

# Area law and OPE blocks in conformal field theory

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## Abstract

This is an introduction to the relationship between area law and OPE blocks in conformal field theory.

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## 2 Contents

|    |  |           |
|----|--|-----------|
| 3  | <b>1 Introduction</b>  | <b>2</b>  |
| 4  | <b>2 Setup</b>   | <b>3</b>  |
| 5  | 2.1 Area law   | 3         |
| 6  | 2.2 OPE block  | 4         |
| 7  | 2.3 Modular Hamiltonian and area law                                   | 6         |
| 8  | 2.4 Deformed reduced density matrix and connected correlation function | 7         |
| 9  | <b>3 Area law</b>  | <b>9</b>  |
| 10 | 3.1 Continuation   | 9         |
| 11 | 3.2 Kinematic information  | 11        |
| 12 | 3.3 UV/IR relation   | 12        |
| 13 | 3.4 Discussion   | 15        |
| 14 | <b>4 Generalizations</b>   | <b>15</b> |
| 15 | <b>5 Summary and outlook</b>   | <b>18</b> |
| 16 | <b>References</b>  | <b>19</b> |

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## 19 1 Introduction

20 This report consists a summary of our recent progress on the relationship between area law and  
21 OPE blocks. Area law has been a continuous topic in physics. The prototype of area law dates  
22 back to black hole physics in general relativity. The unusual property that the thermal entropy  
23 of a black hole is proportional to the event horizon of the black hole [1, 2] has stimulated  
24 various modern idea of theoretical physics, including the famous holographic principle.

25 OPE block [3, 4], on the other hand, is a relatively unexplored topic in conformal field theory,  
26 though it has been defined and discussed at the early stages of conformal field theory [5, 6].  
27 The operator product expansion of two primary operators is equivalent to a summation of OPE  
28 blocks with corresponding three point function coefficients. It is a smeared operator which is  
29 generated from the so-called (quasi-)primary operator, and extends the study of local operators  
30 in CFT to non-local operators.

31 Modular Hamiltonian, the logarithm of the reduced density matrix [7], plays a central role  
32 in the context of geometric entanglement entropy [8–11]. Entanglement entropy is a von  
33 Neumann entropy generated from the reduced density matrix of a subregion of spacetime. It  
34 suffers divergent problem in general. One can introduce a UV cutoff to secure this problem.  
35 An intriguing fact of the entanglement entropy is that it obeys area law in the leading order  
36 of the divergences. Its connection to gravity has been established by the work of Ryu and  
37 Takayanagi [12], in which they proposed that the entanglement entropy of a CFT is equal to  
38 the area of a minimal surface in the bulk AdS spacetime.

39 On the CFT side, the OPE block provides a novel look at the modular Hamiltonian. Modular  
40 Hamiltonian is a special OPE block generated by the stress energy-momentum tensor for a  
41 ball region. As we will show, modular Hamiltonian is related to “area laws” in the context of  
42 entanglement entropy<sup>1</sup>. This leads to the conjecture that similar to the modular Hamiltonian,  
43 general OPE blocks may exhibit area law. Indeed, in a series of papers [13, 14, 16, 17], we  
44 have shown that the quantity which satisfies area law is the type- $(m)$  connected correlation  
45 function (CCF). More explicitly, the leading term of the type- $(m)$  CCF is proportional to the  
46 area of the boundary of the ball. In the subleading terms, we find a logarithmic divergence  
47 with degree  $q$ . In all examples we studied, we found  $q = 0, 1, 2$ , but in general we don’t rule  
48 out the possibility of other values. The coefficient  $p_q$  for the logarithmic term with degree  $q$   
49 is cutoff independent. We establish a relationship between  $p_q$  and the type- $(m - 1, 1)$  CCF of  
50 OPE blocks for two balls which are far away from each other. The coefficient  $p_q$  obeys a cyclic  
51 identity which is independent of the order of the operators.

52 This paper is organised as follows. In section 2, we will introduce some basic concepts and  
53 conventions used in this paper. Section 3 is devoted to the study of the new area law which is  
54 related to the OPE blocks. Various generalizations have been given in section 4. We conclude  
55 in section 5 with a number of general open problems that deserve, in our opinion, more work.

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<sup>1</sup>The “area law” discussed in this paper includes subleading corrections. We use the slogan “area law” following the convention of geometric entanglement entropy.

56 **2 Setup**

57 In this section, we introduce some basic concepts and conventions used in this paper.

58 **2.1 Area law**

59 In any continuous quantum field theory (QFT), physical degrees exist at each point  $(t, x^i), i = 1, \dots, d-1$   
 60 of spacetime  $M$ . At each time slice  $t = t_0$ , the data on the Cauchy surface  $\Sigma$  determines the  
 61 evolution of the fields. One can divide the surface  $\Sigma$  into a spacelike subregion  $A$  and its comple-  
 62 ment  $\bar{A}, \Sigma = A \cup \bar{A}$ . The boundary  $\partial A$  is a codimension 2 surface whose area is  $\mathcal{A}$ . The  
 63 causal development of  $A$  is denoted by  $\mathcal{D}(A)$ . The physical data on  $A$  can only determine the  
 64 evolution of the fields in  $\mathcal{D}(A)$ . The causal development  $\mathcal{D}(A)$  is an independent subsystem of  
 65 the original spacetime  $M$ . Operators in this subsystem are collected to form an algebra  $\mathfrak{a}(A)$ .  
 66 Assume the QFT in the spacetime  $M$  is described by a density matrix  $\rho$ , then by integrating  
 67 out the degrees of freedom in the complement of  $\bar{A}$ , one achieves a reduced density matrix  $\rho_A$

$$\rho_A = \text{tr}_{\bar{A}} \rho. \tag{2.1}$$

68 The reduced density matrix  $\rho_A$  is a special operator in  $\mathfrak{a}(A)$  since it describes the subsystem  
 69  $\mathcal{D}(A)$  effectively. A general quantity  $\mathcal{Q}(A)$  in  $\mathfrak{a}(A)$  is said to obey area law if its leading term is  
 70 proportional to the area of the boundary  $\partial A$ ,

$$\mathcal{Q}(A) \propto \mathcal{A} + \dots. \tag{2.2}$$

71 The area law defined in (2.2) can be extended to general field theory. One typical example is  
 72 the black hole entropy in Einstein gravity. The black hole entropy is proportional to the area  
 73 of its event horizon,

$$S_{bh} = \frac{\mathcal{A}}{4G} \tag{2.3}$$

74 where  $G$  is the Newton constant. At the loop level, black hole entropy requires logarithmic cor-  
 75 rections [18–23]. Usually, the logarithmic correction is in the form  $C \log \mathcal{A}$  where the constant  
 76  $C$  may encode useful information of the black hole.

77 Sometimes the area law is divergent, one typical example is the geometric entanglement en-  
 78 tropy

$$S_A = -\text{tr}_A \rho_A \log \rho_A. \tag{2.4}$$

79 In this case, one should insert a cutoff  $\epsilon > 0$ ,

$$S_A = \gamma \frac{\mathcal{A}}{\epsilon^{d-2}} + \dots. \tag{2.5}$$

80 In the subleading terms, there may be a logarithmic term whose coefficient is independent of  
 81 the cutoff,

$$S_A = \gamma \frac{R^{d-2}}{\epsilon^{d-2}} + \dots + p \log \frac{R}{\epsilon} + \dots \tag{2.6}$$

82 where the parameter  $R$  is the characteristic length of the region  $A$ .

83 In this report, we will present a quantity  $\mathcal{Q}(A)$  which has a slightly different logarithmic be-  
 84 haviour

$$\mathcal{Q}(A) = \gamma \frac{R^{d-2}}{\epsilon^{d-2}} + \dots + p_q \log^q \frac{R}{\epsilon} + \dots. \tag{2.7}$$

85 The maximum power  $q$  of the logarithmic terms is a nonnegative integer. We will call it the  
 86 degree of the quantity  $\mathcal{Q}(A)$ . The coefficient  $p_q$  is cutoff independent and encodes useful

87 information of the theory. There could be logarithmic pieces with smaller power, however, their  
 88 coefficients are not universal under a rescaling  $\epsilon \rightarrow \lambda\epsilon$ . In the special case that the subregion  
 89  $A$  is a ball,  $R$  could be chosen to be its radius. The subregion  $A$  and its causal development  
 90  $\mathcal{D}(A)$  are in one-to-one correspondence, we will not distinguish them in the following.

91 Finally, let's further comment on the area law and logarithmic behaviour studied in this paper.

- 92 • In two dimensions, there is no polynomial term of  $\frac{R}{\epsilon}$ , the modified “area law” is

$$\mathcal{Q}(A) = p_q \log^q \frac{R}{\epsilon} + \dots \tag{2.8}$$

- 93 • In higher dimensions ( $d > 2$ ), the leading term is always proportional to the area. One  
 94 should notice that this term is non-universal and the interesting part is the subleading  
 95 logarithmic term.

## 96 2.2 OPE block

97 In any  $d$  dimensional CFT, operators are classified into (quasi-)primary operators  $\mathcal{O}$  and their  
 98 descendants  $\partial_\mu \partial_\nu \dots \mathcal{O}$ . A general primary operator is characterized by two quantum numbers,  
 99 conformal weight  $\Delta$  and  $so(d-1)$  spin  $J$ . Under a global conformal transformation  $x \rightarrow x'$ , a  
 100 primary spin 0 operator transforms as

$$\mathcal{O}(x) \rightarrow \left| \frac{\partial x'}{\partial x} \right|^{-\Delta/d} \mathcal{O}(x) \tag{2.9}$$

101 where  $|\partial x'/\partial x|$  is the Jacobian of the conformal transformation of the coordinates,  $\Delta$  is the  
 102 conformal weight of the primary operator. Operator product expansion(OPE) of two separated  
 103 primary scalar operators  $\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)$  is to expand their product in a local orthogonal and  
 104 complete basis around a suitable point

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C_{ijk} |x_{12}|^{\Delta_k - \Delta_i - \Delta_j} (\mathcal{O}_k(x_2) + \dots) \tag{2.10}$$

105 where  $\dots$  are descendants of the primary operator  $\mathcal{O}_k$ . Its form is fixed by global conformal  
 106 symmetry, therefore it just contains kinematic information of the CFT. The summation is over  
 107 all possible primary operators of the CFT. Here we expand the product around the point  $x_2$ .  
 108 The distance of any two points  $x_i, x_j$  is written as  $|x_{ij}|$ . The constant  $C_{ijk}$  is called the OPE  
 109 coefficient which is related to the three point function of primary operators

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3) \rangle = \frac{C_{ijk}}{|x_{12}|^{\Delta_{12,3}} |x_{23}|^{\Delta_{23,1}} |x_{13}|^{\Delta_{13,2}}}, \quad \Delta_{ij,k} = \Delta_i + \Delta_j - \Delta_k. \tag{2.11}$$

110 They are the only dynamical parameters in the CFT. The constants  $\Delta_i, \Delta_j, \Delta_k$  are conformal  
 111 weights of the corresponding primary operators. By collecting all kinematic terms in the sum-  
 112 mation, we can rewrite the OPE (2.10) as

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = |x_{12}|^{-\Delta_i - \Delta_j} \sum_k C_{ijk} Q_k^{ij}(x_1, x_2). \tag{2.12}$$

113 The objects  $Q_k^{ij}(x_1, x_2)$  are called OPE blocks [3, 5, 6]. They are non-local operators in the CFT  
 114 and depend on the position  $x_1$  and  $x_2$  of the external operators. The upper index  $i$  and  $j$  show  
 115 that it also depends on the quantum numbers of the external operators  $\mathcal{O}_i$  and  $\mathcal{O}_j$ . It is easy

116 to see that OPE block has dimension zero. Under a global conformal transformation  $x \rightarrow x'$ ,  
 117 an OPE block  $Q_k^{ij}(x_1, x_2)$  will transform as

$$Q_k^{ij}(x_1, x_2) \rightarrow f(x'_1, x'_2) Q_k^{ij}(x'_1, x'_2). \quad (2.13)$$

118 The explicit form of  $f(x'_1, x'_2)$  is not important in this work. When the two external operators  
 119 have the same quantum numbers, we have  $f(x'_1, x'_2) = 1$  and OPE block will be invariant under  
 120 the global conformal transformation. One can also show that the OPE block is independent  
 121 of the external operator in this special case. Due to this reason, we relabel such kind of OPE  
 122 block as

$$Q_A[\mathcal{O}_k] = Q_k^{ii}(x_1, x_2). \quad (2.14)$$

123 The subscript  $A$  denotes the region determined by the two points  $x_1$  and  $x_2$  where the two  
 124 external operators are inserted. The operator in the square bracket reflects the fact that OPE  
 125 block is generated by a primary operator  $\mathcal{O}_k$ . We omit the information of  $i$  since the OPE block  
 126 is insensitive to the external operators in this case. We will classify the primary operators  
 127  $\mathcal{O}_k$  into conserved currents  $\mathcal{J}$  and non-conserved operators  $\mathcal{O}$ . A general symmetric traceless  
 128 primary operator obeys the following unitary bound [24]

$$\begin{cases} \Delta \geq J + d - 2, & J \geq 1, \\ \Delta \geq \frac{d-2}{2}, & J = 0. \end{cases}$$

129 A conserved current  $\mathcal{J}$  with spin  $J(J \geq 1)$  will satisfy  $\Delta = J + d - 2$ . All other primary  
 130 operators are non-conserved operators. Correspondingly, the OPE block (2.14) generated by  
 131 a conserved current  $\mathcal{J}$  will be called a type-J OPE block. On the other hand, the OPE block  
 132 (2.14) generated by a non-conserved operator  $\mathcal{O}$  will be called a type-O OPE block.

133 When two operators are time-like separated, the region  $A$  is a causal diamond. The two oper-  
 134 ators are at the sharp corner of the diamond  $A$ . We can use the conformal transformation to  
 135 fix

$$x_1 = (1, \vec{x}_0), \quad x_2 = (-1, \vec{x}_0), \quad (2.15)$$

136 then the causal diamond  $A$  intersects the  $t = 0$  slice at a unit ball ( $R = 1$ ) which we will also  
 137 denote it as  $A$

$$A = \{(0, \vec{x}) | (\vec{x} - \vec{x}_0)^2 \leq 1\}. \quad (2.16)$$

138 The center of the ball is  $\vec{x}_0$ . The boundary of the ball  $A$  is a unit sphere  $\partial A$ . In the context of  
 139 geometric entanglement entropy, the surface  $\partial A$  is an entanglement surface which separates  
 140 the ball  $A$  and its complement. The leading term of entanglement entropy is proportional to  
 141 the area of the surface  $\partial A$  in general higher dimensions ( $d > 2$ ). In two dimensions, the  
 142 entanglement entropy is logarithmically divergent with the logarithmic degree  $q = 1$ . There  
 143 is a conformal Killing vector  $K$  which preserves the diamond  $A$ ,

$$K^\mu = \frac{1}{2}(1 - (\vec{x} - \vec{x}_0)^2 - t^2, -2t\vec{x}). \quad (2.17)$$

144 The conformal Killing vector  $K$  is null on the boundary of the diamond  $A$ . It generates a  
 145 modular flow of the diamond  $A$ . A type-O OPE block corresponding to the point pair (2.15) or  
 146 the unit ball  $A$  (2.16) is [4]

$$Q_A[\mathcal{O}_{\mu_1 \dots \mu_J}] = c_{\mathcal{O}_{\mu_1 \dots \mu_J}} \int_{\mathcal{D}(A)} d^d x K^{\mu_1} \dots K^{\mu_J} |K|^{\Delta-d-J} \mathcal{O}_{\mu_1 \dots \mu_J}, \quad (2.18)$$

147 where the primary operator  $\mathcal{O}_{\mu_1 \dots \mu_J}$  is non-conserved

$$\partial^{\mu_1} \mathcal{O}_{\mu_1 \dots \mu_J} \neq 0. \quad (2.19)$$

148 It has dimension  $\Delta$  and spin  $J$ . When the operator is a conserved current

$$\partial^{\mu_1} \mathcal{J}_{\mu_1 \dots \mu_J} = 0, \tag{2.20}$$

149 the corresponding type-J OPE block is

$$Q_A[\mathcal{J}_{\mu_1 \dots \mu_J}] = c_{\mathcal{J}_{\mu_1 \dots \mu_J}} \int_A d^{d-1} \vec{x} (K^0)^{J-1} \mathcal{J}_{0 \dots 0}. \tag{2.21}$$

150 It can be obtained from (2.18) by using the conservation law (2.20) and reducing it to a lower  
 151  $d-1$  dimensional integral. The coefficient  $c_{\mathcal{J}_{\mu_1 \dots \mu_J}}$  is also redefined at the same time. In (2.18)  
 152 and (2.21), the coefficients  $c_{\mathcal{O}_{\mu_1 \dots \mu_J}}$  and  $c_{\mathcal{J}_{\mu_1 \dots \mu_J}}$  are free parameters which are fixed by the  
 153 normalization of the corresponding operators, we set them to be 1.

### 154 2.3 Modular Hamiltonian and area law

155 A very special type-J OPE block is the modular Hamiltonian [7, 25] of the ball  $A$ ,

$$H_A = 2\pi \int_A d^{d-1} \vec{x} K^0 T_{00} = 2\pi \int_A d^{d-1} \vec{x} \frac{1 - (\vec{x} - \vec{x}_0)^2}{2} T_{00}(0, \vec{x}). \tag{2.22}$$

156 Modular Hamiltonian is the logarithm of the reduced density matrix  $\rho_A$

$$H_A = -\log \rho_A. \tag{2.23}$$

157 It plays a central role in the context of entanglement entropy,

$$S_A = -\text{tr}_A \rho_A \log \rho_A = \text{tr}_A e^{-H_A} H_A. \tag{2.24}$$

158 More generally, the Rényi entanglement entropy

$$S_A^{(n)} = \frac{1}{1-n} \log \text{tr}_A \rho_A^n \tag{2.25}$$

159 has been shown to satisfy an area law generally

$$S_A^{(n)} = \gamma(n) \frac{\mathcal{A}}{\epsilon^{d-2}} + \dots, \tag{2.26}$$

160 where  $\mathcal{A}$  is the area of the entanglement surface  $\partial A$  and  $\epsilon$  is a UV cutoff. The constant  $\gamma(n)$  is  
 161 cutoff dependent. The subleading terms  $\dots$  contain a logarithmic term with degree  $q = 1$  in  
 162 even dimensions

$$S_A^{(n)} = \gamma(n) \frac{\mathcal{A}}{\epsilon^{d-2}} + \dots + p_1(n) \log \frac{R}{\epsilon} + \dots, \tag{2.27}$$

163 where we have restored the radius  $R$  that was previously set to 1. The area  $\mathcal{A}$  is related to the  
 164 radius  $R$  through the power law

$$\mathcal{A} \sim R^{d-2}. \tag{2.28}$$

165 The coefficient  $p_1(n)$  encodes useful information of the CFT. The relation between modular  
 166 Hamiltonian and area law motivates the conjecture that OPE block maybe related to area  
 167 law in a suitable way. We will give the framework to discuss this problem in the following  
 168 subsection.

169 **2.4 Deformed reduced density matrix and connected correlation function**

170 Given a primary operator  $\mathcal{O}$  in a ball  $A$ , one can always define a corresponding OPE block  
 171  $Q_A[\mathcal{O}]$ . We construct an exponential operator formally [14]

$$\rho_A = e^{-\mu Q_A} \tag{2.29}$$

172 which is still in the subregion  $A$ . The constant  $\mu$  is free. Operators of the form (2.29) is called  
 173 deformed reduced density matrix. Note we use the same symbol  $\rho_A$  to label deformed reduced  
 174 density matrix. Recall that the modular Hamiltonian is a special OPE block, if one replaces the  
 175 OPE block by the modular Hamiltonian (2.29) and set  $\mu = 1$ , the deformed reduced density  
 176 matrix becomes the reduced density matrix exactly. We can relax the definition, namely,  $Q_A$  in  
 177 (2.29) could be a linear superposition of several OPE blocks. Note our definition of deformed  
 178 reduced density matrix is a direction extension of the generalized reduced density matrix in  
 179 the context of the so-called charged Rényi entropy [15]. In that work,  $Q_A$  is a charge which  
 180 is generated by a  $U(1)$  current. The corresponding charged Rényi entropy is holographically  
 181 dual to the thermal entropy of a charged black hole with hyperbolic horizon. However, in our  
 182 definition,  $Q_A$  is just a general OPE block or their linear superposition. As a naive generalization  
 183 of Rényi entanglement entropy, we construct the logarithm of the vacuum expectation value  
 184 of the deformed reduced density matrix,

$$T_A(\mu) = \log\langle\rho_A\rangle = \log\langle e^{-\mu Q_A}\rangle. \tag{2.30}$$

185 When  $Q_A$  is modular Hamiltonian, the above quantity is related to the Rényi entropy for the  
 186 vacuum state.

187 However, a direct computation of  $T_A(\mu)$  is hard in general. A much more severe problem is  
 188 that OPE block has no lower bound in general, therefore the definition is not valid for general  
 189 OPE blocks. To solve this problem, we observe that  $T_A(\mu)$  could be expanded for small  $\mu$ ,

$$T_A(\mu) = \sum_{m=1}^{\infty} \frac{(-\mu)^m}{m!} \langle Q_A^m \rangle_c. \tag{2.31}$$

190 The Taylor expansion coefficient

$$\langle Q_A^m \rangle_c = (-1)^m \frac{\partial^m}{\partial \mu^m} T_A(\mu)|_{\mu \rightarrow 0} \tag{2.32}$$

191 is called Type-(m) connected correlation function (CCF) of the OPE block  $Q_A$ . For each definite  
 192  $m$ , one can always calculate the corresponding CCF without knowing  $T_A(\mu)$ . The first few CCFs  
 193 are

$$\begin{aligned} \langle Q_A^2 \rangle_c &= \langle Q_A^2 \rangle - \langle Q_A \rangle^2, \\ \langle Q_A^3 \rangle_c &= \langle Q_A^3 \rangle - 3\langle Q_A^2 \rangle \langle Q_A \rangle + 2\langle Q_A \rangle^3. \end{aligned} \tag{2.33}$$

194 Using CCF, there is no issue of lower bound of the OPE block. The convergence of the sum-  
 195 mation (2.31) could be a hard problem for general OPE blocks. However, for modular Hamil-  
 196 tonian in two dimensional CFT, one can use the summation to define Rényi entropy. As an  
 197 application of the concept of CCF, we choose the OPE block as the modular Hamiltonian, then  
 198 it is easy to show that CCF of modular Hamiltonian  $H_A$  satisfies area law with logarithmic  
 199 degree  $q = 1$  in even dimensions,

$$\langle H_A^m \rangle_c = \tilde{\gamma} \frac{\mathcal{A}}{\epsilon^{d-2}} + \dots + \tilde{p}_1^{(m)} \log \frac{R}{\epsilon} + \dots, \quad m \geq 1. \tag{2.34}$$

200 The coefficient  $\tilde{p}_1^{(m)}$  is determined from  $p_1(n)$  by

$$\tilde{p}_1^{(m)} = (-1)^m \partial_n^m (1-n)p_1(n)|_{n \rightarrow 1}. \quad (2.35)$$

201 There could be multiple spacelike-separated balls  $A_1, A_2, \dots$ , each region has associate OPE  
 202 block  $Q_{A_i}$ . We insert  $m_i$  OPE blocks into region  $A_i$ , then we can define the corresponding  
 203 type-Y CCF

$$\langle Q_{A_1}^{m_1} Q_{A_2}^{m_2} \dots \rangle_c \quad (2.36)$$

204 where the Young diagram  $Y$  is

$$Y = (m_1, m_2, \dots), \quad m_1 \geq m_2 \geq \dots \geq 1. \quad (2.37)$$

205 The generator of all type-Y CCFs is

$$T_{\cup A_i}(\mu_1, \mu_2, \dots) = \log \frac{\langle e^{-\sum_i \mu_i Q_{A_i}} \rangle}{\prod_i \langle e^{-\mu_i Q_{A_i}} \rangle}. \quad (2.38)$$

206 When there are only two balls  $A$  and  $B$ , the generator is

$$T_{A \cup B}(\mu_1, \mu_2) = \log \frac{\langle e^{-\mu_1 Q_A - \mu_2 Q_B} \rangle}{\langle e^{-\mu_1 Q_A} \rangle \langle e^{-\mu_2 Q_B} \rangle} = \sum_{m_1 \geq 1, m_2 \geq 1} \frac{(-1)^{m_1+m_2} \mu_1^{m_1} \mu_2^{m_2}}{m_1! m_2!} \langle Q_A^{m_1} Q_B^{m_2} \rangle_c. \quad (2.39)$$

207 We parameterize  $A$  and  $B$  as

$$A = \{(0, \vec{x}) | (\vec{x} - \vec{x}_0)^2 \leq 1\}, \quad B = \{(0, \vec{x}) | \vec{x} \leq R'^2\}. \quad (2.40)$$

208 There is only one cross ratio

$$\xi = \frac{4R'}{x_0^2 - (1-R')^2}. \quad (2.41)$$

209 When the two regions  $A$  and  $B$  are spacelike-separated,  $|x_0| > 1 + R'$ , the cross ratio is between  
 210 0 and 1,

$$0 < \xi < 1. \quad (2.42)$$

211 In some cases, it is more convenient to use an equivalent cross ratio

$$\eta = \frac{\xi}{1-\xi} = \frac{4R'}{x_0^2 - (1+R')^2}. \quad (2.43)$$

212 For spacelike-separated regions  $A$  and  $B$ , the range of the cross ratio  $\eta$  is

$$0 < \eta < \infty. \quad (2.44)$$

213 Since the OPE block  $Q_A[\mathcal{O}]$  is invariant under conformal transformation, any type- $(m_1, m_2)$   
 214 CCF should be a function of cross ratio  $\xi$  or  $\eta$ . Actually the OPE block is an eigenvector of the  
 215 conformal Casimir

$$[L^2, Q_A[\mathcal{O}]] = C_{\Delta, J} Q_A[\mathcal{O}] \quad (2.45)$$

216 where  $L^2$  is the Casimir operator of the global conformal group. The eigenvalue  $C_{\Delta, J}$  is

$$C_{\Delta, J} = -\Delta(\Delta - d) - J(J + d - 2). \quad (2.46)$$

217 Therefore, any type- $(m-1, 1)$  CCF should be a conformal block up to a constant <sup>2</sup>

$$\langle Q_A[\mathcal{O}_1] \dots Q_A[\mathcal{O}_{m-1}] Q_B[\mathcal{O}_m] \rangle_c = D^{(d)}[\mathcal{O}_1, \dots, \mathcal{O}_m] G_{\Delta_m, J_m}^{(d)}(\eta). \quad (2.47)$$

<sup>2</sup>See Appendix A of [16] for a detailed discussion. For each spherical space, there is a pair of timelike separated points that live on the tips of its causal diamond [3]. Therefore, for two balls, one can use the two pairs of timelike separated points to define the corresponding cross ratio  $\eta$ .



218 The subscript  $\Delta_m, J_m$  are the conformal weight and spin of the primary operator  $\mathcal{O}_m$ . The  
 219 index ( $d$ ) is used to label the dimension of spacetime. The conformal block  $G_{\Delta_m, J_m}^{(d)}(\eta)$  can be  
 220 constructed explicitly in even dimensions [26, 27]. In this paper, we use the convention that

$$G_{\Delta_m, J_m}^{(d)}(\eta) \rightarrow \eta^{\Delta_m}, \quad \eta \rightarrow 0. \quad (2.48)$$

221 Therefore the overall constant  $D^{(d)}[\mathcal{O}_1, \dots, \mathcal{O}_m]$  is fixed uniquely. When  $A$  and  $B$  are far away  
 222 from each other, the type- $(m-1, 1)$  CCF is dominated by

$$\langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_{m-1}] Q_B[\mathcal{O}_m] \rangle_c \approx D^{(d)}[\mathcal{O}_1, \dots, \mathcal{O}_m] \eta^{\Delta_m}. \quad (2.49)$$

223 For  $m \geq 2$ , the coefficients  $D^{(d)}[\mathcal{O}_1, \dots, \mathcal{O}_m]$  are related to the normalization of the primary  
 224 operators. For any  $m \geq 3$ , it also contains dynamical information of the theory. The explicit  
 225 form of the conformal block can be found in [28]. Any type- $(m_1, m_2)$  CCF with  $m_1 \geq m_2 \geq 2$   
 226 is not a conformal block .

### 227 3 Area law

228 We conjecture that the type- $(m)$  CCF of OPE blocks obeys the following area law

$$\langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_m] \rangle_c = \gamma \frac{R^{d-2}}{\epsilon^{d-2}} + \cdots + p_q \log^q \frac{R}{\epsilon} + \cdots. \quad (3.1)$$

229 The leading term is proportional to the area of the boundary  $\partial A$ . We have restored the radius  
 230  $R$  in the formula to balance the dimension. The small positive constant  $\epsilon$  is the UV cutoff which  
 231 is roughly the distance from the cutoff to the boundary  $\partial A$ . The constant  $\gamma$  depends on the  
 232 choice of the cutoff and the method of regularization, we will not be interested in its explicit  
 233 value. The  $\cdots$  terms are subleading and cutoff dependent. Therefore we omit their forms.  
 234 The degree  $q$  characterizes the maximal power of the logarithmic terms. The coefficient  $p_q$  is  
 235 invariant under the rescaling of the cutoff, therefore it encodes detail universal information of  
 236 the theory<sup>3</sup>. When all the OPE blocks are equal to the modular Hamiltonian, the degree  $q = 1$   
 237 for even dimensions according to (2.34). However, as we will see,  $q$  is not necessarily equal  
 238 to 1 in general. To distinguish different type- $(m)$  CCFs in different dimensions, we write the  
 239 area law (3.1) more explicitly as

$$\langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_m] \rangle_c = \gamma[\mathcal{O}_1, \dots, \mathcal{O}_m] \frac{R^{d-2}}{\epsilon^{d-2}} + \cdots + p_q^{(d)}[\mathcal{O}_1, \dots, \mathcal{O}_m] \log^q \frac{R}{\epsilon} + \cdots. \quad (3.2)$$

#### 240 3.1 Continuation

241 The two formulas (2.47) and (3.2) are actually related to each other through an analytic  
 242 continuation. We use the example of the two dimensional modular Hamiltonian to illustrate  
 243 this relation. For any  $\text{CFT}_2$ , the modular Hamiltonian can be decomposed into the holomorphic  
 244 and anti-holomorphic part, we focus on the holomorphic part

$$H_A = - \int_{-1}^1 dz \frac{1-z^2}{2} T(z+x_0) + c. \quad (3.3)$$

<sup>3</sup>Note the constant  $p_q$  also depends on the operator normalization.

245 The constant  $c$  can be fixed by the normalization condition

$$\text{tr}_A \rho_A = \text{tr}_A e^{-H_A} = 1. \tag{3.4}$$

246 Its value doesn't affect the type-Y CCF with any  $\sum_i m_i \geq 2$ . We also used the convention  
 247  $T(z) = -2\pi T_{zz}$  where the subscript  $z$  is the holomorphic coordinate  $z = t + x$ . The radius of  
 248 the interval  $A$  is 1, we have shifted the variable  $z$  such that the dependence of the center  $x_0$  is  
 249 in the stress tensor. The modular Hamiltonian of region  $B$  can be obtained by setting  $x_0 = 0$   
 250 and restoring the radius  $R'$ . The type- $(m-1, 1)$  CCF of the modular Hamiltonian is

$$\langle H_A^{m-1} H_B \rangle_c = D^{(2)}[T_{\mu_1 \nu_1}, \dots, T_{\mu_m \nu_m}] G_2^{(2)}(\eta). \tag{3.5}$$

251 The two dimensional conformal block for a chiral operator can be labeled by the conformal  
 252 weight  $h$  of the operator

$$G_h^{(2)}(\eta) = (-\eta)^h {}_2F_1(h, h, 2h, -\eta). \tag{3.6}$$

253 We can move the interval  $A$  to  $B$  such that they coincide. In this limit, any type- $(m-1, 1)$  CCF  
 254 should approach a type- $(m)$  CCF. This is equivalent to setting  $\eta \rightarrow -1$ . We can set  $x_0 \rightarrow 0$   
 255 and then take the limit  $R' \rightarrow 1$ ,

$$x_A \rightarrow 0, \quad R' = 1 - \epsilon, \quad \epsilon \rightarrow 0. \tag{3.7}$$

256 The cross ratio  $\xi \rightarrow -\infty$  or  $\eta \rightarrow -1$  by

$$\xi = -\frac{4(1-\epsilon)}{\epsilon^2} \approx -\frac{4}{\epsilon^2}, \quad \eta = -\frac{4(1-\epsilon)}{(2-\epsilon)^2} \approx -1 + \frac{\epsilon^2}{4}. \tag{3.8}$$

257 On the right hand side of (3.5), we find a logarithmically divergent term in this limit

$$G_2^{(2)}(\eta) = 12 \log \frac{2}{\epsilon} + \dots = 12 \log \frac{R}{\epsilon} + \dots \tag{3.9}$$

258 The left hand side of (3.5) approaches type- $(m)$  CCF, therefore

$$\langle H_A^m \rangle_c = 12 D^{(2)}[T_{\mu_1 \nu_1}, \dots, T_{\mu_m \nu_m}] \log \frac{R}{\epsilon} + \dots. \tag{3.10}$$

259 We read out the cutoff independent coefficient

$$p_1^{(2)}[T_{\mu_1 \nu_1}, \dots, T_{\mu_m \nu_m}] = 12 D^{(2)}[T_{\mu_1 \nu_1}, \dots, T_{\mu_m \nu_m}]. \tag{3.11}$$

260 The relation (3.11) is a typical UV/IR relation for the modular Hamiltonian. The left hand side  
 261 is the universal coefficient for  $B$  and  $A$  coinciding (UV). On the right hand side, the  $D$  coefficient  
 262 characterizes the leading order behaviour of CCF when  $B$  and  $A$  are far away from each other  
 263 (IR). They provide equivalent information of the CFT since the constant 12 is completely fixed  
 264 by conformal symmetry. The continuation of the conformal block can be generalized to higher  
 265 dimensions. For example, in four dimensions, the conformal block associated with stress tensor  
 266 becomes divergent as  $A$  approaches  $B$ ,

$$G_{4,2}^{(4)} \approx \tilde{\gamma} \frac{R^2}{\epsilon^2} + \dots - 120 \log \frac{R}{\epsilon} + \dots. \tag{3.12}$$

267 The leading term is exactly proportional to the area of the boundary and the logarithmic di-  
 268 vergent term also appears in the subleading terms. We can read out the type- $(m)$  CCF of the  
 269 modular Hamiltonian in four dimensions

$$\langle H_A^m \rangle_c = \gamma \frac{R^2}{\epsilon^2} + \dots + p_1^{(4)}[T_{\mu_1 \nu_1}, \dots, T_{\mu_m \nu_m}] \log \frac{R}{\epsilon} + \dots \tag{3.13}$$

270 with

$$p_1^{(4)}[T_{\mu_1 \nu_1}, \dots, T_{\mu_m \nu_m}] = -120D^{(4)}[T_{\mu_1 \nu_1}, \dots, T_{\mu_m \nu_m}]. \quad (3.14)$$

271 Note we obtain the area law and the logarithmic behaviour of the type- $(m)$  CCF of the modular  
 272 Hamiltonian without using any knowledge of Rényi entanglement entropy. The method of  
 273 analytic continuation can be applied to general dimensions and OPE blocks. A conformal  
 274 block  $G_{\Delta, J}^{(d)}(\xi)$  obeys area law in the limit  $\xi \rightarrow -\infty$  in even dimensions. It has degree  $q = 1$   
 275 only for  $\Delta = J + d - 2$ ,

$$G_{\Delta, J}^{(d)}(\xi) = \tilde{\gamma} \frac{R^{d-2}}{\epsilon^{d-2}} + \dots + E^{(d)}[\Delta, J] \log \frac{R}{\epsilon} + \dots, \quad \xi \rightarrow -\infty. \quad (3.15)$$

276 This means that type- $(m)$  CCF of type- $J$  OPE blocks may always obey area law with degree  
 277  $q = 1$ , the cutoff independent coefficient is

$$p_q^{(d)}[\mathcal{O}_1, \dots, \mathcal{O}_m] = E^{(d)}[\mathcal{O}_m] \times D^{(d)}[\mathcal{O}_1, \dots, \mathcal{O}_m]. \quad (3.16)$$

278 We have replaced the quantum numbers in E function by the corresponding primary opera-  
 279 tor. For non-conserved operators, the conformal block  $G_{\Delta, J}^{(d)}$  also obeys area law in the limit  
 280  $\xi \rightarrow -\infty$  in even dimension, though it has degree  $q = 2$

$$G_{\Delta, J}^{(d)}(\xi) = \tilde{\gamma} \frac{R^{d-2}}{\epsilon^{d-2}} + \dots + E^{(d)}[\Delta, J] \log^2 \frac{R}{\epsilon} + \dots, \quad \xi \rightarrow -\infty. \quad (3.17)$$

281 Therefore, type- $(m)$  CCF of type- $O$  OPE blocks obeys area law with degree  $q = 2$ . We can  
 282 obtain similar UV/IR relations as (3.16). In odd dimensions, the story is the same. The degree  
 283  $q$  is 0 for type- $(m)$  CCF of type- $J$  OPE blocks and 1 for type- $O$  OPE blocks.

### 284 3.2 Kinematic information

285 The function  $E^{(d)}[\mathcal{O}]$  is completely fixed by conformal symmetry. It can be obtained by reading  
 286 out the coefficient of the logarithmic term with degree  $q$ . For each fixed quantum number  $\Delta$   
 287 and  $J$ , there is a unique number  $E^{(d)}[\mathcal{O}]$ . For any type- $J$  OPE block in two dimensions, the  
 288 primary operator  $\mathcal{O}$  has dimension  $\Delta = J = h$ . The conformal block (3.6) has degree  $q = 1$  in  
 289 the limit  $\eta \rightarrow -1$ . The function  $E^{(2)}[\mathcal{O}]$  is

$$E^{(2)}[\mathcal{O}] = \frac{2\Gamma(2h)}{\Gamma(h)^2}, \quad \Delta = J = h. \quad (3.18)$$

290 For type- $O$  OPE block, the primary operator  $\mathcal{O}$  has dimension  $\Delta = h + \bar{h}$  and spin  $J = h - \bar{h}$ .  
 291 The conformal block has degree  $q = 2$  in the limit  $\eta \rightarrow -1$ . The function  $E^{(2)}[\mathcal{O}]$  is

$$E^{(2)}[\mathcal{O}] = \begin{cases} \frac{2^{4h}\Gamma(h+\frac{1}{2})^2}{\pi\Gamma(h)^2} & J = 0, h > 0 \\ -\frac{4^{2h-1}\Gamma(h-\frac{1}{2})\Gamma(h+\frac{1}{2})}{\pi\Gamma(h-1)\Gamma(h)} & J = 1, h > 1 \\ \frac{4^{2h-3}(h-2)(h-1)(2h-3)(2h-1)\Gamma(h-\frac{3}{2})^2}{\pi\Gamma(h)^2} & J = 2, h > 2 \\ \dots & \dots \end{cases} \quad (3.19)$$

292 In four dimensions, we also find

$$E^{(4)}[\mathcal{O}] = \begin{cases} 12 & \Delta = 3, J = 1 \\ -120 & \Delta = 4, J = 2 \\ 840 & \Delta = 5, J = 3 \\ \dots & \dots \end{cases} \quad (3.20)$$

293 for conserved currents and

$$E^{(4)}[\mathcal{O}] = \begin{cases} -\frac{2^{2\Delta-1}\Gamma(\frac{\Delta-1}{2})\Gamma(\frac{\Delta+1}{2})}{\pi\Gamma(\frac{\Delta-2}{2})^2} & \Delta > 1, J = 0, \\ \frac{2^{2\Delta-1}\Gamma(\frac{\Delta}{2})\Gamma(\frac{\Delta+2}{2})}{\pi\Gamma(\frac{\Delta-3}{2})\Gamma(\frac{\Delta+1}{2})} & \Delta > 3, J = 1, \\ -\frac{4^{\Delta-1}(\Delta-2)\Gamma(\frac{\Delta-3}{2})\Gamma(\frac{\Delta+3}{2})}{\pi\Gamma(\frac{\Delta-4}{2})\Gamma(\frac{\Delta+2}{2})} & \Delta > 4, J = 2, \\ \dots & \dots \end{cases} \quad (3.21)$$

294 for non-conserved operators. In three dimensions, we find

$$E^{(3)}[\mathcal{O}] = \begin{cases} -\frac{2^{2\Delta-1}(\Delta-1)\Gamma(\Delta-\frac{1}{2})}{\sqrt{\pi}\Gamma(\Delta-1)} & \Delta > \frac{1}{2}, J = 0. \\ \frac{2^{\Delta+1}\Delta\Gamma(\Delta-\frac{1}{2})}{\Gamma(\frac{\Delta-2}{2})\Gamma(\frac{\Delta+1}{2})} & \Delta > 2, J = 1, \\ -\frac{2^{2\Delta-1}(\Delta^2-1)\Gamma(\Delta-\frac{1}{2})}{\sqrt{\pi}(\Delta-2)^2\Delta\Gamma(\Delta-3)} & \Delta > 3, J = 2, \\ \dots & \dots \end{cases} \quad (3.22)$$

295 for non-conserved operators. Note for conserved currents in odd dimensions, the function  
 296  $E^{(3)}[\mathcal{O}]$  may depend on explicit choice of the cutoff. For example, a transformation  $\epsilon \rightarrow \epsilon(1+a\epsilon)$   
 297 may shift its value. This is because the degree is 0, there is no logarithmic divergence at all.

### 298 3.3 UV/IR relation

299 The UV/IR relation (3.16) relates type-( $m$ ) CCF to type-( $m - 1, 1$ ) CCF. This relation may  
 300 simplify computation in many cases. To see this point, let's compute the following type-(2)  
 301 CCF in two dimensions

$$\begin{aligned} \langle Q_A[\mathcal{O}]^2 \rangle_c &= \int_{-1}^1 dz_1 \int_{-1}^1 dz_2 \frac{(1-z_1^2)^{h-1}(1-z_2^2)^{h-1}}{(z_1-z_2)^{2h}} \\ &= \frac{(-1)^{-h}\sqrt{\pi}\Gamma(h)}{\Gamma(h+\frac{1}{2})} \int_{-1}^1 dz_1 \frac{1}{1-z_1^2} \\ &= \frac{(-1)^{-h}\sqrt{\pi}\Gamma(h)}{\Gamma(h+\frac{1}{2})} \log \frac{2}{\epsilon}. \end{aligned} \quad (3.23)$$

302 This is a double integral with poles at  $z_1 = z_2$ . We regularize the integral by ignoring these  
 303 poles at the second step. At the last step, we insert a UV cutoff to regularize the integral.  
 304 However, using UV/IR relation, one just need to fix the coefficient  $D$  which is related to the  
 305 large distance behaviour of the type-(1, 1) CCF,

$$\langle Q_A[\mathcal{O}]Q_B[\mathcal{O}] \rangle_c = \int_{-1}^1 dz_1 \int_{-1}^1 dz_2 \frac{(1-z_1^2)^{h-1}(1-z_2^2)^{h-1}}{(z_1-z_2+x_0)^{2h}}. \quad (3.24)$$

306 In the large distance limit,  $x_0 \rightarrow \infty$ , the integral becomes simpler

$$\begin{aligned} \langle Q_A[\mathcal{O}]Q_B[\mathcal{O}] \rangle_c &\approx \int_{-1}^1 dz_1 \int_{-1}^1 dz_2 \frac{(1-z_1^2)^{h-1}(1-z_2^2)^{h-1}}{x_0^{2h}} \\ &= 4^{-h} \left( \frac{\sqrt{\pi}\Gamma(h)}{\Gamma(h+\frac{1}{2})} \right)^2 \eta^h. \end{aligned} \quad (3.25)$$

307 We have used the relation  $\eta \approx \frac{4}{x^2}$  in the large distance limit. Then we can read out

$$D^{(2)}[\mathcal{O}, \mathcal{O}] = (-1)^{-h} 4^{-h} \left( \frac{\sqrt{\pi} \Gamma(h)}{\Gamma(h + \frac{1}{2})} \right)^2. \quad (3.26)$$

308 Combining UV/IR relation and (3.18), we find

$$p_1^{(2)}[\mathcal{O}, \mathcal{O}] = E^{(2)}[\mathcal{O}] \times D^{(2)}[\mathcal{O}, \mathcal{O}] = \frac{(-1)^{-h} \sqrt{\pi} \Gamma(h)}{\Gamma(h + \frac{1}{2})}. \quad (3.27)$$

309 The result is exactly the same as (3.23). We use the UV/IR relation to obtain type-(3) CCF for  
310 type-J OPE blocks in two dimensions, the cutoff independent coefficient is

$$p_1^{(2)}[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3] = \frac{C_{123} \pi^{3/2} (-1)^{\frac{h_1+h_2+h_3}{2}} \Gamma(h_1) \Gamma(h_2) \Gamma(h_3) \kappa}{\Gamma(\frac{1+h_1+h_2-h_3}{2}) \Gamma(\frac{1+h_1+h_3-h_2}{2}) \Gamma(\frac{1+h_2+h_3-h_1}{2}) \Gamma(\frac{h_1+h_2+h_3}{2})}, \quad (3.28)$$

311 where the constant  $\kappa = \frac{1}{2} [1 + (-1)^{h_1+h_2+h_3}]$ . We notice that the result is totally symmetric  
312 under the exchange of any two conformal weights. This is a cyclic identity for  $m = 3$

$$p_q^{(d)}[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3] = p_q^{(d)}[\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_1] = p_q^{(d)}[\mathcal{O}_3, \mathcal{O}_1, \mathcal{O}_2]. \quad (3.29)$$

313 Note that the cyclic identity cannot be assumed to be a priori since we are dealing with the  
314 limits of rather different quantities. However, interestingly, the UV/IR relation and the cyclic  
315 identity could be checked for all the examples in the following. For four dimensional type-( $m$ )  
316 CCF ( $m=2,3$ ), we list the cutoff independent coefficients below [17].

317 • Type-(2). The normalization constants are set to 1.

318 – Spin 1-1 conserved currents.

$$p_1^{(4)}[\mathcal{J}_\mu, \mathcal{J}_\nu] = -\frac{\pi^2}{3}. \quad (3.30)$$

319 – Spin 2-2 conserved currents.

$$p_1^{(4)}[T_{\mu\nu}, T_{\rho\sigma}] = -\frac{\pi^2}{40}. \quad (3.31)$$

320 – Spin 0-0 non-conserved operators.

$$p_2^{(4)}[\mathcal{O}, \mathcal{O}] = -\frac{4\pi^2(\Delta - 1)\Gamma(\Delta - 2)^2\Gamma(\frac{\Delta}{2})^4}{\Gamma(\Delta)^2\Gamma(\Delta - 1)^2}. \quad (3.32)$$

321 – Spin 1-1 non-conserved operators.

$$p_2^{(4)}[\mathcal{O}_\mu, \mathcal{O}_\nu] = -\frac{4^{1-\Delta} \pi^3 \Delta \Gamma(\frac{\Delta-3}{2}) \Gamma(\frac{\Delta+1}{2})}{\Gamma(\frac{\Delta}{2} + 1)^2}, \quad \Delta > 3. \quad (3.33)$$

322 – Spin 2-2 non-conserved operators.

$$p_2^{(4)}[\mathcal{O}_{\mu\nu}, \mathcal{O}_{\rho\sigma}] = -\frac{3\pi^2(\Delta - 2)\Delta^2\Gamma(\frac{\Delta}{2} - 2)^2\Gamma(\frac{\Delta}{2} - 1)^2}{64\Gamma(\Delta - 4)\Gamma(\Delta + 2)}, \quad \Delta > 4. \quad (3.34)$$

323 • Type-(3).

324 – Spin 1-1-2 conserved currents. The three point function of zero components are  
 325 fixed by conformal symmetry

$$\langle T_{00}(x_1)\mathcal{J}_0(x_2)\mathcal{J}_0(x_3)\rangle_c = \frac{C_{T\mathcal{J}\mathcal{J}}}{x_{12}^4 x_{13}^2 x_{23}^2}. \quad (3.35)$$

326 Then the coefficient

$$p_1^{(4)}[\mathcal{J}_\mu, \mathcal{J}_\nu, T_{\rho\sigma}] = -\frac{\pi^3}{2} C_{T\mathcal{J}\mathcal{J}}. \quad (3.36)$$

327 – Spin 2-2-2 conserved currents. The three point function of zero components are  
 328 fixed by conformal symmetry

$$\langle T_{00}(x_1)T_{00}(x_2)T_{00}(x_3)\rangle_c = \frac{C_{TTT}}{x_{12}^4 x_{13}^4 x_{23}^4}. \quad (3.37)$$

329 Then the coefficient

$$p_1^{(4)}[T_{\mu\nu}, T_{\rho\sigma}, T_{\alpha\beta}] = \frac{\pi^3}{12} C_{TTT}. \quad (3.38)$$

330 – Spin 0-0-0 non-conserved currents.

$$\begin{aligned} & p_2^{(4)}[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3] \\ &= -2^{4-\Delta_1-\Delta_2-\Delta_3} \pi^3 C_{123} \int_{\mathbb{D}^2} d\zeta d\bar{\zeta} (\zeta + \bar{\zeta})^2 \int_{\mathbb{D}^2} d\zeta' d\bar{\zeta}' (\zeta' + \bar{\zeta}')^2 \\ & \quad \times (1 - \zeta^2)^{\frac{\Delta_1-4}{2}} (1 - \bar{\zeta}^2)^{\frac{\Delta_1-4}{2}} (1 - \zeta'^2)^{\frac{\Delta_2-4}{2}} (1 - \bar{\zeta}'^2)^{\frac{\Delta_2-4}{2}} \\ & \quad \times \int_0^\pi d\theta \frac{\sin \theta}{(a + b \cos \theta)^{\frac{\Delta_{12,3}}{2}}}. \end{aligned} \quad (3.39)$$

331 Though the expression (3.39) is not symmetric superficially under the exchange of any two  
 332 conformal weights, we checked explicitly that it satisfies the cyclic identity for integer con-  
 333 formal weights. There could be singularities when  $\zeta, \bar{\zeta}, \zeta', \bar{\zeta}'$  are close to the boundary  $-1$   
 334 and  $1$ , we can deal with these singularities for integer conformal weights explicitly. We don't  
 335 find a straightforward way to regularize the integral for non-integer conformal weights. It is  
 336 interesting to find an unambiguous way to define  $p_q$  for general operators.

337 For  $m = 4$ , the cyclic identities are much more harder to check. We considered type-(4)  
 338 CCF for massless free scalar theory [13, 14]. In this theory, one can construct an infinite  
 339 tower of conserved currents with even spin [29]. The four point functions can be calculated  
 340 explicitly. Therefore we can find type-(3, 1) and type-(4) CCFs and read out the corresponding  
 341 coefficients. For example, for spin-2-2-2-4 conserved currents [14],

$$D[2, 2, 2, 4] = \frac{3}{70} D[2, 2, 4, 2]. \quad (3.40)$$

342 Both of them lead to the cutoff independent coefficients

$$p_1^{(2)}[2, 2, 2, 4] = \frac{2\Gamma(8)}{\Gamma(4)^2} D[2, 2, 2, 4] = \frac{2\Gamma(4)}{\Gamma(2)^2} D[2, 2, 4, 2] = p_1^{(2)}[2, 2, 4, 2]. \quad (3.41)$$

343 The cyclic identity is obeyed.

344 **3.4 Discussion**

345 The UV/IR relation should be slightly modified when the CCF contains both type-J and type-O  
 346 OPE blocks. One simple example is the following type-(3) CCF

$$\langle Q_A[\mathcal{J}]Q_A[\mathcal{O}]Q_A[\tilde{\mathcal{O}}] \rangle_c \tag{3.42}$$

347 where  $Q_A[\mathcal{J}]$  is a type-J OPE block while  $Q_A[\mathcal{O}]$  and  $Q_A[\tilde{\mathcal{O}}]$  are type-O OPE blocks. This CCF  
 348 is related to the following two type-(2, 1) CCFs

$$\langle Q_A[\tilde{\mathcal{O}}]Q_A[\mathcal{J}]Q_B[\mathcal{O}] \rangle_c = D^{(d)}[\tilde{\mathcal{O}}, \mathcal{J}, \mathcal{O}]G_{\Delta, J}^{(d)}(\xi), \tag{3.43}$$

$$\langle Q_A[\mathcal{O}]Q_A[\tilde{\mathcal{O}}]Q_B[\mathcal{J}] \rangle_c = D^{(d)}[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}]G_{\Delta', J'}^{(d)}(\xi). \tag{3.44}$$

349 We choose  $d = 4$ . Taking the limit  $A \rightarrow B$  from (3.43), we find a type-(3) CCF with degree  
 350  $q = 2$ , the UV/IR relation reads

$$p_2^{(4)}[\tilde{\mathcal{O}}, \mathcal{J}, \mathcal{O}] = E^{(4)}[\mathcal{O}] \times D^{(4)}[\tilde{\mathcal{O}}, \mathcal{J}, \mathcal{O}] \tag{3.45}$$

351 We can also take the limit  $A \rightarrow B$  from (3.44), then we will find a type-(3) CCF with degree  
 352  $q = 1$ , the UV/IR relation reads

$$p_1^{(4)}[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}] = E^{(4)}[\mathcal{J}] \times D^{(4)}[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}]. \tag{3.46}$$

353 The equations (3.45) and (3.46) are not identical superficially since the subscript  $q$  are not  
 354 equal to each other. However, an explicit calculation for spin 2-0-0 and spin 2-2-0 in four  
 355 dimensions [17] shows that the coefficient  $D^{(4)}[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}]$  is actually divergent logarithmically,  
 356

$$D^{(4)}[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}] = D_{\log}^{(4)}[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}] \log \frac{R}{\epsilon} + \dots \tag{3.47}$$

357 The terms in  $\dots$  are finite and depends on cutoff scale. Due to the logarithmic divergence  
 358 behaviour of the coefficient  $D^{(4)}[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}]$ , the degree of type-(3) CCF from (3.44) increases  
 359 by 1, the modified UV/IR relation becomes

$$p_2^{(4)}[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}] = E^{(4)}[\mathcal{J}] \times D_{\log}^{(4)}[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}]. \tag{3.48}$$

360 We checked explicitly that the two constants (3.45) and (3.48) are equal to each other. The  
 361 cyclic identity is still satisfied after counting the logarithmic divergence of the  $D$  function.

362 **4 Generalizations**

363 The area law and logarithmic behaviour in the subleading terms can be extended in different  
 364 directions. In this section, we mention several extensions.

- 365 • UV/IR relation. In general, one can uplift any type-( $m$ ) CCF to a type-( $p, m - p$ ) CCF

$$\langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_m] \rangle_c \xrightarrow{\text{uplift}} \langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_p] Q_B[\mathcal{O}_{p+1}] \cdots Q_B[\mathcal{O}_m] \rangle_c, \quad 1 \leq p \leq m-1. \tag{4.1}$$

366 When  $p$  is not 1 or  $m - 1$ , the type-( $p, m - p$ ) CCF is not a conformal block. It is still  
 367 a function of cross ratio  $\xi$ , therefore it should reproduce the type-( $m$ ) CCF after taking  
 368 the limit  $A \rightarrow B$ ,

$$\langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_m] \rangle_c = \lim_{\xi \rightarrow -\infty} \langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_p] Q_B[\mathcal{O}_{p+1}] \cdots Q_B[\mathcal{O}_m] \rangle_c. \tag{4.2}$$

369 Obviously, this also defines a UV/IR relation between  $p_q^{(d)}$  and several coefficients in  
 370 the type- $(p, m - p)$  CCF. Since the right hand side is not proportional to any conformal  
 371 block, it is not easy to write out an explicit formula. Nevertheless, one may still check  
 372 the relation (4.2) case by case. One example is to consider the type-(2, 2) CCF of the  
 373 modular Hamiltonian in  $CFT_2$ . By making use of the universal feature of the CCF of the  
 374 stress tensor, one can fix the generator of type- $(m_1, m_2)$  CCFs [14]

$$T_{A \cup B}(\mu_1, \mu_2) = -\frac{c}{2} \text{tr} \log \left[ 1 - \begin{pmatrix} A & C \\ D & B \end{pmatrix} \right], \quad (4.3)$$

375 where the matrices  $A, B, C$  and  $D$  are

$$A_{xx'} = \frac{\eta^2}{4} \int_0^\infty dy \frac{\sqrt{xx'} y \sinh \pi \mu_1 x \sinh \pi \mu_2 y}{\sinh \pi x' \sinh \pi y \sinh \pi(1 + \mu_1)x \sinh \pi(1 + \mu_2)y} \left(\frac{x_{13}}{x_{23}}\right)^{i(x-x')} \mathcal{F}(x, x', y), \quad (4.4)$$

$$B_{xx'} = \frac{\eta^2}{4} \int_0^\infty dy \frac{\sqrt{xx'} y \sinh \pi \mu_1 x \sinh \pi \mu_2 y}{\sinh \pi x' \sinh \pi y \sinh \pi(1 + \mu_1)x \sinh \pi(1 + \mu_2)y} \left(\frac{x_{13}}{x_{23}}\right)^{-i(x-x')} \mathcal{F}(x', x, y), \quad (4.5)$$

$$C_{xx'} = \frac{\eta^2}{4} \int_0^\infty dy \frac{\sqrt{xx'} y \sinh \pi \mu_1 x \sinh \pi \mu_2 y}{\sinh \pi x' \sinh \pi y \sinh \pi(1 + \mu_1)x \sinh \pi(1 + \mu_2)y} \left(\frac{x_{13}}{x_{23}}\right)^{i(x+x')} \mathcal{F}(x, -x', y) \quad (4.6)$$

$$D_{xx'} = \frac{\eta^2}{4} \int_0^\infty dy \frac{\sqrt{xx'} y \sinh \pi \mu_1 x \sinh \pi \mu_2 y}{\sinh \pi x' \sinh \pi y \sinh \pi(1 + \mu_1)x \sinh \pi(1 + \mu_2)y} \left(\frac{x_{13}}{x_{23}}\right)^{-i(x+x')} \mathcal{F}(-x, x', y) \quad (4.7)$$

376 with

$$\begin{aligned} \mathcal{F}(x, x', y) = & {}_2F_1(1 + ix, 1 - iy, 2, -\eta) {}_2F_1(1 - ix', 1 + iy, 2, -\eta) \\ & + {}_2F_1(1 + ix, 1 + iy, 2, -\eta) {}_2F_1(1 - ix', 1 - iy, 2, -\eta). \end{aligned} \quad (4.8)$$

377  $\mathcal{F}$  and its complex conjugate obey

$$\mathcal{F}^*(x, x', y) = \mathcal{F}(x', x, y), \quad \mathcal{F}^*(-x, -x', y) = \mathcal{F}(x, x', y). \quad (4.9)$$

378 so

$$A = B^*, \quad C = D^*. \quad (4.10)$$

379 We read out the first few CCFs

$$\begin{aligned} \langle H_A^m \rangle_c &= \frac{cm!}{12} \log \frac{2}{\epsilon}, \\ \langle H_A^{m-1} H_B \rangle_c &= \frac{cm!}{144} G_2^{(2)}(\eta), \\ \langle H_A^2 H_B^2 \rangle_c &= c \left\{ \frac{1 + \eta}{\eta^2} [4\text{Li}_3(1 + \eta) - 2 \log(1 + \eta) \text{Li}_2(1 + \eta) + \frac{2 \log(1 + \eta)}{3} \text{Li}_2(-\eta) - 4\zeta(3) \right. \\ &\quad \left. + \frac{1 + \eta}{3} \log^2(1 + \eta) - \frac{\pi^2}{3} \log(1 + \eta) \right] + \frac{2 + \eta}{3\eta} [2\text{Li}_2(-\eta) + 3 \log(1 + \eta)] - \frac{4}{3} \right\} \end{aligned} \quad (4.11)$$

380 where the polylogarithm  $\text{Li}_n(z)$  is

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}. \quad (4.12)$$

381 The relation (4.2) can be checked for  $p = 2, m = 4$ . The right hand side is

$$\lim_{\eta \rightarrow -1} \langle H_A^2 H_B^2 \rangle_c = 2c \log \frac{2}{\epsilon} + \dots. \quad (4.13)$$

382 The cutoff independent coefficient  $2c$  matches with the one in  $\langle H_A^4 \rangle_c$ .



- New power law. In the previous discussion, we focus on the case that  $B$  and  $A$  coincide with each other. However, there are other cases that the CCFs are still divergent. One can consider the limit that  $A$  just attaches the edge of  $B$ ,

$$R' = 1, \quad x_0 = 2 + \epsilon, \quad \epsilon \rightarrow 0. \tag{4.14}$$

The cross ratio  $\xi$  does not approach  $-\infty$  but 1

$$\xi = \frac{4}{(2 + \epsilon)^2} = 1 - \epsilon + \dots \tag{4.15}$$

We can define a new CCF which is also divergent from type- $(m - 1, 1)$  CCF

$$\langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_{m-1}] \odot Q_B[\mathcal{O}_m] \rangle_c = \lim_{\xi \rightarrow 1} \langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_{m-1}] Q_B[\mathcal{O}_m] \rangle_c \tag{4.16}$$

The continuation of conformal block tells us that the new CCF obeys a new power law

$$\langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_{m-1}] \odot Q_B[\mathcal{O}_m] \rangle_c = \tilde{\gamma} \left(\frac{R}{\epsilon}\right)^{\frac{d-2}{2}} + \dots + \bar{p}_q^{(d)} \log^q \frac{R}{\epsilon} + \dots \tag{4.17}$$

The leading term is proportional to

$$\mathcal{L} = R^{\frac{d-2}{2}} = \sqrt{A} \tag{4.18}$$

which is the characteristic length of the region  $A$  in four dimensions. In two dimensions, the leading term is a logarithmic term with power  $q$ . In this case, there is a new UV/IR relation between  $\bar{p}_q$  and  $D$  coefficient, we write it schematically

$$\bar{p}_q = \bar{E} \times D. \tag{4.19}$$

The function  $\bar{E}^{(d)}[\mathcal{O}]$  is proportional to  $E^{(d)}[\mathcal{O}]$ . The proportional constant is shown below.

–  $d$  is even.

- \* For conserved current  $\mathcal{O}$  with conformal weight  $\Delta = J + d - 2$ ,

$$\bar{E}^{(d)}[\mathcal{O}] = \frac{(-1)^J}{2} E^{(d)}[\mathcal{O}]. \tag{4.20}$$

- \* For non-conserved current  $\mathcal{O}$  with conformal weight  $\Delta$  and spin  $J$ ,

$$\bar{E}^{(d)}[\mathcal{O}] = \frac{(-1)^J}{4} E^{(d)}[\mathcal{O}]. \tag{4.21}$$

We checked the relation for  $d = 2, 4$  and spin  $J \leq 2$ .

–  $d$  is odd.

- \* For non-conserved current  $\mathcal{O}$  with conformal weight  $\Delta$  and spin  $J$ ,

$$\bar{E}^{(d)}[\mathcal{O}] = \frac{(-1)^J}{2} E^{(d)}[\mathcal{O}]. \tag{4.22}$$

- \* For conserved current  $\mathcal{O}$ , there is no logarithmic divergent term in the CCF.

We checked the relation for  $d = 3$  and spin  $J \leq 2$ .

Since  $D$  function is the same, we find a relation between two cutoff independent coefficients  $p$  and  $\bar{p}$ ,

$$\frac{p}{E} = \frac{\bar{p}}{\bar{E}}. \tag{4.23}$$

## 405 5 Summary and outlook

406 In this report, we have introduced the area law (3.1) of type- $(m)$  CCFs of OPE blocks. It is a  
 407 generalization of the area law of entanglement entropy. We will list several open problems for  
 408 future work.

- 409 • Higher  $m \geq 4$ . In most of the work, we consider type-(2) and type-(3) CCFs. This  
 410 is because the structure of  $m$ -point correlation function of primary operators in CFT is  
 411 fixed up to  $m = 3$ . For  $m = 4$ , we can also extract cutoff independent information for  
 412 two dimensional massless free scalar theory [16].

- 413 • UV/IR relation. The UV/IR relation

$$p = E \times D \quad (5.1)$$

414 has been checked for several examples. A rigorous proof is still lacking.

- 415 • Cyclic identity. The cyclic identity of  $p$  reflects the fact that  $p$  is independent of the way  
 416 to regularize the type- $(m)$  CCF. However, we feel that a direct computation is impossible  
 417 to check this identity.

- 418 • New power law. We generalize the type- $(m_1, m_2)$  CCF to the case that  $A$  and  $B$  just attach  
 419 to each other. The corresponding CCF is divergent with a new power law (4.17). The  
 420 corresponding new UV/IR relation

$$\bar{p} = \bar{E} \times D \quad (5.2)$$

421 also needs understanding.

- 422 • Deformed reduced density matrix. This exponential operator is similar to the “Wilson  
 423 loop” in gauge theories [30, 31] despite the fact that the OPE block has no lower bound  
 424 in general. When the OPE block has a lower bound, the logarithm of the vacuum expec-  
 425 tation value of the deformed reduced density matrix

$$\log \langle e^{-\mu Q_A} \rangle \quad (5.3)$$

426 should also obey area law with logarithmic divergence. There may be a gravitational  
 427 dual for this quantity as [32, 33]. The similarity of the area law between this program  
 428 and black hole entropy implies that the classical part contributes to the area term while  
 429 quantum effects lead to logarithmic corrections.

- 430 • Multiple integrals. According to the method of continuation of conformal block, area  
 431 law of type- $(m)$  CCF is protected by conformal invariance. However, the method of  
 432 continuation itself cannot guarantee that it always leads to the correct result. One has  
 433 to develop other methods to deal with the multiple integrals. In two dimensions, one  
 434 should generalize Selberg integrals [34, 35] to include more parameters [16].

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437 **References**

- 438 [1] J. D. Bekenstein, “Black holes and entropy,” *Phys. Rev. D* **7** (1973), 2333-2346  
439 doi:10.1103/PhysRevD.7.2333
- 440 [2] S. W. Hawking, “Particle Creation by Black Holes,” *Commun. Math. Phys.* **43** (1975),  
441 199-220 doi:10.1007/BF02345020
- 442 [3] B. Czech, L. Lamprou, S. McCandlish, B. Mosk and J. Sully, “A Stereoscopic Look into  
443 the Bulk,” *JHEP* **1607**, 129 (2016) doi:10.1007/JHEP07(2016)129 [arXiv:1604.03110  
444 [hep-th]].
- 445 [4] J. de Boer, F. M. Haehl, M. P. Heller and R. C. Myers, “Entanglement, hologra-  
446 phy and causal diamonds,” *JHEP* **1608**, 162 (2016) doi:10.1007/JHEP08(2016)162  
447 [arXiv:1606.03307 [hep-th]].
- 448 [5] S. Ferrara, A. F. Grillo and R. Gatto, “Manifestly conformal covariant operator-product  
449 expansion,” *Lett. Nuovo Cim.* **2S2**, 1363 (1971) [*Lett. Nuovo Cim.* **2**, 1363 (1971)].  
450 doi:10.1007/BF02770435
- 451 [6] S. Ferrara, A. F. Grillo and R. Gatto, “Manifestly conformal-covariant expansion on the  
452 light cone,” *Phys. Rev. D* **5**, 3102 (1972). doi:10.1103/PhysRevD.5.3102
- 453 [7] R. Haag, “Local quantum physics: fields, particles, algebras ,” Springer, Berlin, Germany  
454 (1992).
- 455 [8] L. Bombelli, R. K. Koul, J. Lee and R. D. Sorkin, “A Quantum Source of Entropy for Black  
456 Holes,” *Phys. Rev. D* **34**, 373 (1986).
- 457 [9] M. Srednicki, “Entropy and area,” *Phys. Rev. Lett.* **71**, 666 (1993)  
458 doi:10.1103/PhysRevLett.71.666 [hep-th/9303048].
- 459 [10] C. G. Callan, Jr. and F. Wilczek, “On geometric entropy,” *Phys. Lett. B* **333**, 55 (1994)  
460 doi:10.1016/0370-2693(94)91007-3 [hep-th/9401072].
- 461 [11] H. Araki, “Relative Entropy of States of Von Neumann Algebras,” *Publ. Res. Inst. Math.*  
462 *Sci. Kyoto* **1976**, 809 (1976).
- 463 [12] S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from  
464 AdS/CFT,” *Phys. Rev. Lett.* **96**, 181602 (2006) doi:10.1103/PhysRevLett.96.181602  
465 [hep-th/0603001].
- 466 [13] J. Long, “Correlation function of modular Hamiltonians,” *JHEP* **11** (2019), 163  
467 doi:10.1007/JHEP11(2019)163 [arXiv:1907.00646 [hep-th]].
- 468 [14] J. Long, “Correlation function with the insertion of zero modes of modular Hamilto-  
469 nians,” *JHEP* **01** (2020), 173 doi:10.1007/JHEP01(2020)173 [arXiv:1911.11487 [hep-  
470 th]].
- 471 [15] A. Belin, L. Y. Hung, A. Maloney, S. Matsuura, R. C. Myers and T. Sierens, “Holographic  
472 Charged Renyi Entropies,” *JHEP* **12** (2013), 059 doi:10.1007/JHEP12(2013)059  
473 [arXiv:1310.4180 [hep-th]].
- 474 [16] J. Long, “Logarithmic behaviour of connected correlation function in CFT,”  
475 [arXiv:2001.05129 [hep-th]].

- 476 [17] J. Long, “Area law of connected correlation function in higher dimensional conformal  
477 field theory,” [arXiv:2007.15380 [hep-th]].
- 478 [18] S. N. Solodukhin, “The Conical singularity and quantum corrections to entropy of  
479 black hole,” Phys. Rev. D **51** (1995), 609-617 doi:10.1103/PhysRevD.51.609 [arXiv:hep-  
480 th/9407001 [hep-th]].
- 481 [19] S. N. Solodukhin, “On ‘Nongeometric’ contribution to the entropy of black hole due to  
482 quantum corrections,” Phys. Rev. D **51** (1995), 618-621 doi:10.1103/PhysRevD.51.618  
483 [arXiv:hep-th/9408068 [hep-th]].
- 484 [20] R. K. Kaul and P. Majumdar, “Logarithmic correction to the Bekenstein-Hawking entropy,”  
485 Phys. Rev. Lett. **84** (2000), 5255-5257 doi:10.1103/PhysRevLett.84.5255 [arXiv:gr-  
486 qc/0002040 [gr-qc]].
- 487 [21] S. Carlip, “Logarithmic corrections to black hole entropy from the Cardy formula,” Class.  
488 Quant. Grav. **17** (2000), 4175-4186 doi:10.1088/0264-9381/17/20/302 [arXiv:gr-  
489 qc/0005017 [gr-qc]].
- 490 [22] T. R. Govindarajan, R. K. Kaul and V. Suneeta, “Logarithmic correction to the Bekenstein-  
491 Hawking entropy of the BTZ black hole,” Class. Quant. Grav. **18** (2001), 2877-2886  
492 doi:10.1088/0264-9381/18/15/303 [arXiv:gr-qc/0104010 [gr-qc]].
- 493 [23] A. Sen, “Logarithmic Corrections to Schwarzschild and Other Non-extremal Black Hole  
494 Entropy in Different Dimensions,” JHEP **04** (2013), 156 doi:10.1007/JHEP04(2013)156  
495 [arXiv:1205.0971 [hep-th]].
- 496 [24] S. Minwalla, “Restrictions imposed by superconformal invariance on quantum field the-  
497 ories,” Adv. Theor. Math. Phys. **2**, 783 (1998) doi:10.4310/ATMP.1998.v2.n4.a4 [hep-  
498 th/9712074].
- 499 [25] H. Casini, M. Huerta and R. C. Myers, “Towards a derivation of holographic entanglement  
500 entropy,” JHEP **1105**, 036 (2011) doi:10.1007/JHEP05(2011)036 [arXiv:1102.0440  
501 [hep-th]].
- 502 [26] F. A. Dolan and H. Osborn, “Conformal four point functions and the operator product  
503 expansion,” Nucl. Phys. B **599**, 459 (2001) doi:10.1016/S0550-3213(01)00013-X [hep-  
504 th/0011040].
- 505 [27] F. A. Dolan and H. Osborn, “Conformal partial waves and the operator product ex-  
506 pansion,” Nucl. Phys. B **678**, 491 (2004) doi:10.1016/j.nuclphysb.2003.11.016 [hep-  
507 th/0309180].
- 508 [28] M. Hogervorst, H. Osborn and S. Rychkov, “Diagonal Limit for Conformal Blocks in  $d$  Di-  
509 mensions,” JHEP **1308**, 014 (2013) doi:10.1007/JHEP08(2013)014 [arXiv:1305.1321  
510 [hep-th]].
- 511 [29] I. Bakas and E. Kiritsis, Nucl. Phys. B **343** (1990), 185-204 [erratum: Nucl. Phys. B **350**  
512 (1991), 512-512] doi:10.1016/0550-3213(90)90600-I
- 513 [30] J. M. Maldacena, “Wilson loops in large N field theories,” Phys. Rev. Lett. **80**, 4859 (1998)  
514 doi:10.1103/PhysRevLett.80.4859 [hep-th/9803002].
- 515 [31] S. J. Rey and J. T. Yee, “Macroscopic strings as heavy quarks in large N gauge theory and  
516 anti-de Sitter supergravity,” Eur. Phys. J. C **22**, 379 (2001) doi:10.1007/s100520100799  
517 [hep-th/9803001].

- 518 [32] D. L. Jafferis and S. J. Suh, “The Gravity Duals of Modular Hamiltonians,” JHEP **09**  
519 (2016), 068 doi:10.1007/JHEP09(2016)068 [arXiv:1412.8465 [hep-th]].
- 520 [33] D. L. Jafferis, A. Lewkowycz, J. Maldacena and S. J. Suh, “Relative entropy  
521 equals bulk relative entropy,” JHEP **06** (2016), 004 doi:10.1007/JHEP06(2016)004  
522 [arXiv:1512.06431 [hep-th]].
- 523 [34] A. Selberg, “Bemerkninger om et multipelt integral,”Norsk.Mat.Tidsskr.24(1944)71-78.
- 524 [35] P. Forrester and S. Warnaar, “The importance of the Selberg integral,”Bull. Amer. Math.  
525 Soc. (N.S.) 45 (2008), 489-534.