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Universal tradeoff relation between speed, uncertainty, and dissipation in nonequilibrium stationary states

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Abstract

We derive universal thermodynamic inequalities that bound from below the moments of first-passage times of stochastic currents in nonequilibrium stationary states and in the limit where the thresholds that define the first-passage problem are large. These inequalities describe a tradeoff between speed, uncertainty, and dissipation in nonequilibrium processes, which are quantified, respectively, with the moments of the first-passage times of stochastic currents, the splitting probability, and the mean entropy production rate. Near equilibrium, the inequalities imply that mean-first passage times are lower bounded by the Van't Hoff-Arrhenius law, whereas far from thermal equilibrium the bounds describe a universal speed limit for rate processes. When the current is the stochastic entropy production, then the bounds are equalities, a remarkable property that follows from the fact that the exponentiated negative entropy production is a martingale.

Contents

1 Introduction

In thermal equilibrium transitions between metastable states are activated by thermal fluctuations. The equilibrium transition rates satisfy the Van't Hoff-Arrhenius law [1, 2]

$$k = \frac{1}{\langle T \rangle} = \nu e^{-\frac{E_{\rm b}}{\mathsf{T}_{\rm env}}}, \tag{1}$$

where the rate k is the inverse of the mean first-passage time $\langle T \rangle$, $E_{\rm b}$ is the energy barrier that separates the two metastable states, $\mathsf{T}_{\rm env}$ is the temperature of the environment, and ν is a prefactor that has been determined, among others, by Kramers [1, 3].

To speed up a process, an external agent can drive a system out of equilibrium. For example, in Fig. 1 we illustrate how external driving can increase the reaction rate in a nonequilibrium version of Kramers' model [3]. Other examples are the reduced travel times of self-propelled particles [4–9], the activated escape of a particle from a metastable state [10], or the enhanced reaction rates of nonequilibrium chemical reactions [11–13]. Since dissipation can increase the rate of a process, one may wonder whether there is a generic speed limit on processes that are driven away from thermal equilibrium.

In the present paper, building on Ref. [14], we show that rate processes are governed by a universal tradeoff between dissipation, speed, and uncertainty. We quantify this tradeoff with generic inequalities on the moments of the first-passage times of stochastic currents with two thresholds. The derived inequalities are reminiscent of the thermodynamic uncertainty relations for first-passage times [15], but there exist also a couple of important distinctions. First, the trade-off relations derived in this paper quantify the uncertainty in the outcome of the process with the splitting probability of the first-passage problem, instead of with the variance of the first-passage time as in Ref. [15]. Second, the derived bounds are equalities when the current is the stochastic entropy production, and hence the derived first-passage inequalities are optimal in this case.

The paper is organised as follows: in Sec. 2, we state the main results of this paper. In Sec. 3, we discuss the system setup for which the main results are derived. Secs. 4 and 5 present the derivations of the main results based on large deviation theory and martingale theory, and Sec. 6 provides an alternative derivation that is based on the theory of sequential hypothesis testing. In Secs. 7 and 8, we relate the main results of this paper to results previously published in the literature and to the Van't Hoff-Arrhenius law, respectively. In Sec. 9, we illustrate with an example the tightness of the first-passage time bounds when the stochastic current is proportional to the stochastic entropy production. The paper ends with a discussion in Sec. 10 and after the discussion there are several appendices that contain technical details on the mathematical derivations.

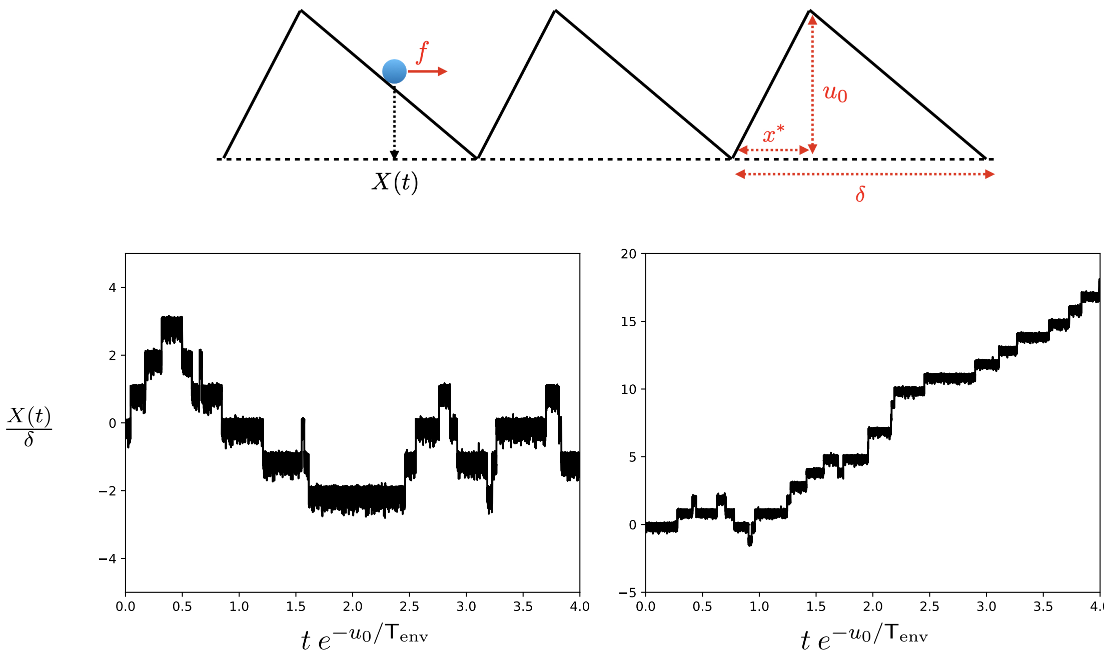

Figure 1: *Nonequilibrium version of Kramer's model demonstrating an increased reaction rate.* Trajectories shown are for a reaction coordinate X that solves the Langevin equation $\partial_t X(t) = (f - \partial_x u(X(t)))/\gamma + \sqrt{2\mathsf{T}_{\rm env}/\gamma}\xi(t)$, where $\xi(t) = {\rm d}W(t)/{\rm d}t$ is a delta-correlated white Gaussian noise term, and where $u(x)$ is a triangular potential with period δ, i.e. $u(x) = u(\pm\delta)$, $u(x) = u_0 x/x^*$ if $x \in [0, x^*]$, and $u(x) = u_0(\delta - x)/(\delta - x^*)$ if $x \in [x^*, \delta]$. Left: equilibrium trajectory with $f = 0$. Right: nonequilibrium trajectory with $f\delta/\mathsf{T}_{\rm env} = 1$. The remaining parameters are set to $\delta = 5$, $\gamma = 1$, $x^* = 1$, $u_0 = 10$, and $\mathsf{T}_{\rm env} = 1$.

2 Main results

The paper contains two main results. The first main result is an inequality that holds for the first-passage times of stochastic currents. The second main result is an equality that holds for first-passage times of stochastic currents that are proportional to the stochastic entropy production.

2.1 Bound for the moments of first-passage times of stochastic currents

Let $J(t)$ be a stochastic current in a nonequilibrium, stationary process $X(t)$ and let

$$T_J = \inf\{t > 0 : J(t) \notin (-\ell_-, \ell_+)\} \tag{2}$$

be the first time when $J(t)$ leaves the open interval $(-\ell_-, \ell_+)$, where $t \geq 0$ is an index that labels the time and where $\ell_-, \ell_+ > 0$ are the threshold values of the first-passage problem.

In this paper we show that in the limit of large thresholds ℓ_- and ℓ_+ it holds that

$$\langle T_J^n \rangle \geq \left(\frac{\ell_+}{\ell_-}\frac{|\log p_-|}{\dot{s}}\right)^n (1 + o_{\ell_{\rm min}}(1)), \tag{3}$$

where

$$p_- = P(J(T_J) \leq -\ell_-) \tag{4}$$

denotes the probability that the current J goes below the negative threshold $-\ell_-$ before exceeding for the first time the positive threshold ℓ_+, where $\dot{s}$ is the entropy production

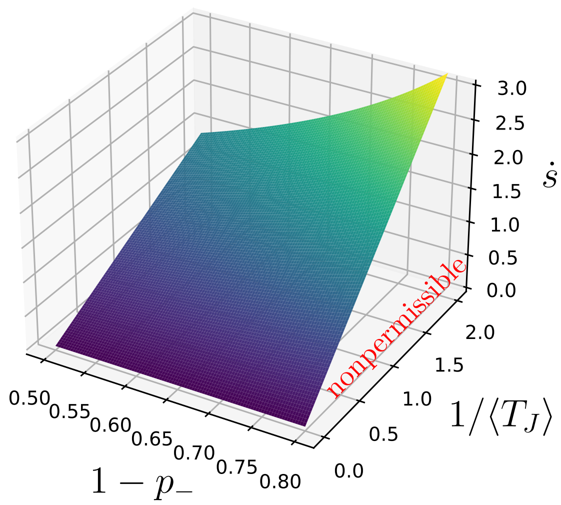

Figure 2: *Universal tradeoff between speed, uncertainty, and dissipation in nonequilibrium processes.* The three axes represent the speed ($1/\langle T_J\rangle$), uncertainty ($1-p_-$), and dissipation ($\dot{s}$) in a nonequilibrium process X. The plotted surface is $\dot{s} = |\log p_-|/\langle T_J\rangle$. Processes that are situated below the surface are physically nonpermissible as they violate the bound Eq. (3).

rate, and where $n \in \mathbb{N}$. The quantity p_- is called the splitting probability. The averages $\langle\cdot\rangle$ are taken over repeated realisations of the stationary process X. We have used the little-o-notation $o_{\ell_{\rm min}}(1)$ to denote a function that converges to zero when $\ell_{\rm min} = \min\{\ell_-, \ell_+\} \to \infty$ while the ratio ℓ_-/ℓ_+ is kept fixed. Equation (3) holds for $\langle J(t)\rangle > 0$; if $\langle J(t)\rangle < 0$, then p_- should be replaced by $p_+ = P(J(T_J) \geq \ell_+)$, ℓ_- with ℓ_+, and vice versa.

The inequality Eq. (3) describes a tradeoff between dissipation $\dot{s}$, speed $\langle T^n\rangle$, and the uncertainty in the outcome of the process that is quantified by p_-. It states that processes that are fast, precise, and have a small entropy production rate are physically not permissible. We can illustrate this trade-off relation graphically by plotting a surface in a three-dimensional space delimiting the parameter regime that is physically not permissible, see Fig. 2.

Near equilibrium $\dot{s} \sim e^{-\frac{E_{\rm b}}{T_{\rm env}}}$ and $p_- \approx \ell_-/(\ell_+ + \ell_-)$. Consequently, Eq. (3) implies that $\langle T_J\rangle$ is lower bounded by the Van't Hoff-Arrhenius law, i.e.,

$$\langle T_J\rangle \geq \frac{1}{\nu} e^{\frac{E_{\rm b}}{T_{\rm env}}}. \tag{5}$$

On the other hand, far from thermal equilibrium the right hand side of Eq. (3) goes below $\frac{1}{\nu}e^{\frac{E_{\rm b}}{T_{\rm env}}}$ implying that dissipation can increase the reaction rate $k = 1/\langle T_J\rangle$, as we illustrate in Fig. 1 for a nonequilibrium version of Kramer's model [3].

Taken together, the Eq. (3) states we can speed up a process by driving it out of equilibrium, but there exists a universal speed limit that is determined by the rate of dissipation and the amount of fluctuations in the process.

2.2 Equality for the moments of first-passage times of entropy production

If $J(t) = S(t)$ with $S(t)$ the stochastic entropy production [16–18], then the inequality Eq. (3) becomes an equality, viz.,

$$\langle T_S^n \rangle = \left(\frac{\ell_+}{\ell_-} \frac{|\log p_-|}{\dot{s}} \right)^n (1 + o_{\ell_{\min}}(1)). \tag{6}$$

This remarkable property follows form the fact that $e^{-S(t)}$ is a martingale [19–21], which implies the formula $p_- = e^{-\ell_-}(1 + o_{\ell_{\min}}(1))$ [20, 21].

The Eq. (6) implies that the bound Eq. (3) is tight when the stochastic current is the entropy production ($J = S$), and this is one of its main advantages with respect to other tradeoff inequalities reported in the literature, such as, the thermodynamic uncertainty relation for first-passage times that quantifies uncertainty in terms of the variance of the first-passage time [15].

3 System setup

3.1 General setup

Let $\vec{X}(t) = (X_1(t), X_2(t), \ldots, X_m(t))$ be a vector of variables $X_i(t) \in \mathcal{X}_i$ that describe the evolution in time of the slow degrees of freedom in a mesoscopic system. We denote the trajectories of $\vec{X}(t)$ over a time interval $[0, t]$ as $\vec{X}_0^t$. In examples for which $m = 1$, we simply write $\vec{X} = X$.

We say that a system satisfies local detailed balance when it is weakly coupled to an environment in thermal equilibrium [22]. For systems that satisfy local detailed balance the stochastic entropy production S can be expressed as [16, 18]

$$S(t) = \log \frac{p(\vec{X}_0^t)}{p(\Theta_t(\vec{X}_0^t))}, \tag{7}$$

where the time-reversal operation Θ_t maps trajectories $\vec{X}_0^t$ on their time-reversed trajectory $(\vec{X}^\dagger)_0^t$ with entries $\vec{X}_i^\dagger(\tau) = \vec{X}_i(t - \tau)$. For simplicity, we will only consider processes with even parity under time-reversal, although it will become clear that the results extend to processes that contain odd parity variables. The quantity $p(\vec{X}_0^t)/p(\Theta_t(\vec{X}_0^t))$ is the ratio between the probability densities of the trajectory $\vec{X}_0^t$ in the forward and backward dynamics, also known as the Radon-Nikodym derivative [20, 23, 24]. For a stationary process $p(\Theta_t(\vec{X}_0^t)) = p(\Theta_0(\vec{X}_0^t))$ as the statistics of the process are invariant under a translation in time. One can verify with examples of Langevin processes and Markov jump processes that Eq. (7) is the stochastic entropy production when the process satisfies local detailed balance [18, 25, 26]. Notice that we use natural units for which the Boltzmann constant is set equal to one.

Since the process is stationary, the entropy production rate $\dot{s}$, or equivalently the rate of dissipation, is given by

$$\langle S(t) \rangle = \dot{s}\, t. \tag{8}$$

Stochastic currents $J(t) = J(\vec{X}_0^t)$ are real-valued functionals defined on the set of trajectories $\vec{X}_0^t$ with the following two properties:

(i) J is time extensive, i.e.,

$$\langle J(t) \rangle = \bar{j}\, t \tag{9}$$

where $\bar{j}$ is the current rate. Without loss of generality we can assume that $\bar{j} > 0$.

(ii) J is odd under time-reversal, i.e.,

$$J(\Theta_t(\vec{X}_0^t)) = -J(\vec{X}_0^t). \tag{10}$$

Note that this implies $J(0) = 0$.

So far, from a mathematical point of view, the system setup has been general. We discuss now two mathematical assumptions on the stochastic current J that we will use to derive the main results Eqs. (3) and (6).

First, we assume that the current J satisfies a large deviation principle. This means that for large enough times t the probability distribution of J/t takes the form [27]

$$p_{J/t}(z) = e^{-t\mathcal{J}(z)(1+o_t(1))}, \tag{11}$$

where $o_t(1)$ is a function that converges to zero when t is large enough and where $\mathcal{J}(z)$ is the large deviation function of the current. Note that in Eq. (11) the normalisation constant is contained in the term $o_t(1)$ in the argument of the exponential. The large deviation function $\mathcal{J}(z) \geq 0$ is a convex function that takes its minimum value when $J/t = \bar{j}$. As shown in Ref. [28], a large deviation principle holds for stochastic currents in Markov jump processes and diffusion processes that are homogeneous and ergodic.

Second, in the derivation of Eq. (3) we assume that at large time scales t the process J behaves up to leading order in t as a drift-diffusion process, i.e.,

$$\mathrm{d}J(t) = \bar{j}(1 + o_t(1))\mathrm{d}t + \sqrt{2d_J}(1 + o_t(1))\mathrm{d}W(t), \tag{12}$$

where $W(t)$ is a standard Wiener process and where the diffusion constant d_J is defined through

$$\lim_{t\to\infty} \left(\langle J(t)J(t-t')\rangle - \bar{j}^2 t(t-t')\right) = 2d_J t'(1 + o_{t'}(1)). \tag{13}$$

Eq. (12) holds when the stochastic current J has a finite memory. Indeed, in this case the process $J(n\Delta t)$, with $n \in \mathbb{N}$ and Δt a fixed time interval that is large enough, is a random walk process on the real line with increments $\Delta J(n) = J(n\Delta t) - J((n-1)\Delta t)$ that are independent and identically distributed variables. As a consequence, the central limit theorem applies and the process $J(n\Delta t)$ converges to a drift-diffusion process in the limit of large t. Examples of processes $\vec{X}$ that contain stochastic currents and for which Eq. (12) applies are Markov processes with a finite phase space $\mathcal{X} = \prod_{i=1}^m \mathcal{X}_i$ or Markov processes with a finite relaxation time.

3.2 Markov jump process

We illustrate the general setup discussed above with the example of a Markov jump process $X(t)$ defined on a discrete set $X(t) \in \mathcal{X}$. The dynamics of $X(t)$ consists of a sequence of jumps with jump rates that are determined by the Markov transition rate matrix $w_{x\to y}$ with $x, y \in \mathcal{X}$ [29].

Stochastic currents in a Markov jump process take the form

$$J(t) = \sum_{x,y\in\mathcal{X}} c_{x,y} J_{x\to y}(t), \tag{14}$$

with coefficients $c_{x,y} \in \mathbb{R}$ and with $c_{x,x} = 0$. The edge currents

$$J_{x\to y}(t) = N_{x\to y}(t) - N_{y\to x}(t) \tag{15}$$

denote the difference between the number of times $N_{x\to y}(t)$ the process has jumped from the x-th state to the y-th state in trajectory X_0^t and the number of reverse jumps $N_{y\to x}(t)$ from the y-th to the x-the state.

The stochastic entropy production is given by

$$S(t) = \frac{1}{2}\sum_{x,y\in\mathcal{X}} \log \frac{p_{\rm ss}(x) w_{x\to y}}{p_{\rm ss}(y) w_{y\to x}} J_{x\to y}(t), \tag{16}$$

where $p_{\rm ss}(x)$ is the probability distribution of $X(t)$ in the stationary state.

3.3 Overdamped Langevin process

As a second illustration, we consider overdamped Langevin processes [30,31]. In this case the dynamics of $\vec{X} \in \mathbb{R}^m$ is governed by [32]

$$\frac{{\rm d}\vec{X}}{{\rm d}t} = \boldsymbol{\mu}\left(-\vec{\nabla}u + \vec{f}\right) + \vec{\nabla}\cdot \mathbf{d}_{\rm th} + \sqrt{2}\boldsymbol{\sigma}\frac{{\rm d}\vec{W}}{{\rm d}t} \tag{17}$$

where $\boldsymbol{\mu}$ is the mobility tensor, u is a potential, $\vec{f}$ is an external force, $\boldsymbol{\sigma}$ is the noise amplitude, and $\mathbf{d}_{\rm th} = \boldsymbol{\sigma}\boldsymbol{\sigma}^{\rm T}$ is the diffusion tensor. The process $\vec{W} = (W_1, W_2, \ldots, W_m)^{\rm T}$ is a vector of m independent, standard Wiener processes W_i. We have left the explicit dependence of $\boldsymbol{\mu}$, u, $\vec{f}$, and $\boldsymbol{\sigma}$ on $\vec{X}$ away not to overload the notation. The process obeys local detailed balance when $\mathbf{d}_{\rm th} = \mathsf{T}_{\rm env}\boldsymbol{\mu}$.

A stochastic current J takes the form [32]

$$J(t) = \sum_{j=1}^{m}\int_0^t c_j(\vec{X}) \circ {\rm d}X_j(s) \tag{18}$$

where $\circ$ denotes the Stratonovich integral.

If $\mathbf{d}_{\rm th} = \mathsf{T}_{\rm env}\boldsymbol{\mu}$, then the stochastic entropy production is given by [30]

$${\rm d}S(t) = \sum_{j=1}^{m}\int_0^t \left(\frac{1}{p_{\rm ss}}\mathbf{d}_{\rm th}^{-1}\vec{j}_{\rm ss} - \frac{1}{\mathsf{T}_{\rm env}}\left(-\vec{\nabla}u + \vec{f}\right)\right)_j \circ {\rm d}X_j(s), \tag{19}$$

where $j_{\rm ss}(\vec{x})$ and $p_{\rm ss}(\vec{x})$ are the stationary probability flux and probability distribution.

4 Bounds for the moments of first-passage times of stochastic currents

We derive the bounds (3) for the moments of first-passage times of stochastic currents.

The derivation consists of three parts. First, in Subsec. 4.1 we use the large deviation principle for J to show that for all $n \in \mathbb{N}$

$$\langle T_J^n \rangle = \left(\frac{\ell_+}{\bar{j}}\right)^n (1 + o_{\ell_{\rm min}}(1)). \tag{20}$$

The Eq. (20) is determined by the events for which $J(t)$ hits the positive boundary. However, to obtain the Eq. (3) we also need to know the statistics of times T_J when $J(t)$ hits the negative boundary. In Subsec. 4.2 we will use the property Eq. (12) and a duality property for the first-passage times of drift-diffusion processes to show that for all $n \in \mathbb{N}$

$$\langle T_J^n \rangle_- = \left(\frac{\ell_-}{\bar{j}}\right)^n (1 + o_{\ell_{\rm min}}(1)), \tag{21}$$

where $\langle T_J^n \rangle_-$ is the average value of T_J^n conditioned on the event $J(T_J) \leq -\ell_-$ that J hits the negative boundary first.

Lastly, we focus on the splitting probability of J. Using a large deviation function bound for stochastic currents [33–35] we show in Subsec. 4.3 that

$$p_- \quad \geq \quad \exp\left(-\frac{\ell_- \dot{s}}{\overline{j}}\right). \tag{22}$$

Combining the Eqs. (20) and (22) we readily obtain Eq. (3), which completes the derivation.

4.1 Moments $\langle T_J^n \rangle$ of first-passage times of stochastic currents

Since $J(t)$ satisfies a large-deviation principle, see Eq. (11), $J(t)$ converges with probability one to a deterministic function $\overline{j}t$, viz.,

$$\frac{J(t)}{t} = \overline{j}(1 + o_t(1)), \tag{23}$$

where the little-o notation $o_t(1)$ denotes a function that decays to zero when $t \gg 1$. Consequently, the first-passage time given by Eq. (2) is deterministic for large values of $\ell_{\min}$, and as $\overline{j} > 0$ we get

$$T_J = \frac{\ell_+}{\overline{j}}(1 + o_{\ell_{\min}}(1)), \tag{24}$$

which implies Eq. (20).

4.2 Moments $\langle T_J^n \rangle_-$ of first-passage times of stochastic currents hitting the negative boundary

To determine the statistical properties of first-passage times at the negative threshold, we use a duality property for first-passage times of drift-diffusion processes [20, 36, 37]. As shown in the Appendix B, a drift-diffusion process satisfies the first-passage duality

$$\langle T_J^n(\ell_-, \ell_+) \rangle_- = \langle T_J^n(\ell_+, \ell_-) \rangle_+, \tag{25}$$

where $T_J(\ell_-, \ell_+)$ denotes the first-passage time of J with threshold values $-\ell_- < 0$ and $\ell_+ > 0$, and where $\langle \cdot \rangle_+$ and $\langle \cdot \rangle_-$ denote the ensemble averages conditioned on the process realisations for which J first hits the positive boundary and negative boundary, respectively.

Since by assumption the statistics of J are up to leading order determined by the drift-diffusion process Eq. (12), and since in the limit of large thresholds $\ell_+, \ell_- \gg 1$ the first-passage times at the positive thresholds are deterministic and given by Eq. (20), we readily obtain Eq. (21). Equivalently, we have that

$$T_J = \frac{\ell_-}{\overline{j}}(1 + o_{\ell_{\min}}(1)) \tag{26}$$

for the realisations of the process that hit the negative threshold. Hence, up to leading order in $\ell_{\min}$ the statistics of T_J at the negative threshold are determined by the first-passage duality Eq. (25), implying that they are deterministic.

4.3 Splitting probability p_-

We derive the inequality given by Eq. (22). Using Eq. (26) we express the splitting probability p_- in terms of the large deviation function of the current J. Using Eqs. (11) and (26), we get

$$p_- = \int_{-\infty}^{-\bar{j}} \exp\left(-\frac{\ell_-}{\bar{j}}\mathcal{J}(z)(1+o_{\ell_-}(1))\right) \mathrm{d}z. \tag{27}$$

Consequently, we use in Eq. (27) the bound

$$\mathcal{J}(z) \leq \frac{\dot{s}}{4}(z/\bar{j}-1)^2 \tag{28}$$

for the large deviation function of the current [33] that has been derived for overdamped Markov processes in nonequilibrium stationary states in Ref. [34], yielding

$$p_- \geq \int_{-\infty}^{-\bar{j}} \exp\left(-\frac{\ell_-}{\bar{j}}\frac{\dot{s}}{4}(z/\bar{j}-1)^2\right) \mathrm{d}z = \exp\left(-\frac{\ell_-}{\bar{j}}\dot{s}\right), \tag{29}$$

which is the bound Eq. (22) that we were meant to derive.

5 Equalities for the moments of first-passage times of entropy production

The derivation of the equality Eq. (6) goes similar as the derivation of the inequality Eq. (3) in the sense that it also relies on the Eqs. (20) and (21) [or equivalently (26)]. However, instead of deriving the inequality Eq. (22) for the splitting probability, we derive the equality

$$p_- = e^{-\ell_-(1+o_{\ell_{\min}}(1))}. \tag{30}$$

We present two different derivations for the equality Eq. (6): (i) based on Eq. (27) and the Gallavotti-Cohen fluctuation relation for entropy production [38] and (ii) based on the martingale property of $e^{-S(t)}$ [19–21, 31].

5.1 Derivation based on the Gallavotti-Cohen fluctuation relation

Let us first derive Eq. (6) using the Gallavotti-Cohen fluctuation relation. For $J = S$, the rate function $\mathcal{J}(z)$ is convex, satisfies $\mathcal{J}(z) \geq 0$, and $\mathcal{J}(\dot{s}) = 0$, and it also satisfies the fluctuation relation [38]

$$\mathcal{J}(z) - \mathcal{J}(-z) = -z. \tag{31}$$

In the limit of large thresholds $\ell_- \gg 1$, the integral in Eq. (27) is a saddle point integral. Since $\mathcal{J}$ is a convex function, we find that in this limit

$$p_- = e^{-\frac{\ell_-}{\dot{s}}\mathcal{J}(-\dot{s})(1+o_{\ell_{\min}}(1))} \tag{32}$$

where we have used that $\bar{j} = \dot{s}$ when $J = S$. Consequently, using the Gallavotti-Cohen fluctuation relation Eq. (31) and the fact that $\mathcal{J}(\dot{s}) = 0$ we obtain

$$p_- = e^{-\ell_-(1+o_{\ell_{\min}}(1))}. \tag{33}$$

Eqs. (33) together with (20) for $J = S$ lead to the equality (6).

5.2 Derivation based on the martingality of $e^{-S(t)}$

The fact that p_- is universal and only depends on the threshold ℓ_- is a remarkable fact that is a direct consequence of the martingale property of $e^{-S(t)}$ [19–21]. Indeed, since the process $e^{-S(t)}$ is a martingale and since T_S is a first-passage time with two thresholds, the integral fluctuation relation at stopping times

$$\langle e^{-S(T_S)} \rangle = 1, \tag{34}$$

applies, see Corollary 2 of the Appendix of Ref. [21]. The Eq. (34) also reads

$$p_- \langle e^{-S(T_S)} \rangle_- + p_+ \langle e^{-S(T_S)} \rangle_+ = 1, \tag{35}$$

where $\langle \cdot \rangle_-$ and $\langle \cdot \rangle_+$ denote averages over those trajectories that terminate at the negative and positive threshold values, respectively. Using that for $\ell_-, \ell_+ \gg 1$, it holds that $S(T_S) = \ell_\pm (1 + o_{\ell_{\min}}(1))$, we obtain

$$p_- e^{\ell_-(1+o_{\ell_{\min}}(1))} + p_+ e^{-\ell_+(1+o_{\ell_{\min}}(1))} = 1, \tag{36}$$

and $\ell_+ \gg 1$ we obtain

$$p_- = e^{-\ell_-(1+o_{\ell_{\min}}(1))}, \tag{37}$$

which implies again the equality (6).

6 First-passage time bounds from the asymptotic optimality of sequential probability ratio tests

As pointed out in Ref. [14], first-passage problems of stochastic currents with two thresholds are sequential hypothesis tests that decide on the arrow of time and first-passage problems for entropy production are sequential probability ratio tests. Therefore, we can use the theory of sequential hypothesis testing to derive bounds on the moments of first-passage times of stochastic currents. We first provide a brief review of the theory of sequential hypothesis testing, focusing on the asymptotic optimality of sequential probability ratio tests, and then show how to use these results to derive the main results Eqs. (3) and (6).

6.1 Review of sequential hypothesis testing

Sequential hypothesis tests are statistical hypothesis tests that take a decision D about the true hypothesis H at a random stopping time T. The general setup goes as follows [39,40]. There is an observation process $\vec{X}(t)$ whose statistics are determined by one of two possible probability measures P_+ or P_- corresponding to two hypotheses $H = +$ and $H = -$, respectively. A sequential hypothesis test is a pair (T, D), where T is a stopping time relative to the process $\vec{X}$, and $D \in \{-1, 1\}$ is a decision variable defined on the set of trajectories $\vec{X}_0^T$ up to the decision time T. The error reliabilities of the test are

$$p_- = P_+(D = -) \quad \text{and} \quad p_+^\dagger = P_-(D = +), \tag{38}$$

where $P_+(D = -) = P(D = -|H = +)$ and $P_-(D = +) = P(D = +|H = -)$.

Given certain maximally allowed error probabilities α_- and α_+, we define the set

$$\mathcal{C}_{\alpha_-,\alpha_+} = \Big\{ (T, D) : p_- \leq \alpha_+,\ p_+^\dagger \leq \alpha_-, \quad \langle T|H = +\rangle < \infty,\ \langle T|H = -\rangle < \infty \Big\} \tag{39}$$

of all sequential hypothesis tests that meet the required constraints on the error reliabilities and with finite expected decision times under both hypotheses. We say that a sequential hypothesis test is optimal if it is an element of $\mathcal{C}_{\alpha_-,\alpha_+}$ and it minimises the mean decision times $\langle T|H=+\rangle$ and $\langle T|H=-\rangle$.

For general observation processes $\vec{X}(t)$, the optimal sequential hypothesis tests is not known. Nevertheless, in the asymptotic limit of small error probabilities α_- and α_+ the optimal test is the sequential probability ratio test [40]. The sequential probability ratio test was first introduced by Wald for observation processes of independent and identically distributed random variables [41], and subsequently, Wald and Wolfowitz proved the optimality of this test in this setup [42]. In a later work [43], Lai proved the the asymptotic optimality of sequential probability ratio tests for general observation processes.

Let

$$\Lambda(t) = \log \frac{p^+(\vec{X}_0^t)}{p^-(\vec{X}_0^t)}, \tag{40}$$

be the log-likehood ratio process, which should be understood as the logarithm of the Radon-Nikodym derivative of the probability measure P_+ with respect to the probability measure P_-, both constrained on the sub-σ-algebra generated by the trajectories X_0^t. Loosely said this is the logarithm of the ratio of the probability densities $p^+(\vec{X}_0^t)$ and $p^-(\vec{X}_0^t)$ associated to the trajectries $\vec{X}_0^t$, which clarifies the notation in Eq. (92). The sequential probability ratio test is then the first-passage problem T_Λ (see Eq. (2)) with thresholds ℓ_- and ℓ_+ that determine the error probabilities p_- and $p_+^\dagger$. When Λ is a continuous process, then

$$\ell_- = \log[(1-p_+^\dagger)/p_-], \quad \ell_+ = \log[(1-p_-)/p_+^\dagger]. \tag{41}$$

We formulate a lemma and a theorem about the asymptotic properties of sequential hypothesis tests and the asymptotic optimality of sequential probability ratio tests. We first consider Lemma 3.4.1 in [40] that derives an asymptotic lower bound for the moments of the decision times of sequential hypothesis tests.

Lemma 1 (Asymptotic lower bounds for the moments of decision times in sequential hypothesis tests)**.** *Let $\delta = (T, D)$ be a sequential hypothesis test in the set $\mathcal{C}_{\alpha_-,\alpha_+}$. We assume that $\Lambda(t) \in \mathbb{R}$ and $1/\Lambda(t) \in \mathbb{R}$ for all $t \geq 0$. We assume that there exists a nonnegative increasing function $\psi(t)$ with $\psi(\infty) = \infty$ such that*

$$\lim_{t\to\infty} \frac{\Lambda(t)}{\psi(t)} = \overline{\lambda}_+, \quad (P_+\text{-almost surely}); \quad \lim_{t\to\infty} \frac{\Lambda(t)}{\psi(t)} = -\overline{\lambda}_-, \quad (P_-\text{-almost surely}) \tag{42}$$

with $\overline{\lambda}_-, \overline{\lambda}_+ \in (0,\infty)$. Moreover, we assume that for all finite τ

$$P_+ \left(\sup_{t\in[0,\tau]} \Lambda(t) < \infty\right) = 1, \quad P_- \left(-\inf_{t\in[0,\tau]} \Lambda(t) < \infty\right) = 1. \tag{43}$$

Under these assumptions, it holds that for all $\epsilon > 0$

$$\lim_{\alpha_{\max}\to 0} \inf_{\delta\in\mathcal{C}(\alpha_-,\alpha_+)} P_+\left(T > (1-\epsilon)\Psi\left(|\log\alpha_-|/\overline{\lambda}_+\right)\right) = 1 \tag{44}$$

$$\lim_{\alpha_{\max}\to 0} \inf_{\delta\in\mathcal{C}(\alpha_-,\alpha_+)} P_-\left(T > (1-\epsilon)\Psi\left(|\log\alpha_+|/\overline{\lambda_-}\right)\right) = 1 \tag{45}$$

where $\Psi(t)$ is the inverse of $\psi(t)$, i.e., $\Psi(\psi(t)) = t$. Moreover, for all $n > 0$

$$\lim_{\alpha_{\max}\to 0} \inf_{\delta\in\mathcal{C}(\alpha_-,\alpha_+)} \langle T^n|H=+\rangle \geq \left(\Psi\left(|\log\alpha_-|/\overline{\lambda}_+\right)\right)^n (1 + o_{\alpha_{\max}}(1)) \tag{46}$$

$$\lim_{\alpha_{\max}\to 0} \inf_{\delta\in\mathcal{C}(\alpha_-,\alpha_+)} \langle T^n|H=-\rangle \geq \left(\Psi\left(|\log\alpha_+|/\overline{\lambda}_+\right)\right)^n (1 + o_{\alpha_{\max}}(1)). \tag{47}$$

Second, we consider Theorem 3.4.2 in [40] for the asymptotic optimality of the sequential probability ratio test. Contrarily to Lemma 1, this theorem provides an equality for the mean first-passage times and therefore we will need to replace the almost sure convergence conditions Eqs. (42) by the stronger r-quick convergence condition. Let

$$L_\epsilon(Y(t)) = \sup\{t > 0 : |Y(t)| > \epsilon\}, \tag{48}$$

be the last entry time of a real-valued stochastic process $Y(t) \in \mathbb{R}$ into an interval $[-\epsilon, \epsilon]$. We say that $Y(t)$ converges r-quickly to 0 in P_+ if $\langle L^r_\epsilon | H = +\rangle < \infty$ for every $\epsilon > 0$.

Theorem 1 (Asymptotic optimality of sequential probability ratio tests)**.** *We assume that*

$$\lim_{t\to\infty} \frac{\Lambda(t)}{\psi(t)} = \overline{\lambda}_+, \quad (r\text{−quickly in } P_+); \quad \lim_{t\to\infty} \frac{\Lambda(t)}{\psi(t)} = -\overline{\lambda}_-, \quad (r\text{−quickly in } P_-). \tag{49}$$

It holds then that

- *for any finite threshold values ℓ_- and ℓ_+,*

$$\langle T^r_\Lambda | H = \pm\rangle < \infty; \tag{50}$$

- *for all $m \in (0, r]$,*

$$\langle T^m_\Lambda | H = \pm\rangle = \left(\Psi\left(\ell_\pm/\overline{\lambda}_\pm\right)\right)^m (1 + o_{\alpha_{\max}}(1)); \tag{51}$$

- *if $\ell_- = |\log p_-|(1 + o_{\ell_{\min}}(1))$ and $\ell_+ = |\log p^\dagger_+|(1 + o_{\alpha_{\max}}(1))$, then for all $m \in (0, r]$*

$$\langle T^m_\Lambda | H = +\rangle = \left(\Psi\left(|\log p^\dagger_+|/\overline{\lambda}_+\right)\right)^m (1 + o_{\alpha_{\max}}(1)) \tag{52}$$

  and

$$\langle T^m_\Lambda | H = -\rangle = \left(\Psi\left(|\log p_-|/\overline{\lambda}_-\right)\right)^m (1 + o_{\alpha_{\max}}(1)). \tag{53}$$

6.2 Using Lemma 1 to derive the first-passage bound Eq. (3)

We derive Eq. (3) by using Lemma 1. As will become evident, Lemma 1 is not sufficient to derive Eq. (3) and we also require the condition that J converges asymptotically to the drift-diffusion process in Eq. (12).

Let P denote the probability measure of events in the forward dynamics and let $P \circ \Theta$ be the probability measure of events in the time-reversed dynamics. Setting $P_+ = P$, $P_- = P \circ \Theta$, and $\psi(t) = t$, we obtain that $\Lambda(t) = S(t)$ and $\overline{\lambda}_+ = \dot{s}$. Since J is a stochastic current it changes sign under time-reversal and therefore the pair (T_J, D_J), with T_J as defined in Eq. (2) and $D_J = \text{sign}(J(T_J))$, is a sequential hypothesis test corresponding to the two probability measures P and $P \circ \Theta$ [14]. Replacing in Eq. (46) the α_- by $p^\dagger_+$ and $o_{\alpha_{\max}}(1)$ by $o_{\ell_{\min}}(1)$, we obtain [14]

$$\langle T^n_J \rangle \geq \left(\frac{|\log p^\dagger_+|}{\dot{s}}\right)^n (1 + o_{\ell_{\min}}(1)). \tag{54}$$

Note that Eq. (54) cannot be interpreted as a tradeoff between dissipation, speed, and uncertainty as the error probability $p^\dagger_+$ relates to the time-reversed process and has therefore not much to say about the fluctuations in the original forward process.

When J converges asymptotically to a drift-diffusion process, i.e., when the condition Eq. (12) holds, then (see Appendix B.3)

$$\ell_+ = |\log p_+^\dagger|(1 + o_{\ell_{\min}}(1)), \quad \text{and} \quad \ell_- = |\log p_-|(1 + o_{\ell_{\min}}(1)). \tag{55}$$

Multiplying the right-hand side of Eq. (54) with

$$1 = \left(\frac{\ell_+}{\ell_-}\frac{|\log p_-|}{|\log p_+^\dagger|}\right)^n, \tag{56}$$

we obtain Eq. (3), which concludes the derivation.

Note that the present derivation of Eq. (3) is more general than the derivation in Sec. 4. Indeed, in Sec. 4. we have used the large deviation function bound Eq. (28) that has been derived for Markov jump processes or overdamped Langevin processes [34]. On the other hand, Lemma 1 does not refer to a specific observation process $\vec{X}$ and therefore Eq. (3) also applies to, e.g., underdamped Langevin processes or periodically driven systems, as long as the asymptotic condition Eq. (12) is satisfied.

6.3 Using Theorem 1 to derive the asymptotic equality Eq. (6)

We set again $P_+ = P$, $P_- = P \circ \Theta$, and $\psi(t) = t$, obtaining $\overline{\lambda}_+ = \dot{s}$ and $\Psi(t) = t$. Therefore, Eq. (52) reads

$$\langle T_S^n \rangle = \left(\frac{|\log p_+^\dagger|}{\dot{s}}\right)^n (1 + o_{\ell_{\min}}(1)). \tag{57}$$

In Sec. 5.2 we have shown that

$$\ell_- = |\log p_-|(1 + o_{\ell_{\min}}(1)), \tag{58}$$

which follows readily from the martingale property of e^{-S}. Analogously, one can show that [20]

$$\ell_+ = |\log p_+^\dagger|(1 + o_{\ell_{\min}}(1)). \tag{59}$$

Multiplying the right-hand side of Eq. (57) with

$$1 = \left(\frac{\ell_+}{\ell_-}\frac{|\log p_-|}{|\log p_+^\dagger|}\right)^n, \tag{60}$$

we obtain Eq. (6), which completes the derivation.

7 Connections between Eq. (3) and other trade-off relations

We point out connections between Eqs. (3) and trade-off inequalities that appeared before in the literature.

7.1 Dissipation-time uncertainty relation

Eq. (3) is related to the so-called dissipation-time uncertainty relation that states

$$\langle T_J \rangle \geq \frac{1}{\dot{s}} \tag{61}$$

in the limit $|\log p_-| \gg 1$ [44].

The dissipation-time uncertainty relation is a loose bound when compared to the bounds Eqs. (3) and Eq. (54). Indeed, comparing Eq. (61) with (3), we conclude that

$$\langle T_J \rangle \geq \frac{c}{\dot{s}}(1 + o_{\ell_{\min}}(1)) \tag{62}$$

holds for any prefactor $c \geq 0$. This is because the prefactor in Eq. (3) is $c = |\log p_-|$ and thus diverges when p_- is small.

7.2 Thermodynamic uncertainty relations

Since the large deviation function bound Eq. (28) implies both the bound Eq. (3) and the thermodynamic uncertainty relations [33, 34, 45–49], one may expect that the bound Eq. (3) is related to the latter.

The thermodynamic uncertainty relation bounds from below the Fano factor of stochastic currents, i.e., [34, 45]

$$\frac{\sigma_J^2}{2\bar{j}^2} \geq \frac{1}{\dot{s}}, \tag{63}$$

where $\bar{j}$ is the current rate and

$$\sigma_J^2 = \lim_{t\to\infty} \frac{1}{t}\left(\langle J^2(t)\rangle - \langle J(t)\rangle^2\right). \tag{64}$$

A first-passage time thermodynamic uncertainty relation was derived in Ref. [15], viz.,

$$\frac{\langle T_J^2\rangle - \langle T_J\rangle^2}{2\langle T_J\rangle} \geq \frac{1}{\dot{s}}(1 + o_{\ell_{\min}}(1)). \tag{65}$$

The bounds Eqs. (3), (63) and (65) all express a nonequilibrium tradeoff between dissipation, speed, and uncertainty. The differences between these bounds is in how they quantify speed and uncertainty. The thermodynamic uncertainty relation Eq. (63) quantifies speed with $\bar{j}$ and uncertainty with σ_J^2, the first-passage time uncertainty relation Eq. (65) quantifies speed with $\langle T_J\rangle$ and uncertainty with $\langle T_J^2\rangle - \langle T_J\rangle^2$, and the bound Eq. (3) quantifies speed with $\langle T_J\rangle$ and uncertainty with p_-.

An important distinction between the thermodynamic uncertainty relations, Eqs. (63) and Eq. (65), and the bound Eq. (3) on the moments of first-passage times, is that the latter is tight when $J = S$ while the former is loose. Indeed, if $J(t) = S(t)(1+o_t(1))$, then Eq. (3) becomes the equality Eq. (6), whereas the Eqs. (63) and Eq. (65) are in general not equalities, even not when $J(t) = S(t)(1 + o_t(1))$ [31, 50].

Remarkably, all the three relations Eqs. (3), (63) and (65) are a consequence of the large deviation function bound Eq. (28). However, Eq. (3) uses the large deviation function bound at a value $z = -\bar{j}$, while Eqs. (63) and (65) use the large deviation function bound at values $z \approx 0$. As observed in Ref. [33], the large deviation function bound Eq. (28) is tight when $J = S$ and $z = -\dot{s}$, see the lower panel of Figure 3 in Ref. [33], while it is in general loose for $z \approx 0$. This clarifies why the bound (3) is tight for $J = S$ while the Eqs. (63) and (65) or loose, even though they all follow from the same inequality Eq. (28).

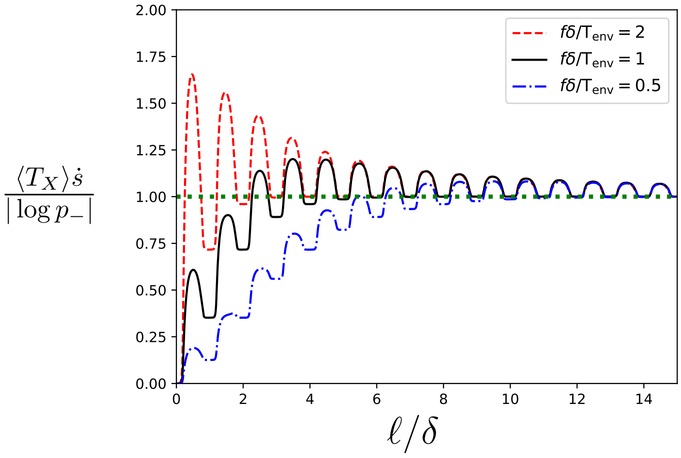

Figure 3: *Asymptotic lower bound on the mean first-passage time.* The ratio $\langle T_X\rangle \dot{s}/|\log p_-|$ is plotted as a function of ℓ/δ, where T_X is the first-passsage time Eq. (2) of the nonequilibrium Kramer process X described by Eq. (66) with triangular potential u given by Eq. (67). Curves shown are for the parameters $\delta = 5$, $x^* = 1$, $u_0 = 10$, $\mathsf{T}_{\rm env} = 1$, and $\gamma = 1$, and the values of f are given in the figure legend.

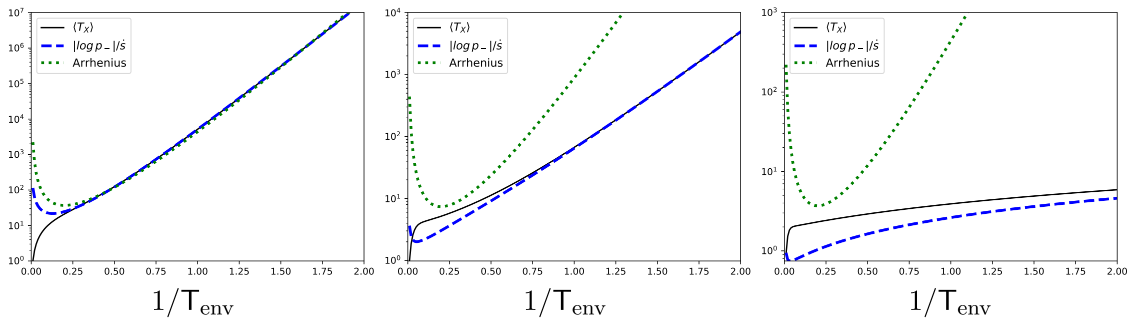

Figure 4: *Extension of the Van't Hoff-Arrhenius law to nonequilibrium stationary states.* The mean-first passage time $\langle T_X\rangle$ (solid black line) of the reaction coordinate X, described by Eq. (66) with triangular potential u given by Eq. (67), is plotted as a function of the inverse temperature $1/\mathsf{T}_{\rm env}$, and $\langle T_X\rangle$ is also compared with its asymptotic value $|\log p_-|/\dot{s}$ for large thresholds ℓ (blue dashed line) and with the Van't Hoff-Arrhenius law Eq. (77) (green dotted line). The model parameters are $\delta = 5$, $x^* = 1$, $u_0 = 10$, $\mathsf{T}_{\rm env} = 1$ and $\gamma = 2$ and the values of f are $f = 1$, $f = 5$ and $f = 10$ (left to right). The threshold for the first-passage time T_X, which is defined in Eq. (70), is $\ell = 10$.

8 Recovering the Van't Hoff-Arrhenius law in the near equilibrium limit

We show that near equilibrium Eq. (3) implies that $1/\langle T_J\rangle$ is smaller or equal than the Van't Hoff-Arrhenius law Eq. (5). To this aim, we consider a nonequilibrium version of Kramer's model [1,3]. Details of the calculations can be found in the Appendices C and D.

We consider a reaction coordinate $X \in \mathbb{R}$ that is described by the overdamped Langevin equation

$$\mathrm{d}X(t) = \frac{f - \partial_x u(X(t))}{\gamma}\,\mathrm{d}t + \sqrt{2\mathsf{T}_{\rm env}/\gamma}\,\mathrm{d}W(t), \tag{66}$$

where $u(x)$ is a periodic potential with period δ, i.e., $u(x+\delta) = u(x) = u(x-\delta)$, f is a nonconservative force, γ is a friction coefficient, $W(t)$ is a standard Wiener process that models the thermal noise, and $\mathsf{T}_{\rm env}$ is the temperature of the environment. We assume that at time $t = 0$, $X(0) = 0$ and $W(0) = 0$.

The variable X models, e.g., a reaction coordinate that tracks the progress of a chemical reaction. In this scenario, $E_{\rm b} = \max_x u(x) - \min_x u(x)$ is the Gibbs free energy barrier that separates two chemical states and the ratio $[X/\delta]$ is the number of cycles of the reaction that have been completed; $[a]$ denotes the largest integer smaller than a.

Figure 1 presents two trajectories generated by Eq. (66) for the special case where $u(x)$ is the triangular potential

$$u(x) = \begin{cases} u_0 \frac{x}{x^*} & \text{if} \quad x \in [0, x^*), \\ u_0 \frac{\delta - x}{\delta - x^*} & \text{if} \quad x \in [x^*, \delta). \end{cases} \tag{67}$$

From Fig. 1 we observe that the dynamics consists of a sequence of jumps between metastable states that are centred at the positions nx^* with $n \in \mathbb{Z}$. In the equilibrium case with $f = 0$ the jumps are activated by thermal fluctuations and the Van't Hoff-Arrhenius law Eq. (5) applies. On the other hand, when $f > 0$, then jumps in one direction over the energy barrier $E_{\rm b}$ are facilitated by the external driving f, while in the reverse direction jumps are less likely. In this case, although the Van't Hoff-Arrhenius law Eq. (5) does not apply, the Eqs. (3) and (6) apply and can thus be considered nonequilibrium versions of the Van't Hoff-Arrhenius law.

For values $f\delta/E_{\rm b} > 0$ the chemical reaction settles into a nonequilibrium stationary state with an entropy production rate (see Appendix C.2)

$$\dot{s} = \frac{f\delta}{\mathsf{T}_{\rm env}} j_{\rm ss}, \tag{68}$$

where $j_{\rm ss}$ is the stationary current (see Appendix C.1)

$$j_{\rm ss} = \frac{\mathsf{T}_{\rm env}}{\gamma} \frac{1 - e^{\frac{-f\delta}{\mathsf{T}_{\rm env}}}}{\int_0^\delta {\rm d}y\, w(y) \left(\int_y^{y+\delta} {\rm d}x' \frac{1}{w(x')} \right)}, \tag{69}$$

and where $w(x) = \exp(-(u(x) - fx)/\mathsf{T}_{\rm env})$.

Consider the first time

$$T_X = \inf \{ t > 0 : X(t) \notin (-\ell, \ell) \} \tag{70}$$

when the reaction has completed a net number $[\ell/\delta]$ of cycles in either the forward or backward direction. Since, (see Appendix C.2)

$$S(t) = \frac{f X(t)}{\mathsf{T}_{\rm env}} + o(t) \tag{71}$$

the equality (6) applies to T_X. In Appendices C.3 and C.4, we derive explicit analytical expressions for the splitting probability p_- and the mean first-passage time $\langle T_X \rangle$, respectively, which we omit here as the expressions are involved. However, as shown in Appendix C.5, in the limit of large ℓ we obtain the formula

$$\frac{|\log p_-|}{\langle T_X \rangle} = \dot{s} + O\left(\frac{1}{\ell}\right), \tag{72}$$

in correspondence with Eq. (6), where O denotes the big-O notation. Hence, in this case, the correction term in Eq. (6) is of order $1/\ell$.

In Fig. 3 we plot $|\log p_-|\dot{s}/\langle T_X\rangle$ as a function of ℓ/δ. The figure demonstrates the convergence of $|\log p_-|\dot{s}/\langle T_X\rangle$ to its universal limit for different values of the nonequilibrium driving $f\delta/\mathsf{T}_{\rm env}$. Observe the oscillations of $|\log p_-|\dot{s}/\langle T_X\rangle$. These oscillations appear because for the parameters selected it holds that $E_{\rm b} \gg \mathsf{T}_{\rm env}$, and therefore the process consists of discrete-like hops over the energy barrier $E_{\rm b}$ that represent the subsequent completion cycles of the chemical reaction.

In the limits $\mathsf{T}_{\rm env} \to 0$ and $f\delta/\mathsf{T}_{\rm env} \to 0$, the Eq. (6) leads to a Van't Hoff-Arrhenius law for $1/\langle T_X\rangle$. Indeed, as shown in Appendix C.6, taking the limits $\mathsf{T}_{\rm env} \to 0$ and $f\delta/\mathsf{T}_{\rm env} \to 0$ in the expression of the stationary current Eq. (69), we obtain

$$j_{\rm ss} = \kappa \frac{f\delta}{\gamma} e^{\frac{-E_{\rm b}}{\mathsf{T}_{\rm env}}}, \tag{73}$$

where the prefactor

$$\kappa = \frac{\sqrt{-u''_{\rm min} u''_{\rm max}}}{2\pi \mathsf{T}_{\rm env}} \tag{74}$$

if the second derivatives $u''_{\rm min}$ and $u''_{\rm max}$ evaluated at the minimum and maximum of $u(x)$, respectively, exist. In the special case of the triangular potential, given by Eq. (67), the second derivatives $u''_{\rm min}$ and $u''_{\rm max}$ do not exist. In this particular case

$$\frac{1}{\kappa} = \left(\frac{1}{u^+_{\rm max}} - \frac{1}{u^-_{\rm max}}\right)\left(\frac{1}{u^+_{\rm min}} - \frac{1}{u^-_{\rm min}}\right)\mathsf{T}^2_{\rm env} \tag{75}$$

where $u^+_{\rm max}$ and $u^-_{\rm max}$ denote the left and right derivatives evaluated at the maximum of $u(x)$. In addition, as shown in Appendix C.6, in the limit of $\mathsf{T}_{\rm env} \to 0$ and $f\delta/\mathsf{T}_{\rm env} \to 0$ the logarithm of the splitting probability is inversely proportional to the temperature, viz.,

$$\log p_- = -\frac{f\ell}{\mathsf{T}_{\rm env}} + O_\ell(1). \tag{76}$$

Combining Eqs. (6), (68), (73), and (76) we obtain the Van't Hoff-Arrhenius law

$$\langle T_X\rangle = \frac{\ell}{\delta}\frac{\gamma}{f\delta}\frac{1}{\kappa} e^{\frac{E_{\rm b}}{\mathsf{T}_{\rm env}}}. \tag{77}$$

In Fig. 4 we compare $\langle T_X\rangle$ with its asymptotic value $|\log p_-|/\dot{s}$, given by Eq. (6), and with the Van't Hoff-Arrhenius law, given by Eq. (77), for three values of the driving force f. We make a few interesting observations: (i) the Van't Hoff-Arrhenius law approximates well $\langle T_X\rangle$ up to moderately large values of $f\delta/\mathsf{T}_{\rm env} < 5$; (ii) for $f\delta/\mathsf{T}_{\rm env} > 25$, $\langle T_X\rangle$ is significantly smaller than what is predicted by the Van't Hoff-Arrhenius law, implying that the nonequilibrium driving speeds up the process. Nevertheless, $\langle T_X\rangle$ is larger than $|\log p_-|/\dot{s}$, which is a consequence of the trade-off between speed, uncertainty, and dissipation as expressed by Eq. (3); (iii) the asymptotic expression $|\log p_-|/\dot{s}$ given by Eq. (6) approximates $\langle T_X\rangle$ already well for relatively small values of the threshold, viz., $\ell/\delta = 2$.

Taken together, we conclude that the Eqs. (3) and (6) reduce to a Van't Hoff-Arrhenius law near equilibrium simply because $\dot{s} \sim \exp(-E_{\rm b}/\mathsf{T}_{\rm env})$ in the limit of small temperatures $\mathsf{T}_{\rm env} \approx 0$ and small driving force $f\delta/\mathsf{T}_{\rm env} \approx 0$. On the other hand, one can can significantly increase the reaction rate $1/\langle T_X\rangle$ by driving a system out of equilibrium, even though the reaction rates are still bounded from above by the inequality Eq. (3) that expresses a tradeoff between speed, uncertainty, and dissipation.

9 Illustration of the tightness of the first-passage time bounds with a biased random walker

As stated before, the bound Eq. (3) is tight for $J = S$, whereas the thermodynamic uncertainty relation Eq. (65) is loose when $J = S$. In this section we compute the moments $\langle T_J^n \rangle$ on an example of a nonequilibrium process to better understand the origin of the tightness of the bound Eq. (3).

We consider a hopping process $X \in \mathbb{Z}$ described by

$$\mathrm{d}X(t) = \mathrm{d}N_+(t) - \mathrm{d}N_-(t), \tag{78}$$

where N_+ and N_- are two counting process with rates k_+ and k_-, respectively. The bias of the process is defined by the ratio

$$b := \frac{k_-}{k_+} = \exp\left(-\frac{a}{\mathsf{T}_{\mathrm{env}}}\right) \tag{79}$$

where a is the thermodynamic affinity and $\mathsf{T}_{\mathrm{env}}$ the temperature of the environment. We assume, without loss of generality, that $k_- < k_+$ so that $b < 1$.

The coordinate X may represent the number of times a chemical reaction has been completed or the position of a molecular motor on a biofilament. In the former, $a = \Delta\mu$ is the difference between the sum of the chemical potentials of the products and the reagents of the chemical reaction, and in the latter $a = f\delta$ is the work performed by the system on the motor when it moves forwards. Hence, the stochastic entropy production S obeys

$$\mathrm{d}S(t) = \frac{a}{\mathsf{T}_{\mathrm{env}}}\mathrm{d}X(t) \tag{80}$$

and

$$\dot{s} = \left\langle \frac{\mathrm{d}S}{\mathrm{d}t} \right\rangle = \frac{a}{\mathsf{T}_{\mathrm{env}}}(k_+ - k_-) \tag{81}$$

is the entropy production rate.

We consider the first passage time

$$T_X = \inf\left\{t > 0 : X(t) - X(0) \notin (-\ell_-, \ell_+)\right\}, \tag{82}$$

which is also the first-passage time T_S of the stochastic entropy production with thresholds $s_- = a\ell_-/\mathsf{T}_{\mathrm{env}}$ and $s_+ = a\ell_+/\mathsf{T}_{\mathrm{env}}$.

The splitting probabilities p_- and p_+ are given by (see Appendix E.3)

$$p_+ = \frac{1 - b^{[\ell_-]}}{1 - b^{[\ell_-]+[\ell_+]}} \quad \text{and} \quad p_- = b^{[\ell_-]}\frac{1 - b^{[\ell_+]}}{1 - b^{[\ell_-]+[\ell_+]}}, \tag{83}$$

where $[\ell_-]$ and $[\ell_+]$ denote the largest integers that are smaller than ℓ_- and ℓ_+, respectively. The generating function

$$g(y) = \langle e^{-yT_X(k_- + k_+)} \rangle \tag{84}$$

is for all $y > 0$ given by (see Appendix E.4)

$$g(y) = \left(\frac{2}{\zeta_+(y)}\right)^{[\ell_+]} \frac{1 - \left(\frac{\zeta_-(y)}{\zeta_+(y)}\right)^{[\ell_-]}}{1 - \left(\frac{\zeta_-(y)}{\zeta_+(y)}\right)^{[\ell_-]+[\ell_+]}} + \left(\frac{\zeta_-(y)}{2}\right)^{[\ell_-]} \frac{1 - \left(\frac{\zeta_-(y)}{\zeta_+(y)}\right)^{[\ell_+]}}{1 - \left(\frac{\zeta_-(y)}{\zeta_+(y)}\right)^{[\ell_-]+[\ell_+]}}, \tag{85}$$

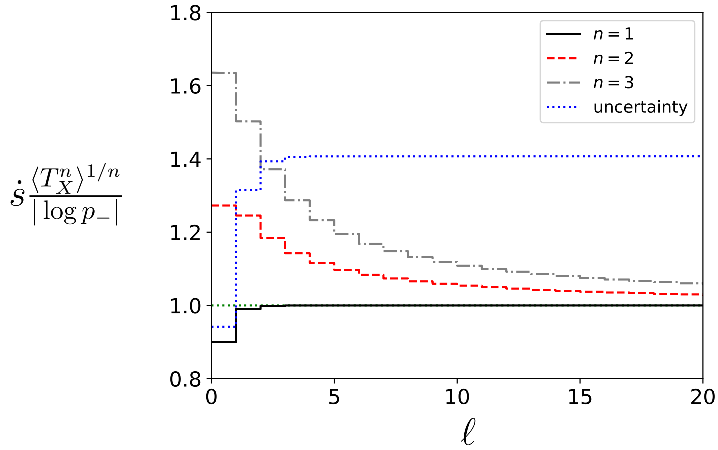

Figure 5: *Comparing the tightness of the first-passage time bounds Eq. (3) with the thermodynamic uncertainty relation Eq. (65).* The ratio $\dot{s}\langle T_X^n\rangle^{1/n}/|\log p_-|$ for $n = 1, 2, 3$ and the uncertainty $\dot{s}(\langle T_X^2\rangle - \langle T_X\rangle^2)/(2\langle T_X\rangle)$ as a function of $\ell = \ell_- = \ell_+$ for a biased random walk process X described by Eq. (78) with $k_+ = 1$ and $b = 0.1$. Note that the inequalities Eq. (3) are tight for $\ell \to \infty$, while the uncertainty relation Eq. (65) is loose.

where

$$\zeta_\pm(y) = \beta(y) \pm \sqrt{-4b + \beta^2(y)} \tag{86}$$

and

$$\beta(y) = (1 + y)(1 + b). \tag{87}$$

The moments of T_X follow from

$$\langle T_X^n\rangle = \left(\frac{-1}{k_- + k_+}\right)^n \frac{\mathrm{d}^n}{(\mathrm{d}y)^n} g(y)\bigg|_{y=0}, \tag{88}$$

where $n \in \mathbb{N}$.

Figure 5 compares the first-passage time bounds Eqs. (3) with the thermodynamic uncertainty relation Eq. (65). The plotted curves are obtained from the explicit analytical expressions for $\dot{s}$ and p_-, given by Eqs. (81) and (83), respectively, and from explicit analytical expressions for $\langle T^n\rangle$ that we have obtained from the Eqs. (84-88) and can be found in the Appendix E.6. The figure shows that for large values of the first-passage thresholds the bounds Eqs. (3) are tight, as predicted by Eq. (6), while the thermodynamic uncertainty relation is loose.

In Fig. 5 we also observe that the first moment $\langle T\rangle$ converges fast to its asymptotic value, while higher order moments $\langle T^2\rangle$ and $\langle T^3\rangle$ converge slowly to their asymptotic values. Using Eqs. (81), (83), and (84-88), we obtain the asymptotics (see Appendices E.7 and E.8)

$$\frac{[\ell_+]}{[\ell_-]}\frac{|\log p_-|}{\langle T_X\rangle} \quad = \quad \dot{s} + O\left(b^{[\ell_-]}\right), \tag{89}$$

and for $n > 1$

$$\frac{[\ell_+]}{[\ell_-]}\frac{|\log p_-|}{\left(\langle T_X^n\rangle\right)^{1/n}} = \dot{s} + O\left(\frac{1}{[\ell_+]}\right). \tag{90}$$

Hence, the first moment converges exponentially fast to the entropy production rate $\dot{s}$, while the higher order moments converge as $1/[\ell_+]$ to their asymptotic value. Consequently, in this example the first moment is more effective for the inference of the entropy production rate $\dot{s}$. However, from Eq. (72) we can conclude that the exponential fast convergence for the first moment is a model specific property.

The asymptotic expression for the thermodynamic uncertainty relation depends on the subleading $O(1/[\ell_+])$ term in Eq. (90), and is given by

$$\frac{2\langle T_X\rangle}{\langle T_X^2\rangle - \langle T_X\rangle^2} = \frac{2(k_+ - k_-)}{\tanh\left(\frac{a}{2\mathsf{T}_{\rm env}}\right)} + O\left(b^{[\ell_-]}\right). \tag{91}$$

Since $\tanh(x) \leq x$, Eq. (65) holds. However, contrary to Eqs. (89) and (90), the thermodynamic uncertainty relation is not tight in the limit of large thresholds and the ratio Eq. (91) depends on the affinity $a/\mathsf{T}_{\rm env}$ of the process.

Taken together, we can conclude that the equality Eq. (6), and thus the tightness of the bound Eq. (3) for $J = S$, follows from the universality of the leading order term in the Eqs. (89) and (90) for $\langle T_X^n\rangle$. On the other hand, the looseness of the thermodynamic uncertainty relation Eq. (65) for $S = J$ is a consequence of the nonuniversality of the subleading term of $\langle T_X^2\rangle$ in the Eqs. (89) and (90) and therefore the right-hand side of Eq. (91) depends on the affinity a of the process.

10 Discussion

Driving a system out of equilibrium can speed up the rate of a chemical reaction. However, there exists a fundamental thermodynamic tradeoff between speed, the fluctuations in the process, and the rate of dissipation. The main contribution of this paper is the derivation of a universal inequality, Eq. (3), that expresses in nonequilibrium stationary states a thermodynamic tradeoff between speed, uncertainty, and dissipation, which are quantified in terms of the mean first passage time $\langle T_J\rangle$, the splitting probability p_-, and the dissipation rate $\dot{s}$, respectively. The main advantage of the inequality (3) with respect to previously published trade-off relations, such as the thermodynamic uncertainty relations [33, 34, 45–49, 51, 52], is that Eq. (3) is an equality when $J = S$, see Eq. (6), and hence the bound is optimal in this case.

From a mathematical point of view, the Eqs. (3) and (6) are interesting as they are related to thermodynamic uncertainty relations, martingale theory, and the theory of sequential hypothesis testing. Indeed, both Eq. (3) and the thermodynamic uncertainty relations follow readily from the large deviation function bound Eq. (28). On the other hand, the equality Eq. (6) follows from martingale theory [19, 20], in particular the integral fluctuation relation at stopping times [21]. In addition, both of the Eqs. (3) and Eq.(6) are can be derived by using results from the theory of sequential hypothesis testing [40, 43], more specifically the asymptotic optimality of sequential probability ratio tests. It is fascinating that all these different research areas are related to each other and certainly more fundamental insights about stochastic thermodynamics can be gained by exploring the links between these areas.

Let us now discuss the conditions under which the main results Eqs. (3) and (6) apply. We will rely on the derivations in Sec. 6 as they are more general than those in Secs. 4 and 5. Both of the Eqs. (3) and (6) require stationarity in the weak sense that $S(t)/t$ converges to a constant limit $\dot{s}$: for the inequality Eq. (3) we require that $S(t)/t$ converges almost surely to $\dot{s}$ and for the equality Eq. (6) we require that $S(t)/t$ converges r-quickly to $\dot{s}$. In addition, for Eq. (3) we need that e^{-S} is a martingale, which is the case whenever it takes the form [19–21]

$$e^{-S(t)} = \frac{p(\vec{X}_0^t)}{\tilde{p}(\vec{X}_0^t)}, \tag{92}$$

with $\tilde{p}$ characterising the statistics of trajectories in the time-reversed process. Note that Eq. (92) and the martingalitiy of $e^{-S(t)}$ holds when $\vec{X}$ obeys local detailed balance, see e.g. [19–21, 53–55]. For the inequality Eq. (3) we need the additional condition that J converges asymptotically to the drift-diffusion process Eq. (12). This condition could be violated by considering for $\vec{X}$, e.g., a nonMarkovian process or a Lévy flight.

We end the paper with a brief discussion of potential applications for the Eqs. (3) and (6). The inequality Eq. (3) could be used to infer dissipation rates from the measurements of first-passage times of stochastic currents. It is difficult to measure the entropy production rate directly as it is related to the heat exchanged with the environment [56]. However, since the mean first-passage time $\langle T_J \rangle$ and the splitting probability p_- are directly measurable quantities, Eq. (3) can be used to bound the entropy production rate from below. When compared with other methods that infer entropy production rates from the measurements of stochastic currents, see e.g. [57–61], the present inequalities may turn out to perform better as they are optimal when $J = S$, although this requires further study. A second interesting application is in the use of the bound Eq. (3) to determine how far molecular systems operate from what is physically nonpermissible. Notable examples are molecular motors that are responsible for copying genetic information in biological cells, such as, ribosomes or polymerases. These motors are known to attain a reliability that is larger than what is possible in equilibrium through kinetic proof reading [62–64], but it is not known how close to the physically nonpermissible limits these motors operate. Another example are transistors that are small enough so that they are prone to noise [65]. Bounds of the form Eq. (3) could be used to understand thermodynamic limitations on computing that are based on the tradeoff between dissipation, speed, and uncertainty in nonequilibrium processes.

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A Martingales

In this appendix we briefly revisit some key properties of martingales that we use in this paper.

A.1 Definition of a martingale

Let Ω be the set of all realisations of a physical process $\vec{X}$, which is endowed with a σ-algebra $\mathscr{F}$. Let P be a probability measure that determines the probabilities $P(\Phi)$ of events $\Phi \in \mathscr{F}$. We denote averages with respect to P by $\langle \cdot \rangle$. Let $\{\mathscr{F}(t)\}_{t\geq 0}$ be the filtration generated by $\vec{X}$, i.e., a sequence of sub-σ-algebras $\mathscr{F}(t)$ that is generated by the trajectories $\vec{X}^t_0$ of the process X.

A martingale $M(t)$ with respect to a filtration $\{\mathscr{F}(t)\}_{t\geq 0}$ is a stochastic process for which (i) the process $M(t)$ is $\mathscr{F}(t)$-measurable (ii) $\langle |M(t)| \rangle < \infty$ (iii) $\langle M(t)|\mathscr{F}(s)\rangle = M(s)$ [66, 67]. The latter condition implies that the martingale M is a driftless process.

A.2 Doob's optional stopping theorem

A stopping time T is a random time $T : \Omega \to \mathbb{R}^+ \cup \{\infty\}$ such that $\{T \leq t\} \in \mathscr{F}(t)$ for all values of $t \in \mathbb{R}^+$. This means that T stops the process X based on a stopping rule that does not anticipate the future or use side information.

One of the key properties of martingales that we use in this paper is described by Doob's optional stopping theorem [67].

Theorem 2 (Doob's optional stopping theorem)**.** *Let $(\Omega, \mathscr{F}, P)$ be a probability space with sample space Ω, σ-algebra $\mathscr{F}$, and probability measure P. Let $X(t)$ with $t \geq 0$ be a $\mathscr{F}$-measurable stochastic process and let $\{\mathscr{F}(t)\}_{t\geq 0}$ be the filtration generated by X. Let M be a martingale process with respect to the filtration $\{\mathscr{F}(t)\}_{t\geq 0}$ and let T be a stopping time relative to the filtration $\{\mathscr{F}(t)\}_{t\geq 0}$. It holds then that*

$$\langle M(T \wedge t)\rangle = \langle M(0)\rangle \tag{93}$$

where $T \wedge t = \min\{T, t\}$.

B First-passage duality for an overdamped Brownian particle in an external force field

We derive the Eq. (25) that expresses a duality for the first-passage times Eq. (2) in drift-diffusion processes of the form

$$\mathrm{d}J(t) = \bar{j}\mathrm{d}t + \sqrt{2d_J}\mathrm{d}W(t), \tag{94}$$

where $W(t)$ is a standard, one-dimensional Wiener process. Since dualities compare first-passage times with different threshold values ℓ_- and ℓ_+, we refer to the first-passage problem of Eq. (2) as $T_J(\ell_-, \ell_+)$ whenever that is necessary to avoid misunderstandings. We derive Eq. (25) by showing that

$$\left\langle e^{-y\frac{T_J(\ell_-,\ell_+)}{\tau}} | D_J = -1 \right\rangle = \left\langle e^{-y\frac{T_J(\ell_+,\ell_-)}{\tau}} | D_J = 1 \right\rangle, \tag{95}$$

where

$$\tau = \frac{d_J}{\bar{j}^2} \tag{96}$$

and where

$$D_J = \mathrm{sign}\left(J(T_J)\right) \tag{97}$$

is the decision variable. The Eq. (95) readily implies the duality relation (25).

B.1 Two martingale equalities

The two processes

$$Z(t) = \exp\left(\frac{\sqrt{z}J(t)}{\bar{j}\tau} - \frac{t\,(z+\sqrt{z})}{\tau}\right), \quad t \geq 0 \tag{98}$$

and

$$\tilde{Z}(t) = \exp\left(\frac{-\sqrt{z}J(t)}{\bar{j}\tau} - \frac{t\,(z-\sqrt{z})}{\tau}\right), \quad t \geq 0, \tag{99}$$

are *martingales* for all values $z > 0$. Indeed, using Itô's formula we readily obtain that

$$\mathrm{d}Z(t) = \sqrt{2z}Z(t)\mathrm{d}W(t), \quad \text{and} \quad \mathrm{d}\tilde{Z}(t) = -\sqrt{2z}\tilde{Z}(t)\mathrm{d}W(t), \tag{100}$$

and the processes Z and $\tilde{Z}$ are thus driftless. Note that in the special case of $z = 1$, the martingale $\tilde{Z}(t)$ is the exponentiated negative entropy production.

Since Z and $\tilde{Z}$ are martingales, we can employ Theorem 2 yielding

$$\langle Z(t \wedge T_J)\rangle = \langle Z(0)\rangle = 1, \quad \text{and} \quad \langle \tilde{Z}(t \wedge T_J)\rangle = \langle \tilde{Z}(0)\rangle = 1. \tag{101}$$

Using the equalities (101), we can derive the splitting probabilities and generating functions for the first-passage time T_J in the interval $(-\ell_-, \ell_+)$. To this aim, we will use the following two equalities.

Proposition 1 (Equality 1)**.** *For all $z > 0$, we obtain*

$$1 = \left\langle 1_{T_J<\infty}1_{D_J=1}e^{\sqrt{z}\frac{\tau_+}{\tau}-(z+\sqrt{z})\frac{T_J}{\tau}} + 1_{T_J<\infty}1_{D_J=-1}e^{-\sqrt{z}\frac{\tau_-}{\tau}-(z+\sqrt{z})\frac{T_J}{\tau}}\right\rangle, \tag{102}$$

where $\tau_+ = \ell_+/\bar{j}$ and $\tau_- = \ell_-/\bar{j}$.

Proof. Since $Z(t \wedge T_J)$ is a martingale, Theorem 2 applies yielding

$$1 = \langle Z(t \wedge T_J)\rangle = \left\langle e^{\sqrt{z}\frac{J(t\wedge T_J)}{\bar{j}\tau}-(z+\sqrt{z})\frac{t\wedge T_J}{\tau}}\right\rangle. \tag{103}$$

Since

$$e^{\sqrt{z}\frac{J(t\wedge T_J)}{\bar{j}\tau}-(z+\sqrt{z})\frac{t\wedge T_J}{\tau}} \leq e^{\sqrt{z}\frac{\tau_+}{\tau}}, \tag{104}$$

the bounded convergence theorem applies, and we can take the limit $t \to \infty$ under the expectation value. We obtain

$$\begin{aligned} 1 &= \left\langle \lim_{t\to\infty} e^{\sqrt{z}\frac{J(t\wedge T_J)}{\bar{j}\tau}-(z+\sqrt{z})\frac{t\wedge T_J}{\tau}}\right\rangle && (105) \\ &= \left\langle 1_{T_J<\infty}1_{D_J=1}e^{\sqrt{z}\frac{\tau_+}{\tau}-(z+\sqrt{z})\frac{T_J}{\tau}} + 1_{T_J<\infty}1_{D_J=-1}e^{-\sqrt{z}\frac{\tau_-}{\tau}-(z+\sqrt{z})\frac{T_J}{\tau}}\right\rangle, && (106) \end{aligned}$$

which completes the proof. □

Proposition 2 (Equality 2)**.** *For all $z > \sqrt{z}$, it holds that*

$$1 = \left\langle 1_{T_J<\infty}1_{D_J=1}e^{-\sqrt{z}\frac{\tau_+}{\tau}-(z-\sqrt{z})\frac{T_J}{\tau}} + 1_{T_J<\infty}1_{D_J=-1}e^{\sqrt{z}\frac{\tau_-}{\tau}-(z-\sqrt{z})\frac{T_J}{\tau}}\right\rangle. \tag{107}$$

Proof. The proof is analogous to the proof of Eq. (102), with the distinction that now we consider the martingale $\tilde{Z}(t \wedge T_J)$.

Using Theorem 2, we obtain

$$1 = \langle \tilde{Z}(t \wedge T_J)\rangle = \langle e^{-\sqrt{z}\frac{J(t\wedge T_J)}{\bar{j}\tau}-(z-\sqrt{z})\frac{t\wedge T_J}{\tau}}\rangle. \tag{108}$$

Since for $z > \sqrt{z}$,

$$e^{-\sqrt{z}\frac{J(t\wedge T_J)}{\bar{j}\tau}-(z-\sqrt{z})\frac{t\wedge T_J}{\tau}} \leq e^{\sqrt{z}\frac{\tau_-}{\tau}} \tag{109}$$

the bounded convergence theorem applies, and

$$\begin{aligned} 1 &= \left\langle \lim_{t\to\infty} \tilde{Z}(t \wedge T_J)\right\rangle = \left\langle \lim_{t\to\infty} e^{-\sqrt{z}\frac{J(t\wedge T_J)}{\bar{j}\tau}-(z-\sqrt{z})\frac{t\wedge T_J}{\tau}}\right\rangle \\ &= \left\langle 1_{T_J<\infty}1_{D_J=1}e^{-\sqrt{z}\frac{\tau_+}{\tau}-(z-\sqrt{z})\frac{T_J}{\tau}} + 1_{T_J<\infty}1_{D_J=-1}e^{\sqrt{z}\frac{\tau_-}{\tau}-(z-\sqrt{z})\frac{T_J}{\tau}}\right\rangle. && (110) \end{aligned}$$

□

B.2 The first-passage time T_J is almost surely finite

.

Proposition 3. *It holds that T_J is almost surely finite, i.e.,*

$$p_- + p_+ = 1. \tag{111}$$

Proof. We take the the limit $z \to 0$ in Eq. (102). Since for $z < 1$ the argument in the expectation value on the right-hand side of Eq. (102) is bounded by $e^{\frac{\tau_+}{\tau}}$, the bounded convergence theorem applies and thus

$$\begin{aligned} 1 &= \lim_{z\to 0} \left\langle 1_{T_J<\infty} 1_{D_J=1} e^{\sqrt{z}\frac{\tau_+}{\tau} - (z+\sqrt{z})\frac{T_J}{\tau}} + 1_{T_J<\infty} 1_{D_J=-1} e^{-\sqrt{z}\frac{\tau_-}{\tau} - (z+\sqrt{z})\frac{T_J}{\tau}} \right\rangle \\ &= \langle 1_{T_J<\infty} 1_{D_J=1} \rangle + \langle 1_{T_J<\infty} 1_{D_J=-1} \rangle = p_- + p_+. \end{aligned}$$

□

B.3 Splitting probabilities

Proposition 4. *The splitting probabilities are given by*

$$p_+ = \frac{1-w_-}{1-w_-w_+}, \quad \text{and,} \quad p_- = w_- \frac{1-w_+}{1-w_-w_+}, \tag{112}$$

where

$$w_- = e^{-\frac{\tau_-}{\tau}}, \quad \text{and} \quad w_+ = e^{-\frac{\tau_+}{\tau}}, \tag{113}$$

where $\tau_- = \ell_-/\bar{j}$, $\tau_+ = \ell_+/\bar{j}$ and $\tau = d_J/\bar{j}^2$.

Proof. We consider the martingale process

$$e^{-S(t)} = e^{-\frac{J(t)}{\bar{j}\tau}}. \tag{114}$$

Theorem 2 implies that

$$\left\langle e^{-\frac{J(t\wedge T_J)}{\bar{j}\tau}} \right\rangle = 1. \tag{115}$$

Since J is a continuous process, it holds that

$$\lim_{t\to\infty} \left\langle e^{-\frac{J(t\wedge T_J)}{\bar{j}\tau}} \right\rangle \leq p_- e^{\frac{\tau_-}{\tau}} + p_+ e^{-\frac{\tau_+}{\tau}} + p_0 e^{\frac{\tau_-}{\tau}} \tag{116}$$

and

$$\lim_{t\to\infty} \left\langle e^{-\frac{J(t\wedge T_J)}{\bar{j}\tau}} \right\rangle \geq p_- e^{\frac{\tau_-}{\tau}} + p_+ e^{-\frac{\tau_+}{\tau}}. \tag{117}$$

According to Proposition 3, it holds that $p_0 = 0$, and thus

$$p_- e^{\frac{\tau_-}{\tau}} + p_+ e^{-\frac{\tau_+}{\tau}} = 1 \tag{118}$$

The solutions to the Eqs. (111) and (118) is given by Eqs. (112) and (113), which completes the proof.

□

B.4 Generating functions of T_J

We derive an explicit expressions for the generating functions

$$g_+(y) = \langle e^{-y\frac{T_J}{\tau}}|D_J = 1\rangle, \quad g_-(y) = \langle e^{-y\frac{T_J}{\tau}}|D_J = -1\rangle, \tag{119}$$

with $\lambda \geq 0$. We will use the notation

$$\kappa(y) = \sqrt{1+4y}. \tag{120}$$

The derivation of g_+ and g_- is based on the following two lemma's.

Lemma 2. *It holds that for $y \geq 0$,*

$$1 = p_+ e^{-\frac{1+\kappa(y)}{2}\frac{\tau_+}{\tau}} g_+(y) + p_- e^{\frac{1+\kappa(y)}{2}\frac{\tau_-}{\tau}} g_-(y) \tag{121}$$

and

$$1 = p_+ e^{\frac{\kappa(y)-1}{2}\frac{\tau_+}{\tau}} g_+(\lambda) + p_- e^{\frac{1-\kappa(y)}{2}\frac{\tau_-}{\tau}} g_-(\lambda). \tag{122}$$

Proof. Using the relation (102) for $z > 0$, we obtain

$$1 = p_+ e^{\sqrt{z}\frac{\tau_+}{\tau}} \langle e^{-(z+\sqrt{z})\frac{T_J}{\tau}}|D_J = 1\rangle + p_- e^{-\sqrt{z}\frac{\tau_-}{\tau}} \langle e^{-(z+\sqrt{z})\frac{T_J}{\tau}}|D_J = -1\rangle. \tag{123}$$

Setting

$$y = z + \sqrt{z} \tag{124}$$

yields the solution

$$z = \frac{1}{2}\left(1 + 2y - \kappa(y)\right). \tag{125}$$

Using Eqs. (124) and (125) in Eq. (123) yields Eq. (122).

Let us now consider the relation (107) for $z > \sqrt{z}$. This relation reads

$$1 = p_+ e^{-\sqrt{z}\frac{\tau_+}{\tau}} \langle e^{-(z-\sqrt{z})\frac{T_J}{\tau}}|D_J = 1\rangle + p_- e^{\sqrt{z}\frac{\tau_-}{\tau}} \langle e^{-(z-\sqrt{z})\frac{T_J}{\tau}}|D_J = -1\rangle. \tag{126}$$

Setting

$$y = z - \sqrt{z} \tag{127}$$

we find that

$$z = \frac{1}{2}\left(1 + 2y + \kappa(y)\right). \tag{128}$$

Using Eqs. (127) and (128) in Eq. (126) yields Eq. (121).

□

Proposition 5. *The generating functions*

$$g_+(y) = \frac{e^{(1-\kappa(y))\frac{\tau_+}{2\tau}}}{p_+}\left[\frac{1 - e^{-\kappa(y)\frac{\tau_-}{\tau}}}{1 - e^{-\kappa(y)\left(\frac{\tau_+ + \tau_-}{\tau}\right)}}\right] \tag{129}$$

and

$$g_-(y) = \frac{e^{-(1+\kappa(y))\frac{\tau_-}{2\tau}}}{p_-}\left[\frac{1 - e^{-\kappa(y)\frac{\tau_+}{\tau}}}{1 - e^{-\kappa(y)\left(\frac{\tau_+ + \tau_-}{\tau}\right)}}\right]. \tag{130}$$

Proof. We solve the Eqs. (121) and (122) towards $p_+g_+(y)$ and $p_-g_-(y)$ to find Eqs. (129) and (130). □

B.5 First-passage duality

Let us denote $g_+(y;\ell_-,\ell_+)$ and $g_-(y;\ell_-,\ell_+)$ for the generating functions of $T_J(\ell_-,\ell_+)$ when $D_J = 1$ or $D_J = -1$, respectively.

It follows readily from the Eqs. (112), (113), (129) and (130) that

$$g_+(y;\ell_-,\ell_+) = g_-(y;\ell_+,\ell_-) \tag{131}$$

and

$$g_-(y;\ell_-,\ell_+) = g_+(y;\ell_+,\ell_-), \tag{132}$$

which is exactly the first-passage duality Eq. (95) that we were meant to show.

C Mean-first passage time for an overdamped Brownian particle in a generic periodic potential and in a uniform force field

In this appendix, we analyse the first-passage problem for a Brownian motion in a generic periodic potential u and a uniform force field f, as described by Eq. (66). In particular, we derive analytical expressions for the mean-first passage time $\langle T_X \rangle$, the splitting probability p_-, and the mean entropy production rate $\dot{s}$, where T_X is defined as in Eq. (70). In the limit of large thresholds $\ell \gg 1$, we show that the main result Eq. (6) holds. In addition, in the near-equilibrium limit and at low temperatures, we show that Eq. (6) is a Van't Hoff-Arrhenius law.

C.1 Stationary distribution and current

We derive Eq. (69) in the main text for the stationary current $j_{\rm ss}$.

The stationary distribution of $X \in \mathbb{R}$ does not exist. However, we can define the process on a ring with periodic boundary conditions such that $X(t) = X(t) + \delta$. The stationary state $p_{\rm ss}$ of the equivalent process defined on a ring exists, and we can use the stationary process on a ring to determine the stationary current $j_{\rm ss}$.

The stationary distribution $p_{\rm ss}$ solves the equation [18, 68]

$$\partial_x j_{\rm ss}(x) = 0 \tag{133}$$

with periodic boundary conditions $p_{\rm ss}(x) = p_{\rm ss}(x+\delta)$, where

$$j_{\rm ss}(x) = \mu(f - \partial_x u(x)) p_{\rm ss}(x) - \frac{\mathsf{T}_{\rm env}}{\gamma} \partial_x p_{\rm ss}(x). \tag{134}$$

The solution to Eq. (133) is given by [21, 69]

$$p_{\rm ss}(x) = \frac{w(x)\left(\int_x^{x+\delta} {\rm d}x' \frac{1}{w(x')}\right)}{\int_0^\delta {\rm d}y\, w(y) \left(\int_y^{y+\delta} {\rm d}x' \frac{1}{w(x')}\right)} \tag{135}$$

with $x \in [0, \delta]$, and where

$$w(x) = e^{-\frac{u(x) - fx}{\mathsf{T}_{\rm env}}}. \tag{136}$$

The expression Eq. (69) for the stationary current $j_{\rm ss}$ follows readily from the Eqs. (134) and (135).

C.2 Entropy production

We derive Eqs. (68) and (71) in the main text for the entropy production rate $\dot{s}$ and the stochastic entropy production S, respectively. We will again use the equivalent process defined on a ring with periodic boundary conditions.

The stochastic entropy production S of X, as defined in Eq. (7), is determined by the stochastic differential equation [30, 31]

$$\mathrm{d}S = v_S(X)\,\mathrm{d}t + \sqrt{2v_S(X)}\,\mathrm{d}W(t), \tag{137}$$

where

$$v_S(x) = \frac{\gamma}{\mathsf{T}_{\rm env}}\frac{j_{\rm ss}^2}{p_{\rm ss}^2(x)} = \frac{\mathsf{T}_{\rm env}}{\gamma}\frac{\left(1-e^{\frac{-f\delta}{\mathsf{T}_{\rm env}}}\right)^2}{w^2(x)\left(\int_x^{x+\delta}\mathrm{d}x'\frac{1}{w(x')}\right)^2}. \tag{138}$$

Alternatively, we can write

$$S(t) = \frac{fX(t) - u(X(t)) + u(X(0))}{\mathsf{T}_{\rm env}} + \log\frac{p_{\rm ss}(X(0))}{p_{\rm ss}(X(t))}. \tag{139}$$

The latter formula implies that for large $t \gg 1$ it holds that

$$S(t) = \frac{fX(t)}{\mathsf{T}_{\rm env}} + o(t), \tag{140}$$

which is Eq. (71) in the main text.

The average stationary entropy production rate is given by

$$\dot{s} \;=\; \frac{\langle S(t)\rangle}{t} = \langle v_S\rangle = \frac{\gamma j_{\rm ss}^2}{\mathsf{T}_{\rm env}}\int_0^\delta \frac{\mathrm{d}x}{p_{\rm ss}(x)}. \tag{141}$$

Since the stationary distribution $p_{\rm ss}$ is given by Eq. (135) and $u(x)$ is a periodic function, we can express this also as

$$\dot{s} \;=\; j_{\rm ss}\left(1-e^{\frac{-f\delta}{\mathsf{T}_{\rm env}}}\right)\int_0^\delta \mathrm{d}x\frac{1}{w(x)\left(\int_0^\delta \mathrm{d}x'\frac{1}{w(x')} - (1-e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}})\int_0^x \mathrm{d}x'\frac{1}{w(x')}\right)}. \tag{142}$$

Introducing the function

$$\int_0^x \mathrm{d}x'\frac{1}{w(x')} = W(x), \tag{143}$$

we find that

$$\dot{s} \;=\; j_{\rm ss}\left(1-e^{\frac{-f\delta}{\mathsf{T}_{\rm env}}}\right)\int_0^{W(\delta)} \mathrm{d}u\frac{1}{\left(W(\delta) - (1-e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}})u\right)}. \tag{144}$$

Integrating yields the expression for $\dot{s}$ given by Eq. (68) in the main text.

C.3 Splitting probabilities

We use the martingale property of $e^{-S(t)}$, see Refs. [20, 21] or Appendix A, to determine the splitting probabilities p_- and p_+. Doob's optional stopping theorem for martingales implies the following integral fluctuation relation at stopping times

$$\langle e^{-S(T_X)}|X(0)=0\rangle = e^{-S(0)} = 1, \tag{145}$$

and since $S(t)$ is continuous as a function of t this implies that, see Refs. [20, 21],

$$p_- = e^{-s_-}\frac{1-e^{-s_+}}{1-e^{-s_--s_+}}, \quad \text{and} \quad p_+ = \frac{1-e^{-s_-}}{1-e^{-s_--s_+}}, \tag{146}$$

where

$$s_- = -\frac{-f\ell - u(-\ell) + u(0)}{\mathsf{T}_{\rm env}} - \log\frac{p_{\rm ss}(0)}{p_{\rm ss}(-\ell)}, \quad \text{and} \quad s_+ = \frac{f\ell - u(\ell)+u(0)}{\mathsf{T}_{\rm env}} + \log\frac{p_{\rm ss}(0)}{p_{\rm ss}(\ell)}. \tag{147}$$

Notice that we have used a slight abuse of notation in the sense that $u(x)$ and $p_{\rm ss}(x)$ are here defined on $x \in \mathbb{R}$ using $u(x) = u(x \pm \delta)$ and $p_{\rm ss}(x) = p_{\rm ss}(x\pm\delta)$.

C.4 Mean first-passage time

Consider the backward Fokker-Planck equation

$$\mu\left(f - \partial_x u(x)\right)\partial_x t(x) + \frac{\mathsf{T}_{\rm env}}{\gamma}\partial_x^2 t(x) = -1 \tag{148}$$

with boundary conditions $t(-\ell) = t(\ell) = 0$. It then holds that, see Ref. [70],

$$\langle T_X | X(0) = x\rangle = t(0). \tag{149}$$

The solution of $t(x)$ to Eq, (148) with boundary conditions $t(-\ell) = t(\ell) = 0$ is given by

$$t(x) = \frac{\gamma}{\mathsf{T}_{\rm env}}\int_{-\ell}^{\ell}{\rm d}y\frac{1}{w(y)}\int_0^y {\rm d}x' w(x')\left(\frac{\int_{-\ell}^{x}{\rm d}y\frac{1}{w(y)}}{\int_{-\ell}^{\ell}{\rm d}y\frac{1}{w(y)}} - \frac{\int_{-\ell}^{x}{\rm d}y\frac{1}{w(y)}\int_0^y{\rm d}x'w(x')}{\int_{-\ell}^{\ell}{\rm d}y\frac{1}{w(y)}\int_0^y{\rm d}x'w(x')}\right), \tag{150}$$

and therefore

$$\langle T_X\rangle = \frac{\gamma}{\mathsf{T}_{\rm env}}\left(\int_{-\ell}^{\ell}{\rm d}y\frac{1}{w(y)}\int_0^y {\rm d}x' w(x')\right)\left(\frac{\int_{-\ell}^{0}{\rm d}y\frac{1}{w(y)}}{\int_{-\ell}^{\ell}{\rm d}y\frac{1}{w(y)}} - \frac{\int_{-\ell}^{0}{\rm d}y\frac{1}{w(y)}\int_0^y{\rm d}x'w(x')}{\int_{-\ell}^{\ell}{\rm d}y\frac{1}{w(y)}\int_0^y{\rm d}x'w(x')}\right). \tag{151}$$

In order to better understand the structure of the expression Eq. (151) for the mean-first passage time, it is useful to express the integrals in Eq. (151) that run over the intervals $[-\ell, \ell]$ and $[-\ell, 0]$ in terms of integrals that run over the interval $[0, \delta]$. Let $n = [\ell/\delta]$ be the largest integer smaller than ℓ/δ, then we can write

$$\ell = n\delta + z, \tag{152}$$

with $z \in [0, \delta]$. Using this decomposition for ℓ, we obtain that

$$\int_{-n\delta-z}^{0}{\rm d}y\frac{1}{w(y)} = e^{n\frac{f\delta}{\mathsf{T}_{\rm env}}}\left\{\left(\frac{1-e^{-n\frac{f\delta}{\mathsf{T}_{\rm env}}}}{1-e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}}\right)\int_0^\delta\frac{{\rm d}x}{w(x)} + e^{\frac{f\delta}{\mathsf{T}_{\rm env}}}\int_{\delta-z}^{\delta}\frac{{\rm d}x}{w(x)}\right\} \tag{153}$$

and

$$\begin{aligned}&\int_{-n\delta-z}^{n\delta+z}{\rm d}y\frac{1}{w(y)} \\ &= e^{n\frac{f\delta}{\mathsf{T}_{\rm env}}}\left\{\left(\frac{1-e^{-2n\frac{f\delta}{\mathsf{T}_{\rm env}}}}{1-e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}}\right)\int_0^\delta\frac{{\rm d}x}{w(x)} + e^{\frac{f\delta}{\mathsf{T}_{\rm env}}}\int_{\delta-z}^{\delta}\frac{{\rm d}x}{w(x)} + e^{-2n\frac{f\delta}{\mathsf{T}_{\rm env}}}\int_0^z\frac{{\rm d}x}{w(x)}\right\}.\end{aligned} \tag{154}$$

In addition,

$$
\begin{aligned}
&\int_0^{n\delta+z} \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x' w(x') \\
&= n\left\{\frac{e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}}{1-e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}} \int_0^\delta \mathrm{d}x \frac{1}{w(x)} \int_0^\delta \mathrm{d}x\, w(x) + \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_0^y w(x)\mathrm{d}x\right\} \\
&\quad -\frac{e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}\left(1-e^{-n\frac{f\delta}{\mathsf{T}_{\rm env}}}\right)}{\left(1-e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}\right)^2} \int_0^\delta \mathrm{d}x\, w(x) \int_0^\delta \mathrm{d}x \frac{1}{w(x)} \\
&\quad + e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}} \frac{1-e^{-n\frac{f\delta}{\mathsf{T}_{\rm env}}}}{1-e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}} \int_0^z \mathrm{d}y \frac{1}{w(y)} \int_0^\delta \mathrm{d}x\, w(x) + \int_0^z \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x\, w(x), \quad (155)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_{-n\delta-z}^0 \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x'\, w(x') \\
&= \frac{1-e^{n\frac{f\delta}{\mathsf{T}_{\rm env}}}}{(1-e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}})(1-e^{\frac{f\delta}{\mathsf{T}_{\rm env}}})} \left(\int_0^\delta \mathrm{d}x\, w(x)\right)\left(\int_0^\delta \mathrm{d}x \frac{1}{w(x)}\right) \\
&\quad + n\left\{\int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_y^\delta \mathrm{d}x\, w(x) - \frac{1}{1-e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}} \left(\int_0^\delta \mathrm{d}x\, w(x)\right)\left(\int_0^\delta \mathrm{d}x \frac{1}{w(x)}\right)\right\} \\
&\quad + \frac{e^{n\frac{f\delta}{\mathsf{T}_{\rm env}}}-1}{1-e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}}} \int_{\delta-z}^\delta \mathrm{d}x \frac{1}{w(x)} \int_0^\delta \mathrm{d}x\, w(x) + \int_{\delta-z}^\delta \mathrm{d}y \frac{1}{w(y)} \int_y^\delta \mathrm{d}x\, w(x). \quad (156)
\end{aligned}
$$

Using the Eqs. (153), (154), (155), and (156) in Eq. (151), we obtain an expression for $\langle T_X \rangle$ that depends only on integrals over the interval $[0,\delta]$.

C.5 Limit of large thresholds

We derive the Eq. (72) that holds in the limit of large ℓ.

C.5.1 Splitting probabilities

In the limit of large thresholds, the linear term in ℓ dominates the Eqs. (147) and therefore

$$
s_- = \frac{f\ell}{\mathsf{T}_{\rm env}} + O_\ell(1), \quad \text{and} \quad s_+ = \frac{f\ell}{\mathsf{T}_{\rm env}} + O_\ell(1). \quad (157)
$$

Using Eq. (157) in the Eqs. (146) for p_- and p_+, we obtain that

$$
\log p_- = -\frac{f\ell}{\mathsf{T}_{\rm env}} + O_\ell(1), \quad \text{and} \quad \log p_+ = 1 + O_\ell(1). \quad (158)
$$

C.5.2 Mean first-passage time

We use that

$$
n = \left[\frac{\ell}{\delta}\right] + O_\ell(1), \quad (159)
$$

where as before $\left[\frac{\ell}{\delta}\right]$ denotes the largest integer that is smaller than $\frac{\ell}{\delta}$.

Taking the asymptotic limit of large ℓ in Eqs. (153) and (154), we obtain that

$$\frac{\int_{-\ell}^{0}\mathrm{d}y\frac{1}{w(y)}}{\int_{-\ell}^{\ell}\mathrm{d}y\frac{1}{w(y)}} = 1-e^{-\left[\frac{\ell}{\delta}\right]\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}\frac{\int_0^\delta \frac{\mathrm{d}x}{w(x)}}{\int_0^\delta \frac{\mathrm{d}x}{w(x)}+(e^{\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}-1)\int_{\delta-z}^{\delta}\frac{\mathrm{d}x}{w(x)}}+O\left(e^{-2\left[\frac{\ell}{\delta}\right]\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}\right). \quad (160)$$

The asymptotic limit of Eq. (155) is

$$\begin{aligned}&\int_0^\ell \mathrm{d}y\frac{1}{w(y)}\int_0^y \mathrm{d}x' w(x') \\ &= \left[\frac{\ell}{\delta}\right]\left\{\frac{e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}}{1-e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}}\int_0^\delta \mathrm{d}x\frac{1}{w(x)}\int_0^\delta \mathrm{d}x w(x)+\int_0^\delta \mathrm{d}y\frac{1}{w(y)}\int_0^y w(x)\mathrm{d}x\right\}+O_\ell(1),\end{aligned} \quad (161)$$

and from Eqs. (155) and (156) it follows that

$$\begin{aligned}&-\int_{-\ell}^{\ell}\mathrm{d}y\frac{1}{w(y)}\int_0^y \mathrm{d}x' w(x') \\ &= e^{\left[\frac{\ell}{\delta}\right]\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}\left\{\frac{\int_0^\delta \mathrm{d}x\, w(x)\int_0^\delta \mathrm{d}x\frac{1}{w(x)}}{(1-e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}})(e^{\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}-1)}+\frac{\int_{\delta-z}^{\delta}\mathrm{d}x\frac{1}{w(x)}\int_0^\delta \mathrm{d}x w(x)}{1-e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}}\right\} \\ &\quad+\left[\frac{\ell}{\delta}\right]\left\{\int_0^\delta \mathrm{d}y\frac{1}{w(y)}\int_y^\delta \mathrm{d}x\, w(x)-\frac{1}{\tanh\left(\frac{f\delta}{2\mathsf{T}_{\mathrm{env}}}\right)}\int_0^\delta \mathrm{d}x\frac{1}{w(x)}\int_0^\delta \mathrm{d}x\, w(x)\right. \\ &\quad\left.-\int_0^\delta \mathrm{d}y\frac{1}{w(y)}\int_0^y \mathrm{d}x\, w(x)\right\}+O_\ell(1).\end{aligned} \quad (162)$$

The Eqs. (161) and (162) imply that the ratio

$$\begin{aligned}&\frac{\int_{-\ell}^{0}\mathrm{d}y\frac{1}{w(y)}\int_0^y \mathrm{d}x' w(x')}{\int_{-\ell}^{\ell}\mathrm{d}y\frac{1}{w(y)}\int_0^y \mathrm{d}x' w(x')} \\ &= 1+\left[\frac{\ell}{\delta}\right]e^{-\left[\frac{\ell}{\delta}\right]\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}\left\{\frac{e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}\left(\int_0^\delta \mathrm{d}x\, w(x)\right)\left(\int_0^\delta \mathrm{d}x\frac{1}{w(x)}\right)+\left(1-e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}\right)\int_0^\delta \mathrm{d}y\frac{1}{w(y)}\int_0^y w(x)\mathrm{d}x}{\frac{\int_0^\delta \mathrm{d}x w(x)\int_0^\delta \mathrm{d}x\frac{1}{w(x)}}{e^{\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}-1}+\int_{\delta-z}^{\delta}\mathrm{d}x\frac{1}{w(x)}\int_0^\delta \mathrm{d}x\, w(x)}\right\} \\ &\quad+O\left(e^{-\left[\frac{\ell}{\delta}\right]\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}\right).\end{aligned} \quad (163)$$

Using Eqs. (160)-(163) in Eq. (151) yields for the mean first-passage time the asymptotic expression

$$\langle T_X\rangle = \frac{\gamma}{\mathsf{T}_{\mathrm{env}}}\left[\frac{\ell}{\delta}\right]\left[\frac{e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}}{1-e^{-\frac{f\delta}{\mathsf{T}_{\mathrm{env}}}}}\left(\int_0^\delta \mathrm{d}x w(x)\right)\left(\int_0^\delta \mathrm{d}x\frac{1}{w(x)}\right)+\int_0^\delta \mathrm{d}y\frac{1}{w(y)}\int_0^y w(x)\mathrm{d}x\right]+O_\ell(1). \quad (164)$$

C.5.3 The ratio $|\log p_-|/\langle T_X\rangle$

It follows from the asymptotic relations for $\langle T_X\rangle$ and $|\log p_-|$, given by Eqs. (164) and (158), respectively, that the ratio

$$\frac{|\log p_-|}{\langle T_X\rangle} = \frac{f\delta}{\gamma}\frac{1-e^{\frac{-f\delta}{\mathsf{T}_{\mathrm{env}}}}}{\int_0^\delta \mathrm{d}y\, w(y)\left(\int_y^{y+\delta}\mathrm{d}x'\frac{1}{w(x')}\right)}+O(1/\ell). \quad (165)$$

Using Eqs. (68) and (69) for $\dot{s}$ and $j_{\rm ss}$, respectively, together with the identities

$$\int_0^\delta {\rm d}y \frac{1}{w(y)} \int_0^y {\rm d}x\, w(x) = \int_0^\delta {\rm d}y\, w(y) \int_y^\delta \frac{1}{w(x)} {\rm d}x \tag{166}$$

and

$$e^{-\frac{f\delta}{\mathsf{T}_{\rm env}}} \int_0^\delta {\rm d}x\, w(x) \int_0^y {\rm d}x \frac{1}{w(x)} = \int_0^\delta {\rm d}x\, w(x) \int_\delta^{y+\delta} {\rm d}x \frac{1}{w(x)}, \tag{167}$$

we readily obtain Eq. (72), which is what we were meant to show.

C.6 Van't Hoff-Arrhenius law near equilibrium

We show that Eq. (72) yields the Van't Hoff-Arrhenius law Eq. (77).

Indeed, if ℓ is large enough, then Eq. (72) together with Eq. (158) yields

$$\langle T_X \rangle = \frac{f\ell}{\mathsf{T}_{\rm env}} \frac{1}{\dot{s}} + O\left(\frac{1}{\ell}\right) \tag{168}$$

where the mean entropy production rate $\dot{s}$ is given by Eq. (68). Since the mean entropy production rate is proportional to the stationary current, given by Eq. (69), we can use saddle point integrals to evaluate the mean current in the limit $\mathsf{T}_{\rm env} \to 0$ and to obtain the Van't Hoff-Arrhenius law.

Let us therefore first revisit the saddle point method, and then apply it to the mean current to obtain the Van't Hoff-Arrhenius law.

C.6.1 Saddle point integrals in the limit of $\mathsf{T}_{\rm env} \to 0$

We first revisit briefly the saddle point method.

Let $v(x)$ be a function defined on the interval $[0, \delta]$. Then we analyse integrals of the form

$$\int_0^\delta {\rm d}x\, e^{\frac{v(x)}{\mathsf{T}_{\rm env}}} f(x) \tag{169}$$

in the limiting case of small $\mathsf{T}_{\rm env}$. In this limiting case,

$$\int_0^\delta {\rm d}x\, e^{\frac{v(x)}{\mathsf{T}_{\rm env}}} f(x) = \kappa f(x_{\rm max}) e^{\frac{v_{\rm max}}{\mathsf{T}_{\rm env}}} + O\left(\frac{\mathsf{T}_{\rm env}}{v_{\rm max}}\right) \tag{170}$$

where κ is a prefactor that depends on the properties of the function v at the maximum. Note that we use the following notation: if $x_{\rm max} = \operatorname{argmax} v(x)$, then $v_{\rm max} = v(x_{\rm max})$, $v'_{\rm max} = v'(x_{\rm max})$, and $v''_{\rm max} = v''(x_{\rm max})$.

There exist four relevant cases:

- $v'_{\rm max} = 0$ and $x_{\rm max} \in (0, \delta)$:

$$\kappa = \sqrt{\frac{2\pi \mathsf{T}_{\rm env}}{-v''_{\rm max}}}; \tag{171}$$

- $v'_{\rm max}$ does not exist (maximum is a cusp) and $x_{\rm max} \in (0, \delta)$:

$$\kappa = \mathsf{T}_{\rm env} \left(\frac{1}{v^+_{\rm max}} - \frac{1}{v^-_{\rm max}}\right) \tag{172}$$

  where

$$v^+_{\rm max} = \lim_{\epsilon\to 0} \frac{v(x_{\rm max}) - v(x_{\rm max} - \epsilon)}{\epsilon}, \quad \text{and} \quad v^-_{\rm max} = \lim_{\epsilon\to 0} \frac{v(x_{\rm max} + \epsilon) - v(x_{\rm max})}{\epsilon}; \tag{173}$$

- $x_{\max} = 0$:

$$\kappa = -\frac{\mathsf{T}_{\rm env}}{v^{-}_{\max}}; \tag{174}$$

- $x_{\max} = \delta$:

$$\kappa = \frac{\mathsf{T}_{\rm env}}{v^{+}_{\max}}. \tag{175}$$

C.6.2 The mean first-passage time in the low temperature limit and the linear response limit

We consider first the near equilibrium limit with $f\delta/\mathsf{T}_{\rm env} \approx 0$, and then we consider the low temperature limit $\mathsf{T}_{\rm env} \approx 0$.

First we take the linear response limit with $f\delta/\mathsf{T}_{\rm env} \approx 0$. It holds then that

$$w(x) = e^{-\frac{u(x)}{\mathsf{T}_{\rm env}}}\left(1 + \frac{fx}{\mathsf{T}_{\rm env}} + O\left(\left(\frac{f\delta}{\mathsf{T}_{\rm env}}\right)^2\right)\right), \tag{176}$$

and

$$\frac{1}{w(x)} = e^{\frac{u(x)}{\mathsf{T}_{\rm env}}}\left(1 - \frac{fx}{\mathsf{T}_{\rm env}} + O\left(\left(\frac{f\delta}{\mathsf{T}_{\rm env}}\right)^2\right)\right), \tag{177}$$

such that

$$j_{\rm ss} = \frac{f\delta}{\gamma}\frac{1}{\int_0^\delta {\rm d}y e^{-\frac{u(y)}{\mathsf{T}_{\rm env}}}\int_0^\delta {\rm d}x e^{\frac{u(x)}{\mathsf{T}_{\rm env}}}} + O\left(\left(\frac{f\delta}{\mathsf{T}_{\rm env}}\right)^2\right). \tag{178}$$

Second, we take the low temperature limit with $\mathsf{T}_{\rm env} \approx 0$. Using the saddle point method, we obtain that

$$j_{\rm ss} = \frac{f\delta}{\gamma}\kappa_1\kappa_2 e^{-\frac{E_{\rm b}}{\mathsf{T}_{\rm env}}} + O\left(\left(\frac{f\delta}{\mathsf{T}_{\rm env}}\right)^2\right) \tag{179}$$

where κ_1 and κ_2 are two prefactors due to the two saddle point integrals in Eq. (178). The entropy production rate follows from Eq. (68) and is given by

$$\dot{s} = \frac{(f\delta)^2}{\gamma\mathsf{T}_{\rm env}}\kappa_1\kappa_2 e^{-\frac{E_{\rm b}}{\mathsf{T}_{\rm env}}} + O\left(\left(\frac{f\delta}{\mathsf{T}_{\rm env}}\right)^3\right). \tag{180}$$

Lastly, using Eq. (168) we obtain the Van't Hoff-Arrhenius law for the mean-first passage time

$$\langle T_X\rangle = \frac{\ell}{\delta}\frac{\gamma}{f\delta}\frac{1}{\kappa_1\kappa_2}e^{\frac{E_{\rm b}}{\mathsf{T}_{\rm env}}}\left(1 + O\left(\frac{f\delta}{\mathsf{T}_{\rm env}}\right)\right). \tag{181}$$

We discuss two relevant cases:

- $u'_{\max} = u'_{\min} = 0$ and $x_{\max}, x_{\min} \in (0, \delta)$:

$$\kappa_1\kappa_2 = \frac{\sqrt{-u''_{\min}u''_{\max}}}{2\pi\mathsf{T}_{\rm env}}; \tag{182}$$

- $u'_{\max} \neq 0$ and $u'_{\min} \neq 0$:

$$\kappa_1\kappa_2 = \left(\frac{1}{u^{+}_{\max}} - \frac{1}{u^{-}_{\max}}\right)^{-1}\left(\frac{1}{u^{+}_{\min}} - \frac{1}{u^{-}_{\min}}\right)^{-1}\frac{1}{\mathsf{T}^2_{\rm env}}. \tag{183}$$

D Mean-first passage time for a Brownian particle in a periodic potential that is triangular and in a uniform force field

Just as in the previous appendix, we consider a Brownian motion in a uniform force field f and a periodic potential u, for which dynamics of the position variable X is described by the overdamped Langevin Eq. (66). However, in this appendix we consider the specific case of the triangular potential given by Eq. (67). We have used this case to generate the curves in the Figs. 1-4.

D.1 Stationary distribution

The stationary probability distribution, given by Eq. (135), for a triangular potential is given by [31]

$$p_{\rm ss}(x) = \begin{cases} a_1 + a_2 e^{\frac{x f_+}{\mathsf{T}_{\rm env}}} & \text{if} \quad x \in [0, x^*], \\ a_3 + a_4 e^{\frac{x f_-}{\mathsf{T}_{\rm env}}} & \text{if} \quad x \in [x^*, \delta], \end{cases} \tag{184}$$

where

$$f_+ = f - \frac{u_0}{x^*}, \quad \text{and} \quad f_- = f + \frac{u_0}{\delta - x^*}, \tag{185}$$

and

$$a_1 = f_+ f_-^2 \frac{e^{\frac{f_- x^*}{\mathsf{T}_{\rm env}}} - e^{\frac{f_-\delta + f_+ x^*}{\mathsf{T}_{\rm env}}}}{\mathcal{N}}, \tag{186}$$

$$a_2 = f_+ f_- (f_- - f_+) \frac{e^{\frac{f_-\delta}{\mathsf{T}_{\rm env}}} - e^{\frac{f_- x^*}{\mathsf{T}_{\rm env}}}}{\mathcal{N}}, \tag{187}$$

$$a_3 = f_+^2 f_- \frac{e^{\frac{f_- x^*}{\mathsf{T}_{\rm env}}} - e^{\frac{f_-\delta + f_+ x^*}{\mathsf{T}_{\rm env}}}}{\mathcal{N}}, \tag{188}$$

$$a_4 = f_+ f_- (f_+ - f_-) \frac{e^{\frac{f_+ x^*}{\mathsf{T}_{\rm env}}} - 1}{\mathcal{N}}, \tag{189}$$

and where the normalisation constant

$$\begin{aligned} \mathcal{N} = {} & \mathsf{T}_{\rm env} (f_+ - f_-)^2 \left(e^{\frac{f_+ x^*}{\mathsf{T}_{\rm env}}} - 1 \right) \left(e^{\frac{f_-\delta}{\mathsf{T}_{\rm env}}} - e^{\frac{f_- x^*}{\mathsf{T}_{\rm env}}} \right) \\ & + f_+ f_- (f_+\delta - f_+ x^* + f_- x^*) \left(e^{\frac{f_- x^*}{\mathsf{T}_{\rm env}}} - e^{\frac{f_-\delta + f_+ x^*}{\mathsf{T}_{\rm env}}} \right). \end{aligned} \tag{190}$$

The stationary current is given by the expression

$$j_{\rm ss} = \frac{f_+ a_1}{\gamma} = \frac{f_- a_3}{\gamma}. \tag{191}$$

In Fig. 6, we plot the stationary distribution $p_{\rm ss}$ for various values of the nonequilibrium driving $f\delta/\mathsf{T}_{\rm env}$. Observe that the distribution concentrates around the values $x \approx 0$ or $x \approx \delta$, and thus the process resembles a hopping process, as is also visible in Fig. 1.

D.2 Mean first-passage time

In the case of the triangular potential we can obtain an explicit expression for $\langle T_X \rangle$ given by Eq. (151). This is because the integrals that appear in the Eqs. (153), (154), (155), and (156) can be solved explicitly.

We obtain explicit expressions for the following integrals:

$$\int_0^z \mathrm{d}x\, w(x) = \begin{cases} \frac{\mathsf{T}_{\mathrm{env}}}{f_+}\left(e^{\frac{f_+ z}{\mathsf{T}_{\mathrm{env}}}} - 1\right) & \text{if} \quad z < x^*, \\ \frac{\mathsf{T}_{\mathrm{env}}}{f_+}\left(e^{\frac{f_+ x^*}{\mathsf{T}_{\mathrm{env}}}} - 1\right) + \frac{\mathsf{T}_{\mathrm{env}}}{f_-} e^{-\frac{u_0}{\mathsf{T}_{\mathrm{env}}}\frac{\delta}{\delta - x^*}}\left(e^{\frac{f_- z}{\mathsf{T}_{\mathrm{env}}}} - e^{\frac{f_- x^*}{\mathsf{T}_{\mathrm{env}}}}\right) & \text{if} \quad z > x^*, \end{cases} \tag{192}$$

$$\int_0^z \frac{\mathrm{d}x}{w(x)} = \begin{cases} \frac{\mathsf{T}_{\mathrm{env}}}{f_+}\left(1 - e^{\frac{-f_+ z}{\mathsf{T}_{\mathrm{env}}}}\right) & \text{if} \quad z < x^*, \\ \frac{\mathsf{T}_{\mathrm{env}}}{f_+}\left(1 - e^{\frac{-f_+ x^*}{\mathsf{T}_{\mathrm{env}}}}\right) + \frac{\mathsf{T}_{\mathrm{env}}}{f_-} e^{\frac{u_0}{\mathsf{T}_{\mathrm{env}}}\frac{\delta}{\delta - x^*}}\left(e^{\frac{-f_- x^*}{\mathsf{T}_{\mathrm{env}}}} - e^{\frac{-f_- z}{\mathsf{T}_{\mathrm{env}}}}\right) & \text{if} \quad z > x^*, \end{cases} \tag{193}$$

and

$$\int_{\delta - z}^{\delta} \frac{\mathrm{d}x}{w(x)} = \begin{cases} \frac{\mathsf{T}_{\mathrm{env}}}{f_+}\left(e^{\frac{-f_+(\delta - z)}{\mathsf{T}_{\mathrm{env}}}} - e^{\frac{-f_+ x^*}{\mathsf{T}_{\mathrm{env}}}}\right) + \frac{\mathsf{T}_{\mathrm{env}}}{f_-} e^{\frac{u_0}{\mathsf{T}_{\mathrm{env}}}\frac{\delta}{\delta - x^*}}\left(e^{\frac{-f_- x^*}{\mathsf{T}_{\mathrm{env}}}} - e^{\frac{-f_- \delta}{\mathsf{T}_{\mathrm{env}}}}\right) & \text{if} \quad \delta - z < x^*, \\ \frac{\mathsf{T}_{\mathrm{env}}}{f_-} e^{\frac{u_0}{\mathsf{T}_{\mathrm{env}}}\frac{\delta}{\delta - x^*}}\left(e^{\frac{-f_-(\delta - z)}{\mathsf{T}_{\mathrm{env}}}} - e^{\frac{-f_- \delta}{\mathsf{T}_{\mathrm{env}}}}\right) & \text{if} \quad \delta - z > x^*. \end{cases} \tag{194}$$

In addition, if $z < x^*$, then

$$\int_0^z \mathrm{d}y \frac{1}{w(y)} \int_0^y w(x)\mathrm{d}y = \frac{\mathsf{T}_{\mathrm{env}}}{f_+} z - \left(\frac{\mathsf{T}_{\mathrm{env}}}{f_+}\right)^2 \left(1 - e^{-\frac{f_+ z}{\mathsf{T}_{\mathrm{env}}}}\right), \tag{195}$$

and if $z > x*$, then

$$\begin{aligned} \int_0^z \mathrm{d}y \frac{1}{w(y)} \int_0^y w(x)\mathrm{d}y \\ = \; & \frac{\mathsf{T}_{\mathrm{env}}}{f_+} x^* + \frac{\mathsf{T}_{\mathrm{env}}}{f_-}(z - x^*) - \left(\frac{\mathsf{T}_{\mathrm{env}}}{f_+}\right)^2 \left(1 - e^{-\frac{f_+ x^*}{\mathsf{T}_{\mathrm{env}}}}\right) - \left(\frac{\mathsf{T}_{\mathrm{env}}}{f_-}\right)^2 \left(1 - e^{\frac{f_-(x^* - z)}{\mathsf{T}_{\mathrm{env}}}}\right) \\ & + \frac{\mathsf{T}_{\mathrm{env}}}{f_-} e^{\frac{u_0}{\mathsf{T}_{\mathrm{env}}}\frac{\delta}{\delta - x^*}} \left(e^{-\frac{f_- x^*}{\mathsf{T}_{\mathrm{env}}}} - e^{-\frac{f_- z}{\mathsf{T}_{\mathrm{env}}}}\right) \frac{\mathsf{T}_{\mathrm{env}}}{f_+}\left(e^{\frac{f_+ x^*}{\mathsf{T}_{\mathrm{env}}}} - 1\right). \end{aligned} \tag{196}$$

Lastly, it holds that

$$\int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_y^\delta \mathrm{d}x w(x) = \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_0^\delta \mathrm{d}x w(x) - \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x w(x) \tag{197}$$

and

$$\begin{aligned} \int_{\delta - z}^\delta \mathrm{d}y \frac{1}{w(y)} \int_y^\delta \mathrm{d}x w(x) \; = \; & \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_0^\delta \mathrm{d}x w(x) - \int_0^\delta \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x w(x) \\ & - \int_0^{\delta - z} \mathrm{d}y \frac{1}{w(y)} \int_0^\delta \mathrm{d}x w(x) + \int_0^{\delta - z} \mathrm{d}y \frac{1}{w(y)} \int_0^y \mathrm{d}x w(x). \end{aligned} \tag{198}$$

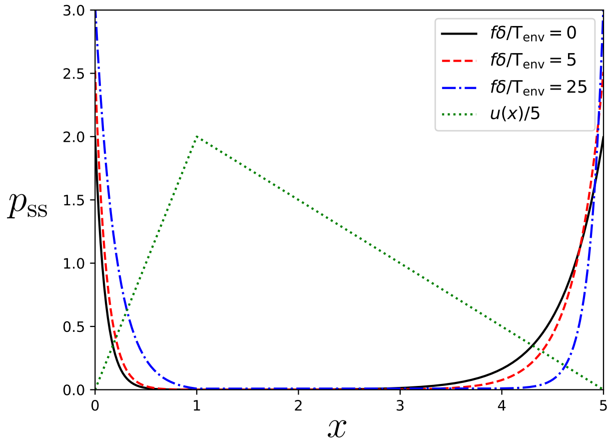

Figure 6: Stationary distribution p_{ss} as a function of x for $\delta = 5$, $x^* = 1$, $u_0 = 10$, $\mathsf{T}_{env} = 1$ and for given values of f. The value of γ is immaterial. Solid lines are results from the Eqs. (184)-(190) while markers are simulation results. The green dotted line plots the potential u divided by 5.

Substituting the above integrals, given by Eqs. (192)-(198), into Eqs. (153), (154), (155), and (156), and consequently using these in Eq. (151) for $\langle T_X \rangle$, we obtain a closed form expression for $\langle T_X \rangle$.

In the Figs. 3 and 4 of the main text we have used this closed form expression of $\langle T_X \rangle$ to plot $\langle T \rangle \dot{s}/|\log p_-|$ as a function of ℓ or $\langle T_X \rangle$ as a function of T_{env}.

D.3 Recovering the Van't Hoff-Arrhenius law

The Eq. (181) in the particular case of a triangular potential leads to

$$\langle T_X \rangle = \frac{\ell \gamma}{f} \frac{\mathsf{T}_{env}^2}{u_0^2} e^{\frac{u_0}{\mathsf{T}_{env}}} \left(1 + O\left(\frac{f\delta}{\mathsf{T}_{env}} \right) \right). \tag{199}$$

We have used this equation to plot the green dotted line in the Fig. 4 of the main text.

E Biased hopping process

In this appendix, we determine the moments of the first-passage time T_X, defined in Eq. (82), of the biased hopping process X determined by Eq. (78). We follow closely the analysis from Appendix B. Just as in Appendix B, we will make use of the decision variable

$$D_X = \text{sign}\left(X(t) - X(0) \right). \tag{200}$$

E.1 Martingales in the biased hopping processes

The processes

$$Z(t) = e^{zX(t) + [(1-e^z)k_+ + (1-e^{-z})k_-]t} \tag{201}$$

are martingales for all values of $z \in \mathbb{R}$ (see Appendix A.1 for the definition of a martingale). Indeed, using Itô's formula for jump processes [71], we obtain

$$\mathrm{dZ}(t) = (e^z - 1) Z(t) \left[\mathrm{d}N_+(t) - k_+ \mathrm{d}t \right] + (e^{-z} - 1) Z(t) \left[\mathrm{d}N_-(t) - k_- \mathrm{d}t \right], \tag{202}$$

which is a martingale process as both $\mathrm{d}N_+(t) - k_+\mathrm{d}t$ and $\mathrm{d}N_-(t) - k_-\mathrm{d}t$ are martingales. In the special case of $z = \ln\frac{k_-}{k_+}$, we obtain that $Z(t) = e^{-S(t)}$ is the exponentiated negative entropy production, which is an example of martingale process [21].

Proposition 6 (A martingale equality). *If $k_+ > k_-$, then for all $z \in \mathbb{R} \setminus [\ln\frac{k_-}{k_+}, 0]$ it holds that*

$$1 = \left\langle 1_{T_X<\infty} 1_{D_X=1} e^{z[\ell_+]+f(z)T_X} + 1_{T_X<\infty} 1_{D_X=-1} e^{-z[\ell_-]+f(z)T_X} \right\rangle, \tag{203}$$

where

$$f(z) = (1-e^{z})k_+ + (1-e^{-z})k_-, \tag{204}$$

and where $[\ell_+]$ and $[\ell_-]$ are the smallest natural numbers that are larger than ℓ_+ and ℓ_-, respectively.

Proof. Since $Z(t)$ is a martingale, we can apply Theorem 2 to $Z(t\wedge T_X)$ giving

$$1 = \langle Z(t\wedge T_X)\rangle = \left\langle e^{zX(t)+f(z)(t\wedge T_X)} \right\rangle. \tag{205}$$

Since for $z \in \mathbb{R} \setminus [\ln\frac{k_-}{k_+}, 0]$ it holds that for $f(z) < 0$

$$e^{zX(t)+f(z)(t\wedge T_X)} < e^{z\ell_+}. \tag{206}$$

Hence, the bounded convergence theorem applies, see e.g. Ref. [72], and we can take the limit $t \to \infty$ under the expectation value to obtain

$$\begin{aligned} 1 &= \langle \lim_{t\to\infty} e^{zX(t)+f(z)(t\wedge T_X)} \rangle && (207)\\ &= \left\langle 1_{T_X<\infty} 1_{D_X=1} e^{z[\ell_+]+f(z)T_X} + 1_{T_X<\infty} 1_{D_X=-1} e^{-z[\ell_-]+f(z)T_X} \right\rangle, && (208) \end{aligned}$$

which completes the proof of the equality (203). □

In what follows, we use this martingale equality to derive various properties T_X.

E.2 The first-passage time T_X is with probability one finite

Proposition 7. *It holds that T_X is almost surely finite, i.e.,*

$$p_- + p_+ = 1. \tag{209}$$

Proof. We take the the limit $z \to 0$ in Eq. (203). Since for $z \in [0, 1]$ the argument in the expectation value is bounded by e^{ℓ_+}, the bounded convergence theorem applies, see e.g. Ref. [72], and

$$\begin{aligned} 1 &= \lim_{z\to 0} \left\langle 1_{T_X<\infty} 1_{D_X=1} e^{z[\ell_+]+f(z)T_X} + 1_{T_X<\infty} 1_{D_X=-1} e^{-z[\ell_-]+f(z)T} \right\rangle \\ &= \langle 1_{T_X<\infty} 1_{D_X=1} + 1_{T_X<\infty} 1_{D_X=-1} \rangle = \langle 1_{T_X<\infty} \rangle = \mathbb{P}(T_X < \infty), \end{aligned}$$

where we have used that $f(0) = 0$. □

E.3 Splitting probabilities

Proposition 8. *It holds that*

$$p_+ = \frac{1-e^{-[\ell_-]\ln\frac{k_+}{k_-}}}{1-e^{-([\ell_+]+[\ell_-])\ln\frac{k_+}{k_-}}}, \quad \text{and} \quad p_- = e^{-[\ell_-]\ln\frac{k_+}{k_-}}\frac{1-e^{-[\ell_+]\ln\frac{k_+}{k_-}}}{1-e^{-([\ell_+]+[\ell_-])\ln\frac{k_+}{k_-}}}, \tag{210}$$

where $[\ell_-]$ and $[\ell_+]$ are the smallest natural numbers that are greater or equal than ℓ_- and ℓ_+, respectively.

Proof. Wee Theorem 2 to the martingale

$$e^{-S(t)} = e^{X(t)\ln\frac{k_-}{k_+}}, \tag{211}$$

yielding

$$\left\langle e^{X(t\wedge T_X)\ln\frac{k_-}{k_+}}\right\rangle = 1. \tag{212}$$

Since X is a jump process on a lattice, it holds that

$$\lim_{t\to\infty}\left\langle e^{X(t\wedge T_X)\ln\frac{k_-}{k_+}}\right\rangle \leq p_- e^{-[\ell_-]\ln\frac{k_-}{k_+}} + p_+ e^{[\ell_+]\ln\frac{k_-}{k_+}} + (1-p_- - p_+)e^{-[\ell_-]\ln\frac{k_-}{k_+}} \tag{213}$$

and

$$\lim_{t\to\infty}\left\langle e^{X(t\wedge T_X)\ln\frac{k_-}{k_+}}\right\rangle \geq p_- e^{-[\ell_-]\ln\frac{k_-}{k_+}} + p_+ e^{[\ell_+]\ln\frac{k_-}{k_+}}. \tag{214}$$

According to Proposition 7, it holds that $p_- + p_+ = 1$, and thus

$$p_- e^{-[\ell_-]\ln\frac{k_-}{k_+}} + p_+ e^{[\ell_+]\ln\frac{k_-}{k_+}} = 1. \tag{215}$$

The solutions to the Eqs. (209) and (215) are given by Eqs. (210), which completes the proof. □

Using $b = k_-/k_+$ in Eq. (210), we obtain the Eq. (83) in the main text.

E.4 Generating function

We derive an explicit formula for the generating function $g(y)$ defined in Eq. (84).

We can write

$$g(y) = p_+ g_+(y) + p_- g_-(y) \tag{216}$$

where g_+ and g_- are the conditional generating functions

$$g_+(y) = \langle e^{-yT_X(k_-+k_+)}|D_X = 1\rangle, \quad \text{and} \quad g_-(y) = \langle e^{-yT_X(k_-+k_+)}|D_X = -1\rangle. \tag{217}$$

Lemma 3. *It holds that*

$$\begin{aligned} 1 = &\left(\frac{1}{2}\left[(1+b)(1+y) + \sqrt{-4b+(1+b)^2(1+y)^2}\right]\right)^{[\ell_+]} p_+ g_+(y) \\ &+ \left(\frac{1}{2}\left[(1+b)(1+y) + \sqrt{-4b+(1+b)^2(1+y)^2}\right]\right)^{-[\ell_-]} p_- g_-(y), \end{aligned} \tag{218}$$

and

$$\begin{aligned} 1 = &\left(\frac{1}{2}\left[(1+b)(1+y) - \sqrt{-4b+(1+b)^2(1+y)^2}\right]\right)^{[\ell_+]} p_+ g_+(y) \\ &+ \left(\frac{1}{2}\left[(1+b)(1+y) - \sqrt{-4b+(1+b)^2(1+y)^2}\right]\right)^{-[\ell_-]} p_- g_-(y), \end{aligned} \tag{219}$$

Proof. We rewrite the relation (203) for $z \notin [\ln \frac{k_-}{k_+}, 0]$ as

$$1 = e^{z[\ell_+]} p_+ \langle e^{f(z)T(k_- + k_+)} | D_X = 1 \rangle + e^{-z[\ell_-]} p_- \langle e^{f(z)T(k_- + k_+)} | D_X = -1 \rangle. \tag{220}$$

Setting

$$y = -f(z) \tag{221}$$

and solving towards z, we obtain two solutions.

First, let us consider the solution branch for $z \geq 0$, which is given by

$$z = \ln\left(\frac{1}{2}\left[(1+b)(1+y) + \sqrt{-4b + (1+b)^2(1+y)^2}\right]\right). \tag{222}$$

Using Eqs. (221) and (222) in (220), we obtain Eq. (220).

Second, let us consider the solution branch for $z \leq \ln b$, namely,

$$z = \ln\left(\frac{1}{2}\left[(1+b)(1+y) - \sqrt{-4b + (1+b)^2(1+y)^2}\right]\right). \tag{223}$$

In this case, using Eqs. (221) and (223) in (220), we obtain the Eq. (219).

□

Proposition 9. *The generating function Eq. (84) is given by Eqs. (85)-(87).*

Proof. We find Eq. (85) readily by solving the Eqs. (219)-(220). □

E.5 Moments of first-passage times

The moments of first passage times follow from taking the derivatives in Eq. (88).

The first moment is given by

$$\langle T_X \rangle = \frac{[\ell_+]p_+ - [\ell_-]p_-}{k_+ - k_-}. \tag{224}$$

The second moment is given by

$$\begin{aligned}
(k_+ - k_-)^2 \langle T_X^2 \rangle \;=\; & \frac{p_+}{1 - b^{[\ell_-]+[\ell_+]}}\left([\ell_+]^2 + [\ell_+]\tanh^{-1}\left(\frac{a}{2\mathsf{T}_{\rm env}}\right)\right) - [\ell_-]^2 p_- \left(\frac{3 + b^{[\ell_-]+[\ell_+]}}{1 - b^{[\ell_-]+[\ell_+]}}\right) \\
& + \frac{p_+ b^{[\ell_-]+[\ell_+]}}{1 - b^{[\ell_-]+[\ell_+]}}\left(3[\ell_+]^2 - [\ell_+]\tanh^{-1}\left(\frac{a}{2\mathsf{T}_{\rm env}}\right)\right) \\
& + [\ell_-]\tanh^{-1}\left(\frac{a}{2\mathsf{T}_{\rm env}}\right)\frac{b^{2[\ell_-]+[\ell_+]}(1 - b^{[\ell_+]})}{(1 - b^{[\ell_-]+[\ell_+]})^2} - 4[\ell_+][\ell_-]\frac{b^{2[\ell_-]+[\ell_+]}}{(1 - b^{[\ell_-]+[\ell_+]})^2} \\
& + \left([\ell_-]\tanh^{-1}\left(\frac{a}{2\mathsf{T}_{\rm env}}\right) + 8[\ell_-][\ell_+]\right)\frac{b^{[\ell_-]}b^{[\ell_+]}}{(1 - b^{[\ell_-]}b^{[\ell_+]})^2} \\
& - [\ell_-]\left(\tanh^{-1}\left(\frac{a}{2\mathsf{T}_{\rm env}}\right) + 4[\ell_+]\right)\frac{b^{[\ell_-]}}{(1 - b^{[\ell_-]}b^{[\ell_+]})^2},
\end{aligned} \tag{225}$$

where $\tanh^{-1}\left(\frac{a}{2\mathsf{T}_{\rm env}}\right) = 1/\tanh\left(\frac{a}{2\mathsf{T}_{\rm env}}\right)$.

We avoid writing down the expression for $\langle T_X^3 \rangle$ given that it is even lengthier than $\langle T_X^2 \rangle$.

E.6 Symmetric thresholds

In the specific case where $\ell_+ = \ell_- = \ell$, we obtain the simpler expression

$$g(y) = \frac{2^{[\ell]} + 2^{-[\ell]}\left(\beta(y) - \sqrt{-4\frac{k_-}{k_+} + \beta^2(y)}\right)^{[\ell]}\left(\beta(y) + \sqrt{-4\frac{k_-}{k_+} + \beta^2(y)}\right)^{[\ell]}}{\left(\beta(y) - \sqrt{-4\frac{k_-}{k_+} + \beta^2(y)}\right)^{[\ell]} + \left(\beta(y) + \sqrt{-4\frac{k_-}{k_+} + \beta^2(y)}\right)^{[\ell]}} \tag{226}$$

for the generating function.

In this case, the first-passage time is

$$\langle T_X \rangle = \frac{[\ell]}{k_+ - k_-}\frac{1 - b^{[\ell]}}{1 + b^{[\ell]}}. \tag{227}$$

and the second moment

$$\langle T_X^2 \rangle = [\ell]\frac{[\ell] + \frac{k_+ + k_-}{k_+ - k_-} - 6[\ell]b^{[\ell]} + b^{2[\ell]}\left([\ell] - \frac{k_+ + k_-}{k_+ - k_-}\right)}{(k_+ - k_-)^2\left(1 + b^{[\ell]}\right)^2} \tag{228}$$

and the third moment,

$$\begin{aligned}\langle T_X^3 \rangle \; = \; & \frac{[\ell]}{k_+^3(1-b)^5(1+b^{[\ell]})^3}\left\{2 + 8b + 2b^2 + 3[\ell](1-b^2) + [\ell]^2(1-b)^2\right. \\ & + b^{[\ell]}(2 + 2b(4+b) + 15(-1+b^2)[\ell] - 23(-1+b)^2[\ell]^2) \\ & + b^{2[\ell]}(-2 - 2b(4+b) + 15(-1+b^2)[\ell] + 23(-1+b)^2[\ell]^2) \\ & \left. + b^{3[\ell]}(2 + 2b(4+b) + 3(-1+b^2)[\ell] + (-1+b)^2[\ell]^2)\right\}. \end{aligned} \tag{229}$$

These are the formulae used in Fig. 5 of the main text.

One readily verifies the thermodynamic uncertainty relation

$$\lim_{[\ell]\to\infty} \frac{\langle T_X^2 \rangle - \langle T_X \rangle^2}{\langle T_X \rangle} = \frac{k_+ + k_-}{(k_+ - k_-)^2} \geq \frac{2}{(k_+ - k_-)\log\frac{k_+}{k_-}} \tag{230}$$

where we used the fact that $\log(x) \geq \frac{x-1}{x} \geq \frac{x-1}{x+1}$ with $x = k_+/k_-$.

E.7 Asymptotics with large thresholds

We consider the limit $\ell_+, \ell_- \gg 1$ with the ratio ℓ_+/ℓ_- fixed to a constant value.

The big-O notation $O(f(\ell_-))$ denotes an arbitrary function $g(\ell_-)$ for which it holds that there exists a constant c such that $g(\ell_-) < cf(\ell_-)$ for ℓ_- large enough.

From Eqs. (83), we obtain for the splitting probabilities that

$$p_- = b^{[\ell_-]} + O(b^{[\ell_+]+[\ell_-]}), \quad \text{and} \quad p_+ = 1 + O(b^{[\ell_-]}). \tag{231}$$

Equation (224) implies that the mean first-passage time

$$\langle T_X \rangle = \frac{[\ell_+]}{k_+ - k_-}\left(1 + O(b^{[\ell_-]})\right), \tag{232}$$

and from Eq. (225) it follows that the second moment

$$\langle T_X^2 \rangle = \frac{[\ell_+]^2}{(k_+ - k_-)^2}\left(1 + \frac{1}{[\ell_+]\tanh\left(\frac{a}{2\mathsf{T}_{\rm env}}\right)} + O(b^{[\ell_-]})\right). \tag{233}$$

The Eqs. (231) and (232) imply that

$$\frac{[\ell_+]}{[\ell_-]}\frac{|\log p_-|}{\langle T_X\rangle} = \frac{a}{\mathsf{T}_{\rm env}}\frac{1}{k_+ - k_-}(1+O(b^{[\ell_-]})). \tag{234}$$

We recognise in the above formula the entropy production rate $\dot{s}$ given by Eq. (81), and thus

$$\frac{[\ell_+]}{[\ell_-]}\frac{|\log p_-|}{\langle T_X\rangle} = \dot{s} + O(b^{[\ell_-]}). \tag{235}$$

Analogously, Eqs. (231) and (233) imply that

$$\frac{[\ell_+]}{[\ell_-]}\frac{|\log p_-|}{\sqrt{\langle T_X^2\rangle}} = \dot{s} + O\left(\frac{1}{[\ell_+]}\right). \tag{236}$$

The thermodynamic uncertainty relation is governed by the subleading $O\left(1/[\ell_+]\right)$ term in Eq. (236). Using Eqs. (231) and (233), we obtain the Eq. (91) in the main text. Since,

$$\frac{1}{\tanh(x/2)} \geq \frac{2}{x} \tag{237}$$

the thermodynamic uncertainty relation [15]

$$\frac{2\langle T_X\rangle}{\langle T_X^2\rangle - \langle T_X\rangle^2} \geq \dot{s} \tag{238}$$

holds.

In order to find asymptotic expressions for the higher order moments, we analyze in the next subsection the probability distribution of T_X in the limit of large thresholds ℓ_- and ℓ_+.

E.8 Probability distribution in the asymptotic limit $\ell_\pm \to \infty$

In order to derive asymptotic expressions for the moments $\langle T^n\rangle$ with $n > 2$, we determine the probability distribution in this limit.

Using that $\zeta_- < \zeta_+$, we obtain in the limit $\ell_{\rm min} \to \infty$,

$$g(y) = \left(\frac{2}{\zeta_+(y)}\right)^{[\ell_+]}\left(1+O\left(\left(\frac{\zeta_-(y)}{\zeta_+(y)}\right)^{[\ell_-]}\right)\right) + \left(\frac{\zeta_-(y)}{2}\right)^{[\ell_-]}\left(1+O\left(\left(\frac{\zeta_-(y)}{\zeta_+(y)}\right)^{[\ell_-]}\right)\right). \tag{239}$$

In the limit $\ell_{\rm min} \to \infty$, we obtain

$$g(y) = \left(\frac{2}{\zeta_+(y)}\right)^{[\ell_+]} + O(b^{[\ell_-]}). \tag{240}$$

Considering that T will be large when both $[\ell_+]$ and $[\ell_-]$ are large, we use that $y \sim \frac{1}{[\ell_{\rm min}]}$. Therefore,

$$\zeta_+(y) = 2 + 2\frac{1+b}{1-b}y + O(y^2). \tag{241}$$

Taking the inverse Laplace transform, we obtain up to leading order

$$p_{T_X}(t) = \frac{((k_+ + k_-)t)^{[\ell_+]-1}}{\Gamma([\ell_+])}\left(\frac{1-b}{1+b}\right)^{[\ell_+]} e^{-t(k_+ + k_-)\frac{1-b}{1+b}} + O(b^{[\ell_-]}), \tag{242}$$

which is the Gamma distribution with shape parameter $[\ell_+]$ and rate $(1-b)/(1+b)$.

If we introduce a new variable,

$$\tau = \frac{(k_+ + k_-)t}{[\ell_+]}, \tag{243}$$

then we obatain

$$p_{\frac{(k_- + k_+)T_X}{[\ell_+]}}(\tau) \sim \exp\left(-[\ell_+]I(\tau) + O_{[\ell_+]}(1)\right) + O(b^{[\ell_-]}) \tag{244}$$

with the large deviation function

$$I(\tau) = \frac{1-b}{1+b}\tau - \log(\tau) - \log\frac{1-b}{1+b} - 1. \tag{245}$$

The minimum is found when

$$\tau^* = \frac{1+b}{1-b} \tag{246}$$

in which case $I(\tau^*) = 0$. Expanding $I(\tau)$ around τ^* we obtain

$$I(\tau) = \frac{\left(\tau - \frac{1+b}{1-b}\right)^2}{2\left(\frac{1+b}{1-b}\right)^2} + O(\tau^3). \tag{247}$$

Hence, the distribution of p_T is

$$p_{\frac{(k_+ + k_-)T_X}{[\ell_+]}}(\tau) = \sqrt{\frac{[\ell_+]}{2\pi(\tau^*)^2}}\exp\left(-[\ell_+]\frac{(\tau-\tau^*)^2}{2(\tau^*)^2} + O(\tau^2)\right) + O(b^{[\ell_-]}). \tag{248}$$

For large $[\ell_+]$, the distribution $p_{\frac{(k_+ + k_-)T_X}{[\ell_+]}}(\tau)$ is centered around $\tau = \tau^*$, and therefore $\frac{(k_+ + k_-)T_X}{[\ell_+]}$ is a deterministic variable in this limit. The moments of T are thus up to leading order terms of the form

$$\langle T_X^n \rangle = [\ell_+]^n \frac{(\tau^*)^n}{(k_+ + k_-)^n} + O([\ell_+]^{n-1}) = \frac{[\ell_+]^n}{(k_+ - k_-)^n} + O([\ell_+]^{n-1}). \tag{249}$$

Using the formula for p_-, given by Eq. (231), and the expression for $\dot{s}$ in Eq. (81), we find thus indeed

$$\frac{[\ell_+]}{[\ell_-]}\frac{|\log p_-|}{(\langle T_X^n \rangle)^{1/n}} = \dot{s} + O\left(\frac{1}{[\ell_+]}\right). \tag{250}$$

Note that obtaining the $1/[\ell_+]$ correction terms is more complicated as we need to consider subleading order terms in Eq. (247). The subleading order terms depend on b and are thus process dependent. Hence, the moments $\langle T_X^n \rangle$ converge for large thresholds to the universal limit given by Eq. (250) since they are governed by the leading order term in the asymptotic behaviour of T_X. On the other hand, the Fano factor

$$\frac{\langle T_X^2 \rangle - \langle T_X \rangle^2}{\langle T_X \rangle} \tag{251}$$

characterising uncertainty depends on the subleading terms and will therefore not converge to a universal limit when the thresholds diverge.

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