Dissipation bounds the moments of first-passage times of dissipative currents in nonequilibrium stationary states

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Abstract

We derive generic thermodynamic bounds on the moments of first-passage times of dissipative currents in nonequilibrium stationary states. These bounds hold generically for nonequilibrium stationary states in the limit where the threshold values of the current that define the first-passage time are large enough. The derived first-passage time bounds describe a tradeoff between dissipation, speed, reliability, and a margin of error and therefore represent a first-passage time analogue of thermodynamic uncertainty relations. For systems near equilibrium the bounds imply that mean first-passage times of dissipative currents are lower bounded by the Van't Hoff-Arrhenius law. In addition, we show that the first-passage time bounds are equalities if the current is the entropy production, a remarkable property that follows from the fact that the exponentiated negative entropy production is a martingale. Because of this salient property, the first-passage time bounds allow for the exact inference of the entropy production rate from the measurements of the trajectories of a stochastic process without knowing the affinities or thermodynamic forces of the process.

Contents

1 Introduction

In thermal equilibrium, transitions between metastable states are activated by thermal fluctuations. The equilibrium transition rates satisfy the Van't Hoff-Arrhenius law [1]

$$k = \frac{1}{\langle T \rangle} = \nu e^{-\frac{E_{\rm b}}{\mathsf{T}_{\rm env}}}, \tag{1}$$

where the rate k is the inverse of the mean first-passage time $\langle T \rangle$, $E_{\rm b}$ is the energy barrier that separates the two metastable states, $\mathsf{T}_{\rm env}$ is the temperature of the environment, and ν is a prefactor that has been determined, among others, by Kramers [1, 2].

Here we focus on nonequilibrium systems that are driven away from equilibrium by an external agent that continually supplies energy to the system. Such systems settle into stationary states that carry currents with nonzero average values. Figure 1 shows the trajectories of a particular example of a nonequilibrium rate process and compares it with Kramer's equilibrium model for a rate process. So far, first-passage times have been determined for specific examples of nonequilibrium stochastic processes, such as, self-propelled particles [3–8], nonequilibrium chemical reactions [9–11], or biological processes [12,13]. However, a generic thermodynamic relation that extends the Van't Hoff-Arrhenius law Eq. (1) to nonequilibrium systems is hitherto not known.

In this paper, we derive a set of generic inequalities for the moments of first-passage times of dissipative currents. These inequalities are remarkable because they hold generically for dissipative currents in stationary stochastic processes. The derived inequalities determine a trade-off between dissipation, speed, and reliability that is reminiscent of kinetic proof reading in biomolecular processes [14] and thus represent first-passage time analogues of the thermodynamic uncertainty relations [15–23]. However, contrary to thermodynamic uncertainty relations, the first-passage bounds we derive have the salient property that they reduce to equalities when the dissipative current is proportional to the stochastic entropy production, and hence can be used to infer exactly the entropy production rate in nonequilibruim stationary states. Near equilibrium, the derived inequalities imply that mean-first passage times of dissipative currents are lower-bounded by the Van't Hoff-Arrhenius law, and hence the bounds also extend the Van't Hoff-Arrhenius law to nonequilibrium stationary states.

The paper is organised as follows: in Sec. 2, we state the main results of this paper. In Sec. 3, we discuss in detail the system setup for which the main results are derived. Secs. 4 and 5 present the derivations of the main results. Sec. 4 derives a universal lower bound for the moments of first-passage times of dissipative currents in terms of the mean dissipation rate, and Sec. 5 derives equalities for the moments of first-passage times of the stochastic entropy production. In Sec. 6, we relate the main results of this paper to results previously published in the literature. In Sec. 7, we show that near equilibrium the derived lower bounds imply that mean first passage times are lower bounded by the Van't Hoff-Arrhenius law. In Sec. 8, we show with an example how the derived bounds can be used to infer exactly the entropy production rate from the measurements of the trajectories of a stochastic process. We also show that this is not possible with the thermodynamic uncertainty relations. The paper ends with a discussion in Sec. 9 and after the discussion there are five appendices that contain technical details on the mathematical derivations.

2 Main results

Let $J(t)$ be a dissipative current in a nonequilibrium, stationary process $X(t)$, and let

$$T_J = \inf\{t > 0 : J(t) \notin (-\ell_-, \ell_+)\} \quad (2)$$

be the first time when $J(t)$ leaves the open interval $(-\ell_-, \ell_+)$, where $t \geq 0$ is an index that labels time. In this paper, we show that in the limit of large thresholds ℓ_- and ℓ_+ it holds that

$$\langle T_J^n \rangle \geq \left(\frac{\ell_+}{\ell_-} \frac{|\log p_-|}{\dot{s}} \right)^n (1 + o_{\ell_{\rm min}}(1)) \quad (3)$$

where

$$p_- = \text{Prob}(J(T_J) < -\ell_-) \quad (4)$$

denotes the probability that the process terminates at the negative threshold $-\ell_-$, $\dot{s}$ is the entropy production rate, and $n \in \mathbb{N}$. The averages $\langle \cdot \rangle$ are taken over repeated realisations of the stationary process X. We have used the little-o-notation $o_{\ell_{\rm min}}(1)$ to denote a function that decays to zero when $\ell_{\rm min} = \min\{\ell_-, \ell_+\} \to \infty$ while the ratio ℓ_-/ℓ_+ is kept fixed. Equation (3) holds for $\langle J(t) \rangle > 0$; if $\langle J(t) \rangle < 0$, then p_- should be replaced by $p_+ = \text{Prob}(J(T_J) > \ell_+)$.

The inequality Eq. (3) describes a tradeoff between dissipation $\dot{s}$, speed $\langle T \rangle$, reliability p_-, and the allowed margin of error ℓ_-/ℓ_+. Therefore, it can be considered a first-passage time analogue of the thermodynamic uncertainty relations [15–23].

For processes near equilibrium, the bound given by Eq. (3) implies that $\langle T_J \rangle$ is lower bounded by the Van't Hoff-Arrhenius law, i.e.,

$$\langle T_J \rangle \geq \frac{1}{\nu} e^{\frac{E_{\rm b}}{\mathsf{T}_{\rm env}}}. \quad (5)$$

This is because $\dot{s} \sim e^{-\frac{E_{\rm b}}{\mathsf{T}_{\rm env}}}$ near equilibrium as we demonstrate explicitly with the nonequilibrium version of Kramer's model illustrated in Fig. 1.

An interesting application of Eq. (3) is the inference of entropy production from the measurements of the trajectories of a stochastic process X [24]. In Eq. (3) the quantities $\langle T_J^n \rangle$, p_-, and the ratio ℓ_+/ℓ_- can be measured directly from an experiment, while the entropy production rate $\dot{s}$ is in general not known. The inequality Eq. (3) has an important advantage with respect to other methods, such as those based on the thermodynamic uncertainty relations [25–28]: If $J(t) = S(t)$ with $S(t)$ the stochastic entropy production [29–31], then the inequality Eq. (3) becomes an equality, viz.,

$$\langle T_S^n \rangle = \left(\frac{\ell_+}{\ell_-} \frac{|\log p_-|}{\dot{s}} \right)^n (1 + o_{\ell_{\rm min}}(1)). \quad (6)$$

This remarkable property follows form the fact that $e^{-S(t)}$ is a martingale [32–34], which implies the formula $p_- = e^{-\ell_-}(1 + o_{\ell_{\rm min}}(1))$ [33, 34]; note that thermodynamic uncertainty relations do not have this tightness properties, see e.g. [35, 36]. Since the result Eq. (6) holds in the limit of large thresholds and $S(t) \in O(t)$, the equality remains valid for dissipative currents of the form $J(t) = aS(t) + o(t)$ with a an arbitrary proportionality constant and $o(t)$ an arbitrary function $f(t)$ for which $f(t)/t \to 0$.

In what follows, we derive the inequalities given by Eq. (3) for stationary states in overdamped, Markovian systems that satisfy local detailed balance, and we derive the equality Eq. (6) for stationary states in systems that satisfy local detailed balance [29–31, 37].

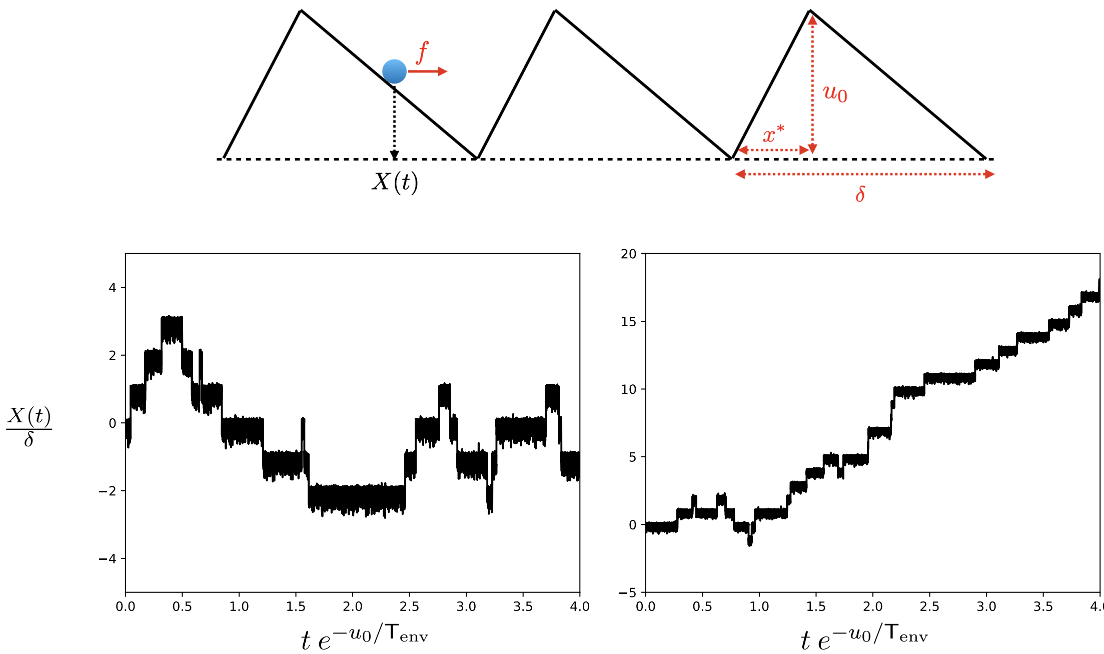

Figure 1: *Example of a nonequilibrium rate process.* Trajectories shown are for a reaction coordinate X that solves the Langevin equation $\partial_t X(t) = (f - \partial_x u(X(t)))/\gamma + \sqrt{2\mathsf{T}_{\rm env}/\gamma}\xi(t)$, where $\xi(t) = {\rm d}W(t)/{\rm d}t$ is a delta-correlated white Gaussian noise term, and where $u(x)$ is a triangular potential with period δ, i.e. $u(x) = u(\pm\delta)$, $u(x) = u_0\, x/x^*$ if $x \in [0, x^*]$, and $u(x) = u_0(\delta - x)/(\delta - x^*)$ if $x \in [x^*, \delta]$. Left: equilibrium trajectory with $f = 0$ Right: nonequilibrium trajectory with $f\delta/\mathsf{T}_{\rm env} = 1$. The remaining parameters are set to $\delta = 5$, $\gamma = 1$, $x^* = 1$, $u_0 = 10$, and $\mathsf{T}_{\rm env} = 1$.

3 System setup

Let $X(t) = (X_1(t), X_2(t), \dots, X_n(t))$ be a stationary process with $t \geq 0$ a time index. We denote trajectories of $X(t)$ over a time interval $[0, t]$ as X_0^t.

For systems that satisfy local detailed balance the stochastic entropy production S can be expressed as [29,31]

$$S(t) = \log \frac{p(X_0^t)}{\tilde{p}(X_0^t)}, \tag{7}$$

where $p(X_0^t)/\tilde{p}(X_0^t)$ is the ratio between the probability densities of the trajectory X_0^t in the forward and backwards dynamics, also known the Radon-Nikodym derivative [33,38,39]. Note that we use natural units for which the Boltzmann constant $k_{\rm B} = 1$. Equation (7) relies on the physical assumption that the system is weakly coupled to an environment in thermal equilibrium [37]. Indeed, an explicit calculation for Langevin processes and Markov jump processes shows that Eq. (7) is the stochastic entropy production [31,40].

Since the process is stationary, the entropy production rate $\dot{s}$, or equivalently the rate of dissipation, is given by

$$\langle S(t)\rangle = \dot{s}\, t(1 + o_t(1)), \tag{8}$$

where the little-o notation $o_t(1)$ denotes a function that decays to zero when $t \gg 1$.

Dissipative currents $J(t) = J(X_0^t)$ are time extensive functionals defined on the set of trajectories X_0^t with the following two properties:

- J is time extensive, i.e.,

$$\langle J(t)\rangle = \overline{j}\, t(1 + o_t(1)) \tag{9}$$

where $\bar{j}$ is the current rate.

- J is odd under time-reversal, i.e.,

$$J(\Theta_t(X_0^t)) = -J(X_0^t). \tag{10}$$

Note that this implies $J(0) = 0$. The time-reversal operation Θ_t maps trajectories X_0^t on their time-reversed trajectory $\tilde{X}_0^t$ such that $\tilde{X}_i(\tau) = \sigma_i X_i(t-\tau)$, where $\sigma_i \in \{-1, 1\}$ denotes the parity of the i-th degree of freedom. For overdamped systems $\sigma_i = 1$ for all values of $i = 1, 2, \ldots, n$.

As an example, consider the case of a Markov jump process for which $X(t)$ takes values in a discrete set $\mathcal{X}$. We denote the Markov transition matrix by $w_{x\to y}$ with $x, y \in \mathcal{X}$ [41]. For a Markov jump process dissipative currents take the form

$$J(t) = \sum_{x,y\in\mathcal{X}} c_{x,y} J_{x\to y}(t) \tag{11}$$

where the coefficients $c_{x,y} \in \mathbb{R}$ and

$$J_{x\to y}(t) = N_{x\to y}(t) - N_{y\to x}(t) \tag{12}$$

denotes the net number of jumps in the trajectory X_0^t between the states x and y. The entropy production is an example of a dissipative current and is given by

$$S(t) = \frac{1}{2} \sum_{x,y\in\mathcal{X}} \log \frac{p_{\rm ss}(x) w_{x\to y}}{p_{\rm ss}(y) w_{y\to x}} J_{x\to y}(t) \tag{13}$$

where $p_{\rm ss}(x)$ is the probability distribution of $X(t)$ in the stationary state.

4 Bounds for the moments of first-passage times of dissipative currents

We derive the bounds given by Eq. (3) for the moments of first passage times of dissipative currents.

First, we use that $J(t)$ converges with probability one to a deterministic function $\bar{j}t$, viz.,

$$\frac{J(t)}{t} = \bar{j}(1 + o_t(1)), \tag{14}$$

where the little-o notation $o_t(1)$ denotes a function that decays to zero when $t \gg 1$. Convergence to a deterministic limit is a natural assumption for time-extensive variables as they satisfy a large-deviation principle [42]. Since the current is deterministic and we have assumed that $\bar{j} > 0$, it holds that also the first-passage time given by definition Eq. (2) is deterministic for large values of $\ell_{\rm min}$, i.e.,

$$T_J = \frac{\ell_+}{\bar{j}}(1 + o_{\ell_{\rm min}}(1)), \tag{15}$$

and therefore also

$$\langle T_J^n \rangle = \left(\frac{\ell_+}{\bar{j}}\right)^n (1 + o_{\ell_{\rm min}}(1)). \tag{16}$$

Note that Eq. (15) describes only the first-passage time of trajectories that terminate at the positive threshold as $p_+ = 1 + o_{\ell_{\rm min}}(1)$.

Second, we use that for trajectories that reach the negative threshold

$$T_J = \frac{\ell_-}{\overline{j}}(1 + o_{\ell_{\rm min}}(1)) = \tau_-. \tag{17}$$

We derive Eq. (17) in the Appendix B using time-reversal arguments that are similar to those used in Ref. [33] in the context of stopping time fluctuation relations for entropy production.

Third, we express the splitting probability p_- in terms of the large deviation function of the current J. Since J is time-extensive, it satisfies a large deviation principle, i.e., in the limit of large t the distribution $p_{J/t}$ of J/t is given by [42]

$$p_{J/t}(z) \sim e^{-t\mathcal{J}(z)} \tag{18}$$

where $\mathcal{J}(z)$ is the large deviation function of the current. For large values of $\ell_{\rm min}$, the first passage time T_J is given by Eq. (17), and therefore,

$$p_- \;=\; \int_{-\infty}^{-\ell_-/\tau_-} e^{-\tau_- \mathcal{J}(z)} {\rm dz}. \tag{19}$$

Fourth, we bound the splitting probability as a function of the entropy production rate $\dot{s}$. To this aim, we use the bound

$$\mathcal{J}(z) \leq \frac{\dot{s}}{4}(z/\overline{j} - 1)^2 \tag{20}$$

for the large deviation function of the current that has been derived in Refs. [16, 17, 43] for overdamped Markov processes in nonequilibrium stationary states. Using Eq. (20) in Eq. (19) yields

$$p_- \;\geq\; \int_{-\infty}^{-\overline{j}} e^{-\frac{\ell_-}{\overline{j}}\frac{\dot{s}}{4}(z/\overline{j}-1)^2} {\rm dz} = e^{-\frac{\ell_-}{\overline{j}}\dot{s}}. \tag{21}$$

Lastly, combining Eqs. (16) and (21) we readily obtain Eq. (3).

5 Equalities for the moments of first-passage times of entropy production

We present two different derivations for the equality Eq. (6): (i) based on Eq. (19) and the Gallavotti-Cohen fluctuation relation for entropy production [44] and (ii) based on the martingale property of e^{-S} [33–35].

5.1 Derivation based on the Gallavotti-Cohen fluctuation relation

Let us first derive Eq. (6) with the Gallavotti-Cohen fluctuation relation. For $J = S$, the rate function $\mathcal{J}(z)$ is convex, satisfies $\mathcal{J}(z) \geq 0$ and $\mathcal{J}(\dot{s}) = 0$, and it also satisfies the fluctuation relation [44]

$$\mathcal{J}(z) - \mathcal{J}(-z) = -z. \tag{22}$$

Hence, from Eq. (19) it follows that

$$p_- = e^{-\frac{\ell_-}{\dot{s}}\mathcal{J}(-\dot{s})} = e^{-\ell_-}. \tag{23}$$

Combining Eqs. (23) and (16) we obtain the equality (6).

5.2 Derivation based on the martingality of e^{-S}

The fact that p_- is universal and only depends on the threshold ℓ_- is a remarkable fact that is a direct consequence of the martingale property of e^{-S} [33], see also Appendix A for a discussion of martingales. Indeed, the process e^{-S} is a martingale and satisfies the integral fluctuation relation at stopping times [34]

$$\langle e^{-S(T_S)} \rangle = 1. \tag{24}$$

The Eq. (24) also reads

$$p_- \langle e^{-S(T_S)} \rangle_- + p_+ \langle e^{-S(T_S)} \rangle_+ = 1, \tag{25}$$

where $\langle \cdot \rangle_-$ and $\langle \cdot \rangle_+$ denote averages over those trajectories that terminate at the negative and positive threshold value, respectively. Using that for $\ell_-, \ell_+ \gg 1$ it holds that $S(T_S) = \ell_\pm(1 + o_{\ell_{\min}}(1))$, we obtain

$$p_- e^{\ell_-} + p_+ e^{-\ell_+} = 1 \tag{26}$$

and for $\ell_+ \gg 1$ thus

$$p_- = e^{-\ell_-}, \tag{27}$$

which implies again the equality (6).

6 Comparison with previously published results

We discuss how the main results Eqs. 3 and 6 are related to previously published results in the literature.

6.1 Bounds on $\langle T_J \rangle$ based on the optimality of sequential probability ratio tests

In the special case of $n = 1$ and symmetric thresholds $\ell_- = \ell_+ = \ell$, the bound Eq. (3) is related to a bound on $\langle T_J \rangle$ that appeared in Ref. [45] and is based on the asymptotic optimality of sequential probability ratio tests in the limit of large thresholds $\ell \gg 1$, in the sense that it minimises the mean time $\langle T_J \rangle$ over the set of all sequential hypothesis tests that satisfy certain prescribed reliability constraints.

Equation (9) in Ref. [45] implies that for $|\log p_-| \gg 1$ it holds that

$$\dot{s} \geq \frac{(1 - 2p_-)\log((1 - p_-)/p_-)}{\langle T_J \rangle}. \tag{28}$$

In the limit $p_- \ll 1$, this is equivalent to Eq. (3) when $n = 1$ and $\ell_- = \ell_+ = \ell$.

6.2 Time-dissipation uncertainty relation

Eq. (3) is also related to the dissipation-time uncertainty relation that states

$$\langle T_J \rangle \geq \frac{1}{\dot{s}} \tag{29}$$

in the limit $|\log p_-| \gg 1$ [46]. One should bear in mind that this bound is loose. Indeed, the prefactor in Eq. (3) is $|\log p_-|$ and hence in the limit $p_- \ll 1$ the dissipation-time uncertainty relation holds for any finite value of $c \geq 0$, i.e.,

$$\langle T_J \rangle \geq \frac{c}{\dot{s}}(1 + o_{\ell_{\min}}(1)). \tag{30}$$

6.3 Thermodynamic uncertainty relations

Lastly, let us discuss connections with thermodynamic uncertainty relations [15–23]. There exists different types of thermodynamic uncertainty relations. All these relations have in common that they express a tradeoff between dissipation, speed and reliability, but they vary in the way that speed and reliability are quantified. In this sense, also Eq. (3) represents a thermodynamic uncertainty relation where now speed is quantified by $\langle T_J \rangle$ and reliability by p_-.

The original thermodynamic uncertainty relation is the lower bound [15, 17]

$$\dot{s} \geq \frac{2\overline{j}^2}{\sigma_J^2}, \tag{31}$$

where $\overline{j}$ is the current rate and

$$\sigma_J^2 = \lim_{t\to\infty} \frac{1}{t} \left(\langle J^2(t) \rangle - \langle J(t) \rangle^2 \right). \tag{32}$$

Hence, the thermodynamic uncertainty relation quantifies speed with $\overline{j}$ and reliability with the variance σ_J^2.

The first-passage time thermodynamic uncertainty relation is the lower bound [47]

$$\dot{s} \geq \frac{2\langle T_J \rangle}{\langle T_J^2 \rangle - \langle T_J \rangle^2}. \tag{33}$$

Here, speed is quantified with $\langle T_J \rangle$ and reliability with $\langle T_J^2 \rangle - \langle T_J \rangle^2$.

An important distinction between the thermodynamic uncertainty relations, Eqs. (31) and Eq. (33), and the bound Eq. (3) on the moments of first-passage times, is that the former are loose bounds while the latter is a tight bound. Indeed, if $J(t) = S(t)(1 + o_t(1))$, then Eq. (3) is an equality, see Eq. (6), whereas the Eqs. (31) and Eq. (33) are in general not equalities, even not when $J(t) = S(t)(1 + o_t(1))$ [35, 36]. In other words, the ratio $|\log p_-|/(\dot{s}\langle T_S \rangle)$ is universal, while the ratio $2\langle T_S \rangle/[\dot{s}(\langle T_S^2 \rangle - \langle T_S \rangle^2)]$ is system dependent.

Lastly, we comment on a connection between the bound Eq. (3) and the inequality [23]

$$e^{-t\dot{s}} \leq \frac{\sigma_J^2}{4\overline{j}^2}, \tag{34}$$

which holds for $t \gg 1$. Consider the first-passage time T_J with symmetric thresholds $\ell_+ = \ell_- = \ell$. In this case, it holds that

$$\langle J(T_J) \rangle = \ell\,(1 - 2p_-)(1 + o_\ell(1)) \tag{35}$$

and

$$\langle J^2(T_J) \rangle = \ell^2 (1 + o_\ell(1)). \tag{36}$$

Equations (35) and (36) yield

$$\frac{\mathrm{var}[J(T_J)]}{\langle J(T_J) \rangle^2} = \frac{1 - (1 - 2p_-)^2}{(1 - 2p_-)^2} = 4p_-(1 + o_\ell(1)), \tag{37}$$

where

$$\mathrm{var}[J(T_J)] = \langle (J(T_J))^2 \rangle - \langle J(T_J) \rangle^2 (1 + o_\ell(1)). \tag{38}$$

Using the bound Eq. (3), we obtain

$$e^{-\langle T_J \rangle \dot{s}} \leq \frac{\mathrm{var}[J(T_J)]}{4\langle J(T_J) \rangle^2}(1 + o_\ell(1)). \tag{39}$$

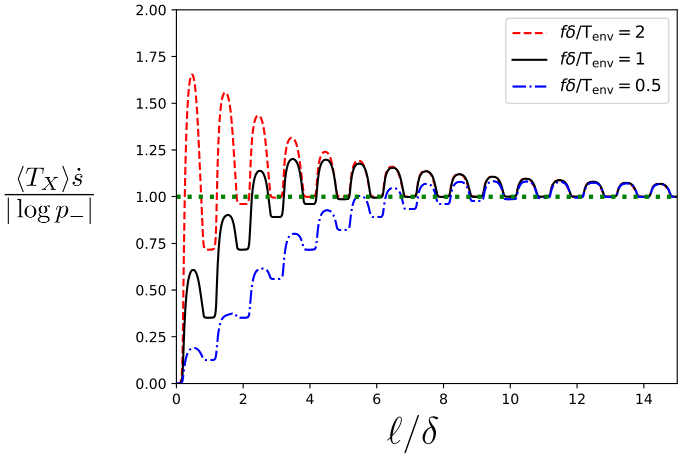

Figure 2: *Asymptotic lower bound on the mean first-passage time.* The ratio $\langle T_X\rangle \dot{s}/|\log p_-|$ is plotted as a function of ℓ/δ, where T_X is the first-passsage time Eq. (2) of the nonequilibrium Kramer process X described by Eq. (40) with triangular potential u given by Eq. (41). Curves shown are for the parameters $\delta = 5$, $x^* = 1$, $u_0 = 10$, $\mathsf{T}_{\rm env} = 1$, and $\gamma = 1$, and the values of f are given in the figure legend.

Equations (34) and (39) are identical when we identify t with $\langle T_J\rangle$, $\langle J(t)\rangle$ with $\langle J(T_J)\rangle$, and $\langle J^2(t)\rangle$ with $\langle J^2(T_J)\rangle$. In the limit of $\ell \gg 1$ the first-passage time T_J converges to a deterministic limit, see Eq. (15), and therefore these identifications can be made. Ref. [23] argues that Eq. (34) provides, in the limit of $t \gg 1$, the best bound on the precision $\overline{j}^2/\sigma_J^2$ that only depends on $\dot{s}$. This is consistent with the tightness of the bound Eq. (3).

7 Extension of the Van't Hoff-Arrhenius law to nonequilibrium stationary states

In this section, we show that near equilibrium Eq. (3) implies that $1/\langle T_J\rangle$ is smaller or equal than the Van't Hoff-Arrhenius law Eq. (1). To this aim, we consider a nonequilibrium version of Kramer's model [1, 2]. Details of the calculations can be found in the Appendices C and D.

We consider a reaction coordinate $X \in \mathbb{R}$ that is described by the overdamped Langevin equation

$$\mathrm{d}X(t) = \frac{f - \partial_x u(X(t))}{\gamma}\,\mathrm{d}t + \sqrt{2\mathsf{T}_{\rm env}/\gamma}\,\mathrm{d}W(t), \tag{40}$$

where $u(x)$ is a periodic potential with period δ, i.e., $u(x+\delta) = u(x) = u(x-\delta)$, f is a nonconservative force, γ is a friction coefficient, $W(t)$ is a standard Wiener process that models the thermal noise, and $\mathsf{T}_{\rm env}$ is the temperature of the environment. We assume that at time $t = 0$, $X(0) = 0$ and $W(0) = 0$.

The variable X models, e.g., a reaction coordinate that tracks the progress of a chemical reaction. In this scenario, $E_{\rm b} = \max_x u(x) - \min_x u(x)$ is the Gibbs free energy barrier that separates two chemical states and the ratio $[X/\delta]$ is the number of cycles of the reaction that have been completed; $[a]$ denotes the largest integer smaller than a.

Figure 1 presents two trajectories generated by Eq. (40) for the special case where $u(x)$

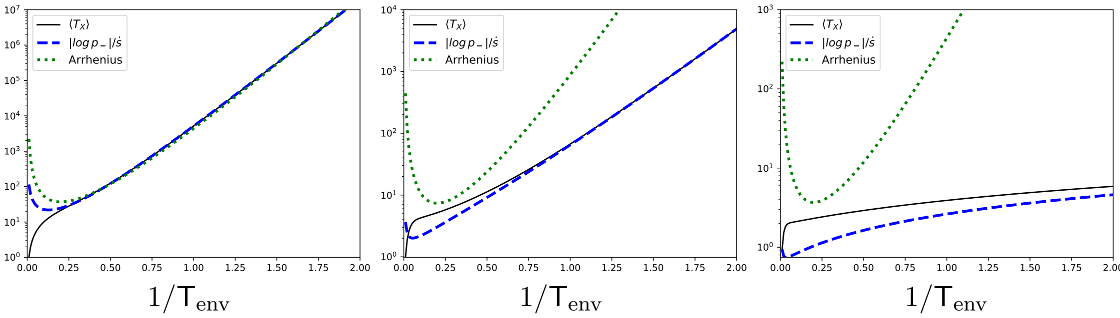

Figure 3: *Extension of the Van't Hoff-Arrhenius law for nonequilibrium stationary states.* The mean-first passage time $\langle T_X \rangle$ (solid black line) of the reaction coordinate X, described by Eq. (40) with triangular potential u given by Eq. (41), is compared with its asymptotic value $|\log p_-|/\dot{s}$ for large thresholds ℓ (blue dashed line) and with the Van't Hoff-Arrhenius law Eq. (51) (green dotted line). The model parameters are $\delta = 5$, $x^* = 1$, $u_0 = 10$, $\mathsf{T}_{\rm env} = 1$ and $\gamma = 2$ and the values of f are $f = 1$, $f = 5$ and $f = 10$ (left to right). The threshold for the first-passage time T_X, which is defined in Eq. (44), is $\ell = 10$.

is the triangular potential

$$u(x) = \begin{cases} u_0 \frac{x}{x^*} & \text{if} \quad x \in [0, x^*), \\ u_0 \frac{\delta - x}{\delta - x^*} & \text{if} \quad x \in [x^*, \delta). \end{cases} \tag{41}$$

From Fig. 1 we observe that the dynamics consists of a sequence of jumps between metastable states that are centred at the positions nx^* with $n \in \mathbb{Z}$. In the equilibrium case with $f = 0$ the jumps are activated by thermal fluctuations and the Van't Hoff-Arrhenius law Eq. (1) applies. On the other hand, when $f > 0$, then jumps in one direction over the energy barrier $E_{\rm b}$ are facilitated by the external driving f, while in the reverse direction jumps are less likely. In this case, although the Van't Hoff-Arrhenius law Eq. (1) does not apply, the Eqs. (3) and (6) apply and can thus be considered nonequilibrium versions of the Van't Hoff-Arrhenius law.

For values $f\delta/E_{\rm b} > 0$ the chemical reaction settles into a nonequilibrium stationary state with an entropy production rate (see Appendix C.2)

$$\dot{s} = \frac{f\delta}{\mathsf{T}_{\rm env}} j_{\rm ss}, \tag{42}$$

where $j_{\rm ss}$ is the stationary current (see Appendix C.1)

$$j_{\rm ss} = \frac{\mathsf{T}_{\rm env}}{\gamma} \frac{1 - e^{\frac{-f\delta}{\mathsf{T}_{\rm env}}}}{\int_0^\delta {\rm d}y\, w(y) \left(\int_y^{y+\delta} {\rm d}x' \frac{1}{w(x')} \right)}, \tag{43}$$

and where $w(x) = \exp(-(u(x) - fx)/\mathsf{T}_{\rm env})$.

Consider the first time

$$T_X = \inf \{ t > 0 : X(t) \notin (-\ell, \ell) \} \tag{44}$$

when the reaction has completed a net number $[\ell/\delta]$ of cycles in either the forward or backward direction. Since, (see Appendix C.2)

$$S(t) = \frac{f X(t)}{\mathsf{T}_{\rm env}} + o(t) \tag{45}$$

the equality (6) applies to T_X. In Appendices C.3 and C.4, we derive explicit analytical expressions for the splitting probability p_- and the mean first-passage time $\langle T_X \rangle$, respectively, which we omit here as the expressions are involved. However, as shown in Appendix C.5, in the limit of large ℓ we obtain the formula

$$\frac{|\log p_-|}{\langle T_X \rangle} = \dot{s} + O\left(\frac{1}{\ell}\right), \tag{46}$$

in correspondence with Eq. (6), where O denotes the big-O notation. Hence, in this case, the correction term in Eq. (6) is of order $1/\ell$.

In Fig. (2) we plot $|\log p_-|\dot{s}/\langle T_X \rangle$ as a function of ℓ/δ. The figure demonstrates the convergence of $|\log p_-|\dot{s}/\langle T_X \rangle$ to its universal limit for different values of the nonequilibrium driving $f\delta/\mathsf{T}_{\rm env}$. Observe the oscillations of $|\log p_-|\dot{s}/\langle T_X \rangle$. These oscillations appear because for the parameters selected it holds that $E_{\rm b} \gg \mathsf{T}_{\rm env}$, and therefore the process consists of discrete-like hops over the energy barrier $E_{\rm b}$ that represent the subsequent completion cycles of the chemical reaction.

In the limits $\mathsf{T}_{\rm env} \to 0$ and $f\delta/\mathsf{T}_{\rm env} \to 0$, the Eq. (6) leads to a Van't Hoff-Arrhenius law for $1/\langle T_X \rangle$. Indeed, as shown in Appendix C.6, taking the limits $\mathsf{T}_{\rm env} \to 0$ and $f\delta/\mathsf{T}_{\rm env} \to 0$ in the expression of the stationary current Eq. (43), we obtain

$$j_{\rm ss} = \kappa \frac{f\delta}{\gamma} e^{\frac{-E_{\rm b}}{\mathsf{T}_{\rm env}}}, \tag{47}$$

where the prefactor

$$\kappa = \frac{\sqrt{-u''_{\rm min} u''_{\rm max}}}{2\pi \mathsf{T}_{\rm env}} \tag{48}$$

if the second derivatives $u''_{\rm min}$ and $u''_{\rm max}$ evaluated at the minimum and maximum of $u(x)$, respectively, exist. In the special case of the triangular potential, given by Eq. (41), the second derivatives $u''_{\rm min}$ and $u''_{\rm max}$ do not exist. In this particular case

$$\frac{1}{\kappa} = \left(\frac{1}{u^+_{\rm max}} - \frac{1}{u^-_{\rm max}}\right)\left(\frac{1}{u^+_{\rm min}} - \frac{1}{u^-_{\rm min}}\right), \mathsf{T}^2_{\rm env} \tag{49}$$

where $u^+_{\rm max}$ and $u^-_{\rm max}$ denote the left and right derivatives evaluated at the maximum of $u(x)$. In addition, as shown in Appendix C.6, in the limit of $\mathsf{T}_{\rm env} \to 0$ and $f\delta/\mathsf{T}_{\rm env} \to 0$ the logarithm of the splitting probability is inversely proportional to the temperature, viz.,

$$\log p_- = -\frac{f\ell}{\mathsf{T}_{\rm env}} + O_\ell(1). \tag{50}$$

Combining Eqs. (6), (42), (47), and (50) we obtain the Van't Hoff-Arrhenius law

$$\langle T_X \rangle = \frac{\ell}{\delta}\frac{\gamma}{f\delta}\frac{1}{\kappa} e^{\frac{E_{\rm b}}{\mathsf{T}_{\rm env}}}. \tag{51}$$

In Fig. 3 we compare $\langle T_X \rangle$ with its asymptotic value $|\log p_-|/\dot{s}$, given by Eq. (6), and with the Van't Hoff-Arrhenius law, given by Eq. (51), for three values of the driving force f. We make a few interesting observations: (i) the Van't Hoff-Arrhenius law approximates well $\langle T_X \rangle$ up to moderately large values of $f\delta/\mathsf{T}_{\rm env} < 5$; (ii) for $f\delta/\mathsf{T}_{\rm env} > 25$ we start to observe significant deviations between the Van't Hoff-Arrhenius law and $\langle T_X \rangle$ (iii) the asymptotic expression $|\log p_-|/\dot{s}$ given by Eq. (6) approximates well $\langle T_X \rangle$ for relatively small values of the threshold, viz., $\ell/\delta = 2$, and it holds for any value of the driving force f.

Taken together, we conclude that the Eqs. (3) and (6) extend the Van't Hoff-Arrhenius law to nonequilibrium stationary states as $\dot{s} \sim \exp(-E_{\rm b}/\mathsf{T}_{\rm env})$ in the limit of small temperatures $\mathsf{T}_{\rm env} \approx 0$ and small driving force $f\delta/\mathsf{T}_{\rm env} \approx 0$.

8 Exact inference of the stationary entropy production

We show with an example how the bound Eq. (3) can be used to infer exactly the entropy production rate $\dot{s}$ in a stationary process X from the measurements of p_- and $\langle T_J^n \rangle$.

We consider a hopping process $X \in \mathbb{Z}$ described by

$$\mathrm{d}X(t) = \mathrm{d}N_+(t) - \mathrm{d}N_-(t), \tag{52}$$

where N_+ and N_- are two counting process with rates k_+ and k_-, respectively. The bias of the process is defined by the ratio

$$b := \frac{k_-}{k_+} = \exp\left(-\frac{a}{\mathsf{T}_{\mathrm{env}}}\right) \tag{53}$$

where a is the thermodynamic affinity and $\mathsf{T}_{\mathrm{env}}$ the temperature of the environment. We assume, without loss of generality, that $k_- < k_+$ so that $b < 1$.

The coordinate X may represent the number of times a chemical reaction has been completed or the position of a molecular motor on a biofilament. In the former, $a = \Delta\mu$ is the difference between the sum of the chemical potentials of the products and the reagents of the chemical reaction, and in the latter $a = f\delta$ is the work performed by the system on the motor when it moves forwards. Hence, the stochastic entropy production S obeys

$$\mathrm{d}S(t) = \frac{a}{\mathsf{T}_{\mathrm{env}}}\mathrm{d}X(t) \tag{54}$$

and

$$\dot{s} = \left\langle \frac{\mathrm{d}S}{\mathrm{d}t} \right\rangle = \frac{a}{\mathsf{T}_{\mathrm{env}}}(k_+ - k_-) \tag{55}$$

is the entropy production rate.

We consider the first passage time

$$T_X = \inf\left\{t > 0 : X(t) - X(0) \notin (-\ell_-, \ell_+)\right\}, \tag{56}$$

which is also the first-passage time T_S of the stochastic entropy production with thresholds $s_- = a\ell_-/\mathsf{T}_{\mathrm{env}}$ and $s_+ = a\ell_+/\mathsf{T}_{\mathrm{env}}$.

The splitting probabilities p_- and p_+ are given by (see Appendix E.3)

$$p_+ = \frac{1 - b^{[\ell_-]}}{1 - b^{[\ell_-]+[\ell_+]}} \quad \text{and} \quad p_- = b^{[\ell_-]}\frac{1 - b^{[\ell_+]}}{1 - b^{[\ell_-]+[\ell_+]}}, \tag{57}$$

and the generating function

$$g(y) = \langle e^{-yT_X(k_-+k_+)} \rangle \tag{58}$$

is for all $y > 0$ given by (see Appendix E.4)

$$\begin{aligned} g(y) = &\left(\frac{2}{\zeta_+(y)}\right)^{[\ell_+]} \frac{1 - \left(\frac{\zeta_-(y)}{\zeta_+(y)}\right)^{[\ell_-]}}{1 - \left(\frac{\zeta_-(y)}{\zeta_+(y)}\right)^{[\ell_-]+[\ell_+]}} \\ &+ \left(\frac{\zeta_-(y)}{2}\right)^{[\ell_-]} \frac{1 - \left(\frac{\zeta_-(y)}{\zeta_+(y)}\right)^{[\ell_+]}}{1 - \left(\frac{\zeta_-(y)}{\zeta_+(y)}\right)^{[\ell_-]+[\ell_+]}}, \end{aligned} \tag{59}$$

where

$$\zeta_\pm(y) = \beta(y) \pm \sqrt{-4b + \beta^2(y)} \tag{60}$$

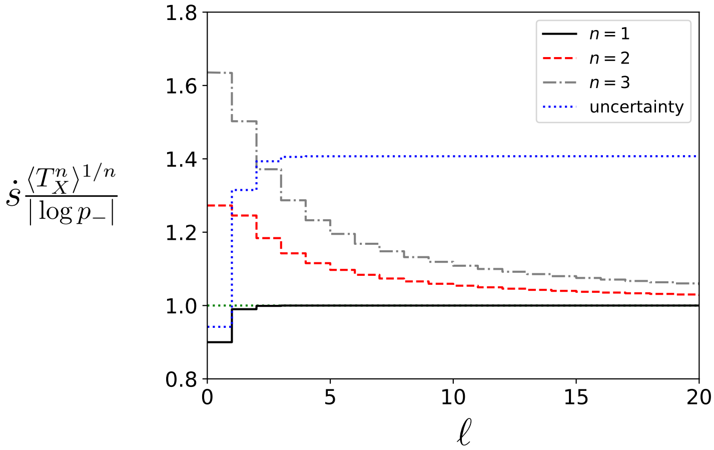

Figure 4: *A comparison between inference of entropy production with the first-passage time bounds Eq. (3) and with the thermodynamic uncertainty relation Eq. (33).* The ratio $\dot{s}\langle T_X^n\rangle^{1/n}/|\log p_-|$ for $n = 1, 2, 3$ and the uncertainty $\dot{s}(\langle T_X^2\rangle - \langle T_X\rangle^2)/(2\langle T_X\rangle)$ as a function of $\ell = \ell_- = \ell_+$ for a biased random walk process X described by Eq. (52) with $k_+ = 1$ and $b = 0.1$. Note that the inequalities Eq. (3) are tight for $\ell \to \infty$, while the uncertainty relation Eq. (33) is loose.

and

$$\beta(y) = (1 + y)(1 + b). \tag{61}$$

The moments of T_X follow from

$$\langle T_X^n\rangle = \left(\frac{-1}{k_- + k_+}\right)^n \frac{\mathrm{d}^n}{(\mathrm{d}y)^n} g(y)\Big|_{y=0}, \tag{62}$$

where $n \in \mathbb{N}$.

Figure 4 compares the first-passage time bounds Eqs. (3) with the thermodynamic uncertainty relation Eq. (33). The plotted curves are obtained from the explicit analytical expressions for $\dot{s}$ and p_-, given by Eqs. (55) and (57), respectively, and from explicit analytical expressions for $\langle T^n\rangle$ that we have obtained from the Eqs. (58-62) and can be found in the Appendix E.6. The figure shows that for large values of the first-passage thresholds the bounds Eqs. (3) are tight, as predicted by Eq. (6), while the thermodynamic uncertainty relation is loose. Hence, we can use moments of first-passage times to infer the exact value of the entropy production rate $\dot{s}$.

In Fig. 4 we also observe that the first moment $\langle T\rangle$ converges fast to its asymptotic value, while higher order moments $\langle T^2\rangle$ and $\langle T^3\rangle$ converge slowly to their asymptotic values. Using Eqs. (55), (57), and (58-62), we obtain the asymptotics (see Appendices E.7 and E.8)

$$\frac{[\ell_+]}{[\ell_-]}\frac{|\log p_-|}{\langle T_X\rangle} = \dot{s} + O\left(b^{[\ell_-]}\right), \tag{63}$$

and for $n > 1$

$$\frac{[\ell_+]}{[\ell_-]}\frac{|\log p_-|}{\left(\langle T_X^n\rangle\right)^{1/n}} = \dot{s} + O\left(\frac{1}{[\ell_+]}\right). \tag{64}$$

Hence, the first moment converges exponentially fast to the entropy production rate $\dot{s}$, while the higher order moments converge as $1/[\ell_+]$ to their asymptotic value. As a consequence, in this example the first moment is more effective for the inference of the entropy production rate $\dot{s}$. However, from Eq. (46) we can conclude that the exponential fast convergence for the first moment is a model specific property.

The asymptotic expression for the thermodynamic uncertainty relation depends on the subleading $O(1/[\ell_+])$ term in Eq. (64), and is given by

$$\frac{2\langle T_X \rangle}{\langle T_X^2 \rangle - \langle T_X \rangle^2} = \frac{2(k_+ - k_-)}{\tanh\left(\frac{a}{2\mathsf{T}_{\mathrm{env}}}\right)} + O\left(b^{[\ell_-]}\right). \tag{65}$$

Since $\tanh(x) \leq x$, Eq. (33) holds. However, contrary to Eqs. (63) and (64), the thermodynamic uncertainty relation is not tight in the limit of large thresholds and the ratio Eq. (65) depends on the affinity $a/\mathsf{T}_{\mathrm{env}}$ of the process.

Taken together, the explicit calculation of the moments $\langle T_X^n \rangle$ of a biased random walker shows that the tightness of the first-passage bounds Eqs. (3) is a consequence of the universality of the leading order term in the moments of the first-passage times, see Eqs. (63) and (64), while the looseness of the thermodynamic uncertainty relation Eq. (33) is a consequence of the nonuniversality of the subleading term of $\langle T_X^2 \rangle$.

9 Discussion

In this paper, we have shown that the rate of dissipation $\dot{s}$ in a nonequilibrium stationary process X bounds the moments of first-passage times $\langle T_J^n \rangle$ of dissipative currents J through the inequality Eq. (3). This bound is interesting for several reasons. First, the bound applies generically to dissipative currents in stationary systems as long as the threshold values ℓ_- and ℓ_+ are large enough. Second, near equilibrium, the bound Eq. (3) implies that inverse mean first-passage times are lower bounded by the Van't Hoff-Arrhenius law Eq. (1). Third, the bounds Eq. (3) describe a trade-off between speed ($\langle T_J^n \rangle$), reliability (p_-), dissipation ($\dot{s}$), and margin of error (ℓ_+/ℓ_-), which is potentially interesting to understand trade-offs in kinetic proof reading [14] or cell-fate decisions [10, 11]. Fourth, the relations can be used for the exact inference of the dissipation rate $\dot{s}$ in a nonequilibrium process. This is possible because the bounds given by Eq. (3) are equalities when $J(t) \sim S(t)$ for t large enough, see Eq. (6).

We overview the assumptions made to derive the main results Eqs. (3) and (6). The derivations of both equations rely on the physical assumption of local detailed balance, which states that the entropy production takes the form given by Eq. (7). In addition, the derivation of the inequality Eq. (3) uses the bound Eq. (20) for the large deviation function of the current, which holds for stationary, Markovian, overdamped processes [16, 17]. On the other hand, the derivation of the equality Eq. (6) relies on the martingality of $\exp(-S)$, which holds generically for stationary processes [33, 34]. Lastly, all results have been derived in the limit of large thresholds, i.e., $\ell_{\min} \gg 1$.

The assumptions on the nature of the process X made to derive the main results Eqs. (3) and (6) can be relaxed significantly. First, since the bound Eq. (3) holds in the asymptotic limit of $\ell_{\min} \gg 1$ and since effects of inertia and memory only play a role at short time scales, we can expect that Eq. (3) also holds for underdamped and nonMarkovian processes. Second, if local detailed balance does not hold, for example because the environment is not in a state of thermal equilibrium [48–52], then the bound Eq. (3) remains valid. Indeed, in this case, we obtain the bound

$$\langle T_J^n \rangle \geq \left(\frac{\ell_+}{\ell_-} \frac{|\log p_-|}{\dot{\Gamma}} \right)^n (1 + o_{\ell_{\min}}(1)), \tag{66}$$

where

$$\dot{\Gamma} = \left\langle \log \frac{p(X_0^t)}{\tilde{p}(X_0^t)} \right\rangle \tag{67}$$

quantifies the irreversibility of the process. Since irreversibility lower bounds the entropy production rate [50,53],

$$\dot{s} \geq \dot{\Gamma}, \tag{68}$$

the inequality Eq. (3) remains valid when replacing $\dot{\Gamma}$ on the right hand side of Eq. (66) by $\dot{s}$. On the other hand, notice that the equality Eq. (6) does not extend to systems that violate local detailed balance. Third, both the inequalities Eq. (3) and the equality (6) can, after reconsidering the definition of p_-, be extended to the case of nonstationary processes that are driven by a time-dependent external force. This is because the bound Eq. (20) on the large deviation function of dissipative currents extends to nonstationary systems driven by a time-dependent external force [54], and also the martingale property of $\exp(-S)$ holds for nonstationary processes [55,56], as it is a direct consequence of the fact that $\exp(-S)$ is a probability ratio, see Eq. (7). Lastly, let us discuss the assumption of large thresholds $\ell_{\rm min} \gg 1$. This assumption is key to the derived results and it is unfortunately not clear how this assumption can be relaxed without losing the simplicity and generality of the bounds given by Eq. (3) and the equality Eq. (6).

Except for relaxing the conditions under which the main results have been derived, we can also envisage to extend the result in other directions, e.g., by tightening the bounds on $\langle T_J^n \rangle$ or by deriving bounds on the moments of first-passage times of observables that are not dissipative currents. In this regard, the affinity and topology dependent bound on the large deviation function of dissipative currents derived in Refs. [16,43] is interesting as it may allow us to derive tighter bounds on $\langle T_J^n \rangle$. Also, the bound on the large deviation function of counting observables derived in Ref. [57] could be potentially interesting to derive bounds on the moments of first-passage times of counting observables.

In the particular case of $n = 1$ and for symmetric thresholds $\ell_- = \ell_+$, the bound Eq. (3) is related to a bound derived in Ref. [45] based on the asymptotic optimality of Wald's sequential probability ratio test, in the sense that it minimises the mean decision time over the set of all sequential hypothesis tests that satisfy certain reliability constraints in the limit of $\ell_{\rm min}$ large enough [58–60]. Since the derivation of the bound Eq. (3) in the present paper is based on the bound Eq. (20) on the large deviation function of dissipative currents, and since this larger deviation function bound is also key in the derivation of thermodynamic uncertainty relations [16,17], there may exist a link between on one hand the optimality of sequential probability ratio tests and on the other hand thermodynamic uncertainty relations. This connection is worthwhile exploring further as it relates two different fields of research. Lastly, notice that although the optimality of Wald's sequential probability ratio test holds for practically any process X [59,60], it also requires that the thresholds $\ell_{\rm min} \gg 1$. This further indicates that extending the bounds given by Eq. (3) and the equality Eq. (6) to small values of $\ell_{\rm min}$ is a challenging problem.

A Martingale theory of entropy production

In this appendix, we briefly revisit some key properties of martingales that we use in this paper.

A.1 Definition of a martingale

Let Ω be the set of all realisations of a physical process X, which is endowed with a σ-algebra $\mathscr{F}$. Let P be a probability measure that determines the probabilities $P(\Phi)$ of events $\Phi \in \mathscr{F}$. We denote averages with respect to P by $\langle \cdot \rangle$. Let $\{\mathscr{F}(t)\}_{t\geq 0}$ be the filtration generated by X, i.e., a sequence of sub-σ-algebras $\mathscr{F}(t)$ that is generated by the trajectories X_0^t of the process X.

A martingale $M(t)$ with respect to a filtration $\{\mathscr{F}(t)\}_{t\geq 0}$ is a stochastic process for which (i) the process $M(t)$ is $\mathscr{F}(t)$-measurable (ii) $\langle |M(t)| \rangle < \infty$ (iii) $\langle M(t) | \mathscr{F}(s) \rangle = M(s)$ [61, 62]. The latter condition implies that the martingale M is a driftless process.

A.2 Doob's optional stopping theorem

A stopping time T is a random time $T : \Omega \to \mathbb{R}^+ \cup \{\infty\}$ such that $\{T \leq t\} \in \mathscr{F}(t)$ for all values of $t \in \mathbb{R}^+$. This means that T stops the process X based on a stopping rule that cannot anticipate the future or use side information.

One of the key properties of martingales is Doob's optional stopping theorem [62].

Theorem 1 (Doob's optional stopping theorem)**.** *Let $(\Omega, \mathscr{F}, P)$ be a probability space with sample space Ω, σ-algebra $\mathscr{F}$, and probability measure P. Let $X(t)$ with $t \geq 0$ be a $\mathscr{F}$-measurable stochastic process and let $\{\mathscr{F}(t)\}_{t\geq 0}$ be the filtration generated by X. Let M be a martingale process with respect to the filtration $\{\mathscr{F}(t)\}_{t\geq 0}$ and let T be a stopping time relative to the filtration $\{\mathscr{F}(t)\}_{t\geq 0}$. It holds then that*

$$\langle M(T \wedge t) \rangle = \langle M(0) \rangle \tag{69}$$

where $T \wedge t = \min\{T, t\}$.

Doob's optional stopping theorem states that a gambler cannot make fortune out of fair game of chance by quitting at an intelligently chosen moment.

A.3 Martingales in stochastic thermodynamics

Martingale theory plays an important role in stochastic thermodynamics as the exponentiated negative entropy production, e^{-S}, is martingale [33, 34]. This is a direct consequence of the definition Eq. (7) of S and the fact that probability ratios are martingales. Applying Doob's optional stopping theorem to e^{-S} we obtain the integral fluctuation relation at stopping times [34]

$$\langle e^{-S(T)} \rangle = 1. \tag{70}$$

This relation holds when either T is bounded or when T is unbounded and $S(t)$ is bounded for all values of $t \in [0, T]$.

B Time required for a dissipative current to reach a negative threshold

In this appendix, we derive Eq. (17).

B.1 Main part of the derivation

Consider the first-passage time

$$\tilde{T}_J = \inf\{t > 0 : J(t) \notin (-\ell_+, \ell_-)\} \tag{71}$$

conjugate to T_J. At the end of this section, we show that in the limit of $\ell_{\rm min} \to \infty$ the following fluctuation relation holds for first-passage times of J, namely,

$$p_{T_J}(t|J(T_J) \leq -\ell_-) = p_{\tilde{T}_J}(t|J(\tilde{T}_J) \geq \ell_-), \tag{72}$$

for all values $t \geq 0$. In other words, the probability distribution of T_J conditioned on the event that J terminates at the negative boundary $-\ell_-$ before it reaches ℓ_+ equals the probability distribution of $\tilde{T}_J$ conditioned on the event that J terminates at the positive boundary ℓ_- before it reaches $-\ell_+$. Equation (15) of the main text applied to $\tilde{T}_j$ yields then

$$\tilde{T}_J = \frac{\ell_-}{\bar{j}}(1 + o_{\ell_{\rm min}}(1)). \tag{73}$$

Equation (17) follows readily from Eqs. (72) and (73).

B.2 First-passage time fluctuation relation for J in the limit of $\ell_{\rm min} \to \infty$

We are still left to demonstrate the first-passage time fluctuation relation Eq. (72). This relation is reminiscent of the first-passage time fluctuation relation for entropy production presented in Ref. [33], with the distinction that Eq. (72) holds for $\ell_{\rm min} \to \infty$, while the fluctuation relation for T_S in Ref. [33] holds for finite threshold values. As a consequence, also the derivation of Eq. (72) is similar to the derivation of the first-passage time fluctuation relation for entropy production in Ref. [33].

We first derive the first-passage time fluctuation relation Eq. (72) for current processes J that have continuous trajectories and for which $J' \neq 0$, as this case is simpler to deal with. Subsequently we will show that the result is also valid for processes with jumps.

Recall that we work on a probability space $(\Omega, \mathscr{F}, P)$ and that all random variables are defined on the set Ω of all physical realisations ω of the process X.

It will be easier to present a mathematical argument based on cumulative distributions then based on probability densities. The probability densities in Eq. (72) are derivatives of cumulative distributions, namely,

$$p_{T_J}(t|J(T_J) \leq -\ell_-) \;=\; \frac{d}{dt}P\left(\Phi^-_{T_J \leq t}\right), \tag{74}$$

and

$$p_{\tilde{T}_J}(t|J(\tilde{T}_J) \geq \ell_-) \;=\; \frac{d}{dt}P\left(\Phi^+_{\tilde{T}_J \leq t}\right), \tag{75}$$

where $\Phi^-_{T_J \leq t}$ is the set of all realisations ω for which $J(T_J) \leq -\ell_-$ and $T_J \leq t$, i.e.,

$$\Phi^-_{T_J \leq t} = \{\omega \in \Omega : T_J(\omega) \leq t \text{ and } J(\omega, T_J(\omega)) \leq -\ell_-\} \tag{76}$$

and analogously,

$$\Phi^+_{\tilde{T}_J \leq t} = \left\{\omega \in \Omega : \tilde{T}_J(\omega) \leq t \text{ and } J(\omega, \tilde{T}_J(\omega)) \geq \ell_-\right\}. \tag{77}$$

The first-passage time fluctuation relation Eq. (72) holds if and only if

$$P\left(\Phi^-_{T_J \leq t}\right) = cP\left(\Phi^+_{\tilde{T}_J \leq t}\right) \tag{78}$$

for all values of $t \geq 0$ and for $\ell_{\rm min} \gg 1$, where c is a constant that is independent of t.

The derivation of the symmetry relation Eq. (78) relies on two key steps:

- A first key step is a symmetry relation that relates the two sets $\Phi^-_{T_J\leq t}$ and $\Phi^+_{\tilde{T}_J\leq t}$. For continuous currents J, the sets $\Phi^-_{T_J\leq t}$ and $\Phi^+_{\tilde{T}_J\leq t}$ are related through a time-reversal operation, namely, it holds that

$$\Theta_{T_J}\left(\Phi^-_{T_J\leq t}\right) = \Phi^+_{\tilde{T}_J\leq t}, \tag{79}$$

where Θ_{T_J} is the time-reversal map that mirrors trajectories relative to the time $T_J(\omega)/2$, which is in general different for each realisation of the process. The symmetry relation Eq. (79) is valid because dissipative currents J change sign under time-reversal. Analogously, it holds that

$$\Theta_{\tilde{T}_J}\left(\Phi^+_{\tilde{T}_J\leq t}\right) = \Phi^-_{T_J\leq t}. \tag{80}$$

Because of Eq. (80), Eq. (78) reads

$$P\left(\Theta_{\tilde{T}_J}\left(\Phi^+_{\tilde{T}_J\leq t}\right)\right) = cP\left(\Phi^+_{\tilde{T}_J\leq t}\right). \tag{81}$$

- A second key step uses the fact that the exponentiated negative entropy production is the Radon-Nikodym derivative between the probability measure P and the time-reversed probability measure $P\circ\Theta_t$, namely,

$$e^{-S(t)} = \frac{d(P\circ\Theta_t)|_{\mathcal{F}(t)}}{dP|_{\mathcal{F}(t)}}, \tag{82}$$

where Θ_t is the time-reversal map that mirrors trajectories relative to the midpoint $t/2$, and where $P|_{\mathcal{F}(t)}$ denotes the restriction of the measure P to the sub-σ-algebra $\mathcal{F}(t)$ generated by trajectories X_0^t. The restriction of P to the sub-σ-algebra $\mathscr{F}(t)$ is defined as

$$P\left(\Phi\right) = \begin{cases} 0 & \text{if} \quad \Phi\notin\mathscr{F}(t), \\ P(\Phi) & \text{if} \quad \Phi\in\mathscr{F}(t). \end{cases} \tag{83}$$

Analogously, at the first-passage time $\tilde{T}_J$, it holds that

$$e^{-S(\tilde{T}_J)} = \frac{d(P\circ\Theta_{\tilde{T}_J})\Big|_{\mathcal{F}(\tilde{T}_J)}}{dP|_{\mathcal{F}(\tilde{T}_J)}}, \tag{84}$$

where $\mathcal{F}(\tilde{T}_J)$ is now the sub-σ-algebra generated by trajectories $X_0^{T_J}$ terminating at the first-passage time $\tilde{T}_J$.

We now use Eq. (84) to express $P\left(\Theta_{\tilde{T}_J}\left(\Phi^+_{\tilde{T}_J\leq t}\right)\right)$ as an expectation value over the measure P, viz.,

$$\begin{aligned} P\left(\Theta_{\tilde{T}_J}\left(\Phi^+_{\tilde{T}_J\leq t}\right)\right) &= \left(P\circ\Theta_{\tilde{T}_J}\right)\left(\Phi^+_{\tilde{T}_J\leq t}\right) && (85)\\ &= \int_{\omega\in\Phi^+_{\tilde{T}_J\leq t}} d(P\circ\Theta_{\tilde{T}_J}) && (86)\\ &= \int_{\omega\in\Phi^+_{\tilde{T}_J\leq t}} d(P\circ\Theta_{\tilde{T}_J})\Big|_{\mathcal{F}(\tilde{T}_J)} && (87)\\ &= \int_{\omega\in\Phi^+_{\tilde{T}_J\leq t}} e^{-S(\tilde{T}_J)}\, dP|_{\mathcal{F}(\tilde{T}_J)}\,. && (88) \end{aligned}$$

In Eq. (86), we have written the weigth of the set $\Phi^+_{\tilde{T}_J\leq t}$ over the measure $P\circ\Theta_{\tilde{T}_J}$ as an integral over a probability space, see Ref. [63]. In Eq. (87), we use that $\Phi^+_{\tilde{T}_J\leq t}\in\mathcal{F}(\tilde{T}_J)$ and therefore we can integrate over all irrelevant degrees of freedom, which boils down to replacing $P\circ\Theta_{\tilde{T}_J}$ by $P\circ\Theta_{\tilde{T}_J}\big|_{\mathcal{F}(\tilde{T}_J)}$. Lastly, in the Eq. (86) we use the Radon-Nikodym theorem, see Ref. [62], together with Eq. (84).

- Finally, we use that in the limit of $\ell_{\rm min}\gg 1$, Eq. (73) holds, and thus

$$P\left(\Theta_{\tilde{T}_J}\left(\Phi^+_{\tilde{T}_J\leq t}\right)\right)=e^{-S(\ell_-/\bar{\jmath})}\int_{\omega\in\Phi^+_{\tilde{T}_J\leq t}}dP|_{\mathcal{F}(\tilde{T}_J)}=e^{-S(\ell_-/\bar{\jmath})}P\left(\Phi^+_{\tilde{T}_J\leq t}\right). \tag{89}$$

Hence, to conclude, Eq. (78) holds with the constant $c=e^{-S(\ell_-/\bar{\jmath})}$, and therefore also Eq. (72) holds, which we were meant to show.

Lastly, let us comment on the case where the current J has jumps. In this scenario the Eqs. (79) and (80) are in general not valid. However, if we assume that J jumps at the initial time $t=0$, then the Eqs. (79) and (80) hold, and therefore also Eq. (72) as the remainder part of the derivation does not depend on the continuity of J. Assuming that a jump of J happens at the time $t=0$ is not going to effect the distribution of T_J in the limit of large thresholds, and therefore the result Eq. (72) holds in the limit $\ell_{\rm min}\gg 1$ also for processes with jumps.

C Mean-first passage time for a Brownian particle in a generic periodic potential and in a uniform force field

In this appendix, we analyse the first-passage problem for a Brownian motion in a generic periodic potential u and a uniform force field f, as described by Eq. (40). In particular, we derive analytical expressions for the mean-first passage time $\langle T_X\rangle$, the splitting probability p_-, and the mean entropy production rate $\dot{s}$, where T_X is defined as in Eq. (44). In the limit of large thresholds $\ell\gg 1$, we show that the main result Eq. (6) holds. In addition, in the near-equilibrium limit and at low temperatures, we show that Eq. (6) is a Van't Hoff-Arrhenius law.

C.1 Stationary distribution and current

We derive Eq. (43) in the main text for the stationary current $j_{\rm ss}$.

The stationary distribution of $X\in\mathbb{R}$ does not exist. However, we can define the process on a ring with periodic boundary conditions such that $X(t)=X(t)+\delta$. The stationary state $p_{\rm ss}$ of the equivalent process defined on a ring exists, and we can use the stationary process on a ring to determine the stationary current $j_{\rm ss}$.

The stationary distribution $p_{\rm ss}$ solves the equation [31,64]

$$\partial_x j_{\rm ss}(x)=0 \tag{90}$$

with periodic boundary conditions $p_{\rm ss}(x)=p_{\rm ss}(x+\delta)$, where

$$j_{\rm ss}(x)=\mu(f-\partial_x u(x))p_{\rm ss}(x)-\frac{\mathsf{T}_{\rm env}}{\gamma}\partial_x p_{\rm ss}(x). \tag{91}$$

The solution to Eq. (90) is given by [34,65]

$$p_{\rm ss}(x)=\frac{w(x)\left(\int_x^{x+\delta}{\rm d}x'\,\frac{1}{w(x')}\right)}{\int_0^\delta {\rm d}y\,w(y)\left(\int_y^{y+\delta}{\rm d}x'\,\frac{1}{w(x')}\right)} \tag{92}$$

with $x \in [0, \delta]$, and where

$$w(x) = e^{-\frac{u(x)-fx}{\mathsf{T}_{\rm env}}}. \tag{93}$$

The expression Eq. (43) for the stationary current $j_{\rm ss}$ follows readily from the Eqs. (91) and (92).

C.2 Entropy production

We derive Eqs. (42) and (45) in the main text for the entropy production rate $\dot{s}$ and the stochastic entropy production S, respectively. We will again use the equivalent process defined on a ring with periodic boundary conditions.

The stochastic entropy production S of X, as defined in Eq. (7), is determined by the stochastic differential equation [35, 66]

$$\mathrm{d}S = v_S(X)\,\mathrm{d}t + \sqrt{2v_S(X)}\,\mathrm{d}W(t), \tag{94}$$

where

$$v_S(x) = \frac{\gamma}{\mathsf{T}_{\rm env}}\frac{j^2_{\rm ss}}{p^2_{\rm ss}(x)} = \frac{\mathsf{T}_{\rm env}}{\gamma}\frac{\left(1-e^{\frac{-f\delta}{\mathsf{T}_{\rm env}}}\right)^2}{w^2(x)\left(\int_x^{x+\delta}\mathrm{d}x'\frac{1}{w(x')}\right)^2}. \tag{95}$$

Alternatively, we can write

$$S(t) = \frac{fX(t) - u(X(t)) + u(X(0))}{\mathsf{T}_{\rm env}} + \log\frac{p_{\rm ss}(X(0))}{p_{\rm ss}(X(t))}. \tag{96}$$

The latter formula implies that for large $t \gg 1$ it holds that

$$S(t) = \frac{fX(t)}{\mathsf{T}_{\rm env}} + o(t), \tag{97}$$

which is Eq. (45) in the main text.

The average stationary entropy production rate is given by

$$\dot{s} \;=\; \frac{\langle S(t)\rangle}{t} = \langle v_S\rangle = \frac{\gamma j^2_{\rm ss}}{\mathsf{T}_{\rm env}}\int_0^\delta \frac{\mathrm{d}x}{p_{\rm ss}(x)}. \tag{98}$$

Since the stationary distribution $p_{\rm ss}$ is given by Eq. (92) and $u(x)$ is a periodic function, we can express this also as

$$\dot{s} \;=\; j_{\rm ss}\left(1-e^{\frac{-f\delta}{\mathsf{T}_{\rm env}}}\right)\int_0^\delta \mathrm{d}x\frac{1}{w(x)\left(\int_0^\delta \mathrm{d}x'\frac{1}{w(x')} - (1-e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}})\int_0^x \mathrm{d}x'\frac{1}{w(x')}\right)}. \tag{99}$$

Introducing the function

$$\int_0^x \mathrm{d}x'\frac{1}{w(x')} = W(x). \tag{100}$$

we find that

$$\dot{s} \;=\; j_{\rm ss}\left(1-e^{\frac{-f\delta}{\mathsf{T}_{\rm env}}}\right)\int_0^{W(\delta)}\mathrm{d}u\frac{1}{\left(W(\delta)-(1-e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}})u\right)}. \tag{101}$$

Integrating yields the expression for $\dot{s}$ given by Eq. (42) in the main text.

C.3 Splitting probabilities

We use the martingale property of $e^{-S(t)}$, see Refs. [33, 34] or Appendix A, to determine the splitting probabilities p_- and p_+. Doob's optional stopping theorem for martingales implies the following integral fluctuation relation at stopping times

$$\langle e^{-S(T_X)}|X(0)=0\rangle = e^{-S(0)} = 1, \tag{102}$$

and since $S(t)$ is continuous as a function of t this implies that, see Refs. [33, 34],

$$p_- = e^{-s_-}\frac{1-e^{-s_+}}{1-e^{-s_- - s_+}}, \quad \text{and} \quad p_+ = \frac{1-e^{-s_-}}{1-e^{-s_- - s_+}}, \tag{103}$$

where

$$s_- = -\frac{-f\ell - u(-\ell)+u(0)}{\mathsf{T}_{\rm env}} - \log\frac{p_{\rm ss}(0)}{p_{\rm ss}(-\ell)}, \quad \text{and} \quad s_+ = \frac{f\ell - u(\ell)+u(0)}{\mathsf{T}_{\rm env}} + \log\frac{p_{\rm ss}(0)}{p_{\rm ss}(\ell)}. \tag{104}$$

Notice that we have used a slight abuse of notation in the sense that $u(x)$ and $p_{\rm ss}(x)$ are here defined on $x \in \mathbb{R}$ using $u(x) = u(x \pm \delta)$ and $p_{\rm ss}(x) = p_{\rm ss}(x \pm \delta)$.

C.4 Mean first-passage time

Consider the backward Fokker-Planck equation

$$\mu\left(f - \partial_x u(x)\right)\partial_x t(x) + \frac{\mathsf{T}_{\rm env}}{\gamma}\partial_x^2 t(x) = -1 \tag{105}$$

with boundary conditions $t(-\ell) = t(\ell) = 0$. It then holds that, see Ref. [67],

$$\langle T_X | X(0) = x\rangle = t(0). \tag{106}$$

The solution of $t(x)$ to Eq, (105) with boundary conditions $t(-\ell) = t(\ell) = 0$ is given by

$$t(x) = \frac{\gamma}{\mathsf{T}_{\rm env}}\int_{-\ell}^{\ell}{\rm d}y\frac{1}{w(y)}\int_0^y {\rm d}x' w(x')\left(\frac{\int_{-\ell}^{x}{\rm d}y\frac{1}{w(y)}}{\int_{-\ell}^{\ell}{\rm d}y\frac{1}{w(y)}} - \frac{\int_{-\ell}^{x}{\rm d}y\frac{1}{w(y)}\int_0^y{\rm d}x' w(x')}{\int_{-\ell}^{\ell}{\rm d}y\frac{1}{w(y)}\int_0^y{\rm d}x' w(x')}\right), \tag{107}$$

and therefore

$$\langle T_X\rangle = \frac{\gamma}{\mathsf{T}_{\rm env}}\left(\int_{-\ell}^{\ell}{\rm d}y\frac{1}{w(y)}\int_0^y {\rm d}x' w(x')\right)\left(\frac{\int_{-\ell}^{0}{\rm d}y\frac{1}{w(y)}}{\int_{-\ell}^{\ell}{\rm d}y\frac{1}{w(y)}} - \frac{\int_{-\ell}^{0}{\rm d}y\frac{1}{w(y)}\int_0^y{\rm d}x' w(x')}{\int_{-\ell}^{\ell}{\rm d}y\frac{1}{w(y)}\int_0^y{\rm d}x' w(x')}\right). \tag{108}$$

In order to better understand the structure of the expression Eq. (108) for the mean-first passage time, it is useful to express the integrals in Eq. (108) that run over the intervals $[-\ell, \ell]$ and $[-\ell, 0]$ in terms of integrals that run over the interval $[0, \delta]$. Let $n = [\ell/\delta]$ be the largest integer smaller than ℓ/δ, then we can write

$$\ell = n\delta + z, \tag{109}$$

with $z \in [0, \delta]$. Using this decomposition for ℓ, we obtain that

$$\int_{-n\delta - z}^{0}{\rm d}y\frac{1}{w(y)} \quad = \quad e^{n\frac{f\delta}{\mathsf{T}_{\rm env}}}\left\{\left(\frac{1-e^{-n\frac{f\delta}{\mathsf{T}_{\rm env}}}}{1-e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}}\right)\int_0^\delta\frac{{\rm d}x}{w(x)} + e^{\frac{f\delta}{\mathsf{T}_{\rm env}}}\int_{\delta-z}^{\delta}\frac{{\rm d}x}{w(x)}\right\} \tag{110}$$

and

$$\int_{-n\delta-z}^{n\delta+z} \mathrm{d}y \frac{1}{w(y)}$$
$$= e^{n\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}} \left\{ \left(\frac{1-e^{-2n\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}}{1-e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}} \right) \int_0^\delta \frac{\mathrm{d}x}{w(x)} + e^{\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}} \int_{\delta-z}^{\delta} \frac{\mathrm{d}x}{w(x)} + e^{-2n\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}} \int_0^z \frac{\mathrm{d}x}{w(x)} \right\}. \tag{111}$$

In addition,

$$\int_0^{n\delta+z} \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x' w(x')$$
$$= n \left\{ \frac{e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}}{1-e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}} \int_0^\delta \mathrm{d}x \frac{1}{w(x)} \int_0^\delta \mathrm{d}x\, w(x) + \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_0^y w(x)\mathrm{d}x \right\}$$
$$- \frac{e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}} \left(1-e^{-n\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}\right)}{\left(1-e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}\right)^2} \int_0^\delta \mathrm{d}x\, w(x) \int_0^\delta \mathrm{d}x \frac{1}{w(x)}$$
$$+ e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}} \frac{1-e^{-n\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}}{1-e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}} \int_0^z \mathrm{d}y \frac{1}{w(y)} \int_0^\delta \mathrm{d}x\, w(x) + \int_0^z \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x\, w(x), \tag{112}$$

and

$$- \int_{-n\delta-z}^{0} \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x'\, w(x')$$
$$= \frac{1-e^{n\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}}{(1-e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}})(1-e^{\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}})} \left(\int_0^\delta \mathrm{d}x\, w(x) \right) \left(\int_0^\delta \mathrm{d}x \frac{1}{w(x)} \right)$$
$$+ n \left\{ \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_y^\delta \mathrm{d}x\, w(x) - \frac{1}{1-e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}} \left(\int_0^\delta \mathrm{d}x\, w(x) \right) \left(\int_0^\delta \mathrm{d}x \frac{1}{w(x)} \right) \right\}$$
$$+ \frac{e^{n\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}} - 1}{1-e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}} \int_{\delta-z}^{\delta} \mathrm{d}x \frac{1}{w(x)} \int_0^\delta \mathrm{d}x\, w(x) + \int_{\delta-z}^{\delta} \mathrm{d}y \frac{1}{w(y)} \int_y^\delta \mathrm{d}x\, w(x). \tag{113}$$

Using the Eqs. (110), (111), (112), and (113) in Eq. (108), we obtain an expression for $\langle T_X \rangle$ that depends only on integrals over the interval $[0, \delta]$.

C.5 Limit of large thresholds

We derive the Eq. (46) that holds in the limit of large ℓ.

C.5.1 Splitting probabilities

In the limit of large thresholds, the linear term in ℓ dominates the Eqs. (104) and therefore

$$s_- = \frac{f\ell}{\mathsf{T}_{\mathrm{env}}} + O_\ell(1), \quad \text{and} \quad s_+ = \frac{f\ell}{\mathsf{T}_{\mathrm{env}}} + O_\ell(1). \tag{114}$$

Using Eq. (114) in the Eqs. (103) for p_- and p_+, we obtain that

$$\log p_- = -\frac{f\ell}{\mathsf{T}_{\mathrm{env}}} + O_\ell(1), \quad \text{and} \quad \log p_+ = 1 + O_\ell(1). \tag{115}$$

C.5.2 Mean first-passage time

We use that

$$n = \left[\frac{\ell}{\delta}\right] + O_\ell(1), \tag{116}$$

where as before $\left[\frac{\ell}{\delta}\right]$ denotes the largest integer that is smaller than $\frac{\ell}{\delta}$.

Taking the asymptotic limit of large ℓ in Eqs. (110) and (111), we obtain that

$$\frac{\int_{-\ell}^{0} \mathrm{d}y \frac{1}{w(y)}}{\int_{-\ell}^{\ell} \mathrm{d}y \frac{1}{w(y)}} = 1 - e^{-\left[\frac{\ell}{\delta}\right]\frac{f\delta}{\mathsf{T}_{\rm env}}} \frac{\int_0^\delta \frac{\mathrm{d}x}{w(x)}}{\int_0^\delta \frac{\mathrm{d}x}{w(x)} + (e^{\frac{f\delta}{\mathsf{T}_{\rm env}}} - 1)\int_{\delta-z}^{\delta} \frac{\mathrm{d}x}{w(x)}} + O\left(e^{-2\left[\frac{\ell}{\delta}\right]\frac{f\delta}{\mathsf{T}_{\rm env}}}\right). \tag{117}$$

The asymptotic limit of Eq. (112) is

$$\begin{aligned}
&\int_0^\ell \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x' w(x') \\
&= \left[\frac{\ell}{\delta}\right] \left\{ \frac{e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}}{1 - e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}} \int_0^\delta \mathrm{d}x \frac{1}{w(x)} \int_0^\delta \mathrm{d}x w(x) + \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_0^y w(x)\mathrm{d}x \right\} + O_\ell(1),
\end{aligned} \tag{118}$$

and from Eqs. (112) and (113) it follows that

$$\begin{aligned}
&-\int_{-\ell}^{\ell} \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x' w(x') \\
&= e^{\left[\frac{\ell}{\delta}\right]\frac{f\delta}{\mathsf{T}_{\rm env}}} \left\{ \frac{\int_0^\delta \mathrm{d}x\, w(x) \int_0^\delta \mathrm{d}x \frac{1}{w(x)}}{(1 - e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}})(e^{\frac{f\delta}{\mathsf{T}_{\rm env}}} - 1)} + \frac{\int_{\delta-z}^{\delta} \mathrm{d}x \frac{1}{w(x)} \int_0^\delta \mathrm{d}x w(x)}{1 - e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}} \right\} \\
&\quad + \left[\frac{\ell}{\delta}\right] \left\{ \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_y^\delta \mathrm{d}x\, w(x) - \frac{1}{\tanh\left(\frac{f\delta}{2\mathsf{T}_{\rm env}}\right)} \int_0^\delta \mathrm{d}x \frac{1}{w(x)} \int_0^\delta \mathrm{d}x\, w(x) \right. \\
&\quad \left. - \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x\, w(x) \right\} + O_\ell(1).
\end{aligned} \tag{119}$$

The Eqs. (118) and (119) imply that the ratio

$$\begin{aligned}
&\frac{\int_{-\ell}^{0} \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x' w(x')}{\int_{-\ell}^{\ell} \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x' w(x')} \\
&= 1 + \left[\frac{\ell}{\delta}\right] e^{-\left[\frac{\ell}{\delta}\right]\frac{f\delta}{\mathsf{T}_{\rm env}}} \left\{ \frac{e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}} \left(\int_0^\delta \mathrm{d}x\, w(x)\right)\left(\int_0^\delta \mathrm{d}x \frac{1}{w(x)}\right) + \left(1 - e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}\right) \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_0^y w(x)\mathrm{d}x}{\frac{\int_0^\delta \mathrm{d}x w(x) \int_0^\delta \mathrm{d}x \frac{1}{w(x)}}{e^{\frac{f\delta}{\mathsf{T}_{\rm env}}} - 1} + \int_{\delta-z}^{\delta} \mathrm{d}x \frac{1}{w(x)} \int_0^\delta \mathrm{d}x\, w(x)} \right\} \\
&\quad + O\left(e^{-\left[\frac{\ell}{\delta}\right]\frac{f\delta}{\mathsf{T}_{\rm env}}}\right).
\end{aligned} \tag{120}$$

Using Eqs. (117)-(120) in Eq. (108) yields for the mean first-passage time the asymptotic expression

$$\langle T_X \rangle = \frac{\gamma}{\mathsf{T}_{\rm env}} \left[\frac{\ell}{\delta}\right] \left[\frac{e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}}{1 - e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}} \left(\int_0^\delta \mathrm{d}x w(x)\right)\left(\int_0^\delta \mathrm{d}x \frac{1}{w(x)}\right) + \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_0^y w(x)\mathrm{d}x \right] + O_\ell(1). \tag{121}$$

C.5.3 The ratio $|\log p_-|/\langle T_X \rangle$

It follows from the asymptotic relations for $\langle T_X \rangle$ and $|\log p_-|$, given by Eqs. (121) and (115), respectively, that the ratio

$$\frac{|\log p_-|}{\langle T_X \rangle} \quad = \quad \frac{f\delta}{\gamma} \frac{1 - e^{\frac{-f\delta}{\mathsf{T}_{\rm env}}}}{\int_0^\delta \mathrm{d}y\, w(y) \left(\int_y^{y+\delta} \mathrm{d}x' \frac{1}{w(x')} \right)} + O(1/\ell). \tag{122}$$

Using Eqs. (42) and (43) for $\dot{s}$ and $j_{\rm ss}$, respectively, together with the identities

$$\int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x\, w(x) = \int_0^\delta \mathrm{d}y\, w(y) \int_y^\delta \frac{1}{w(x)} \mathrm{d}x \tag{123}$$

and

$$e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}} \int_0^\delta \mathrm{d}x\, w(x) \int_0^y \mathrm{d}x \frac{1}{w(x)} = \int_0^\delta \mathrm{d}x\, w(x) \int_\delta^{y+\delta} \mathrm{d}x \frac{1}{w(x)}, \tag{124}$$

we readily obtain Eq. (46), which is what we were meant to show.

C.6 Van't Hoff-Arrhenius law near equilibrium

We show that Eq. (46) yields the Van't Hoff-Arrhenius law Eq. (51).

Indeed, if ℓ is large enough, then Eq. (46) together with Eq. (115) yields

$$\langle T_X \rangle = \frac{f\ell}{\mathsf{T}_{\rm env}} \frac{1}{\dot{s}} + O\left(\frac{1}{\ell}\right) \tag{125}$$

where the mean entropy production rate $\dot{s}$ is given by Eq. (42). Since the mean entropy production rate is proportional to the stationary current, given by Eq. (43), we can use saddle point integrals to evaluate the mean current in the limit $\mathsf{T}_{\rm env} \to 0$ and to obtain the Van't Hoff-Arrhenius law.

Let us therefore first revisit the saddle point method, and then apply it to the mean current to obtain the Van't Hoff-Arrhenius law.

C.6.1 Saddle point integrals in the limit of $\mathsf{T}_{\rm env} \to 0$

We first revisit briefly the saddle point method.

Let $v(x)$ be a function defined on the interval $[0, \delta]$. Then we analyse integrals of the form

$$\int_0^\delta \mathrm{d}x\, e^{\frac{v(x)}{\mathsf{T}_{\rm env}}} f(x) \tag{126}$$

in the limiting case of small $\mathsf{T}_{\rm env}$. In this limiting case,

$$\int_0^\delta \mathrm{d}x\, e^{\frac{v(x)}{\mathsf{T}_{\rm env}}} f(x) = \kappa f(x_{\rm max}) e^{\frac{v_{\rm max}}{\mathsf{T}_{\rm env}}} + O\left(\frac{\mathsf{T}_{\rm env}}{v_{\rm max}}\right) \tag{127}$$

where κ is a prefactor that depends on the properties of the function v at the maximum. Note that we use the following notation: if $x_{\rm max} = \mathrm{argmax}\, v(x)$, then $v_{\rm max} = v(x_{\rm max})$, $v'_{\rm max} = v'(x_{\rm max})$, and $v''_{\rm max} = v''(x_{\rm max})$.

There exists four relevant cases:

- $v'_{\rm max} = 0$ and $x_{\rm max} \in (0, \delta)$:

$$\kappa = \sqrt{\frac{2\pi \mathsf{T}_{\rm env}}{-v''_{\rm max}}}; \tag{128}$$

- $v'_{\rm max}$ does not exist (maximum is a cusp) and $x_{\rm max} \in (0,\delta)$:

$$\kappa = \mathsf{T}_{\rm env}\left(\frac{1}{v^+_{\rm max}} - \frac{1}{v^-_{\rm max}}\right) \tag{129}$$

where

$$v^+_{\rm max} = \lim_{\epsilon\to 0}\frac{v(x_{\rm max}) - v(x_{\rm max}-\epsilon)}{\epsilon}, \quad \text{and} \quad v^-_{\rm max} = \lim_{\epsilon\to 0}\frac{v(x_{\rm max}+\epsilon) - v(x_{\rm max})}{\epsilon}; \tag{130}$$

- $x_{\rm max} = 0$:

$$\kappa = -\frac{\mathsf{T}_{\rm env}}{v^-_{\rm max}}; \tag{131}$$

- $x_{\rm max} = \delta$:

$$\kappa = \frac{\mathsf{T}_{\rm env}}{v^+_{\rm max}}. \tag{132}$$

C.6.2 The mean first-passage time in the low temperature limit and the linear response limit

We consider first the near equilibrium limit with $f\delta/\mathsf{T}_{\rm env} \approx 0$, and then we consider the low temperature limit $\mathsf{T}_{\rm env} \approx 0$.

First we take the linear response limit with $f\delta/\mathsf{T}_{\rm env} \approx 0$. It holds then that

$$w(x) = e^{-\frac{u(x)}{\mathsf{T}_{\rm env}}}\left(1 + \frac{fx}{\mathsf{T}_{\rm env}} + O\left(\left(\frac{f\delta}{\mathsf{T}_{\rm env}}\right)^2\right)\right), \tag{133}$$

and

$$\frac{1}{w(x)} = e^{\frac{u(x)}{\mathsf{T}_{\rm env}}}\left(1 - \frac{fx}{\mathsf{T}_{\rm env}} + O\left(\left(\frac{f\delta}{\mathsf{T}_{\rm env}}\right)^2\right)\right), \tag{134}$$

such that

$$j_{\rm ss} = \frac{f\delta}{\gamma}\frac{1}{\int_0^\delta {\rm d}y e^{-\frac{u(y)}{\mathsf{T}_{\rm env}}}\int_0^\delta {\rm d}x e^{\frac{u(x)}{\mathsf{T}_{\rm env}}}} + O\left(\left(\frac{f\delta}{\mathsf{T}_{\rm env}}\right)^2\right). \tag{135}$$

Second, we take the low temperature limit with $\mathsf{T}_{\rm env} \approx 0$. Using the saddle point method, we obtain that

$$j_{\rm ss} = \frac{f\delta}{\gamma}\kappa_1\kappa_2 e^{-\frac{E_{\rm b}}{\mathsf{T}_{\rm env}}} + O\left(\left(\frac{f\delta}{\mathsf{T}_{\rm env}}\right)^2\right) \tag{136}$$

where κ_1 and κ_2 are two prefactors due to the two saddle point integrals in Eq. (135). The entropy production rate follows from Eq. (42) and is given by

$$\dot{s} = \frac{(f\delta)^2}{\gamma\mathsf{T}_{\rm env}}\kappa_1\kappa_2 e^{-\frac{E_{\rm b}}{\mathsf{T}_{\rm env}}} + O\left(\left(\frac{f\delta}{\mathsf{T}_{\rm env}}\right)^3\right). \tag{137}$$

Lastly, using Eq. (125) we obtain the Van't Hoff-Arrhenius law for the mean-first passage time

$$\langle T_X\rangle = \frac{\ell}{\delta}\frac{\gamma}{f\delta}\frac{1}{\kappa_1\kappa_2}e^{\frac{E_{\rm b}}{\mathsf{T}_{\rm env}}}\left(1 + O\left(\frac{f\delta}{\mathsf{T}_{\rm env}}\right)\right). \tag{138}$$

We discuss two relevant cases:

- $u'_{\rm max} = u'_{\rm min} = 0$ and $x_{\rm max}, x_{\rm min} \in (0, \delta)$:

$$\kappa_1\kappa_2 = \frac{\sqrt{-u''_{\rm min}u''_{\rm max}}}{2\pi \mathsf{T}_{\rm env}}; \tag{139}$$

- $u'_{\rm max} \neq 0$ and $u'_{\rm min} \neq 0$:

$$\kappa_1\kappa_2 = \left(\frac{1}{u^+_{\rm max}} - \frac{1}{u^-_{\rm max}}\right)^{-1}\left(\frac{1}{u^+_{\rm min}} - \frac{1}{u^-_{\rm min}}\right)^{-1}\frac{1}{\mathsf{T}^2_{\rm env}}. \tag{140}$$

D Mean-first passage time for a Brownian particle in a periodic potential that is triangular and in a uniform force field

We consider again a Brownian motion in a uniform force field f and a periodic potential u, for which dynamics of the position variable X is described by the overdamped Langevin Eq. (40). However, now we consider the specific case of the triangular potential given by Eq. (41). We have used this case to generate the curves in the Figs. 1-3.

D.1 Stationary distribution

The stationary probability distribution, given by Eq. (92), for a triangular potential is given by [35]

$$p_{\rm ss}(x) = \begin{cases} a_1 + a_2 e^{\frac{x f_+}{\mathsf{T}_{\rm env}}} & \text{if} \quad x \in [0, x^*], \\ a_3 + a_4 e^{\frac{x f_-}{\mathsf{T}_{\rm env}}} & \text{if} \quad x \in [x^*, \delta], \end{cases} \tag{141}$$

where

$$f_+ = f - \frac{u_0}{x^*}, \quad \text{and} \quad f_- = f + \frac{u_0}{\delta - x^*}, \tag{142}$$

and

$$\begin{aligned} a_1 &= f_+ f_-^2 \frac{e^{\frac{f_- x^*}{\mathsf{T}_{\rm env}}} - e^{\frac{f_-\delta + f_+ x^*}{\mathsf{T}_{\rm env}}}}{\mathcal{N}}, & (143)\\ a_2 &= f_+ f_-(f_- - f_+)\frac{e^{\frac{f_-\delta}{\mathsf{T}_{\rm env}}} - e^{\frac{f_- x^*}{\mathsf{T}_{\rm env}}}}{\mathcal{N}}, & (144)\\ a_3 &= f_+^2 f_- \frac{e^{\frac{f_- x^*}{\mathsf{T}_{\rm env}}} - e^{\frac{f_-\delta + f_+ x^*}{\mathsf{T}_{\rm env}}}}{\mathcal{N}}, & (145)\\ a_4 &= f_+ f_-(f_+ - f_-)\frac{e^{\frac{f_+ x^*}{\mathsf{T}_{\rm env}}} - 1}{\mathcal{N}}, & (146) \end{aligned}$$

and where the normalisation constant

$$\begin{aligned} \mathcal{N} = \mathsf{T}_{\rm env}(f_+ - f_-)^2 &\left(e^{\frac{f_+ x^*}{\mathsf{T}_{\rm env}}} - 1\right)\left(e^{\frac{f_-\delta}{\mathsf{T}_{\rm env}}} - e^{\frac{f_- x^*}{\mathsf{T}_{\rm env}}}\right) \\ &+ f_+ f_-(f_+\delta - f_+ x^* + f_- x^*)\left(e^{\frac{f_- x^*}{\mathsf{T}_{\rm env}}} - e^{\frac{f_-\delta + f_+ x^*}{\mathsf{T}_{\rm env}}}\right). \end{aligned} \tag{147}$$

The stationary current is given by the expression

$$j_{\rm ss} = \frac{f_+ a_1}{\gamma} = \frac{f_- a_3}{\gamma}. \tag{148}$$

In Fig. 5, we plot the stationary distribution p_{ss} for various values of the nonequilibrium driving $f\delta/\mathsf{T}_{env}$. Observe that the distribution concentrates around the values $x \approx 0$ or $x \approx \delta$, and thus the process resembles a hopping process, as is also visible in Fig. 1.

D.2 Mean first-passage time

In the case of the triangular potential we can obtain an explicit expression for $\langle T_X \rangle$ given by Eq. (108). This is because the integrals that appear in the Eqs. (110), (111), (112), and (113) can be solved explicitly.

We obtain explicit expressions for the following integrals:

$$\int_0^z \mathrm{d}x\, w(x) = \begin{cases} \frac{\mathsf{T}_{env}}{f_+}\left(e^{\frac{f_+ z}{\mathsf{T}_{env}}} - 1\right) & \text{if} \quad z < x^*, \\ \frac{\mathsf{T}_{env}}{f_+}\left(e^{\frac{f_+ x^*}{\mathsf{T}_{env}}} - 1\right) + \frac{\mathsf{T}_{env}}{f_-} e^{-\frac{u_0}{\mathsf{T}_{env}}\frac{\delta}{\delta - x^*}}\left(e^{\frac{f_- z}{\mathsf{T}_{env}}} - e^{\frac{f_- x^*}{\mathsf{T}_{env}}}\right) & \text{if} \quad z > x^*, \end{cases} \tag{149}$$

$$\int_0^z \frac{\mathrm{d}x}{w(x)} = \begin{cases} \frac{\mathsf{T}_{env}}{f_+}\left(1 - e^{\frac{-f_+ z}{\mathsf{T}_{env}}}\right) & \text{if} \quad z < x^*, \\ \frac{\mathsf{T}_{env}}{f_+}\left(1 - e^{\frac{-f_+ x^*}{\mathsf{T}_{env}}}\right) + \frac{\mathsf{T}_{env}}{f_-} e^{\frac{u_0}{\mathsf{T}_{env}}\frac{\delta}{\delta - x^*}}\left(e^{\frac{-f_- x^*}{\mathsf{T}_{env}}} - e^{\frac{-f_- z}{\mathsf{T}_{env}}}\right) & \text{if} \quad z > x^*, \end{cases} \tag{150}$$

and

$$\begin{aligned} &\int_{\delta - z}^{\delta} \frac{\mathrm{d}x}{w(x)} \\ &= \begin{cases} \frac{\mathsf{T}_{env}}{f_+}\left(e^{\frac{-f_+(\delta - z)}{\mathsf{T}_{env}}} - e^{\frac{-f_+ x^*}{\mathsf{T}_{env}}}\right) + \frac{\mathsf{T}_{env}}{f_-} e^{\frac{u_0}{\mathsf{T}_{env}}\frac{\delta}{\delta - x^*}}\left(e^{\frac{-f_- x^*}{\mathsf{T}_{env}}} - e^{\frac{-f_- \delta}{\mathsf{T}_{env}}}\right) & \text{if} \quad \delta - z < x^*, \\ \frac{\mathsf{T}_{env}}{f_-} e^{\frac{u_0}{\mathsf{T}_{env}}\frac{\delta}{\delta - x^*}}\left(e^{\frac{-f_-(\delta - z)}{\mathsf{T}_{env}}} - e^{\frac{-f_- \delta}{\mathsf{T}_{env}}}\right) & \text{if} \quad \delta - z > x^*. \end{cases} \end{aligned} \tag{151}$$

In addition, if $z < x^*$, then

$$\int_0^z \mathrm{d}y \frac{1}{w(y)} \int_0^y w(x)\mathrm{d}y = \frac{\mathsf{T}_{env}}{f_+} z - \left(\frac{\mathsf{T}_{env}}{f_+}\right)^2 \left(1 - e^{-\frac{f_+ z}{\mathsf{T}_{env}}}\right), \tag{152}$$

and if $z > x*$, then

$$\begin{aligned} &\int_0^z \mathrm{d}y \frac{1}{w(y)} \int_0^y w(x)\mathrm{d}y \\ &= \frac{\mathsf{T}_{env}}{f_+} x^* + \frac{\mathsf{T}_{env}}{f_-}(z - x^*) - \left(\frac{\mathsf{T}_{env}}{f_+}\right)^2 \left(1 - e^{-\frac{f_+ x^*}{\mathsf{T}_{env}}}\right) - \left(\frac{\mathsf{T}_{env}}{f_-}\right)^2 \left(1 - e^{\frac{f_-(x^* - z)}{\mathsf{T}_{env}}}\right) \\ &\quad + \frac{\mathsf{T}_{env}}{f_-} e^{\frac{u_0}{\mathsf{T}_{env}}\frac{\delta}{\delta - x^*}} \left(e^{-\frac{f_- x^*}{\mathsf{T}_{env}}} - e^{-\frac{f_- z}{\mathsf{T}_{env}}}\right) \frac{\mathsf{T}_{env}}{f_+}\left(e^{\frac{f_+ x^*}{\mathsf{T}_{env}}} - 1\right). \end{aligned} \tag{153}$$

Lastly, it holds that

$$\int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_y^\delta \mathrm{d}x w(x) = \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_0^\delta \mathrm{d}x w(x) - \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x w(x) \tag{154}$$

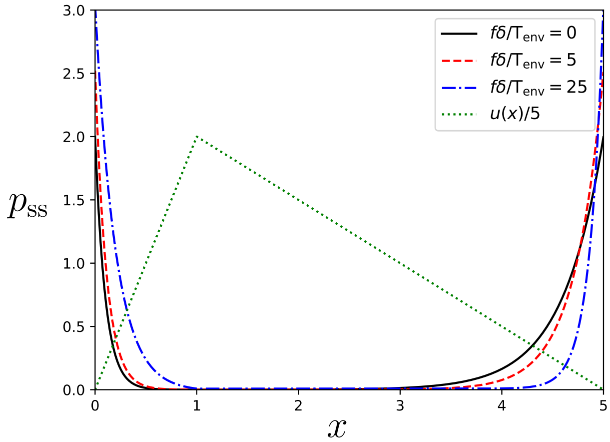

Figure 5: Stationary distribution $p_{\rm ss}$ as a function of x for $\delta = 5$, $x^* = 1$, $u_0 = 10$, $\mathsf{T}_{\rm env} = 1$ and for given values of f. The value of γ is immaterial. Solid lines are results from the Eqs. (141)-(147) while markers are simulation results. The green dotted line plots the potential u divided by 5.

and

$$\int_{\delta-z}^{\delta} {\rm d}y \frac{1}{w(y)} \int_{y}^{\delta} {\rm d}x w(x) = \int_{0}^{\delta} {\rm d}y \frac{1}{w(y)} \int_{0}^{\delta} {\rm d}x w(x) - \int_{0}^{\delta} {\rm d}y \frac{1}{w(y)} \int_{0}^{y} {\rm d}x w(x) - \int_{0}^{\delta-z} {\rm d}y \frac{1}{w(y)} \int_{0}^{\delta} {\rm d}x w(x) + \int_{0}^{\delta-z} {\rm d}y \frac{1}{w(y)} \int_{0}^{y} {\rm d}x w(x). \tag{155}$$

Substituting the above integrals, given by Eqs. (149)-(155), into Eqs. (110), (111), (112), and (113), and consequently using these in Eq. (108) for $\langle T_X \rangle$, we obtain a closed form expression for $\langle T_X \rangle$.

In the Figs. 2 and 3 of the main text we have used this closed form expression of $\langle T_X \rangle$ to plot $\langle T \rangle \dot{s}/|\log p_-|$ as a function of ℓ or $\langle T_X \rangle$ as a function of $\mathsf{T}_{\rm env}$.

D.3 Recovering the Van't Hoff-Arrhenius law

The Eq. (138) in the particular case of a triangular potential leads to

$$\langle T_X \rangle = \frac{\ell \gamma}{f} \frac{\mathsf{T}_{\rm env}^2}{u_0^2} e^{\frac{u_0}{\mathsf{T}_{\rm env}}} \left(1 + O\left(\frac{f\delta}{\mathsf{T}_{\rm env}} \right) \right). \tag{156}$$

We have used this equation to plot the green dotted line in the Fig. 3 of the main text.

E Biased hopping process

In this appendix, we determine the moments of the first-passage time T_X, defined in Eq. (56), of the biased hopping process X determined by Eq. (52).

E.1 Martingales in the biased hopping processes

The processes

$$Z(t) = e^{zX(t)+[(1-e^{z})k_+ + (1-e^{-z})k_-]t} \tag{157}$$

are martingales for all values of $z \in \mathbb{R}$ (see Appendix A.1 for the definition of a martingale). Indeed, using Itô's formula for jump processes [68], we obtain

$$\mathrm{dZ}(t) = (e^z - 1)Z(t)\,[\mathrm{d}N_+(t) - k_+\mathrm{d}t] + (e^{-z} - 1)Z(t)\,[\mathrm{d}N_-(t) - k_-\mathrm{d}t]\,. \tag{158}$$

In the special case of $z = \ln \frac{k_-}{k_+}$, we obtain that $Z(t) = e^{-S(t)}$ is the exponentiated negative entropy production, which is indeed known to be a martingale [34].

Proposition 1 (A martingale equality)**.** *If $k_+ > k_-$, then for all $z \in \mathbb{R} \setminus [\ln \frac{k_-}{k_+}, 0]$ it holds that*

$$1 = \left\langle 1_{T_X<\infty} 1_{D=1} e^{z[\ell_+]+f(z)T_X} + 1_{T_X<\infty} 1_{D=-1} e^{-z[\ell_-]+f(z)T_X} \right\rangle, \tag{159}$$

where

$$f(z) = (1 - e^z)k_+ + (1 - e^{-z})k_-. \tag{160}$$

and where $[\ell_+]$ and $[\ell_-]$ are the smallest natural numbers that are larger than ℓ_+ and ℓ_-, respectively.

Proof. Since $Z(t \wedge T_X)$ is a martingale, we can use Doob's optional stopping theorem, see Appendix A.2, and thus

$$1 = \langle Z(t \wedge T_X) \rangle = \left\langle e^{zX(t)+f(z)(t\wedge T_X)} \right\rangle. \tag{161}$$

Since for $z \in \mathbb{R} \setminus [\ln \frac{k_-}{k_+}, 0]$ it holds that $f(z) < 0$, we have that

$$e^{zX(t)+f(z)(t\wedge T_X)} < e^{z\ell_+}. \tag{162}$$

Hence, the bounded convergence theorem applies, see e.g. Ref. [63], and we can take the limit $t \to \infty$ under the expectation value to obtain

$$\begin{aligned} 1 &= \langle \lim_{t\to\infty} e^{zX(t)+f(z)(t\wedge T_X)} \rangle && (163) \\ &= \left\langle 1_{T_X<\infty} 1_{D=1} e^{z[\ell_+]+f(z)T_X} + 1_{T_X<\infty} 1_{D=-1} e^{-z[\ell_-]+f(z)T_X} \right\rangle, && (164) \end{aligned}$$

which completes the proof of the equality (159). □

In what follows, we use this martingale equality to derive various properties T_X.

E.2 The first-passage time T_X is with probability one finite

Proposition 2. *It holds that T_X is almost surely finite, i.e.,*

$$p_- + p_+ = 1. \tag{165}$$

Proof. We take the the limit $z \to 0$ in Eq. (159). Since for $z \in [0, 1]$ the argument in the expectation value is bounded by e^{ℓ_+}, the bounded convergence theorem applies, see e.g. Ref. [63], and

$$\begin{aligned} 1 &= \lim_{z\to 0} \left\langle 1_{T_X<\infty} 1_{D=1} e^{z[\ell_+]+f(z)T_X} + 1_{T_X<\infty} 1_{D=-1} e^{-z[\ell_-]+f(z)T} \right\rangle \\ &= \langle 1_{T_X<\infty} 1_{D=1} + 1_{T_X<\infty} 1_{D=-1} \rangle = \langle 1_{T_X<\infty} \rangle = \mathbb{P}(T_X < \infty), \end{aligned}$$

where we have used that $f(0) = 0$. □

E.3 Splitting probabilities

Proposition 3. *It holds that*

$$p_+ = \frac{1 - e^{-[\ell_-]\ln\frac{k_+}{k_-}}}{1 - e^{-([\ell_+]+[\ell_-])\ln\frac{k_+}{k_-}}}, \quad \text{and} \quad p_- = e^{-[\ell_-]\ln\frac{k_+}{k_-}} \frac{1 - e^{-[\ell_+]\ln\frac{k_+}{k_-}}}{1 - e^{-([\ell_+]+[\ell_-])\ln\frac{k_+}{k_-}}}, \tag{166}$$

where $[\ell_-]$ *and* $[\ell_+]$ *are the smallest natural numbers that are greater or equal than* ℓ_- *and* ℓ_+*, respectively.*

Proof. We apply Doob's optional stopping theorem, see Appendix A.2, to the martingale

$$e^{-S(t)} = e^{X(t)\ln\frac{k_-}{k_+}}, \tag{167}$$

yielding

$$\left\langle e^{X(t\wedge T_X)\ln\frac{k_-}{k_+}} \right\rangle = 1. \tag{168}$$

Since X is a jump process on a lattice, it holds that

$$\lim_{t\to\infty} \left\langle e^{X(t\wedge T_X)\ln\frac{k_-}{k_+}} \right\rangle \leq p_- e^{-[\ell_-]\ln\frac{k_-}{k_+}} + p_+ e^{[\ell_+]\ln\frac{k_-}{k_+}} + (1 - p_- - p_+) e^{-[\ell_-]\ln\frac{k_-}{k_+}} \tag{169}$$

and

$$\lim_{t\to\infty} \left\langle e^{X(t\wedge T_X)\ln\frac{k_-}{k_+}} \right\rangle \geq p_- e^{-[\ell_-]\ln\frac{k_-}{k_+}} + p_+ e^{[\ell_+]\ln\frac{k_-}{k_+}}. \tag{170}$$

According to Proposition 2, it holds that $p_- + p_+ = 1$, and thus

$$p_- e^{-[\ell_-]\ln\frac{k_-}{k_+}} + p_+ e^{[\ell_+]\ln\frac{k_-}{k_+}} = 1. \tag{171}$$

The solutions to the Eqs. (165) and (171) are given by Eqs. (166), which completes the proof. □

Using $b = k_-/k_+$ in Eq. (166), we obtain the Eq. (57) in the main text.

E.4 Generating function

We derive an explicit formula for the generating function $g(y)$ defined in Eq. (58).

We can write

$$g(y) = p_+ g_+(y) + p_- g_-(y) \tag{172}$$

where g_+ and g_- are the conditional generating functions

$$g_+(y) = \langle e^{-yT_X(k_-+k_+)} | D = 1 \rangle, \quad \text{and} \quad g_-(y) = \langle e^{-yT_X(k_-+k_+)} | D = -1 \rangle. \tag{173}$$

Lemma 1. *It holds that*

$$1 = \left(\frac{1}{2} \left[(1+b)(1+y) + \sqrt{-4b + (1+b)^2(1+y)^2} \right] \right)^{[\ell_+]} p_+ g_+(y)$$
$$+ \left(\frac{1}{2} \left[(1+b)(1+y) + \sqrt{-4b + (1+b)^2(1+y)^2} \right] \right)^{-[\ell_-]} p_- g_-(y), \tag{174}$$

and

$$1 = \left(\frac{1}{2} \left[(1+b)(1+y) - \sqrt{-4b + (1+b)^2(1+y)^2} \right] \right)^{[\ell_+]} p_+ g_+(y)$$
$$+ \left(\frac{1}{2} \left[(1+b)(1+y) - \sqrt{-4b + (1+b)^2(1+y)^2} \right] \right)^{-[\ell_-]} p_- g_-(y), \tag{175}$$

Proof. We rewrite the relation (159) for $z \notin [\ln \frac{k_-}{k_+}, 0]$ as

$$1 = e^{z[\ell_+]} p_+ \langle e^{f(z)T(k_- + k_+)} | D = 1 \rangle + e^{-z[\ell_-]} p_- \langle e^{f(z)T(k_- + k_+)} | D = -1 \rangle. \tag{176}$$

Setting

$$y = -f(z) \tag{177}$$

and solving towards z, we obtain two solutions.

First, let us consider the solution branch for $z \geq 0$, which is given by

$$z = \ln\left(\frac{1}{2}\left[(1+b)(1+y) + \sqrt{-4b + (1+b)^2(1+y)^2}\right]\right). \tag{178}$$

Using Eqs. (177) and (178) in (176), we obtain Eq. (176).

Second, let us consider the solution branch for $z \leq \ln b$, namely,

$$z = \ln\left(\frac{1}{2}\left[(1+b)(1+y) - \sqrt{-4b + (1+b)^2(1+y)^2}\right]\right). \tag{179}$$

In this case, using Eqs. (177) and (179) in (176), we obtain the Eq. (175).

□

Proposition 4. *The generating function Eq. (58) is given by Eqs. (59)-(61).*

Proof. We find Eq. (59) readily by solving the Eqs. (175)-(176). □

E.5 Moments of first-passage times

The moments of first passage times follow from taking the derivatives in Eq. (62).

The first moment is given by

$$\langle T_X \rangle = \frac{[\ell_+]p_+ - [\ell_-]p_-}{k_+ - k_-}. \tag{180}$$

The second moment is given by

$$\begin{aligned}(k_+ - k_-)^2 \langle T_X^2 \rangle \quad = \quad & \frac{p_+}{1 - b^{[\ell_-]+[\ell_+]}}\left([\ell_+]^2 + [\ell_+]\tanh^{-1}\left(\frac{a}{2\mathsf{T}_{\rm env}}\right)\right) - [\ell_-]^2 p_- \left(\frac{3 + b^{[\ell_-]+[\ell_+]}}{1 - b^{[\ell_-]+[\ell_+]}}\right) \\ & + \frac{p_+ b^{[\ell_-]+[\ell_+]}}{1 - b^{[\ell_-]+[\ell_+]}}\left(3[\ell_+]^2 - [\ell_+]\tanh^{-1}\left(\frac{a}{2\mathsf{T}_{\rm env}}\right)\right) \\ & + [\ell_-]\tanh^{-1}\left(\frac{a}{2\mathsf{T}_{\rm env}}\right)\frac{b^{2[\ell_-]+[\ell_+]}(1 - b^{[\ell_+]})}{(1 - b^{[\ell_-]+[\ell_+]})^2} - 4[\ell_+][\ell_-]\frac{b^{2[\ell_-]+[\ell_+]}}{(1 - b^{[\ell_-]+[\ell_+]})^2} \\ & + \left([\ell_-]\tanh^{-1}\left(\frac{a}{2\mathsf{T}_{\rm env}}\right) + 8[\ell_-][\ell_+]\right)\frac{b^{[\ell_-]}b^{[\ell_+]}}{(1 - b^{[\ell_-]}b^{[\ell_+]})^2} \\ & - [\ell_-]\left(\tanh^{-1}\left(\frac{a}{2\mathsf{T}_{\rm env}}\right) + 4[\ell_+]\right)\frac{b^{[\ell_-]}}{(1 - b^{[\ell_-]}b^{[\ell_+]})^2},\end{aligned} \tag{181}$$

where $\tanh^{-1}\left(\frac{a}{2\mathsf{T}_{\rm env}}\right) = 1/\tanh\left(\frac{a}{2\mathsf{T}_{\rm env}}\right)$.

We avoid writing down the expression for $\langle T_X^3 \rangle$ given that it is even lengthier than $\langle T_X^2 \rangle$.

E.6 Symmetric thresholds

In the specific case where $\ell_+ = \ell_- = \ell$, we obtain the simpler expression

$$g(y) = \frac{2^{[\ell]} + 2^{-[\ell]}\left(\beta(y) - \sqrt{-4\frac{k_-}{k_+} + \beta^2(y)}\right)^{[\ell]}\left(\beta(y) + \sqrt{-4\frac{k_-}{k_+} + \beta^2(y)}\right)^{[\ell]}}{\left(\beta(y) - \sqrt{-4\frac{k_-}{k_+} + \beta^2(y)}\right)^{[\ell]} + \left(\beta(y) + \sqrt{-4\frac{k_-}{k_+} + \beta^2(y)}\right)^{[\ell]}} \tag{182}$$

for the generating function.

In this case, the first-passage time is

$$\langle T_X \rangle = \frac{[\ell]}{k_+ - k_-}\frac{1 - b^{[\ell]}}{1 + b^{[\ell]}}. \tag{183}$$

and the second moment

$$\langle T_X^2 \rangle = [\ell]\frac{[\ell] + \frac{k_+ + k_-}{k_+ - k_-} - 6[\ell]b^{[\ell]} + b^{2[\ell]}\left([\ell] - \frac{k_+ + k_-}{k_+ - k_-}\right)}{(k_+ - k_-)^2\left(1 + b^{[\ell]}\right)^2} \tag{184}$$

and the third moment,

$$\begin{aligned}\langle T_X^3 \rangle \quad = \quad & \frac{[\ell]}{k_+^3(1-b)^5(1+b^{[\ell]})^3}\Big\{2 + 8b + 2b^2 + 3[\ell](1-b^2) + [\ell]^2(1-b)^2 \\ & + b^{[\ell]}(2 + 2b(4+b) + 15(-1+b^2)[\ell] - 23(-1+b)^2[\ell]^2) \\ & + b^{2[\ell]}(-2 - 2b(4+b) + 15(-1+b^2)[\ell] + 23(-1+b)^2[\ell]^2) \\ & + b^{3[\ell]}(2 + 2b(4+b) + 3(-1+b^2)[\ell] + (-1+b)^2[\ell]^2)\Big\}.\end{aligned} \tag{185}$$

These are the formulae used in Fig. 4 of the main text.

One readily verifies the thermodynamic uncertainty relation

$$\lim_{[\ell]\to\infty}\frac{\langle T_X^2 \rangle - \langle T_X \rangle^2}{\langle T_X \rangle} = \frac{k_+ + k_-}{(k_+ - k_-)^2} \geq \frac{2}{(k_+ - k_-)\log\frac{k_+}{k_-}} \tag{186}$$

where we used the fact that $\log(x) \geq \frac{x-1}{x} \geq \frac{x-1}{x+1}$ with $x = k_+/k_-$.

E.7 Asymptotics with large thresholds

We consider the limit $\ell_+, \ell_- \gg 1$ with the ratio ℓ_+/ℓ_- fixed to a constant value.

The big-O notation $O(f(\ell_-))$ denotes an arbitrary function $g(\ell_-)$ for which it holds that there exists a constant c such that $g(\ell_-) < cf(\ell_-)$ for ℓ_- large enough.

From Eqs. (57), we obtain for the splitting probabilities that

$$p_- = b^{[\ell_-]} + O(b^{[\ell_+]+[\ell_-]}), \quad \text{and} \quad p_+ = 1 + O(b^{[\ell_-]}). \tag{187}$$

Equation (180) implies that the mean first-passage time

$$\langle T_X \rangle = \frac{[\ell_+]}{k_+ - k_-}\left(1 + O(b^{[\ell_-]})\right), \tag{188}$$

and from Eq. (181) it follows that the second moment

$$\langle T_X^2 \rangle = \frac{[\ell_+]^2}{(k_+ - k_-)^2}\left(1 + \frac{1}{[\ell_+]\tanh\left(\frac{a}{2\mathsf{T}_{\text{env}}}\right)} + O(b^{[\ell_-]})\right). \tag{189}$$

The Eqs. (187) and (188) imply that

$$\frac{[\ell_+]}{[\ell_-]}\frac{|\log p_-|}{\langle T_X\rangle} = \frac{a}{\mathsf{T}_{\rm env}}\frac{1}{k_+ - k_-}(1 + O(b^{[\ell_-]})). \tag{190}$$

We recognize in the above formula the entropy production rate $\dot{s}$ given by Eq. (55), and thus

$$\frac{[\ell_+]}{[\ell_-]}\frac{|\log p_-|}{\langle T_X\rangle} = \dot{s} + O(b^{[\ell_-]}). \tag{191}$$

Analogously, Eqs. (187) and (189) imply that

$$\frac{[\ell_+]}{[\ell_-]}\frac{|\log p_-|}{\sqrt{\langle T_X^2\rangle}} = \dot{s} + O\left(\frac{1}{[\ell_+]}\right). \tag{192}$$

The thermodynamic uncertainty relation is governed by the subleading $O\left(1/[\ell_+]\right)$ term in Eq. (192). Using Eqs. (187) and (189), we obtain the Eq. (65) in the main text. Since,

$$\frac{1}{\tanh(x/2)} \geq \frac{2}{x} \tag{193}$$

the thermodynamic uncertainty relation [47]

$$\frac{2\langle T_X\rangle}{\langle T_X^2\rangle - \langle T_X\rangle^2} \geq \dot{s} \tag{194}$$

holds.

In order to find asymptotic expressions for the higher order moments, we analyze in the next subsection the probability distribution of T_X in the limit of large thresholds ℓ_- and ℓ_+.

E.8 Probability distribution in the asymptotic limit $\ell_\pm \to \infty$

In order to derive asymptotic expressions for the moments $\langle T^n\rangle$ with $n > 2$, we determine the probability distribution in this limit.

Using that $\zeta_- < \zeta_+$, we obtain in the limit $\ell_{\rm min} \to \infty$,

$$g(y) = \left(\frac{2}{\zeta_+(y)}\right)^{[\ell_+]}\left(1 + O\left(\left(\frac{\zeta_-(y)}{\zeta_+(y)}\right)^{[\ell_-]}\right)\right) + \left(\frac{\zeta_-(y)}{2}\right)^{[\ell_-]}\left(1 + O\left(\left(\frac{\zeta_-(y)}{\zeta_+(y)}\right)^{[\ell_-]}\right)\right). \tag{195}$$

In the limit $\ell_{\rm min} \to \infty$, we obtain

$$g(y) = \left(\frac{2}{\zeta_+(y)}\right)^{[\ell_+]} + O(b^{[\ell_-]}). \tag{196}$$

Considering that T will be large when both $[\ell_+]$ and $[\ell_-]$ are large, we use that $y \sim \frac{1}{[\ell_{\rm min}]}$. Therefore,

$$\zeta_+(y) = 2 + 2\frac{1+b}{1-b}y + O(y^2). \tag{197}$$

Taking the inverse Laplace transform, we obtain up to leading order

$$p_{T_X}(t) = \frac{((k_+ + k_-)t)^{[\ell_+]-1}}{\Gamma([\ell_+])}\left(\frac{1-b}{1+b}\right)^{[\ell_+]} e^{-t(k_+ + k_-)\frac{1-b}{1+b}} + O(b^{[\ell_-]}), \tag{198}$$

which is the Gamma distribution with shape parameter $[\ell_+]$ and rate $(1-b)/(1+b)$.

If we introduce a new variable,

$$\tau = \frac{(k_+ + k_-)t}{[\ell_+]}, \tag{199}$$

then we obatain

$$p_{\frac{(k_- + k_+)T_X}{[\ell_+]}}(\tau) \sim \exp\left(-[\ell_+]I(\tau) + O_{[\ell_+]}(1)\right) + O(b^{[\ell_-]}) \tag{200}$$

with the large deviation function

$$I(\tau) = \frac{1-b}{1+b}\tau - \log(\tau) - \log\frac{1-b}{1+b} - 1. \tag{201}$$

The minimum is found when

$$\tau^* = \frac{1+b}{1-b} \tag{202}$$

in which case $I(\tau^*) = 0$. Expanding $I(\tau)$ around τ^* we obtain

$$I(\tau) = \frac{\left(\tau - \frac{1+b}{1-b}\right)^2}{2\left(\frac{1+b}{1-b}\right)^2} + O(\tau^3). \tag{203}$$

Hence, the distribution of p_T is

$$p_{\frac{(k_+ + k_-)T_X}{[\ell_+]}}(\tau) = \sqrt{\frac{[\ell_+]}{2\pi(\tau^*)^2}} \exp\left(-[\ell_+]\frac{(\tau-\tau^*)^2}{2(\tau^*)^2} + O(\tau^2)\right) + O(b^{[\ell_-]}). \tag{204}$$

For large $[\ell_+]$, the distribution $p_{\frac{(k_+ + k_-)T_X}{[\ell_+]}}(\tau)$ is centered around $\tau = \tau^*$, and therefore $\frac{(k_+ + k_-)T_X}{[\ell_+]}$ is a deterministic variable in this limit. The moments of T are thus up to leading order terms of the form

$$\langle T_X^n \rangle = [\ell_+]^n \frac{(\tau^*)^n}{(k_+ + k_-)^n} + O([\ell_+]^{n-1}) = \frac{[\ell_+]^n}{(k_+ - k_-)^n} + O([\ell_+]^{n-1}). \tag{205}$$

Using the formula for p_-, given by Eq. (187), and the expression for $\dot{s}$ in Eq. (55), we find thus indeed

$$\frac{[\ell_+]}{[\ell_-]} \frac{|\log p_-|}{\left(\langle T_X^n \rangle\right)^{1/n}} = \dot{s} + O\left(\frac{1}{[\ell_+]}\right). \tag{206}$$

Note that obtaining the $1/[\ell_+]$ correction terms is more complicated as we need to consider subleading order terms in Eq. (203). The subleading order terms depend on b and are process dependent. Hence, the moments $\langle T_X^n \rangle$ converge for large thresholds to the universal limit given by Eq. (206) since they are governed by the leading order term in the asymptotic behaviour of T_X. On the other hand, the Fano factor

$$\frac{\langle T_X^2 \rangle - \langle T_X \rangle^2}{\langle T_X \rangle} \tag{207}$$

characterising uncertainty depends on the subleading terms and will therefore not converge to a universal limit when the thresholds diverge. This clarifies why the first-passage time relations in the present paper can be used for the exact inference of the dissipation rate $\dot{s}$, while this is not possible with thermodynamic uncertainty relations.

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