lm}, \tag{86c}$$

$$\text{where}\quad \mathcal{L}^{(K)}_{lm}\equiv\Lambda^{(K)}_{lm}(g_{\rm eff})^{tt}_{l+K\,m}\partial_t\phi^*_{l+K\,m}\partial_t\phi_{lm}\quad\text{for } K=0,2,4, \tag{86d}$$

l_{max} is finally taken to ∞. The ϕ_{lm} with l larger than l_{max} are zero, since no such ϕ_{lm} exist by definition. Calculation from (80) to (86) proceeds irrelevantly of either tt- or rr-part.

Focusing on (86a), we write its integrand as

$$\Omega_{lm}\Theta^{(0)}_{lm}\left(\partial_t\phi^*_{lm}\partial_t\phi_{lm}+\left(\frac{(g_{\rm eff})^{tt}_{l+2m}}{(g_{\rm eff})^{tt}_{lm}}\frac{\Lambda^{(2)}_{lm}}{\Theta^{(0)}_{lm}}\partial_t\phi^*_{l+2m}+\frac{(g_{\rm eff})^{tt}_{l+4m}}{(g_{\rm eff})^{tt}_{lm}}\frac{\Lambda^{(4)}_{lm}}{\Theta^{(0)}_{lm}}\partial_t\phi^*_{l+4m}\right)\partial_t\phi_{lm}\right), \tag{87}$$
$$\Omega_{lm}\equiv(2(m_{\rm eff})_{lm})^2(g_{\rm eff})^{tt}_{lm}.$$

$\Theta^{(0)}_{lm}$ are in (85a), and we defined Ω_{lm} to shorten the expression. Rescaling as[7]

$$\phi_{lm}\to\frac{\phi_{lm}}{\left(\Theta^{(0)}_{lm}\right)^{1/2}}\quad\text{for all } l,\, m, \tag{88}$$

we can rewrite (87) as

$$\begin{aligned}(87)\;=\;\Omega_{lm}\Bigg(\partial_t\phi^*_{lm}\partial_t\phi_{lm}+\Bigg(&\frac{(g_{\rm eff})^{tt}_{l+2\,m}}{(g_{\rm eff})^{tt}_{lm}}\frac{\Lambda^{(2)}_{lm}}{\left(\Theta^{(0)}_{lm}\Theta^{(0)}_{l+2\,m}\right)^{1/2}}\partial_t\phi^*_{l+2\,m}\\&+\frac{(g_{\rm eff})^{tt}_{l+4\,m}}{(g_{\rm eff})^{tt}_{lm}}\frac{\Lambda^{(4)}_{lm}}{\left(\Theta^{(0)}_{lm}\Theta^{(0)}_{l+4\,m}\right)^{1/2}}\partial_t\phi^*_{l+4\,m}\Bigg)\partial_t\phi_{lm}\Bigg).\end{aligned} \tag{89}$$

We here would like to look at the calculation from (87) to (89) via (88) in the rr-part, since $\Theta^{(0)}_{lm}$ depend on r as can be seen in (85a), and at (89) in the calculation of the rr-part, the following equation appears, and which can be calculated as

$$\begin{aligned}&\frac{(g_{\rm eff})^{rr}_{l+K\,m}}{(g_{\rm eff})^{rr}_{lm}}\frac{\Lambda^{(K)}_{lm}}{\Theta^{(0)}_{lm}}\partial_r\left(\frac{\phi_{l+K\,m}}{\left(\Theta^{(0)}_{l+K\,m}\right)^{1/2}}\right)\partial_r\left(\frac{\phi_{lm}}{\left(\Theta^{(0)}_{lm}\right)^{1/2}}\right)\\
=&\frac{(g_{\rm eff})^{rr}_{l+K\,m}}{(g_{\rm eff})^{rr}_{lm}}\frac{\Lambda^{(K)}_{lm}}{\Theta^{(0)}_{lm}}\left(\frac{\partial_r\Theta^{(0)}_{l+K\,m}\phi_{l+K\,m}}{2\left(\Theta^{(0)}_{l+K\,m}\right)^{3/2}}-\frac{\partial_r\phi_{l+K\,m}}{\left(\Theta^{(0)}_{l+K\,m}\right)^{1/2}}\right)\left(\frac{\partial_r\Theta^{(0)}_{lm}\phi_{lm}}{2\left(\Theta^{(0)}_{lm}\right)^{3/2}}-\frac{\partial_r\phi_{lm}}{\left(\Theta^{(0)}_{lm}\right)^{1/2}}\right)\\
=&\frac{(g_{\rm eff})^{rr}_{l+K\,m}}{(g_{\rm eff})^{rr}_{lm}}\frac{\Lambda^{(K)}_{lm}}{\Theta^{(0)}_{lm}}\frac{\partial_r\phi_{l+K\,m}}{\left(\Theta^{(0)}_{l+K\,m}\right)^{1/2}}\frac{\partial_r\phi_{lm}}{\left(\Theta^{(0)}_{lm}\right)^{1/2}}+O(\varepsilon^3).\end{aligned} \tag{90}$$

where $K=2,4$ and

$$\frac{(g_{\rm eff})^{rr}_{l+K\,m}}{(g_{\rm eff})^{rr}_{lm}}\sim1+\varepsilon^2,\quad\Lambda^{(K)}_{lm}\sim\varepsilon^{K/2},\quad\Theta^{(0)}_{lm}\sim1+\varepsilon+\left(1+r^{-1}\right)\varepsilon^2,\quad\frac{\Lambda^{(K)}_{lm}}{\Theta^{(0)}_{lm}}\sim\varepsilon^{K/2}, \tag{91}$$

from the definitions of (81b), (81a) and (85a). Therefore, the extra terms drop and the rr-part at (89) can be obtained in the same way with (89) except ∂_t and $(g_{\rm eff})^{tt}_{lm}$.

[7] To be exact, writing as $(\Theta^{(0)}_{lm})^{-1/2}(\Theta^{(0)}_{lm})^{1/2}\phi_{lm}$, treated $(\Theta^{(0)}_{lm})^{1/2}\phi_{lm}$ as ϕ_{lm} as just a difference of configurations in the path-integral.

Then, for the parts in (89), the following calculation can be held in ε^2-order:

$$\frac{(g_{\rm eff})^{tt}_{l+K\,m}}{(g_{\rm eff})^{tt}_{lm}}\frac{\Lambda^{(K)}_{lm}}{\big(\Theta^{(0)}_{lm}\Theta^{(0)}_{l+K\,m}\big)^{1/2}}=\frac{\Lambda^{(K)}_{lm}}{\big(\Theta^{(0)}_{lm}\Theta^{(0)}_{l+K\,m}\big)^{1/2}}+O(\varepsilon^3)\equiv 2\overline{\Lambda}^{(K)}_{lm} \tag{92}$$

for all l, m, where $K=2,4$.

The one above can be actually checked with the $(g_{\rm eff})^{tt}_{lm}$, $\Lambda^{(K)}_{lm}$ and $\Theta^{(0)}_{lm}$ given in (81b), (81a) and (85a) respectively, and can hold in the case of rr, namely if $(g_{\rm eff})^{tt}_{l+K\,m}$ and $(g_{\rm eff})^{tt}_{l\,m}$ are $(g_{\rm eff})^{rr}_{l+K\,m}$ and $(g_{\rm eff})^{rr}_{l\,m}$. Using (92), we can write (89) as

$$(89)\;=\;\Omega_{lm}\big(\partial_t\phi^*_{lm}\partial_t\phi_{lm}+2\big(\overline{\Lambda}^{(2)}_{lm}\partial_t\phi^*_{l+2\,m}+\overline{\Lambda}^{(4)}_{lm}\partial_t\phi^*_{l+4\,m}\big)\partial_t\phi_{lm}\big). \tag{93}$$

We here define

$$\Gamma^{(2)}_{lm}\equiv\overline{\Lambda}^{(2)}_{lm}\partial_t\phi_{l+2\,m},\quad \Gamma^{(4)}_{lm}\equiv\overline{\Lambda}^{(2)}_{lm}\partial_t\phi_{l+2\,m}+\overline{\Lambda}^{(4)}_{lm}\partial_t\phi_{l+4\,m}, \tag{94}$$

to shorten the expression of the equations ($\Gamma^{(2)}_{lm}$ is not used immediately). Then,

$$(93)\;=\;\Omega_{lm}\big(\big|\partial_t\phi_{lm}+\Gamma^{(4)}_{lm}\big|^2-\big(\overline{\Lambda}^{(2)}_{lm}\big)^2\partial_t\phi^*_{l+2m}\partial_t\phi_{l+2m}\big), \tag{95}$$

where $\overline{\Lambda}^{(K)}_{lm}\sim\varepsilon^{K/2}$ from (92) and $\Gamma^{(4)*}_{lm}\Gamma^{(4)}_{lm}\;=\;\big(\overline{\Lambda}^{(2)}_{lm}\big)^2\partial_t\phi^*_{l+2\,m}\partial_t\phi_{l+2\,m}+O(\varepsilon^3)$.

Performing the calculation regarding (86b) likewise, we can write (86) as

$$\begin{aligned}(89)\;=\;&\sum_{l=0}^{l_{max}-4}\sum_{|m|=0}^{l}\int d^2x\,\Omega_{lm}\big(\big|\partial_t\phi_{lm}+\Gamma^{(4)}_{lm}\big|^2-\big(\overline{\Lambda}^{(2)}_{lm}\big)^2\partial_t\phi^*_{l+2m}\partial_t\phi_{l+2m}\big)\\&+\sum_{l=l_{max}-3}^{l_{max}-2}\sum_{|m|=0}^{l}\int d^2x\,\Omega_{lm}\big(\big|\partial_t\phi_{lm}+\Gamma^{(2)}_{lm}\big|^2-\big(\overline{\Lambda}^{(2)}_{lm}\big)^2\partial_t\phi^*_{l+2m}\partial_t\phi_{l+2m}\big)\\&+\sum_{l=l_{max}-1}^{l_{max}}\sum_{|m|=0}^{l}\int d^2x\,\Omega_{lm}\partial_t\phi^*_{lm}\partial_t\phi_{lm}+O(\varepsilon^3).\end{aligned} \tag{96}$$

Calculation for the rr-part from (89) to (96) can be proceeded without problems, and the rr-part at (96) is also obtained basically same with (96).

Now we consider to do uniformly slide each "$\Omega_{lm}\big(\overline{\Lambda}^{(2)}_{lm}\big)^2\partial_t\phi^*_{l+2m}\partial_t\phi_{l+2m}$" appearing in the line of l to the line of $l+2$ in (96). For this, let us check $\Omega_{lm}\big(\overline{\Lambda}^{(2)}_{lm}\big)^2$:

$$\Omega_{lm}\big(\overline{\Lambda}^{(2)}_{lm}\big)^2=\frac{405m^3(\mathcal{I}^{A_2}_{lm})^2}{2\pi(r-2m)}\varepsilon^2+O(\varepsilon^3), \tag{97}$$

where $\mathcal{I}^{A_2}_{lm}$ are numbers in (79a). Thus we can write $\Omega_{lm}\big(\overline{\Lambda}^{(2)}_{lm}\big)^2$ changing its l to $l+2$ as

$$\Omega_{lm}\big(\overline{\Lambda}^{(2)}_{lm}\big)^2=\frac{(\mathcal{I}^{A_2}_{lm})^2}{(\mathcal{I}^{A_2}_{l+2\,m})^2}\Omega_{l+2\,m}\big(\overline{\Lambda}^{(2)}_{l+2\,m}\big)^2+O(\varepsilon^3)\equiv\Omega_{l+2\,m}\Xi_{l+2\,m}, \tag{98}$$

where

$$\frac{(\mathcal{I}^{A_2}_{lm})^2}{(\mathcal{I}^{A_2}_{l+2\,m})^2}=\frac{(2l+9)(2l+7)^2\left((l+1)^2-m^2\right)\left((l+2)^2-m^2\right)}{(2l+1)(2l+3)^2\left((l+3)^2-m^2\right)\left((l+4)^2-m^2\right)}, \tag{99a}$$

$$\Xi_{l+2\,m}\equiv\frac{(\mathcal{I}^{A_2}_{lm})^2}{(\mathcal{I}^{A_2}_{l+2\,m})^2}\big(\overline{\Lambda}^{(2)}_{l+2\,m}\big)^2. \tag{99b}$$

With (98), we can replace as

$$\Omega_{lm}\big(\overline{\Lambda}^{(2)}_{lm}\big)^2\partial_t\phi^*_{l+2\,m}\partial_t\phi_{l+2\,m}\to\Omega_{l+2\,m}\Xi_{l+2\,m}\partial_t\phi^*_{l+2\,m}\partial_t\phi_{l+2\,m}\quad\text{for all } l,\ m. \tag{100}$$

Therefore, uniformly sliding each "$\Omega_{lm}\big(\overline{\Lambda}^{(2)}_{lm}\big)^2\partial_t\phi^*_{l+2m}\partial_t\phi_{l+2m}$" by 2 regarding l in (96),

$$\begin{aligned}
(96)\ =\ & \sum_{l=0}^{1}\sum_{|m|=0}^{l}\int d^2x\,\Omega_{lm}\big|\partial_t\phi_{lm}+\Gamma^{(4)}_{lm}\big|^2\\
&+\sum_{l=2}^{l_{max}-4}\sum_{|m|=0}^{l-2}\int d^2x\,\Omega_{lm}\big(\big|\partial_t\phi_{lm}+\Gamma^{(4)}_{lm}\big|^2-\Xi_{lm}\,\partial_t\phi^*_{lm}\partial_t\phi_{lm}\big)\\
&+\sum_{l=2}^{l_{max}-4}\sum_{|m|=l-1}^{l}\int d^2x\,\Omega_{lm}\big|\partial_t\phi_{lm}+\Gamma^{(4)}_{lm}\big|^2\\
&+\sum_{l=l_{max}-3}^{l_{max}-2}\sum_{|m|=0}^{l-2}\int d^2x\,\Omega_{lm}\big(\big|\partial_t\phi_{lm}+\Gamma^{(2)}_{lm}\big|^2-\Xi_{lm}\,\partial_t\phi^*_{lm}\partial_t\phi_{lm}\big)\\
&+\sum_{l=l_{max}-3}^{l_{max}-2}\sum_{|m|=l-1}^{l}\int d^2x\,\Omega_{lm}\big|\partial_t\phi_{lm}+\Gamma^{(2)}_{lm}\big|^2\\
&+\sum_{l=l_{max}-1}^{l_{max}}\sum_{|m|=0}^{l-2}\int d^2x\,\Omega_{lm}\big(\partial_t\phi^*_{lm}\partial_t\phi_{lm}-\Xi_{lm}\,\partial_t\phi^*_{lm}\partial_t\phi_{lm}\big)\\
&+\sum_{l=l_{max}-1}^{l_{max}}\sum_{|m|=l-1}^{l}\int d^2x\,\Omega_{lm}\partial_t\phi^*_{lm}\partial_t\phi_{lm}+O(\varepsilon^3).
\end{aligned}\tag{101}$$

We once again perform the rescaling of the fields as

$$\phi_{l\,m}\to\frac{\phi_{l\,m}}{(1-\Xi_{lm})^{1/2}}\qquad\begin{array}{l}\text{for } l=2,3,\cdots,l_{max}\ (l=0,1\text{ are not included})\\ \text{and } |m|=0,1,\cdots,l-2\text{ for each } l.\end{array}\tag{102}$$

This rescaling is possible by the same reason in the footnote of (88). At this time, $\overline{\Lambda}^{(K)}_{lm}\sim\varepsilon^{K/2}$ (see under (95)) and $\Gamma^{(K)}_{lm}$ and $\partial_t\phi\,\Gamma^{(K)*}_{lm}$ can stay same in ε^2-order as

$$\Gamma^{(K)}_{lm}\to\Gamma^{(K)}_{lm}+O(\varepsilon^3),\quad \partial_t\phi\,\Gamma^{(K)*}_{lm}\to\partial_t\phi\,\Gamma^{(K)*}_{lm}+O(\varepsilon^3)\tag{103}$$

for all l and m, where $K=2,4$.

Therefore, we can exchange the lines in (101) with the squared form as

$$\big|\partial_t\phi_{lm}+\Gamma^{(K)}_{lm}\big|^2-\Xi_{lm}\,\partial_t\phi^*_{lm}\partial_t\phi_{lm}\to\big|\partial_t\phi_{lm}+\Gamma^{(K)}_{lm}\big|^2,\tag{104}$$

where "$\to$" meand the rescaling (102). Therefore, we can write (101) as

$$\begin{aligned}
(101)\ =\ & \sum_{l=0}^{l_{max}-4}\sum_{m=-l}^{l}\int d^2x\,\Omega_{lm}\big|\partial_t\phi_{lm}+\Gamma^{(4)}_{lm}\big|^2\\
&+\sum_{l=l_{max}-3}^{l_{max}-2}\sum_{m=-l}^{l}\int d^2x\,\Omega_{lm}\big|\partial_t\phi_{lm}+\Gamma^{(2)}_{lm}\big|^2\\
&+\sum_{l=l_{max}-1}^{l_{max}}\sum_{m=-l}^{l}\int d^2x\,\Omega_{lm}\partial_t\phi^*_{lm}\partial_t\phi_{lm}.
\end{aligned}\tag{105}$$

We here would like to give attention to the rr-part. The point to be checked between (96) and (105) is the manipulation (102): whether $\partial_r\big(\frac{\phi_{l\,m}}{(1-\Xi_{lm})^{1/2}}\big) = \frac{\partial_r\phi_{l\,m}}{(1-\Xi_{lm})^{1/2}} + O\big(\varepsilon^3\big)$ can be held or not at (102) in the calculation of the rr-part. For this, let us check the r-dependence of Ξ_{lm}: $\Xi_{lm} = \frac{(\mathcal{I}^{A_2}_{lm})^2}{(\mathcal{I}^{A_2}_{l+2\,m})^2}\Big(\frac{\Lambda^{(2)}_{lm}}{2\big(\Theta^{(0)}_{lm}\Theta^{(0)}_{l+2\,m}\big)^{1/2}}\Big)^2$, where $\frac{(\mathcal{I}^{A_2}_{lm})^2}{(\mathcal{I}^{A_2}_{l+2\,m})^2}$ are numbers, $\Lambda^{(2)}_{lm} \sim \varepsilon$ and $\Theta^{(0)}_{lm} \sim 1 + \varepsilon + (1 + r^{-1})\varepsilon^2$. Therefore, Ξ_{lm} is independent of r in ε^2-order. If so, the equation above can be held, and the rr-part at (105) can be obtained as (105) as well as the tt-part without the difference of ∂_r and $(g_{\text{eff}})^{rr}_{lm}$.

Since it can be written as follows:

$$\partial_t\phi_{lm} + \Gamma^{(4)}_{lm} = \partial_t(\phi_{lm} + \overline{\Lambda}^{(2)}_{lm}\phi_{l+2\,m} + \overline{\Lambda}^{(4)}_{lm}\phi_{l+4\,m}), \tag{106a}$$

$$\partial_t\phi_{lm} + \Gamma^{(2)}_{lm} = \partial_t(\phi_{lm} + \overline{\Lambda}^{(2)}_{lm}\phi_{l+2\,m}), \tag{106b}$$

let us perform the redefinition of the fields as

- $\varphi_{lm} \equiv \phi_{lm} + \overline{\Lambda}^{(2)}_{lm}\phi_{l+2\,m} + \overline{\Lambda}^{(4)}_{lm}\phi_{l+4\,m}$ for $l = 0, 1, \cdots, l_{max} - 4,$ (107a)
- $\varphi_{lm} \equiv \phi_{lm} + \overline{\Lambda}^{(2)}_{lm}\phi_{l+2\,m}$ for $l = l_{max} - 3, l_{max} - 2,$ (107b)
- $\varphi_{lm} \equiv \phi_{lm}$ for $l = l_{max} - 1, l_{max},$ (107c)

where m above are $0, \pm1, \cdots, \pm(l-2)$ for each l. The leadings of $\overline{\Lambda}^{(K)}_{lm}$ is $\varepsilon^{K/2}$.

(88) and (102) are rescaling which can be absorbed as configurations of the path-integral for ϕ_{lm}, however (107) is recombinations. Therefore, the Jacobian for $\phi_{lm} \to \varphi_{lm}$, should be checked. Forming a matrix according to (107) We can check it gives unit.

With φ_{lm} above, we can finally obtain the decoupled 2D effective action which is equivalent with (71) as a action in the range of ε^2 as

$$(105) = \sum_{l=0}^{l_{max}}\sum_{|m|=0}^{l}\int d^2x\,\Phi_{lm}((g_{\text{eff}})^{tt}_{lm}\,\partial_t\varphi^*_{lm}\partial_t\varphi_{lm} + (g_{\text{eff}})^{rr}_{lm}\,\partial_r\varphi^*_{lm}\partial_r\varphi_{lm}), \tag{108}$$

where **1)** $\Phi_{lm} = \big(2(m_{\text{eff}})_{lm}\big)^2$ and the 2D effective metrices are given in (81). **2)** Einstein equation can be satisfied with these effective metrices. **3)** Since the labels distinguishing the effective metrices are irrelevant of the spins, the effective metrices would not be changed if we considered fermions [58–60] and higher spin fields [61].

Lastly, the behavior (91) is critical in the feasibility of the analysis in this subsection.

C Hawking Temperature in the effective 2D

We have obtained the 2D effective metrices, which are labeled by spherical harmonics modes. From these, we can naively expect **1)** existence of various Hawking temperatures for each effective metric, **2)** correspondingly, breaking of the zeroth law of the blackhole thermodynamics. (Furthermore, **3)** $(m_{\text{eff}})_{kn,\,lm}$ get depended on r as in (81d), though this is not problems in the analysis's order.) Hence, let us check the Hawking temperature.

We can obtain the position of the horizon in the 2D picture from $(g_{\text{eff}})_{tt,\,lm} = 0$ as

$$(r_{h,2D})_{lm} = 2m + \frac{15m\,\mathcal{I}^{C_0}_{lm}}{8\pi}\,\varepsilon^2 + O(\varepsilon^3). \tag{109}$$

As this is labeled by spherical harmonics modes, we can expect the points above. However, the Hawking temperature obtained from the 2D effective metrices with the one above is

$$T_H = 1/8\pi m + O(\varepsilon^3). \tag{110}$$

This is just that in the 4D Schwarzschild and free from the concerns above. The reason of this is the same with those written in Sec.B.2, where it is considered replacing with (82).

The original 4D and effective 2D spacetimes are different each other. However the Hawking temperature in the effective 2D spacetime is generally considered to coincide with the one in the original 4D spacetime. Actually the one above coincides with (67).

D Hawking flux by anomaly cancellation

We call the anomaly cancellation method as "anomaly cancellation". Since our U(1) gauge field does not arise chiral anomalies, we ignore it as mentioned under (70b). Hence, we do not consider the Hawking flux of the electric charged current.

D.1 Set up of the radial direction

The key point in the anomaly cancellation is the fact of no outgoing modes on the horizon at the classical level. To treat this situation in the anomaly cancellation, some interval from $(r_{h,2D})_{lm}$ in the radial direction are sharply divided as follows:[8]

$$(r_{h,2D})_{lm} \le r \le (r_{h,2D})_{lm} + \epsilon_{lm}, \tag{111a}$$

$$(r_{h,2D})_{lm} + \epsilon_{lm} < r \le (r_o)_{lm}. \tag{111b}$$

- ϵ_{lm} represent the divided points, which are finally taken to zero,
- $(r_o)_{lm}$ mean the positions put by hand reasonably supposing that it is the maximum of the r to where the description by the 2D effective action (108) is possible.
- (111a) is the region where supposed only ingoing modes exist at the classical level,
- (111b) is the region where both ingoing and outgoing modes exist at classical level.

We refer to the two regions, (111a) and (111b), as the regions $\mathcal{H}$ and $\mathcal{O}$, respectively. In what follows we suppose the following corresponding in the 2D effective picture:

the outgoing modes $\to$ the right-hand modes,
the ingoing modes $\to$ the left-hand modes.

D.2 Hawking flux of the energy-momentum tensors

We consider the distribution function in the region $(r_{h,2D})_{lm} \le r \le (r_o)_{lm}$ as

$$Z\left[(g_{\rm eff})^{\mu\nu}_{lm}, \Phi_{lm}\right] = \int \mathcal{D}\varphi_{lm}\, \exp iS_{\rm 2D}((g_{\rm eff})^{\mu\nu}_{lm}, \Phi_{lm}, \varphi_{lm}), \tag{112}$$

where $(g_{\rm eff})^{\mu\nu}_{lm}$, Φ_{lm} and φ_{lm} are those in (108). Then, consider general coordinate transformation in the region as

$$x^\mu \mapsto x'^\mu = x^\mu - \eta^\mu\left(x^\mu\right). \tag{113}$$

Variation toward general coordinate transformations can be written as

$$(\delta Z)_{lm} = \Big(\delta_L (g_{\rm eff})^{\mu\nu}_{lm} \frac{\delta}{\delta_L (g_{\rm eff})^{\mu\nu}_{lm}} + \delta_L A_{\mu\, lm} \frac{\delta}{\delta_L A_{\mu,\, lm}} + \delta_L \Phi_{lm} \frac{\delta}{\delta_L \Phi_{lm}}\Big) Z \tag{114}$$

[8] Radial direction is sharply divided with ϵ in all the papers of the anomaly cancellation, which is unnatural. This problem is commented in Chap.4 in [57]. There is one more artificial point in the anomaly cancellation, which is to use two anomalies, the gravitational and consistent anomalies. [64] care this point.

$$\text{with}\quad \delta_L(g_{\rm eff})^{\mu\nu}_{lm} = -(\nabla^\mu_{lm}\eta^\nu + \nabla^\nu_{lm}\eta^\mu_{lm}), \tag{115a}$$
$$\delta_L A_{\mu,\,lm} = \nabla_{\mu,\,lm}\eta^\nu,\ A_{\nu\,lm} + \eta^\nu\nabla_{\nu,\,lm}A_{\mu,\,lm}, \tag{115b}$$
$$(\delta_L A^\mu_{lm} = -\nabla^\mu_{lm}\eta^\nu,\ A_{\nu\,lm} + \eta^\nu\nabla_{\nu,\,lm}A^\mu_{lm})$$
$$\delta_L\Phi_{lm} = \eta^\mu\partial_\mu\Phi_{lm}, \tag{115c}$$

where δ_L means Lie derivative. l and m are not summed. We keep $A_{\mu,\,lm}$ just in case.

We can obtain the conservation laws for the energy-momentum tensors at the classical level from $(\delta Z)_{lm} = 0$. Aside from these, quantum anomalies exist as [65]

$$\nabla_\mu T^\mu{}_{\nu,\,lm} = \pm\frac{1}{96\pi(-(g_{\rm eff})_{lm})^{1/2}}\epsilon^{\beta\delta}\partial_\delta\partial_\alpha\Gamma^\alpha_{\nu\beta,\,lm} \equiv \mathscr{A}^\pm_{\nu,\,lm}, \tag{116a}$$
$$(+/- \to \text{left- / right-hand modes's contributions})$$
$$\nabla_\mu\widetilde{T}^\mu{}_{\nu,\,lm} = \mp\frac{1}{96\pi(-(g_{\rm eff})_{lm})^{1/2}}\epsilon_{\mu\nu}\partial^\mu R_{lm} \equiv \widetilde{\mathscr{A}}^\mp_{\nu,\,lm}, \tag{116b}$$
$$(-/+ \to \text{left- / right-hand modes's contributions})$$

where $\epsilon^{tr} = 1$ and $\epsilon_{\mu\nu} = (g_{\rm eff})_{\mu\alpha,\,lm}(g_{\rm eff})_{\nu\beta,\,lm}\epsilon^{\alpha\beta}$. Top and bottom are the consistent and covariant anomalies. $\widetilde{T}^\mu{}_{\nu,\,lm}$ follow the boundary condition as

$$(\widetilde{T}_H)_{\mu\nu,\,lm}\big|_{r=(r_{h,2D})_{lm}} = 0. \tag{117}$$

The conservation laws in the anomaly cancellation are given combining these as

$$\nabla_\mu T^\mu{}_{\nu,\,lm} = F_{\mu\nu,\,lm}J^\mu_{lm} + A_{\nu,\,lm}\nabla_\mu J^\mu_{lm} - \frac{\partial_\nu\Phi_{lm}}{(-(g_{\rm eff})_{lm})^{1/2}}\frac{\delta S_{\rm 2D}}{\delta_L\Phi_{lm}} + \text{both/either } \mathscr{A}^\pm_{\nu,\,lm}, \tag{118a}$$
$$\nabla_\mu \widetilde{T}^\mu{}_{\nu,\,lm} = F_{\mu\nu,\,lm}J^\mu_{lm} + A_{\nu,\,lm}\nabla_\mu J^\mu_{lm} - \frac{\partial_\nu\Phi_{lm}}{(-(g_{\rm eff})_{lm})^{1/2}}\frac{\delta S_{\rm 2D}}{\delta_L\Phi_{lm}} + \text{both/either } \widetilde{\mathscr{A}}^\mp_{\nu,\,lm}, \tag{118b}$$

where $J^\mu_{lm} \equiv \frac{1}{(-(g_{\rm eff})_{lm})^{1/2}}\frac{\delta S_{\rm 2D}}{\delta_L A_{\mu,\,lm}}$ and $T_{\mu\nu,\,lm} \equiv \frac{2}{(-(g_{\rm eff})_{lm})^{1/2}}\frac{\delta S_{\rm 2D}}{\delta_L(g_{\rm eff})^{\mu\nu}_{lm}}$. "both" or "either" is taken according to both left- and ight-hand mode exist or not. Anomalies vanish in "both" as the left- and right-hand modes cancel each other.

We show (118) in our case by calculating these for the case $\nu = t$ and r respectively using (81b), (81c) and (81b) etc as

$$\partial_r T^r{}_{t,\,lm} = \text{both/either } \mathscr{A}^\pm_{t,\,lm}(= \pm\partial_r N^r{}_{t,\,lm}),\quad \partial_r T^r{}_{r,\,lm} = 0, \tag{119a}$$
$$\partial_r \widetilde{T}^\mu{}_{t,\,lm} = \text{both/either } \widetilde{\mathscr{A}}^\mp_{t,\,lm}(= \pm\partial_r \widetilde{N}^r{}_{t,\,lm}),\quad \partial_r \widetilde{T}^r{}_{r,\,lm} = 0, \tag{119b}$$
$$N^r{}_{t,\,lm} = (f'^2 + ff'')/192\pi,\quad \widetilde{N}^r{}_{t,\,lm} = (ff'' - (f')^2/2)/96\pi,$$

where f means $-(g_{\rm eff})_{tt,\,lm}$ and $'$ means ∂_r. We have used the facts that our gauge fields are ignoble (see under (70b)) and our dilaton is time-independent with our killing vector.

We give the expressions of the energy-momentum tensors we employ as

$$T^\mu{}_{\nu,\,lm} = (T_o)^\mu{}_{\nu,\,lm}\Theta_{lm} + (T_H)^\mu{}_{\nu,\,lm}H_{lm}, \tag{120a}$$
$$\widetilde{T}^\mu{}_{\nu,\,lm} = (\widetilde{T}_o)^\mu{}_{\nu,\,lm}\Theta_{lm} + (\widetilde{T}_H)^\mu{}_{\nu,\,lm}H_{lm}, \tag{120b}$$

where Θ_{lm} mean the step function $\theta\left(r - ((r_{h,2D})_{lm} + \epsilon_{lm})\right)$ and H_{lm} is $1-\Theta_{lm}$. Therefore,

$$(T_H)^\mu{}_{\nu,\,lm} \text{ include only the right-hand modes,} \tag{121a}$$
$$(T_o)^\mu{}_{\nu,\,lm} \text{ include both hand modes, which leads no anomalies.} \tag{121b}$$

$(\widetilde{T}_H)^{\mu}{}_{\nu,\,lm}$ and $(\widetilde{T}_o)^{\mu}{}_{\nu,\,lm}$ are likewise. Sharp expressions of (120) is rooted in setting (111).

From (119) and (120) with (121), we can obtain the identities we consider as

$$\partial_r (T_H)^r{}_{t,\,lm} = \mathscr{A}^{+}_{t,\,lm} = \partial_r N^r{}_{t,\,lm} \quad \text{and} \quad \partial_r (T_o)^r{}_{t,\,lm} = 0, \tag{122a}$$

$$\partial_r (\widetilde{T}_H)^r{}_{t,\,lm} = \widetilde{\mathscr{A}}^{-}_{t,\,lm} - \partial_r \widetilde{N}^r{}_{t,\,lm} \quad \text{and} \quad \partial_r (\widetilde{T}_o)^r{}_{t,\,lm} = 0, \tag{122b}$$

From these, we can get the expressions of $(T_{H,\,o})^r{}_{t,\,lm}$ and $(\widetilde{T}_{H,\,o})^r{}_{t,\,lm}$ as

$$(T_H)^r{}_{t,\,lm} = (c_H)^r{}_{t,\,lm} + \int_{(r_{h,2D})_{lm}}^{r} dr\, \partial_r N^r{}_{t,\,lm}, \quad (T_o)^r{}_{t,\,lm} = (c_o)^r{}_{t,\,lm}, \tag{123a}$$

$$(\widetilde{T}_H)^r{}_{t,\,lm} = (\widetilde{c}_H)^r{}_{t,\,lm} + \int_{(r_{h,2D})_{lm}}^{r} dr\, \partial_r \widetilde{N}^r{}_{t,\,lm}, \quad (\widetilde{T}_o)^r{}_{t,\,lm} = (\widetilde{c}_o)^r{}_{t,\,lm}, \tag{123b}$$

where $((c_H)^r{}_{t,lm},\, (\widetilde{c}_H)^r{}_{t,lm})$ and $((\widetilde{c}_o)^r{}_{t,lm},\, (\widetilde{c}_o)^r{}_{t,lm})$ are integral constants. The former two are the values of those at horizon, the latter two are the values of those at $r = (r_o)_{lm}$. $(\widetilde{c}_o)^r{}_{t,\,lm}$ is identified with the total amount of the Hawking flux (e.g. [54, 62][9]).

We consider an equation obtained from (123) as[10]

$$(\widetilde{T}_H)^r{}_{t,\,lm} - (T_H)^r{}_{t,\,lm} = (ff'' - 2(f')^2)/192\pi. \tag{124}$$

We can obtain the value of $(c_H)^r{}_{t,\,lm}$ from (124) with (117) as

$$(c_H)^r{}_{t,\,lm} = -\frac{1}{192\pi}(ff'' - 2(f')^2)\big|_{r=(r_{h,2D})_{lm}} = \frac{(f')^2}{96\pi}\Big|_{r=(r_{h,2D})_{lm}} = \frac{\pi}{6}T_H^2, \tag{125}$$

where $f'|_{r=(r_{h,2D})_{lm}} = 4\pi T_H$, ($T_H$ is (110)). Variation for (113) can be written as

$$\begin{aligned}(\delta S_{\rm 2D})_{lm} &= -\int d^2x (-(g_{\rm eff})_{lm})^{1/2}\, \eta^{\nu} \nabla_{\mu,\,lm} T^{\mu}{}_{\nu,\,lm} \\ &= -\int dx^2\, \eta^t \big(\; ((T_o)^r{}_{t,\,lm} - (T_H)^r{}_{t,\,lm} + N^r{}_{t,\,lm})\delta\left(r - ((r_{h,2D})_{lm} + \epsilon_{lm})\right) \\ &\qquad\qquad + \partial_r (N^r{}_{t,\,lm} H)\big).\end{aligned} \tag{126}$$

ϵ_{lm} are taken to zero as the near-horizon limit. The last term will vanish [53, 54, 66][11].

$(\delta S_{\rm 2D})_{lm}$ should vanish, from which $(c_o)^r{}_{t,\,lm}$, the total amount of the Hawking flux, are determined as

$$(c_o)^r{}_{t,\,lm} = (c_H)^r{}_{t,\,lm} - N^r{}_{t,\,lm}\big|_{r=(r_{h,2D})_{lm}} = \pi T_H^2/12. \tag{127}$$

[9] $(\widetilde{c}_o)^r{}_{t,\,lm}$ can be identified with the value of the total amount of the black-body radiation through the identification of $T^r{}_t$ with that ((28) in [54]), where the fermion case is considered in [54, 62] to avoid the problem of superradiance supposing that it would be the same with the bosonic case.

Once one has checked that the value of $T^r{}_t$ can agree to the black-body radiations in the 3 kinds of the fundamental 4D blackholes (Schwarzschild, Kerr and charged), all the papers concerning the anomaly cancellation compute the value of $T^r{}_t$ in various blackholes, and consider that it always represents the total amount of the black-body radiation. We in this study also follow this way.

[10] There is one point. We can see a quantity: $(\widetilde{T}_H)^r{}_{t,\,lm} - ((\widetilde{c}_o)^r{}_{t,\,lm} - (c_o)^r{}_{t,\,lm})$ appears when obtaining (124) from (123). We can see we should redefine it as a new $(\widetilde{T}_H)^r{}_{t,\,lm}$ so that new $(\widetilde{T}_H)^r{}_{t,\,lm}$ can vanish at the horizon as in (117) by appropriately taking the integral constants, $(\widetilde{c}_o)^r{}_{t,\,lm}$ and $(c_o)^r{}_{t,\,lm}$.

This is because $(\widetilde{T}_H)^r{}_{t,\,lm}$ should vanish at the horizon to get (124) (or (24) in [54] or (36) in [62]), however it does not if it is as it is. We can see this as $(\widetilde{T}_H)^r{}_{t,\,lm}\big|_{r=(r_{h,2D})_{lm}} = -\widetilde{N}^r{}_{t,\,lm}\big|_{r=(r_{h,2D})_{lm}} \neq 0$ with $f'\big|_{r=(r_{h,2D})_{lm}} \neq 0$. (This is not written in any papers such as [54, 62]. Further, [65] is referred at (36) in [62], so look it. Then its (6.21) corresponds to (124). There should be some integral constants there when $\mathcal{P}_{\mu\nu}$ is obtained by performing integration in (6.22), however no comment about this point there.)

[11] $\int_{(r_{h,2D})_{lm}}^{(r_{h,2D})_{lm}+\epsilon} dr\, \partial_r (\cdots) \to 0$ with $\epsilon \to 0$.

This result is the same with just the Schwarzschild [53, 54, 62]. The reason of this is that the Hawking flux is determined from the f, f' and f'' at $r = (r_{h,2D})_{lm}$ as in (124) and (125), however these are not changed from just the Schwarzschild as can be seen from (82). This is the same situation with the Hawking temperature in Sec.B.2 and C.

E Comment on result in terms of the information paradox

As mentioned under (110), the result (127) would be the one in the original 4D blackhole, and if the correction is to ε^2-order and ϕ-independent, we could conclude by the logic in Sec.6 the black-body radiation of the supertranslated blackholes would be always thermal.

Important problem for us is the information paradox. As an insight obtained from this work, the Hawking temperature and flux could not be the solution as we have found there is no breaking of the thermal flux in the range of the analysis in this paper.

Supertranslated blackhole spacetimes would be normal in reality and how to be supertranslated is determined by the initial configuration [6]. Therefore, the information of the initial configuration would be stored in the configuration of the asymptotic region of the spacetime, again which would be the key of the information paradox.

References

[1] https://inspirehep.net/conferences/1772425

[2] F. L. Lin and S. Takeuchi, "Hawking Flux from a Black Hole with Nonlinear Supertranslation Hair," Phys. Rev. D **102**, no.4, 044004 (2020) [arXiv:2004.07474 [hep-th]].

[3] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner, "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems," Proc. Roy. Soc. Lond. A **A269**, 21-52 (1962)

[4] R. K. Sachs, "Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times," Proc. Roy. Soc. Lond. A **A270**, 103-126 (1962)

[5] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," Commun. Math. Phys. **104**, 207-226 (1986)

[6] G. Compere and J. Long, "Classical static final state of collapse with supertranslation memory," Class. Quant. Grav. **33**, no. 19, 195001 (2016) [arXiv:1602.05197 [gr-qc]].

[7] A. Strominger, "Lectures on the Infrared Structure of Gravity and Gauge Theory," [arXiv:1703.05448 [hep-th]].

[8] P. D. Lasky, E. Thrane, Y. Levin, J. Blackman and Y. Chen, "Detecting gravitational-wave memory with LIGO: implications of GW150914," Phys. Rev. Lett. **117**, no.6, 061102 (2016) [arXiv:1605.01415 [astro-ph.HE]].

[9] R. van Haasteren and Y. Levin, "Gravitational-wave memory and pulsar timing arrays," Mon. Not. Roy. Astron. Soc. **401**, 2372 (2010) [arXiv:0909.0954 [astro-ph.IM]].

[10] J. B. Wang, G. Hobbs, W. Coles, R. M. Shannon, X. J. Zhu, D. R. Madison, M. Kerr, V. Ravi, M. J. Keith, R. N. Manchester, Y. Levin, M. Bailes, N. D. R. Bhat, S. Burke-Spolaor, S. Dai, S. Oslowski, W. van Straten, L. Toomey, N. Wang and L. Wen, "Searching for gravitational wave memory bursts with the Parkes Pulsar Timing Array," Mon. Not. Roy. Astron. Soc. **446**, 1657-1671 (2015) [arXiv:1410.3323 [astro-ph.GA]].

[11] S. Pasterski, A. Strominger and A. Zhiboedov, "New Gravitational Memories," JHEP **12**, 053 (2016) [arXiv:1502.06120 [hep-th]].

[12] M. Pate, A. M. Raclariu and A. Strominger, "Color Memory: A Yang-Mills Analog of Gravitational Wave Memory," Phys. Rev. Lett. **119**, no.26, 261602 (2017) [arXiv:1707.08016 [hep-th]].

[13] L. Bieri and D. Garfinkle, "An electromagnetic analogue of gravitational wave memory," Class. Quant. Grav. **30**, 195009 (2013) [arXiv:1307.5098 [gr-qc]].

[14] L. Susskind, "Electromagnetic Memory," [arXiv:1507.02584 [hep-th]].

[15] S. Pasterski, "Asymptotic Symmetries and Electromagnetic Memory," JHEP **09**, 154 (2017) [arXiv:1505.00716 [hep-th]].

[16] A. Strominger, "Asymptotic Symmetries of Yang-Mills Theory," JHEP **07**, 151 (2014) [arXiv:1308.0589 [hep-th]].

[17] T. He, P. Mitra, A. P. Porfyriadis and A. Strominger, "New Symmetries of Massless QED," JHEP **10**, 112 (2014) [arXiv:1407.3789 [hep-th]].

[18] A. Strominger, "On BMS Invariance of Gravitational Scattering," JHEP **07**, 152 (2014) [arXiv:1312.2229 [hep-th]].

[19] T. He, V. Lysov, P. Mitra and A. Strominger, "BMS supertranslations and Weinberg!G s soft graviton theorem," JHEP **05**, 151 (2015) [arXiv:1401.7026 [hep-th]].

[20] A. Strominger and A. Zhiboedov, "Gravitational Memory, BMS Supertranslations and Soft Theorems," JHEP **01**, 086 (2016) [arXiv:1411.5745 [hep-th]].

[21] S. W. Hawking, M. J. Perry and A. Strominger, "Superrotation Charge and Supertranslation Hair on Black Holes," JHEP **1705**, 161 (2017) [arXiv:1611.09175 [hep-th]].

[22] F. Cachazo and A. Strominger, "Evidence for a New Soft Graviton Theorem," [arXiv:1404.4091 [hep-th]].

[23] D. Kapec, V. Lysov, S. Pasterski and A. Strominger, "Semiclassical Virasoro symmetry of the quantum gravity $\mathcal{S}$-matrix," JHEP **08**, 058 (2014) [arXiv:1406.3312 [hep-th]].

[24] S. Pasterski, S. H. Shao and A. Strominger, "Gluon Amplitudes as 2d Conformal Correlators," Phys. Rev. D **96**, no.8, 085006 (2017) [arXiv:1706.03917 [hep-th]].

[25] S. Pasterski, S. H. Shao and A. Strominger, "Flat Space Amplitudes and Conformal Symmetry of the Celestial Sphere," Phys. Rev. D **96**, no.6, 065026 (2017) [arXiv:1701.00049 [hep-th]].

[26] A. Bagchi, R. Basu, A. Kakkar and A. Mehra, "Flat Holography: Aspects of the dual field theory," JHEP **12**, 147 (2016) [arXiv:1609.06203 [hep-th]].

[27] S. Pasterski and S. H. Shao, "Conformal basis for flat space amplitudes," Phys. Rev. D **96**, no.6, 065022 (2017) [arXiv:1705.01027 [hep-th]].

[28] E. Berti, V. Cardoso and C. M. Will, "On gravitational-wave spectroscopy of massive black holes with the space interferometer LISA," Phys. Rev. D **73**, 064030 (2006) [gr-qc/0512160].

[29] • T. Morita, Eur. Phys. J. C **80**, no.4, 331 (2020) [arXiv:1801.00967 [hep-th]]. • T. Morita, Phys. Rev. Lett. **122**, no.10, 101603 (2019) [arXiv:1902.06940 [hep-th]].

[30] • N. Sasakura and S. Takeuchi, Eur. Phys. J. C **80**, no.2, 118 (2020) [arXiv:1907.06137 [hep-th]]. • D. Obster and N. Sasakura, PTEP **2020**, no.7, 073B06 (2020) [arXiv:2004.03152 [hep-th]]. • N. Sasakura, [arXiv:2008.07726 [hep-th]].

[31] • E. Conde and P. Mao, Phys. Rev. D **95**, no.2, 021701 (2017) [arXiv:1605.09731 [hep-th]]. • E. Conde and P. Mao, JHEP **05**, 060 (2017) [arXiv:1612.08294 [hep-th]]. • P. Mao, H. Ouyang, J. B. Wu and X. Wu, Phys. Rev. D **95**, no.12, 125011 (2017) [arXiv:1703.06588 [hep-th]]. • P. Mao and J. B. Wu, Phys. Rev. D **96**, no.6, 065023 (2017) [arXiv:1704.05740 [hep-th]]. • P. Mao and H. Ouyang, Phys. Lett. B **774**, 715-722 (2017) [arXiv:1707.07118 [hep-th]].

[32] • E. Witten and K. Yonekura, [arXiv:1909.08775 [hep-th]]. • C. T. Hsieh, Y. Tachikawa and K. Yonekura, [arXiv:2003.11550 [hep-th]].

[33] • N. Banerjee, A. Bhattacharjee, I. Lodato and T. Neogi, JHEP **01**, 115 (2019) [arXiv:1807.06768 [hep-th]]. • N. Banerjee, A. Bhattacharjee, Neetu and T. Neogi, JHEP **11**, 122 (2019) [arXiv:1905.10239 [hep-th]]. • N. Banerjee, S. Khandelwal and P. Shah, Phys. Rev. D **100**, no.10, 105013 (2019) [arXiv:1907.05866 [hep-th]].

[34] S. Atul Bhatkar, [arXiv:2007.03627 [hep-th]].

[35] • D. Giataganas, JHEP **07**, 031 (2012) [arXiv:1202.4436 [hep-th]]. • D. Giataganas, PoS **Corfu2012**, 122 (2013) [arXiv:1306.1404 [hep-th]]. • K. Bitaghsir Fadafan, D. Giataganas and H. Soltanpanahi, JHEP **11**, 107 (2013) [arXiv:1306.2929 [hep-th]]. • D. Giataganas and H. Soltanpanahi, Phys. Rev. D **89**, no.2, 026011 (2014) [arXiv:1310.6725 [hep-th]]. • D. Giataganas and H. Soltanpanahi, JHEP **06**, 047 (2014) [arXiv:1312.7474 [hep-th]]. • D. Giataganas, U. Gürsoy and J. F. Pedraza, Phys. Rev. Lett. **121**, no.12, 121601 (2018) [arXiv:1708.05691 [hep-th]]. • D. Giataganas, Phys. Rev. D **98**, no.10, 106010 (2018) [arXiv:1805.08245 [hep-th]]. • C. S. Chu and D. Giataganas, Phys. Rev. D **101**, no.4, 046007 (2020) [arXiv:1906.09620 [hep-th]].

[36] • G. Compère, A. Fiorucci and R. Ruzziconi, JHEP **11**, 200 (2018) [erratum: JHEP **04**, 172 (2020)] [arXiv:1810.00377 [hep-th]]. • G. Compère, A. Fiorucci and R. Ruzziconi, Class. Quant. Grav. **36**, no.19, 195017 (2019) [arXiv:1905.00971 [gr-qc]]. • G. Compère, A. Fiorucci and R. Ruzziconi, [arXiv:2004.10769 [hep-th]].

[37] • D. Giataganas and N. Tetradis, Phys. Lett. B **796**, 88-92 (2019) [arXiv:1904.13119 [hep-th]]. • N. Tetradis, Phys. Lett. B **807**, 135552 (2020) [arXiv:1910.10587 [hep-th]].

[38] K. Hajian, [arXiv:2001.07540 [gr-qc]].

[39] • P. Romatschke, Phys. Rev. Lett. **122**, no.23, 231603 (2019) [erratum: Phys. Rev. Lett. **123**, no.20, 209901 (2019)] [arXiv:1904.09995 [hep-th]]. • O. DeWolfe and P. Romatschke, JHEP **10**, 272 (2019) [arXiv:1905.06355 [hep-th]]. • P. Romatschke, Phys. Rev. Lett. **123**, no.24, 241602 (2019) [arXiv:1908.02758 [hep-th]].

[40] • C. Keeler, T. Manton and N. Monga, JHEP **08**, 147 (2020) [arXiv:2005.04242 [hep-th]]. • A. Luna, R. Monteiro, I. Nicholson and D. O'Connell, Class. Quant. Grav. **36**, 065003 (2019) [arXiv:1810.08183 [hep-th]]. • I. Bredberg, C. Keeler, V. Lysov and A. Strominger, JHEP **07**, 146 (2012) [arXiv:1101.2451 [hep-th]].

[41] Prof.Strominger's related papers would be referred here, but these are referred at other points in this article.

[42] • R. Bousso and M. Porrati, Class. Quant. Grav. **34**, no.20, 204001 (2017) [arXiv:1706.00436 [hep-th]]. • R. Javadinezhad, U. Kol and M. Porrati, JHEP **01**, 089 (2019) [arXiv:1808.02987 [hep-th]]. • U. Kol and M. Porrati, Phys. Rev. D **100**, no.4, 046019 (2019) [arXiv:1907.00990 [hep-th]]. • U. Kol and M. Porrati, Phys. Rev. D **101**, no.12, 126009 (2020) [arXiv:2003.09054 [hep-th]].

[43] • E. Hijano, JHEP **07**, 132 (2019) [arXiv:1905.02729 [hep-th]]. • E. Hijano and D. Neuenfeld, [arXiv:2005.03667 [hep-th]].

[44] • R. Monteiro, D. O'Connell and C. D. White, JHEP **12**, 056 (2014) [arXiv:1410.0239 [hep-th]]. • R. Monteiro, D. O'Connell and C. D. White, Int. J. Mod. Phys. D **24**, no.09, 1542008 (2015) • A. Luna, R. Monteiro, D. O'Connell and C. D. White, Phys. Lett. B **750**, 272-277 (2015) [arXiv:1507.01869 [hep-th]]. • A. Luna, R. Monteiro, I. Nicholson, D. O'Connell and C. D. White, JHEP **06**, 023 (2016) [arXiv:1603.05737 [hep-th]]. • A. Luna, R. Monteiro, I. Nicholson, A. Ochirov, D. O'Connell, N. Westerberg and C. D. White, JHEP **04**, 069 (2017) [arXiv:1611.07508 [hep-th]].

[45] • P. Chen, M. Sasaki and D. H. Yeom, Eur. Phys. J. C **79**, no.7, 627 (2019) [arXiv:1806.03766 [hep-th]]. • P. Chen, M. Sasaki and D. h. Yeom, Int. J. Mod. Phys. D **29**, no.12, 2050086 (2020) [arXiv:2005.07011 [gr-qc]].

[46] • S. De, S. Dey and B. R. Majhi, Phys. Rev. D **99**, no.12, 124024 (2019) [arXiv:1901.05735 [hep-th]]. • S. Dey, S. De and B. R. Majhi, Phys. Rev. D **102**, no.6, 064003 (2020) [arXiv:2002.06801 [hep-th]].

[47] E. A. Bergshoeff, J. Gomis, J. Rosseel, C. Şimşek and Z. Yan, J. Phys. A **53**, no.1, 014001 (2020) [arXiv:1907.10668 [hep-th]].

[48] • T. Harmark, J. Hartong and N. A. Obers, Phys. Rev. D **96**, no.8, 086019 (2017) [arXiv:1705.03535 [hep-th]]. • D. Hansen, J. Hartong and N. A. Obers, Phys. Rev. Lett. **122**, no.6, 061106 (2019) [arXiv:1807.04765 [hep-th]]. • T. Harmark, J. Hartong, L. Menculini, N. A. Obers and G. Oling, JHEP **11**, 071 (2019) [arXiv:1907.01663 [hep-th]].

[49] • C. Krishnan, V. Patil and J. Pereira, [arXiv:2005.02993 [hep-th]]. • C. Krishnan, [arXiv:2007.06551 [hep-th]].

[50] • M. Geracie, M. Goykhman and D. T. Son, JHEP **04**, 103 (2016) [arXiv:1511.04772 [hep-th]]. • S. Minwalla, A. Mishra and N. Prabhakar, [arXiv:2008.00024 [hep-th]].

[51] K. Kim, K. Lee, R. Monteiro, I. Nicholson and D. Peinador Veiga, JHEP **02**, 046 (2020) doi:10.1007/JHEP02(2020)046 [arXiv:1912.02177 [hep-th]].

[52] E. Chacón, H. García-Compeán, A. Luna, R. Monteiro and C. D. White, [arXiv:2008.09603 [hep-th]].

[53] S. P. Robinson and F. Wilczek, "A Relationship between Hawking radiation and gravitational anomalies," Phys. Rev. Lett. **95**, 011303 (2005) [gr-qc/0502074].

[54] S. Iso, H. Umetsu and F. Wilczek, "Hawking radiation from charged black holes via gauge and gravitational anomalies," Phys. Rev. Lett. **96**, 151302 (2006) [arXiv:hep-th/0602146 [hep-th]].

[55] D. Lust and W. Vleeshouwers, "Black Hole Information and Thermodynamics," [arXiv:1809.01403 [gr-qc]].

[56] https://en.wikipedia.org/wiki/Clebsch-Gordan_coefficients

[57] K. Umetsu, "Recent Attempts in the Analysis of Black Hole Radiation," arXiv:1003.5534 [hep-th].

[58] R. Li, S. Li and J. R. Ren, "Hawking Radiation of Fermionic Field and Anomaly in 2+1 Dimensional Black Holes," Class. Quant. Grav. **27**, 155011 (2010) [arXiv:1005.3615 [hep-th]].

[59] R. Becar and P. A. Gonzalez, "Hawking Radiation for Scalar and Dirac Fields in Five Dimensional Dilatonic Black Hole via Anomalies," Int. J. Mod. Phys. D **21**, 1250030 (2012) [arXiv:1104.0356 [gr-qc]].

[60] P. J. Mao, R. Li, L. Y. Jia and J. R. Ren, "Hawking radiation of Dirac particles from the Myers-Perry black hole," Eur. Phys. J. C **71**, 1527 (2011).

[61] S. Iso, T. Morita and H. Umetsu, "Hawking radiation via higher-spin gauge anomalies," Phys. Rev. D **77**, 045007 (2008) [arXiv:0710.0456 [hep-th]].

[62] S. Iso, H. Umetsu and F. Wilczek, "Anomalies, Hawking radiations and regularity in rotating black holes," Phys. Rev. D **74**, 044017 (2006) [arXiv:hep-th/0606018 [hep-th]].

[63] K. Umetsu, "Ward Identities in the derivation of Hawking radiation from Anomalies," Prog. Theor. Phys. **119**, 849 (2008) [arXiv:0804.0963 [hep-th]].

[64] R. Banerjee and S. Kulkarni, "Hawking radiation, effective actions and covariant boundary conditions," Phys. Lett. B **659**, 827 (2008) [arXiv:0709.3916 [hep-th]].

[65] R. A. Bertlmann and E. Kohlprath, "Two-dimensional gravitational anomalies, Schwinger terms and dispersion relations," Annals Phys. **288**, 137 (2001) [hep-th/0011067].

[66] K. Murata and J. Soda, "Hawking radiation from rotating black holes and gravitational anomalies," Phys. Rev. D **74**, 044018 (2006) [hep-th/0606069].