Thermal Ionization in Stellar Core Based on Ideal Fermi Gas Model

Shiladitya Debnath¹.
1. Dept. of Electronics and Communication Engineering,
   WBUT- Kolkata India. Email: sd.debnath157@gmail.com

Abstract: This paper examines the thermal ionization in the core of a main-sequence stable star, with an assumption that the main-sequence stellar core behaves like an ideal 3-Dimensional (D) Fermi gas. This assumption has been based on the fact that the stellar core persists as a region of very high temperature, typically in a range between $15 \times 10^6$ K and density near $150 \text{ g cm}^{-3}$, like that of our Sun where the classical gas description fails and Fermi-Dirac (F-D) distribution becomes important. Finally we compare our ionization equation with the Saha's thermal ionization equation based on classical Maxwell-Boltzmann (M-B) distribution. The mathematical calculation provides us with the result that the ionization fraction is exponentially proportional to the Fermi energy and has volume dependency, under the 3-D ideal Fermi gas consideration.

Key Words: Thermal Ionization, 3-D Fermi gas, stellar core.

1. Introduction

At the super high temperature inside a star's core, the electrons are ionized from their parent atoms and an energetic state of charged particle is formed, known as plasma. The constituents of this plasma state are identical particles with half integral spins. These particles are fermions which obey the Pauli’s exclusion principle, according to which no two fermions can have all identical quantum energy states. The statistical consideration of Fermi-Dirac (F-D) distribution in stellar cores was first realized by Fowler (1926) and later developed under relativistic transformation by Chandrasekhar (1931, 1932) to apply it in the structure and stability of white-dwarfs. Because of very high density at the core, fermions (in this case electrons) are packed very closely to each other creating a degenerate electron gas which in addition to thermo-nuclear pressure, generates an electron degeneracy pressure (although it is negligible in the core of a main-sequence stable star) which counteracts the inward gravitational collapse to maintain the so called hydrostatic equilibrium of the stellar core. The thermal ionization formulation in turn provides us with an initial understanding of the mechanism of energy transfer (in addition to another important factor of conduction method which will be discussed in future research) from the star’s core to the inter-atmospheric envelopes.

Here, in the following discussions we have considered the outer envelope of the stellar core as 3-D Fermi gas in a weakly degenerate state where the dominant particle distribution is governed by the F-D distribution and formulated a non-relativistic thermal ionization equation for a main-sequence stellar core.

The main-sequence stars are characterized by their energy generating mechanism deep inside their cores where primarily four hydrogen nuclei combine to form a helium nucleus through thermonuclear fusion. Once a star of mass comparable to that of the Sun forms, after a period of 10 million years, the star’s core reaches a state of thermal equilibrium and becomes radiative. This results in the energy transportation by radiation (also through conduction) rather than convection. Thus, based on the above argument, we provide a thermal ionization equation to understand the radiative energy properties in a main-sequence stellar core. The contents of the paper is as follows: we initially describe the basic properties of a 3-D Fermi sphere w.r.t F-D statistics, then we
proceed to obtain the ionization phenomena in analytical approach, then we compare our result with Saha’s original ionization equation and finally we draw some major conclusion for its application in stellar structure in future perspective.

2. Analytical methodology
In this section we are going to discuss some important quantum-statistical nature of isotropic 3-D Fermi gas and will proceed to derive a thermal ionization equation for a main-sequence stellar core non-relativistically.

2.1 Fundamental ideas regarding 3-dimensional Fermi gas
A 3-D isotropic homogenous and non-relativistic Fermi gas is called a Fermi sphere. This can be considered as a 3-D infinite square potential (i.e. a cubical box) well of infinite length $L$, where the potential can be given as

$$\tilde{V}(x,y,z) = \begin{cases} 
0, & \forall (x,y,z) \in \left( -\frac{L}{2}, \frac{L}{2} \right)^3 \\
\infty, & \text{for any other value of } x
\end{cases} \quad (1)$$

For this model, applying the standard formulation of quantum mechanics, it can be shown that the energy for energy levels $n_x, n_y, n_z$ is given by

$$E_n = E_0 + \left( \frac{\hbar^2 \pi^2}{2mL^2} \right) (n_x^2 + n_y^2 + n_z^2) \quad (2)$$

Where, $(n_x, n_y, n_z) \in \mathbb{Z}^+$, for the $n^{th}$ energy level. $E_0$ is the ground-state potential energy, $m$ is the mass of the single fermion and $\hbar$ is the reduced Plank’s constant. Because fermions obey Pauli’s exclusion principle, for $N$ Fermions with $\frac{1}{2}$ integral spin in the square well potential, two fermi particles cannot have exactly all similar quantum numbers. The 3-D Fermi energy for a Fermi sphere is given by

$$E_{F}^{3-D} \equiv E_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3}, \text{ where } V \text{ is volume when replace by } L^2 \rightarrow V^{2/3}. \text{ Hence the total energy of a Fermi sphere for } N \text{ fermions is given by } (\text{En. wikipedia.org.2020. Fermi Gas})$$

$$E_T = \frac{3}{5} E_F N + E_0 N \equiv \left( \frac{3}{5} E_F + E_0 \right) N \quad (3)$$

Within a thermodynamic limit (i.e. the total number of particles $N$ are so large that the quantum number number $n$ may be treated as a continuous variable. In this case, the overall number density profile in the box is indeed uniform), the degenerate degree can be calculated as

$$g(E) = \frac{1}{V} \frac{\partial N(E)}{\partial E} = \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} (E - E_0)^{1/2} \quad (3a)$$

Where the number of particles as a function of energy $N(E)$ is obtained by substituting equation (3) by a varying energy $(E - E_0)$ as

$$N(E) = \frac{V}{3\pi^2} \left[ \frac{2m}{\hbar^2} (E - E_0) \right]^{3/2} \quad (3b)$$

Thus, we begin our derivation based upon the above stated assumption regarding the 3-D Fermi gas model of the stellar core.

2.2 Derivation of thermal equation for 3-D Fermi gas model
The Fermi energy for the thermodynamic limit in 3-D Fermi gas is given according to equation (3) with an assumption where $E_F \equiv E_0$, where $E_0$ is the ground-state Fermi energy. Let the probability (entropy) function be defined as $S(N_e, N)$ for a Fermi gas that
has $N_e$ electrons out of $N$ Fermi particles in a given ensemble. The grand canonical ensemble as calculated by Kelly (2002) for such a case can be written as

$$Z_q = \sum \exp[-\beta(E_q - \mu_F N_q)]$$

where $\beta = \frac{1}{k_b T}$ and $k_b$ is the Boltzmann’s constant and $T$ is the absolute temperature of the system. Where $q$ indexes the ensemble of all possible microstate for electrons ($e$), protons ($p$) and hydrogen atoms ($H$). Thus for $e$,

$$Z_e = \sum \exp[-\beta(E_e - \mu_F N_e)]$$

Similar expressions could be found out for H and p. The chemical potential $\mu_F$ (Fermi level) of the three-dimensional ideal Fermi gas as given by Kelly (2002) is related to the zero temperature Fermi energy $E_F$ by a Sommerfield Expansion

$$\mu_F = E_0 + E_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_b T}{E_F}\right)^2 - \frac{\pi^4}{80} \left(\frac{k_b T}{E_F}\right)^4 + \cdots\right]$$

Considering a semi-classical limit, the entropy (probability) function for indistinguishable Fermions as calculated by Ghosh et al. (2019) is

$$S(N_e, N) = \left(\frac{Z_e}{N_e!}\right)^{\frac{N}{N_e!}}$$

For the exact momentum distribution calculation of a Fermi gas, we assume that the fermions have an anti-symmetric wave function $(\psi f)$ which are the functions in Fermi space $C_F$ and $C_B$ (for Bose field). Thus the resulting wave functions for fermions are as

$$\psi_i(C_F) = \psi_i(C_F)\psi_j(C_B) - \psi_j(C_F)\psi_i(C_B)$$

The exact momentum distribution calculated by Setlur (2020) is

$$< C_F, C_B > = n_F(k) + (2\pi k_F) \int^{\infty}_{-\infty} \frac{dq_1}{2\pi} \left[\Lambda_{k_F} \left(\frac{m^2}{q_1^2}\right)^{\frac{1}{2}} \left(\cosh(\lambda(q)) - 1\right)\right] -$$

$$(2\pi k_F) \int^{\infty}_{-\infty} \frac{dq_1}{2\pi} \left[\Lambda_{k_F} \left(\frac{m^2}{q_1^2}\right)^{\frac{1}{2}} \left(\cosh(\lambda(q)) - 1\right)\right]$$

where $\lambda(q) = (2\pi q)\nu(q)$, where $\nu(q)$ is the fermi velocity of q particle. And

$$\omega_R = \left(\frac{2q}{m}\right) \sqrt{\left(k_F + \frac{q}{2}\right)^2 - \left(k_F - \frac{q}{2}\right)^2} \exp(-\frac{\lambda(q)}{2}) / (1 - \exp(-\lambda(q)))$$

$\Lambda_k \equiv \frac{V}{(2\pi^2 k)^d}$ where $V$ is the volume of $d$ dimensional phase ($\phi$)-space and $k_F$ is the radius of the Fermi sphere. Thus from equation (8) (Setlur, 2020), we can say that the Fermi momentum ($p_x = E_F/k_F$) is continuous and the sum of the grand canonical distribution function is the integral

$$Z_i = \int g_{i}^{\beta} (\xi) \exp \left[- \frac{p_x^2}{2 k_F m} \left(\frac{d^nx d^n p}{h^3}\right)\right]$$

where $g_{i}^{\beta}$ is the (statistical weight) degenerate fermi level , $\xi_i$ are the individual fermions’ energy states. We now proceed to calculate $Z_i$ within thermodynamical limit in a 6-dimensional ($d$) ($\phi$)-space, so we can assume $n = 3$. Thus $d^x \phi$ is the volume $V$ and $d^p$ is the momentum, $(dp, dp, dp) \equiv d^p = 4\pi p^3 dp$. Now, taking $p_x \to p$, equation (9) can be re-written as

$$Z_i = \frac{4\pi g_{i}^{\beta}}{h^3} \int p^2 \exp \left[- \frac{p_x^2}{2 k_F m} \right] d^3p.$$
\[ Z_i = \frac{4\pi g_i^d}{h^3} \int d^3x \int_0^\infty p^2 \exp \left[ \frac{-p^2}{2mk_BT} \right] d^3p \]  
(10)

Taking \( t^2 = \frac{p^2}{2mk_BT} \Rightarrow 2dt = \frac{p}{mk_BT} dp \Rightarrow 2tdmk_BT = pdp \)  
(11)

and making equation (11) as substitution, equation (10) can be re-written as

\[ Z_i = \frac{4\pi g_i^d}{h^3} (2mk_BT) \int_0^\infty (2mk_BT) \frac{t^2 e^{-t^2}}{\sqrt{\pi}} dt^2 \]

\[
=> Z_i = \frac{4\pi g_i^d}{h^3} (2mk_BT) \frac{3}{\sqrt{\pi}} \frac{\sqrt{\pi}}{4} = \frac{g_i^d}{h^3} (2mk_BT)^{\frac{3}{2}}
\]

(12)

Because electrons and protons are both fermions the degeneracy for electrons and protons is \( g_F^d(p) \equiv g_F^d(e) = 2 \). Thus, based on equation (12) the grand canonical partition function for \( e \) and \( p \) can be written as

\[ Z_e = \frac{2V}{h^3} (2m_e k_BT)^{\frac{3}{2}} \text{ and } Z_p = \frac{2V}{h^3} (2m_p k_BT)^{\frac{3}{2}} \]

(13)

Now, the derivation for \( Z_H \) is identical as of previous one for \( e \) or \( p \) except for the fact that hydrogen atoms are not fermions so the inclusion of Fermi (\( E_F = E_F^{(0)} \)) (binding) energy is necessary in equation (12) as the derivation is carried out in a Fermi sphere as a 3-D Fermi gas along with the zero-level Bohr energy and taking \( -l \equiv \frac{\pi}{n} \) (\(-13.6 \) e.v. The degenerate state for \( H \) which if calculated as \( g_F^d(H) = \sum_{l=0}^{n-1} (2l + 1) \) where \( l \) is the angular momentum quantum number. This degeneracy equation results in series expansion in \( n^2 \) which for excited hydrogen atom is \( 2 \) i.e. \( n=2 \) thus \( n^2 = 4 \). Hence the degeneracy for excited hydrogen atom is given by \( g_F^d(H) = 4 \). So, the grand canonical partition function for hydrogen atom is

\[ Z_H = \frac{4V}{h^3} (2m_H k_BT)^{\frac{3}{2}} e^{\frac{\left[-E_F^{(e)}+(-l)\right]}{k_BT}} \]

(14)

Returning to equation (7) and taking logarithms of both the sides and using the Stirling’s approximation we have

\[ \ln S = N_e \ln Z_e + N_p \ln Z_p + N_H \ln Z_H - N_p \ln N_p - N_p \ln N_p - N_e \ln N_e + N_e + N_p + N_H \]

(15)

As the system is electronically neutral under the influence of electrons and ions, so we can write \( N_e \equiv N_P \) and \( N_H = N - N_p \). Then differentiating equation (15), we have

\[ \frac{d}{dN_e}(\ln Z) = 0 \], which implies

\[ \ln \frac{Z_eZ_p(N - N_p)}{Z_HN_e^2} = 0 \]

(16)

which implies

\[ \frac{Z_pZ_e}{Z_H} = \frac{N_e^2}{N - N_p} \]

(17)

Now, taking \( Z_p, Z_e \) and \( Z_H \) from equations (13) and (14) and substituting in equation (17), we find

\[ \frac{2V}{h^3} (2m_e k_BT)^{\frac{3}{2}} \left[ \frac{2V}{h^3} (2m_p k_BT)^{\frac{3}{2}} \right] \frac{N_e^2}{N - N_p} = \frac{N_e^2}{N - N_p} \]

(18)

Now, simplifying the equation (17) and cancelling the similar terms and taking \( m_p \approx m_n \), we find

\[ \frac{V}{h^3} (2m_e k_BT)^{\frac{3}{2}} e^{\frac{\left[E_F^{(e)}+(-l)\right]}{k_BT}} = \frac{N_e^2}{N - N_p} \]

which implies,
\[
\frac{N_e^2}{N-N_e} = \frac{V}{\hbar^3} (2m_e k_B T)^{\frac{3}{2}} e\left[ \frac{E_F - \mu}{k_B T} \right]^{\frac{3}{2}}
\]  
which further simplifies to

\[
\frac{N_e^2}{N-N_e} = V \left( \frac{m_e k_B T}{2 \pi \hbar^2} \right) \exp \left[ \frac{E_F - \mu}{k_B T} \right]^{\frac{3}{2}}
\]

(19)

(20)

Where, \( N \) is a function of \( T \) as \( N(T) = -\left( \frac{\partial \Omega}{\partial \mu} \right)_{V,T} \) and \( \Omega(V,T,\mu) \) is the grand canonical potential and is defined as

\[
\Omega = -g_F^B \Lambda B \int \ln \left[ 1 + 1 + e^{\beta (\mu - e(p))} \right] \rho^d d\rho.
\]

As moving towards the upper atmospheric zones of high temperature and relatively low density range, and taking the classical limit of M-B distribution in equation (15) and; putting \( E_F \to 0 \) and \( E \equiv -1 \) in \( H \) partition function and taking \( \chi = \frac{n_e}{n} \), we obtain the stellar atmospheric thermal ionization version of Saha’s equation (Saha, 1920)

\[
\frac{\chi^2}{1-2\chi} = \frac{1}{n} \left( \frac{2\pi m_e}{\hbar^2} \right)^{\frac{3}{2}} \exp \left( -\frac{1}{k_B T} \right).
\]

(21)

3. Result, Discussion and Conclusion

From equation (20), we have derived the expression for thermal ionization for a main-sequence stellar core considering it as an ideal and weakly degenerate 3-D Fermi gas which is mathematically similar to that of the Saha’s equation (21). Comparing equations (20) and (21), we find a difference due the fact that in the Fermi version, the ionization factor is exponentially proportional to the zero (ground)-level Fermi energy which in contrast for Saha’s equation is proportional to the zero-level Bohr energy for the classical Maxwellian gas. We also find that ionization energy for the F-D distribution is more energetic as compared to the semi-classical ionization formulation. Importantly, the energy of a lone hydrogen atom is nothing to do with the Fermi energy of the surrounding electronic medium. Notably, the situation is different when we deal with degenerate and fully ionized hydrogen atom. The inclusion of ground Fermi energy in \( H \) partition function is not merely to show distinction from that of surrounding electronic medium but also to show the contribution of Fermi energy on the zero-point energy for \( H \) atom which gives an additional boost of \( 10^{-4} \) e.v. to the 13.6 e.v. of the ordinary \( H \) atom when the entire calculations are carried out in an degenerate Fermi sphere. Moreover, it is assumed that the degenerate hydrogen at the stellar core is in ionized state and all its electrons are collectivized. The electron subsystem is considered in the random phase approximation with regard to the exchange interaction and the correlation of electrons and also the proton–proton interaction is taken into account in the Fermi-sphere in the local-field approximation. Since the thermodynamic potentials of hydrogen is monotonically increasing functions of density and temperature it’s Fermi energy effect becomes quite significant at the high temperature “hot” stellar cores like main-sequence in the formation of metallic hydrogen as theorized by the application of Thomas-Fermi model. Alternatively the above stated reason can be immediately be drawn by the Rayleigh-Schrödinger perturbation expansion which finally lead to an additional term in the ground state calculation of \( H \) atom in a Fermi sphere which is in close approximation to the Fermi energy. This Fermi energy which gives a boost to negative 13.6 e.v. can be expressed in terms of anisotropic constants. Finally, the inclusion of \( E_F^{(0)} \) in \( Z_H \) yields an energy result for \( H \) in degenerate state close to the calculations carried by Wigner and Huntington for degenerate \( H \) atom. However, limitations arise when we compare a complex stellar core with a 3-D ideal Fermi gas, neglecting all other phenomenon like such as plasma collisions, Magnetohydrodynamic
(MHD) turbulence and relativistic motion of ionized particles. As argued by Adams et al. (2004), pressure ionization due to relativistic electron degeneracy can safely be ignored in the above presented discussion because in the core of a main-sequence star, the hydrogen and helium is fully ionized and is weakly degenerate. This theoretical modelling is proceeded by considering simple assumptions regarding the quantum-statistical behaviour of a stellar core ionization but provides a first-step towards the modification of actual stellar-core thermal ionization behaviour for a main-sequence star.

In conclusion, we summarize the fact that considering a stellar core as weakly degenerate 3-D fermi (ideal and homogenous) gas, we have devised a theoretical model where we find that the degree of ionization at the star’s core depends on the net volume associated with the fermi gas and energy associated with such an ionization process is exponentially proportional to the Fermi energy for the constituent particles. In future, we consider to conduct a research paper considering the relativistic ionization and thermal conduction properties along with MHD in interior of main-sequence stars on the basis of 3-dimensional Fermi gas model.

Acknowledgement
The author is highly grateful to Prof. B Majumder, Tripura Govt. College; for many helpful discussions.

Data Availability Statement
No new data were generated or analysed in this research.

References
Zeitschrift fuer Astrophysik 5, no. 5 , pp. 321-27
Adams, C., Laughlin, G. and Graves, G. J. M.,2004.Red Dwarfs and the End of the Main Sequence, Revista Mexicana de Astronomía y Astrofísica,[online]. Available at: https://ui.adsabs.harvard.edu/abs/2004RMxAC..22...46A.