Zeeman term for the Néel vector in a two sublattice antiferromagnet

S. Dasgupta\textsuperscript{1,2}, J. Zou\textsuperscript{3},

\textsuperscript{1} Department of Physics and Astronomy & Stewart Blusson Quantum Matter Institute, University of British Columbia, Vancouver, British Columbia V6T 1Z1, Canada
\textsuperscript{2} Institute for Solid State Physics, University of Tokyo, Kashiwa 277-8581, Japan
\textsuperscript{3} Department of Physics and Astronomy, University of California, Los Angeles, California 90095, USA
* sayak.dasgupta@mail.ubc.ca

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Abstract

We investigate the dynamics of solitons in two sublattice antiferromagnets under external perturbations, focusing on the effect of Dzyaloshinsky-Moriya (DM) interactions. To this end, we construct a micromagnetic field theory for the antiferromagnet in the presence of the external magnetic field, DM interaction, and adiabatic spin-transfer torque. In particular, we show that the DM vector and external magnetic field can be combined to form a Zeeman field for the Néel vector. We also study perturbations, like strains which can modify (externally) the inertia of the Néel vector field.

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1 Introduction

Antiferromagnets hold a promise for a faster spintronics platform. The spin wave dynamics of an antiferromagnetic system is controlled by an energy scale $\propto J$, where $J$ is the antiferromagnetic exchange. For ferromagnets the same scale is $\propto \sqrt{KJ}$, where $K$ is a local anisotropy. In most materials $J \gg K$. The energy scale for the antiferromagnet translates to a frequency scale of a few THz. Antiferromagnets offer another significant advantage over ferromagnetic devices. Since the net magnetic moment largely cancels over a unit cell, they do not produce stray fields. This is particularly important in device design, where we would like our individual memory components to be isolated from one another [1–5].

However, these advantages also present a significant handicap--of coupling antiferromagnetic solitons to external probes. The absence of a local spin density implies a minor response to spin currents. The response to external magnetic fields is also tuned down by a factor of the exchange strength. One way to manipulate these solitons is to transfer linear momentum, exploiting the inertial dynamics of the solitons [6,7]. This can be achieved, for instance, by using magnons to scatter from the domain walls. Other methods involve creating a local Berry phase which can then be coupled to an external spin current field. This technique was used in Ref. [8] to generate a Magnus force for an antiferromagnetic vortex.

We know from the classic work of Schryer and Walker [9] that, in a collective coordinate picture [10], an external magnetic field acts like a force on the ferromagnetic domain wall in one dimension. This construction can be extended generically to any spatial dimension. In the ferromagnetic case, the gyroscopic dynamics causes the force to act in the angular momentum channel, leading to a precession of the domain wall.

In antiferromagnets, a local density of magnetization is energetically costly. The dynamics is expressed in terms of soft modes, which are spin configurations with vanishing net spin density. In the case of a two sublattice antiferromagnet, this is the Néel field. The magnetization density follows the soft mode dynamics and renders an inertial mass to the soft modes. Thus the dynamics in the antiferromagnet is inertial—A force produces a linear acceleration, not a precession [7,11].

In order to propel antiferromagnetic domain walls easily, one may then hope to use the analogue of the Zeeman field for the Néel vector. One question naturally arises--what would be equivalent to the magnetic field for the antiferromagnet? This question was addressed by Gomonay et al [12] for the two sublattice case. They pointed out that a Néel spin-orbit field, induced by an electrical current [13,14], has a Zeeman-like coupling to the Néel vector (staggered magnetization), which they utilized to drive the one-dimensional domain wall efficiently.

In this paper, we find another situation where such a Zeeman-like coupling emerges in an antiferromagnet. In particular, we show that the Dzyaloshinsky-Moriya (DM) [15,16] interaction is the key ingredient. The DM interaction creates a local magnetization density which can then respond to both external magnetic fields and spin currents through Zeeman-like terms. In addition to this, we investigate the effects of straining the local lattice on the staggered magnetization field. The presence of a nonzero strain tensor would modify the inertia of the Néel field. Thus strain can potentially function as a handle on the dynamics of antiferromagnetic solitons.

Our approach will be that of collective coordinates, developed for describing the slow dynamics of magnetic textures in ferromagnets [10] and antiferromagnets [7]. The dynamics of the texture is described through a set of coordinates $q_i$, which represent soft modes of the texture. These are usually restricted to the position and orientations of the soliton.

The kinetic energy of an antiferromagnet is expressed as $M_{ij} \dot{q}_i \dot{q}_j$, where $M_{ij}$ is a symmetric
inertia tensor. The generalized force conjugate to the coordinate \( q_i \) is \( F_i = -\partial U/\partial q_i \) with 
\( U \) being the total potential energy. The dissipative force is given by \( F_i^v = -D_{ij} \dot{q}_j \). The 
inertia and dissipation tensors are proportional to each other \( D_{ij} = M_{ij}/T; \) the relaxation 
time \( T \) is inversely proportional to Gilbert damping constant \( \alpha \) [6].

Although we use collective coordinates as our degrees of freedom, we shall not use the 
Landau-Lifshitz equations for the individual sublattices. Instead, we take the micromagnetic 
field theory picture presented in Ref. [6] [8] and figure out the potential energies (or gauge 
theories) that are spawned by adding external perturbations. To facilitate this, we briefly 
review the micromagnetic field theory for two sublattice antiferromagnets in Sec. 2. We 
then move onto the effects of the individual perturbations: namely a magnetic field, a DM 
interaction, and a spin-transfer torque in Sec. 3. The meat of our discussion lies in Sec. 4 
where we deal with the effect of simultaneous perturbations. This construction is essential 
for a propulsion mechanism. Finally we gather our results in Sec. 5.

2 Two sublattice micromagnetics

In this section, we derive the micromagnetic Lagrangian for the two sublattice antiferro-
magnet along the lines of Ref. [7]. Our description is in terms of the magnetization field 
represented by the unit vectors \( \mathbf{m}(\mathbf{r},t) \). The length of the magnetization, \( \mathcal{M} \), is a constant 
and is connected to the underlying spin density \( \mathcal{J} \) through the relation \( \mathcal{M} = \gamma \mathcal{J} \) with 
gyromagnetic ratio \( \gamma \).

For antiferromagnets, each magnetic unit cell comprises two or more magnetization 
fields \( \mathbf{m}_i \) which are constrained by the exchange interaction to follow \( \sum_i \mathbf{m}_i = 0 \). To make 
this explicit, we convert the nearest neighbour exchange into:

\[
H_{\text{exchange}} = J \sum_{<i,j>} \mathbf{S}_i \cdot \mathbf{S}_j 
\]

\[
= \frac{JS^2}{2} \sum_{\alpha} \left( \sum_i \mathbf{m}_i \right)_{\alpha}^2 - \frac{N}{2} \sum_{\alpha} S^2. 
\]

Here \( \sum_i \mathbf{m}_i \) is a sum over all the spins that constitute the antiferromagnetic unit cell—if 
there are \( N \) sublattices, the sum is over \( N \) spins. The other sum \( \alpha \) is over the lattice, 
broken down into the magnetic unit cell clusters. The second term is dropped as it is 
constant and does not enter equations of motion.

In general, to get to the continuum model, we express the vector fields \( \mathbf{m}_i \) in terms of 
the appropriate normal modes of the systems, dictated by the point group symmetry of 
the order, and expand the exchange interaction (and the other energies) in them [11].

The particular construction of the field theory depends on the specific lattice geometry. 
However, generically they all stem from labeling the sublattice magnetizations as individual 
fields and then putting them together by expressing the respective magnetization fields in 
terms of the normal modes. These are of two kinds—soft modes which do not break the 
constraint \( \sum_i \mathbf{m}_i = 0 \), and hard modes which do, inducing a net magnetization per unit 
cell.

Solitonic dynamics in ferromagnets is dominated by gyroscopic effects generated by the 
local angular momentum density. Thus, to propel a ferromagnetic vortex in the \( xy \) direction 
of the \( xy \) plane, one applies a force in the \( y \) direction [17]. Similarly, exerting a force to 
a domain wall in a uniaxial ferromagnet primarily generates a precession about the long 
axis [9]. To propel it forward, one has to apply a torque to it, for example through the 
adiabatic spin-transfer torque [18,19]. This is not the situation in antiferromagnets where
a net angular momentum density is usually a secondary effect from local anisotropy and
fights with a much larger exchange interaction.

A continuum theory of a collinear antiferromagnet with two sublattices operates with
two slowly varying (in space) fields $\mathcal{M}\mathbf{m}_1(\mathbf{r})$ and $\mathcal{M}\mathbf{m}_2(\mathbf{r})$. $\mathcal{M}$ is the moment size and
$\mathbf{m}_1, \mathbf{m}_2$ are unit vector fields. In a state of equilibrium, $\mathbf{m}_1(\mathbf{r}) = -\mathbf{m}_2(\mathbf{r})$. More generally, the two sublattice fields are expressed in terms of dominant staggered magnetization $\mathbf{n} = (\mathbf{m}_1 - \mathbf{m}_2)/2$ and small uniform magnetization $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$. The constraints $|\mathbf{m}_1|^2 = 1$ and $|\mathbf{m}_2|^2 = 1$ translate into

$$\mathbf{m} \cdot \mathbf{n} = 0, \quad |\mathbf{n}|^2 = 1 - |\mathbf{m}|^2/4 \approx 1; \quad (2)$$

the last approximation is valid as long as $|\mathbf{m}|^2 \ll 1$.

2.1 The kinetic term and spin wave spectrum

We demonstrate the calculation of the spin wave spectrum for a two sublattice antiferromagnet on a square lattice of side length $a$. The only interaction present is the nearest neighbour Heisenberg exchange with strength $J$. The kinetic term for the antiferromagnet emerges from the Berry phases of the two sublattice magnetizations $\mathbf{m}_{1,2}$ [20]. The total Berry phase for the unit cell:

$$\mathcal{L}_B = \mathcal{J}(\mathbf{a}_1.\mathbf{m}_1 + \mathbf{a}_2.\mathbf{m}_2). \quad (3)$$

Here $\mathcal{J} = S/(2a^2)$ is the density of angular momentum in two dimensions with $S$ as the moment (spin) length. While choosing the vector potentials $\mathbf{a}_{1,2}$ for the two sublattices, we adopt different gauges, such that the Dirac string of the two monopoles lie on opposite hemispheres of the magnetization sphere. This ensures that neither $\mathbf{m}_{1,2}$ is near a Dirac string. The convenient choice is $\mathbf{a}_1(\mathbf{m}) = \mathbf{a}(\mathbf{m})$ and $\mathbf{a}_2(\mathbf{m}) = \mathbf{a}(-\mathbf{m})$ [6,21,22].

In the equilibrium state when $\mathbf{m}_1 = -\mathbf{m}_2$, the Berry phases of the two sublattices cancel exactly. This can be seen for the standard gauge choice of the vector potential $a_\theta = 0$ and $a_\phi = (\cos \theta \pm 1)/\sin \theta$. The Dirac string carries a ‘flux’ of $+4\pi$ either through

Figure 1: This figure shows the two dimensional two sublattice antiferromagnet. Red sites have their spins out of plane and blue spins have spins into the plane. The unit cell for each sublattice is marked in dashed lines. The exchanges are isotropic and are marked.
the north or south pole. If we put the string through the south pole for \( \mathbf{m}_2 \) and through the north pole for \( \mathbf{m}_2 \) we have in equilibrium \( L_B = J [\mathbf{a}(\mathbf{n}) - \mathbf{a}(-\mathbf{n})] \cdot \dot{\mathbf{n}} = 0. \)

The lowest non-vanishing kinetic terms are obtained by expanding the vector potentials using \( [\mathbf{m}] \) as a small parameter. Individually, \( \mathbf{a}_1 \cdot \mathbf{m}_1 = \mathbf{a}_1 \mathbf{(m}/2 + \mathbf{n}) \cdot (\mathbf{m}/2 + \dot{\mathbf{n}}) \) and \( \mathbf{a}_2 \cdot \mathbf{m}_2 = \mathbf{a}_2 \mathbf{(m}/2 - \mathbf{n}) \cdot (\mathbf{m}/2 - \dot{\mathbf{n}}). \) Expanding to quadratic order in \( [\mathbf{m}] \) and \( [\dot{\mathbf{n}}] \), the kinetic term Eq. (3) yields the following:

\[
\mathcal{L}_B/\mathcal{J} = \left[ a_1(\mathbf{n}) + a_2(-\mathbf{n}) \right] \cdot \frac{\dot{\mathbf{m}}}{2} + \left[ a_1(\mathbf{n}) - a_2(-\mathbf{n}) \right] \cdot \dot{\mathbf{n}} + \frac{m_i}{2} \left[ \frac{\partial a_1(\mathbf{n})}{\partial n_i} - \frac{\partial a_2(-\mathbf{n})}{\partial n_i} \right] \cdot \frac{\dot{\mathbf{m}}}{2} + \frac{m_i}{2} \left[ \frac{\partial a_1(\mathbf{n})}{\partial n_i} + \frac{\partial a_2(-\mathbf{n})}{\partial n_i} \right] \cdot \dot{\mathbf{n}}
\]

We have the identity \( \partial_n a_1(\mathbf{n}) - \partial_n a_2(-\mathbf{n}) = 0 \), from the definition of the vector potentials. This cancels the second and third terms. In the first term, we now transfer the time derivative to \( \mathbf{a} \) using an integration by parts and combine with the corresponding vector potential term from the last line to get:

\[
m_i \dot{n}_k \left[ \frac{\partial a_k(\mathbf{n})}{\partial n_i} - \frac{\partial a_k(\mathbf{n})}{\partial n_k} \right] = \dot{\mathbf{n}} \cdot (\nabla \times \mathbf{m}),
\]

where we have used \( \nabla \times \mathbf{a} = -\mathbf{n} \).

The potential energy is obtained from the Heisenberg exchange:

\[
U = JS^2 \sum_{\langle i,j \rangle} \mathbf{m}_i \cdot \mathbf{m}_j,
\]

\[
= JS^2 \sum_{\alpha} (\mathbf{m}_1 + \mathbf{m}_2)_{\alpha}^2
\]

\[
= \int dV \frac{JS^2}{2} \left[ \frac{2m^2}{a^2} + (\partial_i \mathbf{n})^2 + \frac{1}{2} \partial_i^2 \mathbf{m}^2 \right],
\]

where \( J \) is the Heisenberg exchange strength and in the second line we have dropped the constant term. In the second line, we have expressed the summation over nearest neighbours in terms of summation over two site magnetic unit cells \( \alpha \). We can see that the uniform magnetization picks up an energy contribution from the exchange interaction at the zeroth order in gradients and is hence a hard mode. The Néel field \( \mathbf{n} \) only appears through gradients and is the typical example of a soft mode in antiferromagnetic systems.

### 2.2 Spin Waves

The procedure to obtain the effective spin wave field theory is similar to the planar ferromagnet [23]: we integrate out the hard field and express the theory in terms of the soft field. This process generates an inertia for the soft mode. Since \( \mathbf{m} \) is hard, we shall drop its gradient terms. Let us carry this out explicitly:

\[
\mathcal{L} = \frac{S}{2a^2} \dot{\mathbf{n}} \cdot (\nabla \times \mathbf{m}) - JS^2 \left[ \frac{2m^2}{a^2} + (\partial_i \mathbf{n})^2 \right].
\]

Now we can solve for the hard field \( \mathbf{m} = (\dot{\mathbf{n}} \times \mathbf{m})/(4JS) \), implying \( \mathbf{m} \) is a slave variable to the Néel field \( \mathbf{n} \) in this treatment. Substituting this solution back into the Lagrangian, we
obtain a field theory for the soft Néel field:

\[ \mathcal{L} = \frac{\rho}{2} \mathbf{n}^2 - \frac{JS^2}{2} (\nabla \mathbf{n})^2, \] (8)

with \( \rho = 1/(8Ja^2) \). Here we have used \((\mathbf{n} \times \mathbf{n})^2 = \mathbf{n}^2 \) as \( \mathbf{n} \cdot \mathbf{n} = 0 \), following from the unit vector constraint of \( \mathbf{n} \).

The ordered ground state \( \mathbf{n}_0(\theta, \phi) \) spontaneously breaks the \( SO(3) \) symmetry of the system up to \( SO(2) \). Hence in this case, there are two Goldstone modes, residing in the coset space \( S^2 = SO(3)/SO(2) \), one for each continuous degree of freedom, dispersing linearly according to \( \omega = \pm (2\sqrt{2}JSa) \). They classically correspond to the opposite circular polarizations of the small-angle oscillations of \( \delta \mathbf{n} \perp \mathbf{n}_0 \).

2.3 Strain

The strain to moment coupling is expressed through the energy density \([24]\):

\[ \mathcal{U}_{ME} = S^2 \sum_{ij} \left[ \frac{\partial J(u)}{\partial u_{ab}} \cdot \delta u_{ab} \right] m_i \cdot m_j, \] (9)

where \( u_{ab} = u_a - u_b \) with \( u \) as the lattice displacement field. On the nearest neighbour square lattice, the only strain components that couple to the Heisenberg Hamiltonian are \( \epsilon_{xx} \) and \( \epsilon_{yy} \), where \( \epsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2 \) is the linear strain tensor. If the system has next-nearest neighbour interactions, we can couple to those using the off-diagonal strain \( \epsilon_{xy} \).

The off diagonal strain will appear in two dimensions for non-collinear magnetic ordering, for instance the \( Mn_3X \) group of 120° ordered antiferromagnets \([25–27]\).

To lowest order in gradients, the strain couples to the uniform magnetization \( m \) and gradients of the Néel vector \( \partial_i \mathbf{n} \). The dominant effect is through a coupling to the uniform magnetization \( m \). This produces an energy density:

\[ \mathcal{U}_{ME} = J'S^2 \left( \epsilon_{xx} + \epsilon_{yy} \right) \frac{m^2}{2a^2}, \] (10)

where \( J' = (\partial_i J) \) and we have assumed \( \partial_x J = \partial_y J \) from the local cubic symmetry. This modifies the inertia for the Néel field:

\[ \frac{1}{\rho'} = \frac{1}{\rho} \left[ 1 + \frac{J'(\epsilon_{xx} + \epsilon_{yy})}{2J} \right]. \] (11)

It serves as an external handle on the Néel field inertia which can be exploited to control its dynamical properties, especially in the case of solitons (see Fig. 2).

The next higher order coupling is to the gradients of the soft Néel field. This coupling modifies the spin wave velocity and makes it anisotropic. This is expected since strains induce an additional two fold anisotropy in the plane. The velocities are now given by:

\[ c = c \left( 1 + \frac{3J'}{2J} \epsilon_{xx} + \frac{J'}{2J} \epsilon_{yy}, 1 + \frac{3J'}{2J} \epsilon_{yy} + \frac{J'}{2J} \epsilon_{xx} \right), \] (12)

with \( c = \pm (2\sqrt{2}JSa) \).

2.4 Solitons

We are interested in the situations where the only spatial dependence of the staggered magnetization field \( \mathbf{n} \) is at the location of topological defects. These regions are characterized by a skyrmion density defined using the Néel vector field:

\[ N_{sk} = \frac{1}{4\pi} \int dq_i dq_j \mathbf{n} \cdot \left( \frac{\partial \mathbf{n}}{\partial q_i} \times \frac{\partial \mathbf{n}}{\partial q_j} \right). \] (13)
(q_i, q_j) are collective coordinates conjugate to each other. Typical examples for the two sublattice case are—domain walls characterized by the conjugate set of location and orientation of the domain wall \((Z, \Phi)\), and the vortex with its core center \((X, Y)\) serving as the conjugate set.

**Uniaxial domain wall:** The uniaxial domain wall is produced by an easy axis anisotropy. Choosing this to lie along the \(z\) axis we get:

\[
U(n) = \frac{A}{2} \left| \frac{\partial n}{\partial z} \right|^2 + \frac{K}{2} |e_3 \times n|^2.
\] (14)

Here \(A > 0\) characterizes the strength of exchange, \(K > 0\) is the easy axis anisotropy, and \(e_3 = (0, 0, 1)\). This system has two uniform ground states \(n = \pm e_3\), linear excitations in the form of spin waves with the dispersion \(\omega^2 = (K + Ak^2)/\rho\), and nonlinear solitons in the form of domain walls which interpolate between the two ground states. Static domain walls in \(n = (\sin \theta(z) \cos \phi, \sin \theta(z) \sin \phi, \cos \theta(z))\) have width \(\lambda = \sqrt{A/K}\) and are parametrized in spherical angles \(\theta(z)\) and \(\phi(z)\) as follows:

\[
\cos \theta(z) = \pm \tanh \frac{z - Z}{\lambda}, \quad \phi(z) = \Phi.
\] (15)

Position \(Z\) and azimuthal angle \(\Phi\) represent the two zero modes of the system associated with the global symmetries of translation and rotation see Fig. \[\text{3}\]. Weak or local external perturbations do not alter the shape of the soliton significantly and mostly induce the dynamics of \(Z\) and \(\Phi\).

**Planar vortex:** This topological feature is stable in two spatial dimensions with an easy plane anisotropy, \(K < 0\) in Eq. (14). The uniform ground states are \(n = (\cos \phi, \sin \phi, 0)\). A vortex centered at the origin is parametrized as:

\[
ie_{\phi}(r) = \left( \frac{x + iy}{|x + iy|} \right)^n, \quad \cos \theta(r) = \pm f_n(r/\lambda).
\] (16)

Here \(n \in \mathbb{Z}\) is the vortex winding number. The magnetization leaves the plane at the cores and this is captured by the function \(f(\zeta)\) with \(f_n(0) = 1\) and \(f_n(\infty) = 0\). The core size is controlled by the same length scale as the domain wall, \(\lambda = \sqrt{A/|K|}\).
Figure 3: On the top we show the constituent sublattice magnetizations $m_{1,2}$. These sublattices combine to form the antiferromagnet. A typical soliton in one dimension is a domain wall shown on the bottom. The domain wall is a soliton interpolating between the two unidirectional ground states of the one dimensional antiferromagnet.

3 External Perturbations

We now consider the situation where the only spatial dependence of the staggered magnetization field $\mathbf{n}$ is at the location of defects. The theory we work with is

$$
\mathcal{L} = J \mathbf{n} \cdot (\mathbf{n} \times \dot{\mathbf{m}}) - \left( \frac{M^2}{2\chi} \right) \mathbf{m}^2 - U_{\text{ext}}[\zeta, \mathbf{n}, \mathbf{m}],
$$

(17)

where we have absorbed the Heisenberg exchange strength into a spin susceptibility $\chi$. $\zeta$ in the theory is an external (pseudo)vector field (it can be a general tensor field, such as the strain tensor we have discussed). Our main objective is to see how $\zeta$ modifies the Lagrangian density, in particular how it couples to the soft mode $\mathbf{n}$. Once we have an understanding of these couplings, we can study their effects on solitons in the staggered magnetization order, such as uniaxial domain walls and planar vortices. We outline the manner in which these solitons can be effectively moved in space by coupling to the order parameter.

These external vector fields couple either to the uniform magnetization $\mathbf{m}(\mathbf{r}, t)$ or the staggered magnetization $\mathbf{n}(\mathbf{r}, t)$ in the Lagrangian. This is broadly guided by symmetries like time reversal and mirror planes of the spin Hamiltonian. Fields, which couple to $\mathbf{m}$, produce a gauge coupling to $\dot{\mathbf{n}}$, on integrating out $\mathbf{m}$. This is the case with perturbations like an external magnetic field $\mathbf{h}(\mathbf{r}, t)$ or a spin transfer torque characterized by the electron drift velocity $\mathbf{u}(\mathbf{r}, t)$. Such terms require a spatial or temporal variation of the external vector field to produce solitonic motion [8,28].

The coupling to $\mathbf{n}$ gives rise to terms like $(A_{ij}\zeta\mathbf{n}_j)^n$, where $n = 1, 2$ is the cases we study. Here $\zeta_i$ represents an external field sourced from a combination of terms like the Dzyaloshinski-Moriya interaction, external magnetic fields, or combinations. This term acts as a potential energy density which can generate a force (or torque) on a soliton. Note here, that an antiferromagnetic soliton by virtue of Eq. (8) is inertial, i.e. a force propels an antiferromagnetic domain wall instead of making it precess. We show that Dzyaloshinski-Moriya interactions generate such terms and can be used to propel solitons.

In the course of working out these contributions to the energy density, one particularly useful identity we repeatedly use is:

$$
(\dot{\mathbf{n}} \times \mathbf{n}) \cdot (A \times \mathbf{n}) = A \cdot [\mathbf{n} \times (\dot{\mathbf{n}} \times \mathbf{n})] = A \cdot \dot{\mathbf{n}}.
$$

(18)
To get an idea of what kind of anisotropy this term induces, let us take a look at the energy with the inertia \( \rho \) which violates the constraint \( m \cdot n = 0 \). To ensure the perpendicularity, we resolve \( h \) into a component perpendicular to \( n \), \( h_\perp = n \times (h \cdot n) \) which enters the Zeeman coupling \( m \cdot h_\perp \) to produce a term \((n \times H) \cdot (n \times m)\).

Now on solving for \( m \), we obtain \( m = \chi \mathcal{J} (n \times n)/M^2 + \chi (n \times h) \times n/M \). Substituting this into the Lagrangian we obtain:

\[
\mathcal{L}(n) = \frac{\rho (\dot{n} - \gamma h \times n)^2}{2},
\]

with the inertia \( \rho = \chi / \gamma^2 \). The Lagrangian is identical to that of a particle in a rotating frame with an angular velocity \( \gamma |h| \), causing a texture in \( n \) to precess. There is an additional contribution to the energy in the form of \( U_h = -\rho |\gamma h \times n|^2/2 \), which adds to the crystal anisotropy term in the energy functional and resembles the potential energy that leads to the centrifugal force in the rotating frame.

Let us take a closer look at each of the terms in Eq. (20). The term \( \rho \dot{n}^2/2 \) is the kinetic energy of staggered magnetization, which endows antiferromagnetic solitons with an inertial mass. Supposing a soliton is parametrized by a set of collective coordinates \( q = \{q_1, q_2, \ldots \} \) such as the position of a domain wall, the coordinates of a vortex core etc., the variation of \( n \) in time is mediated by the change of these collective coordinates: \( \dot{n} = \dot{q}_i \partial_{q_i} n \). The soliton’s kinetic energy is then \( M_{ij} \dot{q}_i \dot{q}_j/2 \), where \( M_{ij} = \rho \int d^3q n \cdot \partial_{q_i} n \cdot \partial_{q_j} n \) is the inertia tensor [29].

The potential term \( \rho |\gamma h \times n|^2/2 \) in Eq. (20) expresses local anisotropy favouring the direction of \( n \) orthogonal to the effective field \( h \). This term modifies the potential landscape \( U(q) \) of a soliton:

\[
U(q, h(r)) = U(q, 0) - \int dV \frac{\rho |\gamma h \times n|^2}{2}.
\]

To get an idea of what kind of anisotropy this term induces, let us take a look at the energy density for the uniaxial domain wall in Eq. (15) with the easy axis along \( \hat{z} \) as shown in Fig. 3.

We now introduce a magnetic field \( h = h_0(\cos \varphi, \sin \varphi, 0) \) modifying the energy density:

\[
\mathcal{U} = -\frac{K}{2} \cos^2 \theta - \frac{\rho \gamma^2}{2} (n \times h)^2,
\]

which leads to

\[
-\frac{K}{2} \cos^2 \theta - \frac{\rho (\gamma h_0)^2}{2} \left[ \cos^2 \theta + \sin^2 \theta \sin^2 (\phi - \varphi) \right],
\]

with \( K > 0 \).

The magnetic field chooses the azimuthal plane for the Néel domain wall and hence acts as an angle-selector. For a particular direction of the field (cos \( \theta \)) the minimum energy occurs when \( |\phi - \varphi| = \pi/2 \). In the figure (Fig. 3) we point the magnetic field along \( \hat{x} \) which
prefers a Néel wall in the $yz$ plane. The easy axis anisotropy, however, is unaffected in this configuration. This leaves the soliton size unchanged.

To modulate the size of the soliton $\lambda = \sqrt{A/K}$ we need to apply a field along the easy axis $\mathbf{h} = h_0 \hat{z}$. In this configuration the anisotropy $K$ defined in Eq. (15) is modified to $K \to \tilde{K} = K - \rho \gamma h_0^2$. Now for the easy axis scenario since $K > 0$ this leads to an expansion, while for the easy plane scenario where $K < 0$ this leads to a constriction of $\lambda$. Thus the magnetic field breaks the $SO(3)$ symmetry of the Néel vector and allows an external control of the soliton size.

We remark that, in the easy axis case, the soliton profile is no longer stable when $\tilde{K} \to 0$ as the applied magnetic field increases; the system undergoes a spin-flop transition into a spin-flop phase, where the Néel vector lies within the plane perpendicular to the magnetic field. In the easy plane case, one can utilize the magnetic field to enhance the easy-plane anisotropy, which is essential for the conservation of spin winding and thus is applicable in energy storages [30,31] and related transport experiments [32–34].

It is notable that the cross term $\rho \gamma h \cdot (\dot{\mathbf{n}} \times \mathbf{n})$ in Eq. (20) is linear in the time derivative $\dot{n}$ and thus quantifies the effective geometric phase for the dynamics of staggered magnetization. This is analogous to the Coriolis effect in a rotating frame. In the Lagrangian of a soliton, it turns into $A_i \dot{q}_i$, a coupling to an external gauge field

$$A_i(q) = \int dV \rho \gamma h \cdot \left( \frac{\partial \mathbf{n}}{\partial q_i} \times \mathbf{n} \right). \quad (23)$$

The equations of motion for an antiferromagnetic soliton have the form of Newton’s second law for a particle of unit electric charge in this gauge field:

$$M_{ij} \ddot{q}_j = -\partial U/\partial q_i + E_i + F_{ij} \dot{q}_i - M_{ij} \dot{q}_j/T. \quad (24)$$

The “magnetic field” $F_{ij} = -F_{ji}$ is the curl of the gauge potential:

$$F_{ij} = \frac{\partial A_j}{\partial q_i} - \frac{\partial A_i}{\partial q_j} = -2 \int dV \rho \gamma h \cdot \left( \frac{\partial \mathbf{n}}{\partial q_i} \times \frac{\partial \mathbf{n}}{\partial q_j} \right). \quad (25)$$

The “electric field”

$$E_i = -\int dV \rho \gamma \dot{h} \cdot (nq_i \times n), \quad (26)$$

arises when $h$ depends on time explicitly.

### 3.2 Dzyaloshinski-Moriya Interaction

We now examine the effect of adding the antisymmetric exchange or DM interaction [15,16] to the Lagrangian. This interaction exists in an antiferromagnet with broken inversion symmetry intrinsically or at interfaces like sample edges and extended domain walls. It is characterized by the energy density $U_{\text{DMI}} = D \cdot (\mathbf{S}_i \times \mathbf{S}_j) = J D \cdot (\mathbf{m}_i \times \mathbf{m}_j)$ where the direction of the DM vector $\mathbf{D}$ is given by the Moriya rules [16].

Their net effect is to induce a weak ferromagnetism in the material, which then couples to external torques and fields. The emergence of a ferromagnetic moment also implies the existence of a non zero gyrotropic force in these systems. In the presence of a homogeneous DM interaction, the theory takes the form:

$$\mathcal{L} = J \dot{n} \cdot (\mathbf{n} \times \mathbf{m}) - \left( \frac{M^2}{2\chi} \right) |\mathbf{m}|^2 - J^2 \mathbf{D} \cdot (\mathbf{n} \times \mathbf{m}). \quad (27)$$
This adds an extra term to the solution for the staggered magnetization \( \mathbf{m} = \chi \mathcal{J} \mathbf{n} \times \mathbf{n}/M^2 - \chi \mathbf{D} \times \mathbf{n}/5^2 \). On integrating out the uniform magnetization we obtain:

\[
\mathcal{L} = \frac{\rho (\mathbf{n} - \mathcal{J} \mathbf{D})^2}{2}.
\]  

(28)

Note that here, unlike in the case of the external magnetic field, there is no additional anisotropy induced by the DM vector. The Lagrangian suggests a steady-state translation for the Néel soliton \( \mathbf{n}' \equiv \mathbf{n} - \mathcal{J} \mathbf{D} \) with a velocity \( \mathbf{v} = \mathcal{J} \mathbf{D} \). In other words it acts as a potential for \( \mathbf{n} \).

The cross term with the kinetic term gives rise to a vector potential of the form:

\[
A_{\text{DM}}^i = -\rho \mathcal{J} \int dV \frac{\partial \mathbf{n}}{\partial q_i} \cdot \mathbf{D}.
\]  

(29)

For the material bulk where the DM vector is a constant, this does not produce an electromagnetic field density \( F_{ij} \). However, there are two situations where an exception occurs. One is when \( \partial_i \partial_j \mathbf{n} - \partial_j \partial_i \mathbf{n} \neq 0 \) as in the case of the antiferromagnetic vortex core where \( \mathbf{n} \) is singular [23]. In this case, the vector potential \( A_i \) yields a density \( F_{XY} = (-2\pi n_2 \mathcal{J}) \mathbf{e}_\phi \cdot \mathbf{D} \). Here \( n \) is the vorticity density and \( \mathbf{e}_\phi \) is the azimuthal unit vector.

It is unlikely that this effect is finite in the two-sublattice case as the DM vector tends to point out of the plane. However, it might be present in non collinear antiferromagnets like Mn₃Ge. The other situation occurs at interfaces where the DM vector can become space dependent. In that case, the electromagnetic tensor strength is given by \( F_{ij} = \rho \mathcal{J} (\partial_j \mathbf{n} \cdot \partial_i \mathbf{D} - \partial_i \mathbf{n} \cdot \partial_j \mathbf{D}) \).

### 3.3 Spin-Transfer Torque

For metallic antiferromagnets, we can transfer angular momentum to each individual sublattice through a conduction band electron current [35]. The local magnetic moments couple to the electron spins through an s-d exchange [36]. The coupling polarizes the conduction band to follow the orientation of spins on individual sublattices. This mechanism gives rise to the adiabatic spin transfer torque.

For the ferromagnet, the adiabatic spin transfer torque modifies the time derivative in the Landau-Lifshitz equation to a convective derivative \( \partial_t \rightarrow \partial_t + \mathbf{u} \cdot \nabla \) [37]. Here \( \mathbf{u} \) is the drift velocity of electrons related to the electric current \( \mathbf{j} = e n \mathbf{u} \) with \( n \) as the concentration of electrons.

This correction can be extended to the two sublattice antiferromagnet [38]. The kinetic term:

\[
\mathbf{m} \cdot (\dot{\mathbf{n}} \times \mathbf{n}) \rightarrow \mathbf{m} \cdot [(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{n} \times \mathbf{n}].
\]  

(30)

This correction modifies the induced magnetic moment \( \mathbf{m} = (\chi \mathcal{J}/M^2)(\mathbf{n} + \mathbf{u} \cdot \nabla) \mathbf{n} \times \mathbf{n} \), which suggests that nonuniform Néel fields will induce a magnetization in the presence of a spin current. The Lagrangian reads:

\[
\mathcal{L} = \frac{\rho (\mathbf{n} + \mathbf{u} \cdot \nabla) \mathbf{n})^2}{2}.
\]  

(31)

The most immediate effect of this coupling is to modify the spin wave velocities. Comparing this with Eq. [7], we can see that the potential energy density is now:

\[
\mathcal{U} = \frac{JS^2}{2} (\nabla \mathbf{n})^2 - \frac{\rho}{2} (\mathbf{u} \cdot \nabla \mathbf{n})^2.
\]  

(32)
Consider an adiabatic spin current of the form \( \mathbf{u} = (u_x, 0) \). This modifies the spin wave velocity in the \( \hat{x} \) direction to \( c_x = c[1 - (u_x/(2c^2))] \), where \( c = 2\sqrt{2}JSa \). Thus for a generic current direction the spin wave will no longer be isotropic in the plane and will get corrections of the order of \( |\mathbf{u}|/c^2 \). This, along with strain can be used to modify spin wave magnitudes and polarizations in the two sublattice antiferromagnet.

The adiabatic spin transfer torque needs a local Berry phase density to effect propulsion of a soliton. This implies that the spin transfer torque needs to be applied in addition to a perturbation that creates a local magnetization density to propel an antiferromagnetic soliton. For instance, in Dasgupta et al. [8] an external magnetic field was used to generate a local Berry phase density. This coupled to the spin transfer torque to produce a Magnus force for the antiferromangetic vortex.

4 Combined Interactions

Single perturbations couple to the Néel field in Eq. (20), Eq. (28), and Eq. (31) through \( \dot{n} \). This gives rise to vector potentials. Under certain circumstances where the perturbation is itself nonuniform in time or space, this leads to a finite electromagnetic tensor. However, as shown in [8], a perturbation that is nonuniform in time does not produce a net propulsion of a soliton. Spatially nonuniform magnetic fields do seem to produce a propulsion [28].

A better alternative for antiferromagnetic solitons is to use a combination of two (or more) perturbations. This is the situation which we now turn to. The theme of two of these combinations is similar. If we have a magnetic field \( \mathbf{h}(r, t) \) or a DM interaction \( \mathbf{D}(r, t) \) locally (at the location of the soliton) inducing a small magnetic moment which the spin current \( \mathbf{u}(r, t) \), latches on to and generates a displacement of the soliton. The other combination, a DM interaction and an external magnetic field, will lead to an energy density which we show is structurally identical to Néel spin orbit torque used in Ref. [12].

**DM interaction and external magnetic field:** If these two types of terms are simultaneously present in the system, the Lagrangian density takes the form:

\[
\mathcal{L} = \frac{\rho}{2} \left[ \dot{n} + \gamma (n \times h) - \frac{(M/\gamma) \cdot D}{2} \right] - \mathcal{U}(n, D, h). \tag{33}
\]

The cross term of interest is:

\[
U_{\text{DM-h}} = -\rho M \mathbf{n} \cdot (\mathbf{D} \times \mathbf{h}). \tag{34}
\]

This term acts as a ‘Zeeman’ term but for the staggered magnetization with an effective magnetic field \( \mathbf{h}_{\text{eff}} = (\mathbf{D} \times \mathbf{h}) \). Note that in the presence of a DM interaction the extra uniform magnetization that is induced is \( \mathbf{m} \propto (\mathbf{D} \times \mathbf{n}) \). It is this extra induced ferromagnetic moment that ‘Zeeman’ couples with the external magnetic field.

This coupling expressed in collective coordinates acts like a potential energy for the domain wall and will produce force on the wall itself. However, to cause a net displacement in the position of the wall, we require: \( \langle \mathbf{D} \times \mathbf{h} \rangle_{\text{easy-axis}} \neq 0 \), such that the force is in the correct channel. Here the easy axis points along the length of the domain wall. This requires in particular a DM vector that is not aligned along the easy axis. In our example domain wall this implies \( D_x, D_y \neq 0 \) and \( D_z = 0 \).

To illustrate this idea, let us consider the familiar example of an easy-axis antiferromagnet in one dimension (see Fig. 4) with potential energy density Eq. (14). We adopt the static domain walls parametrized in spherical angles \( \theta(z) \) and \( \phi(z) \) as follows:

\[
\cos \theta(z) = \tanh \frac{z - Z}{\lambda}, \quad \phi(z) = \Phi, \tag{35}
\]
We assume a simple configuration with $Dz$ where we have parametrized the one-dimensional antiferromagnet with $\rho M D h \lambda$, for the domain wall in the presence of DM vector $D = D \hat{y}$ and magnetic field $h = h \hat{z}$. Here we set the size of the domain wall to be $\lambda = 1$ and set the system to be $Z \in [-10, 10]$. We see the potential is nearly linear and only bends close to two boundaries. The resultant constant force acting on the domain wall, balanced with the dissipative force, leads to a steady velocity $v_{\text{steady}}$.

where position $Z$ and azimuthal angle $\Phi$ are two collective coordinates, standing for two zero modes of the system. We now expand the first term in the Lagrangian $\mathcal{L}$:

$$\mathcal{L} = \frac{\rho}{2} \dot{\hat{n}}^2 + \frac{\rho}{2} \gamma |n \times h|^2 + \frac{\rho}{2} \left( \frac{M D}{\gamma} \right)^2 + \rho \gamma \dot{n} \cdot (n \times h) - \rho M \gamma \dot{\hat{n}} \cdot D - \rho M D \cdot (n \times h).$$

We assume a simple configuration with $D = D \hat{y}$ and $h = h \hat{z}$. Both $D$ and $h$ are constants. Here $\rho \dot{n}^2 / 2$ endows the domain wall with a mass $M$. As shown before, the magnetic field modifies the easy axis anisotropy. The term proportional to $D^2$ is a constant and thus does not enter the equation of motion of the domain wall.

The total “electromagnetic” force acting on the domain wall $\mathcal{F}$ along $\hat{z}$ direction, derived from the vector potential in the second line of (36), vanishes in this situation. The last line in Lagrangian (36) gives rise to a potential energy for the domain wall:

$$U(Z) \equiv \rho M D h \int_{-L}^{L} d z \, n_z = \rho M D h \lambda \ln \frac{\cosh((Z - L)/\lambda)}{\cosh((Z + L)/\lambda)},$$

where we have parametrized the one dimensional antiferromagnet with $z \in [-L, L]$. One can therefore write down the equation of motion for the domain wall:

$$M \ddot{Z} = -M \dot{Z}/T + F,$$
where $F \equiv -dU/dZ$. We consider the situation that the domain wall is far away from two boundaries of the 1D antiferromagnet. The force due to the potential $U(Z)$ is a constant $F = 2\rho M Dh$, independent of the position of the domain wall, in this scenario (see Fig. [4]).

The domain wall mass $M$ is

$$M = \rho \int dz |nZ|^2 = \frac{2\rho}{\lambda}. \tag{39}$$

From Eq. (38), we can read off the velocity of steady motion:

$$v_{\text{steady}} = \mathcal{M} Dh T \lambda = 2\rho M \frac{DhT}{M}, \tag{40}$$

which is linearly proportional to the strength of DM interaction, applied magnetic field, viscous relaxation time, and is inversely proportional to the mass of domain wall, as one may expect. This is one of the main results of this work. Note that the mass $M$ has a lower bound $\rho/L$, set by the system size. We also remark that one cannot crank up the magnetic field incontinently, as it also contributes to magnetization (recall $m = \chi h/M + \cdots$, when $h \perp n$), which would ultimately invalidate our description at large fields.

**DM interaction and adiabatic spin transfer torque**: A combination of these two interactions produces the Lagrangian density

$$\mathcal{L} = \frac{\rho (\dot{n} + \mathbf{u} \cdot \nabla n + \mathbf{J} \mathbf{D})^2}{2}. \tag{41}$$

The cross term generated here is proportional to $u_i (\partial_i n_j) D_j$. This is clearly a total derivative term which has no effect in the bulk of a material where the DM vector is constant. However, at all interfaces and edges of the sample where the DM vector changes direction or magnitude or both, this term has a finite contribution. Across a sample boundary $\perp' r$ to $x_i$, this term adds an energy:

$$U_{\text{boundary}} = \rho J u_i D \cdot (\Delta, n). \tag{42}$$

Depending on the sign of the DM interaction, the system will then choose to have the Néel vector along a boundary to orient $\parallel$ to or $\perp'$ to the DM vector. Note that this boundary anisotropy is controlled by the direction of the adiabatic spin transfer torque $\mathbf{u}$, as the gradient is attached to that term.

## 5 Conclusion

In this paper we studied the two sublattice antiferromagnet in the presence of external perturbations. The method we employed was to write derive a field theory for the Néel field $\mathbf{n}$ and the uniform magnetization $\mathbf{m}$. The perturbations can then couple to these fields. One of our primary points is that to effectively move antiferromagnetic solitons we need to consider a combination of external perturbations. In all of this, our main motive is to identify avenues through which internal features like the inertia of $\mathbf{n}$, the location of solitons and their sizes can be controlled externally.

We work out these couplings for strain fields which modifies the inertia of the Néel vector. It does so by coupling through a magneto-elastic coupling. An external magnetic field can be put to multiple uses. A static field modifies the shape of the soliton and its configuration. It can also be used to create a local Berry curvature which can be coupled to using a spin current [8]. A dynamical magnetic field, $\mathbf{h}(\mathbf{r}, t)$, can be used to generate an effective electromagnetic tensor and propel domain walls [28]. Crucially, what we find,
is that in combination with a Dzyaloshinsky-Moriya interaction a magnetic field acts to provide a Zeeman like interaction for the Néel vector which can be used to directly drive the antiferromagnetic soliton.

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