Jacobi–Lie T-plurality

Jose J. Fernández-Melgarejo[a*] and **Yuho Sakatani**[b†]

[a] *Departamento de Física, Universidad de Murcia, Campus de Espinardo, E-30100 Murcia, Spain*

[b] *Department of Physics, Kyoto Prefectural University of Medicine, 1-5 Shimogamohangi-cho, Sakyo-ku, Kyoto, Japan*

Abstract

We propose a Leibniz algebra, to be called DD^{+}, which is a generalization of the Drinfel'd double. We find that there is a one-to-one correspondence between a DD^{+} and a Jacobi–Lie bialgebra, extending the known correspondence between a Lie bialgebra and a Drinfel'd double. We then construct generalized frame fields $E_A{}^M \in \text{O}(D,D) \times \mathbb{R}^{+}$ satisfying the algebra $\hat{\mathcal{L}}_{E_A} E_B = -X_{AB}{}^C\, E_C$, where $X_{AB}{}^C$ are the structure constants of the DD^{+} and $\hat{\mathcal{L}}$ is the generalized Lie derivative in double field theory. Using the generalized frame fields, we propose the Jacobi–Lie T-plurality and show that it is a symmetry of double field theory. We present several examples of the Jacobi–Lie T-plurality with or without Ramond–Ramond fields and the spectator fields.

*E-mail address: `melgarejo@um.es`

†E-mail address: `yuho@koto.kpu-m.ac.jp`

Contents

1 Introduction

Recently the Poisson–Lie T-duality [1,2] or T-plurality [3] and their U-duality extensions [4–11] have been studied and developed by using the duality-covariant formulations, such as double field theory (DFT) [12–15] and its U-duality extensions. The Poisson–Lie T-duality is based on a Lie algebra called the Drinfel'd double while the U-duality variant is based on the exceptional Drinfel'd algebra (EDA) [4–7, 9, 16], which is an extension of the Drinfel'd double. Unlike the Drinfel'd double, the structure constants $X_{AB}{}^{C}$ of EDA do not necessarily have the antisymmetry, $X_{AB}{}^{C} \neq -X_{BA}{}^{C}$, and it is a Leibniz algebra rather than a Lie algebra. In this paper, we study a minimal extension of the Drinfel'd double by allowing the structure constants to admit the symmetric part $X_{(AB)}{}^{C} \neq 0$. Using this new Leibniz algebra, we study an extension of the Poisson–Lie T-duality, which we call the Jacobi–Lie T-plurality.[1]

The proposed Leibniz algebra has the form

$$
\begin{aligned}
&T_a \circ T_b = f_{ab}{}^{c}\, T_c\,, \qquad T^a \circ T^b = f_c{}^{ab}\, T^c\,,\\
&T_a \circ T^b = \bigl(f_a{}^{bc} + 2\,\delta_a^b\, Z^c - 2\,\delta_a^c\, Z^b\bigr)\, T_c - f_{ac}{}^{b}\, T^c + 2\, Z_a\, T^b\,,\\
&T^a \circ T_b = -f_b{}^{ac}\, T_c + 2\, Z^a\, T_b + \bigl(f_{bc}{}^{a} + 2\,\delta_b^a\, Z_c - 2\,\delta_c^a\, Z_b\bigr)\, T^c\,,
\end{aligned}
\tag{1.1}
$$

where $a = 1, \dots, D$, and this reduces to the Lie algebra of the Drinfel'd double if $Z_a = Z^a = 0$. This Leibniz algebra admits a symmetric bilinear form

$$
\langle T_a,\, T^b\rangle = \delta_a^b\,, \qquad \langle T_a,\, T_b\rangle = \langle T^a,\, T^b\rangle = 0\,, \tag{1.2}
$$

and two subalgebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ (generated by $\{T_a\}$ and $\{T^a\}$, respectively) are maximally isotropic with respect to this bilinear form. Unlike the case of the Drinfel'd double, the "adjoint-invariance" is relaxed as follows by allowing for a scale transformation:

$$
\delta_A\langle T_B,\, T_C\rangle \equiv \langle T_A \circ T_B,\, T_C\rangle + \langle T_B,\, T_A \circ T_C\rangle = 2\, Z_A\, \langle T_B,\, T_C\rangle\,, \tag{1.3}
$$

where $T_A \equiv (T_a,\, T^a)$ $(A = 1, \dots, 2D)$ and $Z_A \equiv (Z_a,\, Z^a)$. Since this Leibniz algebra is an extension of the Drinfel'd double by admitting the scale symmetry $\mathbb{R}^+$, we call this extended Drinfel'd algebra DD^+. It turns out that this $\mathbb{R}^+$ symmetry provides a scale factor similar to the trombone symmetry in supergravity [19].

In this paper, we show that the DD^+ provides an alternative way to define the Jacobi–Lie algebra, and explain how to construct geometric objects such as the Jacobi–Lie structures from a given DD^+. We also show that we can systematically construct the generalized frame

[1] The Jacobi–Lie T-duality studied in [17, 18] is very similar to our proposal, and this paper is strongly inspired by these papers. However, our identification of the supergravity fields is different from the one given in [17, 18]. The details are explained in sections 3 and 4.

fields ${E_A}^M$ satisfying the frame algebra

$$\hat{\mathcal{L}}_{E_A} E_B = -X_{AB}{}^C\, E_C\,, \tag{1.4}$$

where $\hat{\mathcal{L}}$ denotes the generalized Lie derivative in DFT and $X_{AB}{}^C$ are the structure constants of the DD$^+$. Similar to the recent studies on the Poisson–Lie T-duality/T-plurality in the context of DFT [20–22], exploiting the relation (1.4), we show that the Jacobi–Lie T-plurality is a symmetry of type II DFT.

To exhibit the O(D,D) covariance of the DFT equations of motion, we are forced to turn off the structure constants Z^a. The standard Poisson–Lie T-duality is an exchange of the generators $T_a \leftrightarrow T^a$ and this results in $Z_a \leftrightarrow Z^a$ in our setup. Then if we require $Z^a = 0$ both in the original and the dual frame, we are forced to assume $Z_a = Z^a = 0$ and the DD$^+$ reduces to the standard Drinfel'd double. Accordingly, in this paper, instead of considering the T-duality $T_a \leftrightarrow T^a$, we consider O(D,D) transformations which do not produce Z^a while keeping Z_a non-zero. This is the reason why we call this symmetry the Jacobi–Lie T-plurality, rather than the Jacobi–Lie T-duality.

At the level of the supergravity (or more precisely, DFT), the proposed Jacobi–Lie T-duality is indeed a symmetry of the equations of motion even if the Ramond–Ramond (R–R) fields or spectator fields are present. However, at the level of the string sigma model, due to the presence of the scale factor, we find difficulty in showing the covariance of the equations of motion under the Jacobi–Lie T-plurality. We discuss this issue from several approaches and also discuss the relation to the Jacobi–Lie T-duality proposed in [17].

This paper is organized as follows. In section 2, after introducing the Leibniz algebra DD$^+$, we explain how to construct the Jacobi–Lie structures and the generalized frame fields from the DD$^+$. Here, $Z^a = 0$ is not assumed and we find that the generalized frame fields ${E_A}^M$ have a dependence on the dual coordinates $\tilde{x}_m$ of the doubled space. We also consider several examples of DD$^+$ and explicitly construct the Jacobi–Lie structures and the generalized frame fields ${E_A}^M$. A relation between the DD$^+$ and embedding tensors in gauged supergravities is also briefly discussed. In section 3, we provide a definition of the Jacobi–Lie symmetric backgrounds and show that the equations of motion of DFT have a manifest symmetry under the Jacobi–Lie T-plurality. For convenience, we provide several concrete examples of the Jacobi–Lie T-plurality with and without the R–R fields or the spectator fields. In section 4, we discuss the issue of the Jacobi–Lie T-plurality in the string sigma model. Section 5 is devoted to conclusion and discussion.

2 Jacobi–Lie structures

In this section, we propose a Leibniz algebra DD^+ and construct several quantities, such as the Jacobi–Lie structure, which play an important role in the Jacobi–Lie T-plurality. In section 2.3, we clarify the relation between the DD^+ and the Jacobi–Lie bialgebra studied in [23–26]. Several examples are given in section 2.4. In section 2.5, we comment on a relation between DD^+ and embedding tensors in half-maximal 7D gauged supergravity.

2.1 Algebra

A (classical) Drinfel'd double can be defined as a $2D$-dimensional Lie algebra $\mathfrak{d}$ which admits an adjoint-invariant metric $\langle\cdot,\,\cdot\rangle$ and allows a decomposition $\mathfrak{d}=\mathfrak{g}\oplus\tilde{\mathfrak{g}}$, where $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ form Lie subalgebras that are maximally isotropic with respect to $\langle\cdot,\,\cdot\rangle$. We choose the basis $T_a\in\mathfrak{g}$ and $T^a\in\tilde{\mathfrak{g}}$ such that the metric becomes $\langle T_a,\,T^b\rangle=\delta_a^b$, and denote the subalgebras as $[T_a,\,T_b]=f_{ab}{}^c\,T_c$ and $[T^a,\,T^b]=f_c{}^{ab}\,T^c$. Then, from the adjoint invariance

$$\langle[T_A,\,T_B],\,T_C\rangle+\langle T_B,\,[T_A,\,T_C]\rangle=0\,,\tag{2.1}$$

we can determine the mixed-commutator as

$$[T_a,\,T^b]=f_a{}^{bc}\,T_c-f_{ac}{}^b\,T^c\,.\tag{2.2}$$

The adjoint-invariant metric can be expressed as

$$\langle T_A,\,T_B\rangle=\eta_{AB}\,,\qquad \eta_{AB}=\begin{pmatrix}0&\delta_a^b\\ \delta_b^a&0\end{pmatrix},\tag{2.3}$$

and we raise or lower the indices A,B by using η_{AB} and its inverse η^{AB}.

Now, let us introduce the Leibniz algebra DD^+,

$$T_A\circ T_B=X_{AB}{}^C\,T_C\,.\tag{2.4}$$

We keep assuming that $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are maximally-isotropic Lie subalgebras but relax the adjoint-invariance as in Eq. (1.3). We then find that the structure constants should have the form

$$X_{AB}{}^C\equiv F_{AB}{}^C+Z_A\,\delta_B^C-Z_B\,\delta_A^C+\eta_{AB}\,Z^C\,,\tag{2.5}$$

where $F_{AB}{}^C=F_{ABD}\,\eta^{DC}$, $F_{ABC}=F_{[ABC]}$, and F_{ABC} has the only non-vanishing components $F_{ab}{}^c$ and $F_a{}^{bc}$. Defining $f_{ab}{}^c$ and $f_c{}^{ab}$ through $T_a\circ T_b=f_{ab}{}^c\,T_c$ and $T^a\circ T^b=f_c{}^{ab}\,T^c$, we can parameterize F_{ABC} as

$$F_{abc}=0\,,\quad F_{ab}{}^c=f_{ab}{}^c-Z_a\,\delta_b^c+Z_b\,\delta_a^c\,,\quad F_a{}^{bc}=f_a{}^{bc}-\delta_a^b\,Z^c+Z^b\,\delta_a^c\,,\quad F^{abc}=0\,,\tag{2.6}$$

where $Z_A = (Z_a,\, Z^a)$. By substituting these into Eq. (2.4), we obtain the algebra (1.1).

The closure conditions, or the Leibniz identities,

$$T_A \circ (T_B \circ T_C) = (T_A \circ T_B) \circ T_C + T_B \circ (T_A \circ T_C)\,, \tag{2.7}$$

require the following identities for the structure constants:

$$f_{[ab}{}^{e}\, f_{c]e}{}^{d} = 0\,, \qquad f_{e}{}^{[ab}\, f_{d}{}^{c]e} = 0\,, \tag{2.8}$$

$$4\, f_{[a}{}^{e[c}\, f_{b]e}{}^{d]} - f_{ab}{}^{e}\, f_{e}{}^{cd} + 4\, f_{[a}{}^{cd}\, Z_{b]} + 4\, f_{ab}{}^{[c}\, Z^{d]} + 8\, f_{e[a}{}^{[c}\, \delta^{d]}_{b]}\, Z^{e} - 16\, Z_{[a}\, \delta^{[c}_{b]}\, Z^{d]} = 0\,, \tag{2.9}$$

$$f_{ab}{}^{c}\, Z_c = 0\,, \qquad f_{a}{}^{bc}\, Z_c = f_{ac}{}^{b}\, Z^c\,, \qquad Z^c\, f_{c}{}^{ab} = 0\,, \qquad Z^a\, Z_a = 0\,. \tag{2.10}$$

2.2 Generalized frame fields

Here we construct the generalized frame fields $E_A{}^M$. We introduce a group element $g = \mathrm{e}^{x^a\, T_a}$ and define the left-/right-invariant 1-forms as

$$\ell = \ell^a_m\, \mathrm{d}x^m\, T_a = g^{-1}\, \mathrm{d}g\,, \qquad r = r^a_m\, \mathrm{d}x^m\, T_a = \mathrm{d}g\, g^{-1}\,. \tag{2.11}$$

Their inverse matrices are denoted as v^m_a and e^m_a ($v^m_a\, \ell^b_m = \delta^b_a = e^m_a\, r^b_m$). We then consider the adjoint-like action as

$$g \triangleright T_A \equiv \mathrm{e}^{x^b\, T_b \circ}\, T_A = T_A + x^b\, T_b \circ T_A + \tfrac{1}{2!}\, x^b\, T_b \circ \bigl(x^c\, T_c \circ T_A\bigr) + \cdots\,, \tag{2.12}$$

and define

$$g^{-1} \triangleright T_A \equiv M_A{}^B(g)\, T_B\,. \tag{2.13}$$

It turns out that this matrix $M_A{}^B$ can be parameterized as

$$M_A{}^B \equiv \begin{pmatrix} a_a{}^b & 0 \\ -\pi^{ac}\, a_c{}^b & \mathrm{e}^{-2\Delta} (a^{-1})_b{}^a \end{pmatrix}, \tag{2.14}$$

where π^{ab} is an antisymmetric field: $\pi^{ab} = -\pi^{ba}$.

Similar to the case of the Drinfel'd double [27] (see also [9] for a general discussion), we find that $a_a{}^b$, π^{ab}, and Δ satisfy the algebraic identities

$$f_{ab}{}^{c} = a_a{}^{d}\, a_b{}^{e}\, (a^{-1})_f{}^{c}\, f_{de}{}^{f}\,, \tag{2.15}$$

$$f_{d}{}^{[ab}\, \pi^{c]d} + f_{de}{}^{[a}\, \pi^{b|d|}\, \pi^{c]e} - 2\, \pi^{[ab}\, \pi^{c]d}\, Z_d + 2\, \pi^{[ab}\, Z^{c]} = 0\,, \tag{2.16}$$

$$f_{a}{}^{bc} = \mathrm{e}^{-2\Delta}\, a_a{}^{d}\, (a^{-1})_e{}^{b}\, (a^{-1})_f{}^{c}\, f_{d}{}^{ef} + 2\, f_{ad}{}^{[b}\, \pi^{c]d} + 6\, \delta^{[b}_{a}\, \pi^{cd]}\, Z_d\,, \tag{2.17}$$

$$a_a{}^{b}\, Z_b = Z_a\,, \qquad Z^a + \pi^{ab}\, Z_b = \mathrm{e}^{-2\Delta} (a^{-1})_b{}^{a}\, Z^b \qquad \bigl(\Leftrightarrow\ M_A{}^B\, Z_B = Z_A\bigr)\,, \tag{2.18}$$

and the differential identities

$$D_a\Delta = Z_a\,, \qquad D_a a_b{}^c = -f_{ab}{}^d\, a_d{}^c\,, \tag{2.19}$$

$$D_a\pi^{bc} = f_a{}^{bc} + 2\, f_{ad}{}^{[b}\,\pi^{|d|c]} - 2\, Z_a\,\pi^{bc} - 4\, Z^{[b}\,\delta_a^{c]}\,, \tag{2.20}$$

where $D_a \equiv e_a^m\,\partial_m$. Combining these identities, we also find

$$\pounds_{v_a}\Delta = Z_a\,, \qquad \pounds_{v_a} a_b{}^c = -a_b{}^d\, f_{ad}{}^c\,, \tag{2.21}$$

$$\pounds_{v_a}\pi^{mn} = \bigl(f_a{}^{bc} + 2\,\delta_a^b\, Z^c - 2\,\delta_a^c\, Z^b\bigr)\, v_b^m\, v_c^n + 2\, Z_a\,\pi^{mn}\,. \tag{2.22}$$

Here we have defined

$$\pi^{mn} \equiv \mathrm{e}^{2\Delta}\,\pi^{ab}\, e_a^m\, e_b^n\,, \tag{2.23}$$

which turns out to be a Jacobi–Lie structure.

Now we define the generalized frame fields as

$$E_A{}^M \equiv M_A{}^B\, V_B{}^M\,, \qquad V_A{}^M \equiv \begin{pmatrix} v_a^m & 0 \\ 0 & \ell_m^a \end{pmatrix}, \tag{2.24}$$

and obtain

$$E_A{}^M = \begin{pmatrix} e_a^m & 0 \\ -\pi^{ab}\, e_b^m & \mathrm{e}^{-2\Delta}\, r_m^a \end{pmatrix}. \tag{2.25}$$

If $Z^a = 0$, these generalized frame fields satisfy the relation

$$\hat{\pounds}_{E_A} E_B{}^M = -X_{AB}{}^C\, E_C{}^M\,, \tag{2.26}$$

by means of the generalized Lie derivative in DFT,

$$\hat{\pounds}_V W^M \equiv V^N\,\partial_N W^M - (\partial_N V^M - \partial^M V_N)\, W^N\,. \tag{2.27}$$

In the presence of Z^a , we need to modify the generalized frame fields as

$$E_A{}^M \equiv \begin{pmatrix} e_a^m & 0 \\ -\pi^{ac}\, e_c^m & \mathrm{e}^{-2\Delta}\,\tilde{\sigma}\, r_m^a \end{pmatrix}. \tag{2.28}$$

If this $\tilde{\sigma}$ satisfies

$$\partial_m\tilde{\sigma} = 0\,, \qquad \tilde{\partial}^m\tilde{\sigma} \equiv -2\, Z^m \equiv -2\, Z^a\, v_a^m\,, \tag{2.29}$$

we find that the new generalized frame fields satisfy the desired relation (2.26).

Since the modified generalized frame fields have the dependence on the dual coordinates $\tilde{x}_m$, one may be concerned about the section condition (i.e., a consistency condition in DFT).

However, we can easily show that the section condition is not broken. As we discuss later, the supergravity fields are constructed from $E_A{}^M$ which is composed of the fields $\{\Delta,\,\tilde{\sigma},\,e_a^m,\,\pi^{mn}\}$. Using $\pounds_Z = Z^a\,\pounds_{v_a}$, the differential identities, and the Leibniz identities, we find

$$\begin{aligned}\pounds_Z\Delta = Z^a\,Z_a = 0\,,\quad \pounds_Z e_a^m = Z^b\,\pounds_{v_b} e_a^m = 0\,,\\ \pounds_Z\pi^{mn} = Z^a\,\big(f_a{}^{bc} + 2\,\delta_a^b\,Z^c - 2\,\delta_a^c\,Z^b\big)\,v_b^m\,v_c^n + 2\,Z^a\,Z_a\,\pi^{mn} = 0\,.\end{aligned} \tag{2.30}$$

Therefore, Z is a Killing vector field and we can choose the coordinate system such that $Z = \partial_z$ is realized. Then all of the fields ϕ are independent of the coordinate z. In this coordinate system, we can explicitly find $\tilde{\sigma} = \tilde{z}$, and then the section condition reduces to

$$0 = \eta^{MN}\,\partial_M\tilde{\sigma}\,\partial_N\phi = \partial_z\phi\,. \tag{2.31}$$

This is indeed satisfied because ϕ is independent of z.

Let us also show several properties of the bi-vector field $\pi \equiv \frac{1}{2}\,\pi^{mn}\,\partial_m\wedge\partial_n$. By using the differential and algebraic identities, we can show

$$[\pi,\,\pi]_S = 2\,E\wedge\pi\,,\qquad [E,\,\pi]_S = 0\,, \tag{2.32}$$

where $E \equiv -2\,Z^a\,e_a$ and we have defined the Schouten–Nijenhuis bracket for a p-vector v and a q-vector w as

$$\begin{aligned}[v,\,w]_S^{m_1\cdots m_{p+q-1}} \equiv&\ \tfrac{(p+q-1)!}{(p-1)!\,q!}\,v^{p[m_1\cdots m_{p-1}}\,\partial_p w^{m_p\cdots m_{p+q-1}]}\\ &+\tfrac{(-1)^{pq}(p+q-1)!}{(q-1)!\,p!}\,w^{p[m_1\cdots m_{q-1}}\,\partial_p v^{m_q\cdots m_{p+q-1}]}\,,\end{aligned} \tag{2.33}$$

or more explicitly,

$$[\pi,\,\pi]_S \equiv \pi^{q[m}\,\partial_q\pi^{np]}\,\partial_m\wedge\partial_n\wedge\partial_p\,,\qquad [E,\,\pi]_S \equiv \tfrac{1}{2!}\,\pounds_E\pi^{mn}\,\partial_m\wedge\partial_n\,. \tag{2.34}$$

The first property is equivalent to the absence of the non-geometric R-flux

$$X^{abc} = 3\,\pi^{d[a}\,D_d\pi^{bc]} + 3\,f_{de}{}^{[a}\,\pi^{b|d|}\,\pi^{c]e} - 6\,\pi^{[ab}\,\pi^{c]d}\,D_d\Delta - 3\,a_d{}^{[a}\,\pi^{bc]}\,r_m^d\,\tilde{\partial}^m\,\tilde{\sigma} = 0\,, \tag{2.35}$$

and the second one follows from

$$\pounds_{e_a}\pi^{mn} = \mathrm{e}^{2\Delta}\big(f_a{}^{bc} - 4\,Z^{[b}\,\delta_a^{c]}\big)\,e_b^m\,e_c^n\,. \tag{2.36}$$

These two properties show that the bi-vector field π^{mn} is a Jacobi structure and the vector field E may be called the Reeb vector field. Combining this with the property (2.22), we can conclude that the bi-vector field π constructed from a DD$^+$ is always a Jacobi–Lie structure.

As it has been studied in [23,24,26], the Leibniz identity (2.9) can be regarded as a cocycle condition, and it is automatically satisfied if we consider the coboundary ansatz

$$f_a{}^{bc} = 2\,r^{[b|d|}\,f_{ad}{}^{c]} - 2\,Z_a\,r^{bc} + 4\,Z^{[b}\,\delta_a^{c]}\,, \tag{2.37}$$

where r^{ab} is a skew-symmetric constant matrix. The other Leibniz identities (under $f_{[ab}{}^{e}\, f_{c]e}{}^{d} = 0$ and $f_{ab}{}^{c}\, Z_c = 0$) are equivalent to[2]

$$r^{ab}\, Z_b = Z^a\,, \quad Z^c\, f_{cd}{}^{[a}\, r^{b]d} = 0\,, \quad \mathrm{CYBE}^{abc} \equiv 3\, f_{de}{}^{[a}\, r^{b|d|}\, r^{c]e} - 6\, Z^{[a}\, r^{bc]} = 0\,. \qquad (2.38)$$

For this type of algebra, we can find the solution of the differential equation (2.22) as

$$\pi^{mn} = r^{ab}\left(v_a^m\, v_b^n - \mathrm{e}^{2\Delta}\, e_a^m\, e_b^n\right). \qquad (2.39)$$

We note that this type of Jacobi–Lie structure (associated with the coboundary-type algebras) has been studied in [24] (see also [17, 26]).

2.3 Jacobi–Lie bialgebra

Let us explain the relation between DD$^+$ and the Jacobi–Lie bialgebra studied in [23–26]. We begin with a Lie algebra $\mathfrak{g}$ with commutation relation $[T_a,\, T_b] = f_{ab}{}^{c}\, T_c$. We introduce the dual space $\mathfrak{g}^*$ spanned by $\{T^a\}$ and suppose that they form a Lie algebra $[T^a,\, T^b] = f_c{}^{ab}\, T^c$. We introduce the differentials d and d_* which acts on $\mathfrak{g}^*$ and $\mathfrak{g}$ as

$$\mathrm{d}T^a = -\tfrac{1}{2}\, f_{bc}{}^{a}\, T^b \wedge T^c\,, \qquad \mathrm{d}_* T_a = -\tfrac{1}{2}\, f_a{}^{bc}\, T_b \wedge T_c\,, \qquad (2.40)$$

and 1-cocycles $X_0 \in \mathfrak{g}$ and $\phi_0 \in \mathfrak{g}^*$ satisfying $\mathrm{d}_* X_0 = 0$ and $\mathrm{d}\phi_0 = 0$. We then define

$$\mathrm{d}_{*X_0} \equiv \mathrm{d}_* + X_0 \wedge\,, \qquad (2.41)$$

and a bracket $[\cdot,\, \cdot]_{\phi_0}$ for $x \in \wedge^p \mathfrak{g}$ and $y \in \wedge^q \mathfrak{g}$ as

$$[x,\, y]_{\phi_0} = [x,\, y] + (-1)^{p-1}(p-1)\, x \wedge \iota_{\phi_0} y - (q-1)\, \iota_{\phi_0} x \wedge y\,, \qquad (2.42)$$

where $[\cdot,\, \cdot]$ is the algebraic Schouten bracket and ι_{ϕ_0} denotes the contraction. Using these, we can define a Jacobi–Lie bialgebra as a pair $((\mathfrak{g},\, \phi_0),\, (\mathfrak{g}^*,\, X_0))$ which satisfies

$$\begin{aligned} &\mathrm{d}_{*X_0}[x,\, y] = [x, \mathrm{d}_{*X_0} y]_{\phi_0} - [y,\, \mathrm{d}_{*X_0} x]_{\phi_0}\,, \\ &\langle \phi_0,\, X_0 \rangle = 0\,, \qquad \iota_{\phi_0}(\mathrm{d}_* x) + [X_0,\, x] = 0\,, \end{aligned} \qquad (2.43)$$

for any elements $x, y \in \mathfrak{g}$. If we expand X_0 and ϕ_0 as

$$X_0 = \alpha^a\, T_a\,, \qquad \phi_0 = \beta_a\, T^a\,, \qquad (2.44)$$

the 1-cocycle conditions $\mathrm{d}_* X_0 = 0$ and $\mathrm{d}\phi_0 = 0$ are equivalent to

$$\alpha^a\, f_a{}^{bc} = 0\,, \qquad \beta_a\, f_{bc}{}^{a} = 0\,, \qquad (2.45)$$

[2] The first equation is implied by $\left(f_{ac}{}^{b} - 2\, Z_a\, \delta_c^b\right)\left(Z^c - r^{cd}\, Z_d\right) = 0$. The last equation can be relaxed as $f_{de}{}^{[a}\, \mathrm{CYBE}^{|e|bc]} = 0$ if $Z_a = 0$. Indeed, in the case of six-dimensional Jacobi–Lie bialgebras [26], an algebra satisfying $\mathrm{CYBE}^{abc} \neq 0$ (i.e., a quasitriangular coboundary Jacobi–Lie bialgebra) is realized only when $Z_a = 0$.

and the conditions (2.43) can be expressed as

$$\begin{aligned}&4\,f_{[a}{}^{e[c}\,f_{b]e}{}^{d]} - f_{ab}{}^{e}\,f_{e}{}^{cd} + 2\,f_{[a}{}^{cd}\,\beta_{b]} + 2\,f_{ab}{}^{[c}\,\alpha^{d]} + 4\,f_{e[a}{}^{[c}\,\delta^{d]}_{b]}\,\alpha^{e} - 4\,\beta_{[a}\,\delta^{[c}_{b]}\,\alpha^{d]} = 0\,,\\ &\alpha^{a}\,\beta_{a} = 0\,, \qquad \alpha^{c}\,f_{ca}{}^{b} - \beta_{c}\,f_{a}{}^{cb} = 0\,.\end{aligned} \tag{2.46}$$

They are exactly the same as the Leibniz identities of the DD^{+} under the identification

$$\alpha^{a} = 2\,Z^{a}\,, \qquad \beta_{a} = 2\,Z_{a}\,. \tag{2.47}$$

This shows that there is a one-to-one correspondence between a Leibniz algebra DD^{+} and a Jacobi–Lie bialgebra. In [25], by using a generalized Courant bracket, commutation relations

$$\begin{aligned}&[T_a,\,T_b] = f_{ab}{}^{c}\,T_c\,, \qquad [T^{a},\,T^{b}] = f_{c}{}^{ab}\,T^{c}\,,\\ &[T_a,\,T^{b}] = \bigl(f_{a}{}^{bc} + \tfrac{1}{2}\,\alpha^{c}\,\delta^{b}_{a} - \alpha^{b}\,\delta^{c}_{a}\bigr)\,T_c + \bigl(f_{ca}{}^{b} - \tfrac{1}{2}\,\beta_{c}\,\delta^{b}_{a} + \beta_{a}\,\delta^{b}_{c}\bigr)\,T^{c}\,,\end{aligned} \tag{2.48}$$

are introduced, but in general, this does satisfy the Jacobi identities and is not a Lie algebra. Rather, this can be regarded as the antisymmetric part of the Leibniz algebra DD^{+},

$$[T_A,\,T_B] \equiv \tfrac{1}{2}\,\bigl(T_A\circ T_B - T_B\circ T_A\bigr)\,. \tag{2.49}$$

As we discussed in section 2.2, a DD^{+} allows us to systematically construct the Jacobi–Lie structure π^{mn} for a general Jacobi–Lie bialgebra. In [17], a similar construction has been attempted by using the commutation relations (2.48). However, due to the absence of the symmetric part $X_{(AB)}{}^{C}$ of the structure constants, it was not successful, and only the coboundary-type algebras have been studied, where π^{mn} has the simple expression (2.39). A DD^{+} also allows us to obtain the scale factor Δ from a straightforward computation of the matrix $M_A{}^{B}$, and these are the advantage of our approach based on the Leibniz algebra. In the next subsection, as a demonstration, we explicitly compute the Jacobi–Lie structures for several concrete examples.

2.4 Examples of Jacobi–Lie structures

The low-dimensional Jacobi–Lie groups have been classified in [25], and in particular, classifications of the coboundary-type Jacobi–Lie groups have been given in [26]. For the coboundary-type algebras, there is a general formula (2.39) for the Jacobi–Lie structures, and here we consider two examples of Leibniz algebras that are not of the coboundary type.

(I) $((\mathrm{IV}, -\epsilon\tilde{X}^{1}), (\mathrm{IV.i}, -\epsilon\alpha X_3))$

Let us consider $((\mathrm{IV}, -\epsilon\tilde{X}^{1}), (\mathrm{IV.i}, -\epsilon\alpha X_3))$ $(\alpha > 0)$ in Table 6 of [25], which corresponds to

$$f_{12}{}^{2} = -f_{12}{}^{3} = f_{13}{}^{3} = -1\,, \quad f_{1}{}^{13} = f_{2}{}^{23} = \alpha\,, \quad f_{1}{}^{23} = 1\,, \quad Z^{3} = -\tfrac{\epsilon\,\alpha}{2}\,, \quad Z_1 = -\tfrac{\epsilon}{2}\,. \tag{2.50}$$

The Leibniz identities require $\epsilon = 1$ or $\epsilon = 2$. While $\epsilon = 1$ gives a coboundary algebra, here we consider the non-coboundary case $\epsilon = 2$.

Using $g = e^{x\,T_1}\, e^{y\,T_2}\, e^{z\,T_3}$, the left-/right-invariant vectors are found as

$$\begin{aligned} &v_1 = \partial_x + y\,\partial_y + (z-y)\,\partial_z\,, \quad v_2 = \partial_y\,, \quad v_3 = \partial_z\,, \\ &e_1 = \partial_x\,, \quad e_2 = e^{x}(\partial_y - x\,\partial_z)\,, \quad e_3 = e^{x}\,\partial_z\,, \end{aligned} \tag{2.51}$$

and by computing the matrix $M_A{}^B$, we find

$$\pi = \big[\alpha\,(e^{-x} - 1)\,\partial_x + (x - \alpha\,y)\,\partial_y\big] \wedge \partial_z\,, \qquad e^{-2\Delta} = e^{2x}\,. \tag{2.52}$$

From $\tilde{\partial}^m\tilde{\sigma} = -2\,Z^a\,v_a^m$ we can easily find

$$\tilde{\sigma} = 2\,\alpha\,\tilde{z}\,, \tag{2.53}$$

and then we find that the generalized frame fields enjoy the algebra $\hat{\pounds}_{E_A}E_B = -X_{AB}{}^C\,E_C$.

(II) $((\mathrm{III}, -2\,\tilde{X}^1), (\mathrm{III.ii}, -(X_2 + X_3)))$

Another example is $((\mathrm{III}, -2\,\tilde{X}^1), (\mathrm{III.ii}, -(X_2 + X_3)))$ of [25], which corresponds to

$$f_{12}{}^2 = f_{12}{}^3 = f_{13}{}^2 = f_{13}{}^3 = -1\,, \quad f_1{}^{12} = f_1{}^{13} = 1\,, \quad Z^2 = Z^3 = -\tfrac{1}{2}\,, \quad Z_1 = -1\,. \tag{2.54}$$

Using $g = e^{x\,T_1}\, e^{y\,T_2}\, e^{z\,T_3}$, the left-/right-invariant vectors are found as

$$\begin{aligned} &v_1 = \partial_x + (y+z)\,(\partial_y + \partial_z)\,, \qquad v_2 = \partial_y\,, \qquad v_3 = \partial_z\,, \\ &e_1 = \partial_x\,, \quad e_2 = e^{x}(\cosh x\,\partial_y + \sinh x\,\partial_z)\,, \quad e_3 = e^{x}(\sinh x\,\partial_y + \cosh x\,\partial_z)\,. \end{aligned} \tag{2.55}$$

From the matrix $M_A{}^B$ and $\tilde{\partial}^m\tilde{\sigma} = -2\,Z^a\,v_a^m$, we find

$$\pi = (z-y)\,\partial_y \wedge \partial_z\,, \qquad e^{-2\Delta} = e^{2x}\,, \qquad \tilde{\sigma} = \tilde{y} + \tilde{z}\,, \tag{2.56}$$

and then the generalized frame fields satisfy the algebra $\hat{\pounds}_{E_A}E_B = -X_{AB}{}^C\,E_C$.

In this way, for a given Leibniz algebra, we can easily compute the Jacobi–Lie structure and the generalized frame fields.

2.5 Embedding tensor in half-maximal 7D gauged supergravity

As a side remark, we here clarify the relation between six-dimensional DD$^+$s and the embedding tensors in half-maximal 7D gauged supergravity. In [28], the embedding tensor in half-maximal 7D gauged supergravity has been classified, where the duality group is $\mathrm{O}(3,3)\times\mathbb{R}^+$. In our convention, their embedding tensor can be expressed as

$$\begin{aligned} X_{AB}{}^C &\equiv F_{AB}{}^C + Z_A\,\delta_B^C - Z_B\,\delta_A^C + \eta_{AB}\,Z^C\,, \\ F_{abc} &= H_{abc}\,, \quad F_{ab}{}^c = f_{ab}{}^c - Z_a\,\delta_b^c + Z_b\,\delta_a^c\,, \\ F_a{}^{bc} &= f_a{}^{bc} - \delta_a^b\,Z^c + Z^b\,\delta_a^c\,, \quad F^{abc} = R^{abc}\,, \end{aligned} \tag{2.57}$$

where the non-vanishing components are

$$\begin{aligned} &H_{123} = Q_{11}\,, \quad f_1{}^{23} = Q_{22}\,, \quad f_2{}^{13} = -Q_{33}\,, \quad f_3{}^{12} = Q_{44}\,, \quad Z_1 = -\xi_0\,, \\ &R^{123} = \tilde{Q}^{11}\,, \quad f_{23}{}^1 = \tilde{Q}^{22}\,, \quad f_{13}{}^2 = -\tilde{Q}^{33}\,, \quad f_{12}{}^3 = \tilde{Q}^{44}\,, \quad f_{12}{}^2 = f_{13}{}^3 = -\xi_0\,. \end{aligned} \tag{2.58}$$

The possible values of Q_{ij}, $\tilde{Q}^{ij}$, and ξ_0 have been classified in Table 2 of [28].

Due to the presence of F_{abc} and F^{abc}, this is not exactly the algebra of a DD$^+$, but by performing an O$(3,3)$ redefinition of generators, we can obtain a DD$^+$. As an example, let us take orbit 10 of [28], where the gauge group is CSO$(1,1,2)$, and Q_{ii} and $\tilde{Q}^{ii}$ are

$$\tfrac{Q_{ii}}{\cos\alpha} = (1,-1,0,0)\,, \qquad \tfrac{\tilde{Q}^{ii}}{\sin\alpha} = (0,0,1,-1) \qquad \bigl(-1\le \xi_0 \le 1\,,\ -\tfrac{\pi}{4} < \alpha \le \tfrac{\pi}{4}\bigr)\,. \tag{2.59}$$

Now if we perform a redefinition of the generators $T_A \to C_A{}^B\, T_B$ with

$$C_A{}^B = \begin{pmatrix} -\frac{1}{\cos\alpha} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & -\cos\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}, \tag{2.60}$$

we find that the structure constants become

$$f_{12}{}^3 = -1\,, \quad f_{13}{}^2 = -1\,, \quad f_{12}{}^2 = f_{13}{}^3 = \frac{\xi_0 - \sin\alpha}{\cos\alpha}\,, \quad Z_1 = \frac{\xi_0}{\cos\alpha}\,. \tag{2.61}$$

For example, if $\xi_0 = \sin\alpha$ or $\frac{\xi_0-\sin\alpha}{\cos\alpha} = -1$ is realized, this is equivalent to a Jacobi–Lie bialgebra $((\mathrm{VI}_0, b\,X_3), (\mathrm{I}, 0))$ or $((\mathrm{III}, b\,X_1), (\mathrm{I}, 0))$ given in Table 7 of [25], respectively. By choosing another matrix $C_A{}^B$, we will also find another Jacobi–Lie bialgebra classified in [25].

In this way, an embedding tensor classified in [28] can be mapped to several Jacobi–Lie bialgebras or DD$^+$s. Then, similar to the previous section, we can systematically construct the generalized frame fields (or twist matrix) by using the Jacobi–Lie structure. Despite some of the embedding tensor configurations may not be mapped to any DD$^+$, it may be interesting to identify which gaugings of [28] admit a description as Jacobi–Lie bialgebras. Similarly, this analysis can be carried out for any (half-)maximal d-dimensional supergravities, because the T-duality-covariant flux F_{ABC} is always contained in the embedding tensor and the role of Z_A can be played by the trombone gauging [29–31] or the dilaton flux. In particular, the half-maximal $d = 6, 5, 4$ supergravities explicitly contain an O$(10-d, 10-d)$ vector ξ_A (or ξ_{+A}) which potentially plays the role of Z_A. There, the Leibniz identities Eqs. (2.8)–(2.10) appear as some components of the quadratic constraints studied in [28, 32, 33].

3 Jacobi–Lie T-duality

In [20–22], the Poisson–Lie T-duality/T-plurality has been proven to be a symmetry of DFT. As a natural extension, non-Abelian U-duality associated with EDA has been discussed in [4–10, 16], and several examples of the non-Abelian U-duality have been found in [11]. Here, restricting ourselves to the case $Z^a = 0$, we show that the non-Abelian duality based on a DD^+, i.e., the Jacobi–Lie T-plurality, is a symmetry of the DFT equations of motion. Due to our assumption $Z^a = 0$, the bi-vector field π^{mn} is a Poisson structure rather than the Jacobi structure, but due to the existence of the scale factor Δ, this is not a Poisson–Lie structure and we keep calling π^{mn} the Jacobi–Lie structure.

In type II DFT, the bosonic fields in the NS–NS sector are the generalized metric and the DFT dilaton

$$\mathcal{H}_{MN} \equiv \begin{pmatrix} g_{mn} - B_{mp}\, g^{pq}\, B_{qn} & B_{mp}\, g^{pn} \\ -g^{mp}\, B_{pn} & g^{mn} \end{pmatrix}, \qquad \mathrm{e}^{-2d} \equiv \sqrt{|\det g_{mn}|}\, \mathrm{e}^{-2\Phi}, \tag{3.1}$$

and the R–R fields can be described as an $\mathrm{O}(D,D)$ spinor $|F\rangle$. By making a certain ansatz for these bosonic fields, we show the covariance of the equations of motion under the Jacobi–Lie T–plurality, which is an $\mathrm{O}(D,D)$ rotation discussed below.

Let us begin with a simple case where the R–R fields and the spectator fields y^μ (which do not transform under the $\mathrm{O}(D,D)$ rotation) are not present. We consider an ansatz for the NS–NS sector fields,

$$\mathcal{H}_{MN}(x) = \mathcal{E}_M{}^A(x)\, \mathcal{E}_N{}^B(x)\, \hat{\mathcal{H}}_{AB}\,, \qquad \mathrm{e}^{-2d(x)} = \mathrm{e}^{-2\varphi(x)}\, \mathrm{e}^{-\Delta(x)} |\det \ell^a_m(x)|\,, \tag{3.2}$$

where $\hat{\mathcal{H}}_{AB}$ is constant, $\varphi(x)$ is a certain function, and we have defined

$$\mathcal{E}_A{}^M \equiv \mathrm{e}^{\Delta(x)}\, E_A{}^M(x) = \mathrm{e}^{\Delta} \begin{pmatrix} e_a^m & 0 \\ -\pi^{ac}\, e_c^m & \mathrm{e}^{-2\Delta}\, r^a_{\ m} \end{pmatrix} \in \mathrm{O}(D,D)\,. \tag{3.3}$$

When the target space is of this form, this background is called Jacobi–Lie symmetric.

If we parameterize the constant matrix $\hat{\mathcal{H}}_{AB}$ as

$$\hat{\mathcal{H}}_{AB} \equiv \begin{pmatrix} \hat{g}_{ab} & -(\hat{g}\,\hat{\beta})_a{}^b \\ (\hat{\beta}\,\hat{g})^a{}_b & (\hat{g}^{-1} - \hat{\beta}\,\hat{g}\,\hat{\beta})^{ab} \end{pmatrix}, \tag{3.4}$$

by comparing the parameterization (3.1) with (3.2), the metric and the B-field can be expressed as $g_{mn} + B_{mn} = \mathcal{E}_{mn}$ where $\mathcal{E}_{mn}$ is the inverse matrix of

$$\mathcal{E}^{mn} \equiv \mathrm{e}^{2\Delta}\big(\hat{\mathcal{E}}^{ab} + \pi^{ab}\big)\, e_a^m\, e_b^n = \mathrm{e}^{2\Delta}\, \hat{\mathcal{E}}^{ab}\, e_a^m\, e_b^n + \pi^{mn}\,, \qquad \hat{\mathcal{E}}^{ab} \equiv \hat{g}^{ab} + \hat{\beta}^{ab}\,. \tag{3.5}$$

The can be also expressed as

$$g_{mn} + B_{mn} = \mathrm{e}^{-2\Delta}\,\mathcal{E}_{ab}\,r_m^a\,r_n^b\,, \qquad (\mathcal{E}_{ab}) \equiv (\hat{\mathcal{E}}^{ab} + \pi^{ab})^{-1}\,. \tag{3.6}$$

The standard dilaton Φ can be found as

$$\mathrm{e}^{-2\Phi} = \sqrt{|\det \hat{g}_{ab}|}\,\mathrm{e}^{-2\varphi(x)}|\det(\hat{\mathcal{E}}^{ab} + \pi^{ab})\det(a_a{}^b)|\,. \tag{3.7}$$

The structure constants Z_a , which are not present in the Poisson–Lie T-duality, produce the overall factor $\mathrm{e}^{-2\Delta}$ both in the metric and the B-field. We find that $\mathcal{E}_{mn}$ satisfies

$$\pounds_{v_a}\mathcal{E}_{mn} + 2\,Z_a\,\mathcal{E}_{mn} = -\big(f_a{}^{bc} + 2\,\delta_a^b\,Z^c - 2\,\delta_a^c\,Z^b\big)\,\mathcal{E}_{mp}\,v_b^p\,v_c^q\,\mathcal{E}_{qn}\,. \tag{3.8}$$

Here, let us comment on the difference between our proposal and the one studied in [17]. In [17], the metric and the B-field are identified as

$$g_{mn} + B_{mn} = E_{mn}\,, \qquad E_{mn} \equiv \mathrm{e}^{2\Delta}\,\mathcal{E}_{mn} = \mathcal{E}_{ab}\,r_m^a\,r_n^b\,, \tag{3.9}$$

for which we have

$$\pounds_{v_a}E_{mn} = -\,\mathrm{e}^{2\Delta}\big(f_a{}^{bc} + 2\,\delta_a^b\,Z^c - 2\,\delta_a^c\,Z^b\big)\,E_{mp}\,v_b^p\,v_c^q\,E_{qn}\,. \tag{3.10}$$

The difference is only in the overall factor. Below, we show the covariance of the equations of motion under the Jacobi–Lie T-plurality by adopting the former choice $g_{mn} + B_{mn} = \mathcal{E}_{mn}$ and using the dilaton (3.7).

The generalized fluxes associated with $\mathcal{E}_A{}^M$ are defined as

$$\begin{aligned}
&\mathcal{F}_{ABC} \equiv 3\,\mathcal{W}_{[ABC]}\,, \qquad \mathcal{F}_A \equiv \mathcal{W}^B{}_{AB} + 2\,\mathcal{D}_A d\,,\\
&\mathcal{W}_{ABC} \equiv -\mathcal{D}_A\mathcal{E}_B{}^M\,\mathcal{E}_{MC}\,, \qquad \mathcal{D}_A \equiv \mathcal{E}_A{}^M\,\partial_M\,.
\end{aligned} \tag{3.11}$$

Using the algebraic and the differential identities, we find

$$\mathcal{F}_{ABC} = \mathrm{e}^{\Delta}\,F_{ABC}\,, \quad \mathcal{F}_A = \mathcal{E}_A{}^M\,F_M\,, \quad F_M = 2\,\partial_M d + \begin{pmatrix} \partial_m \ln|\det \ell_m^a| - \partial_m\Delta \\ -f_b{}^{ba}\,v_a^m \end{pmatrix}, \tag{3.12}$$

where F_{ABC} is the one given in (2.6). If $f_b{}^{ba}$ does not vanish, we make a replacement [21]

$$\partial_M d \to \partial_M d + \boldsymbol{X}_M\,, \qquad \boldsymbol{X}_M = \big(0, \tfrac{1}{2}\,f_b{}^{ba}\,v_a^m\big)\,, \tag{3.13}$$

which corresponds to introducing the dual coordinate dependence into the dilaton [34, 35] and the background becomes a solution of the generalized supergravity equations of motion [35–37].[3] Then, by using (3.2), the single-index flux becomes

$$\mathcal{F}_A = \mathrm{e}^{\Delta}\,F_A\,, \qquad F_A = 2\,E_A{}^M\,\partial_M\varphi = 2\,\big(D_a\varphi,\, -\pi^{ab}\,D_b\varphi\big)\,, \tag{3.14}$$

[3]We do not consider such examples in this paper, but in general, choosing the coordinates such that the Killing vector $I^m = \frac{1}{2}\,f_b{}^{ba}\,v_a^m$ becomes $I = \partial_z$, we add $\tilde{z}$ to the DFT dilaton d given in Eq. (3.2).

and we suppose that $F_A \equiv E_A{}^M F_M$ is constant.

In general, the equations of motion of DFT are given by

$$\mathcal{R} = 0\,, \qquad \mathcal{G}^{AB} = 0\,. \tag{3.15}$$

Here, $\mathcal{R}$ and $\mathcal{G}^{AB}$, under the section condition, can be expressed as

$$\begin{aligned}
\mathcal{R} &\equiv \hat{\mathcal{H}}^{AB}\,\bigl(2\,\mathcal{D}_A\mathcal{F}_B - \mathcal{F}_A\,\mathcal{F}_B\bigr) + \tfrac{1}{12}\,\hat{\mathcal{H}}^{AD}\,\bigl(3\,\eta^{BE}\,\eta^{CF} - \hat{\mathcal{H}}^{BE}\,\hat{\mathcal{H}}^{CF}\bigr)\,\mathcal{F}_{ABC}\,\mathcal{F}_{DEF}\,, \qquad (3.16)\\
\mathcal{G}^{AB} &\equiv 2\,\hat{\mathcal{H}}^{D[A}\,\mathcal{D}^{B]}\mathcal{F}_D - \tfrac{1}{2}\,\hat{\mathcal{H}}^{DE}\,(\eta^{AF}\,\eta^{BG} - \hat{\mathcal{H}}^{AF}\,\hat{\mathcal{H}}^{BG})\,\bigl(\mathcal{F}_D - \mathcal{D}_D\bigr)\,\mathcal{F}_{EFG}\\
&\quad - \hat{\mathcal{H}}_E{}^{[A}\,\bigl(\mathcal{F}_D - \mathcal{D}_D\bigr)\,\mathcal{F}^{B]DE} + \tfrac{1}{2}\,\bigl(\eta^{CE}\,\eta^{DF} - \hat{\mathcal{H}}^{CE}\,\hat{\mathcal{H}}^{DF}\bigr)\,\hat{\mathcal{H}}^{G[A}\,\mathcal{F}_{CD}{}^{B]}\,\mathcal{F}_{EFG}\,. \qquad (3.17)
\end{aligned}$$

In our setup, we find $\mathcal{D}_D\mathcal{F}_{ABC} = \mathrm{e}^{2\Delta}\,Z_D\,F_{ABC}$ and $\mathcal{D}_D\mathcal{F}_A = \mathrm{e}^{2\Delta}\,Z_D\,F_A$, and we obtain

$$\mathcal{R} = \mathrm{e}^{2\Delta}\,R\,, \qquad \mathcal{G}^{AB} = \mathrm{e}^{2\Delta}\,G^{AB}\,, \tag{3.18}$$

where R and G^{AB} are constants of the form

$$\begin{aligned}
R &\equiv \hat{\mathcal{H}}^{AB}\,(2\,Z_A\,F_B - F_A\,F_B) + \tfrac{1}{12}\,\hat{\mathcal{H}}^{AD}\,(3\,\eta^{BE}\,\eta^{CF} - \hat{\mathcal{H}}^{BE}\,\hat{\mathcal{H}}^{CF})\,F_{ABC}\,F_{DEF}\,, \qquad (3.19)\\
G^{AB} &\equiv 2\,\hat{\mathcal{H}}^{D[A}\,F^{B]}\,F_D - \tfrac{1}{2}\,\hat{\mathcal{H}}^{DE}\,(\eta^{AF}\,\eta^{BG} - \hat{\mathcal{H}}^{AF}\,\hat{\mathcal{H}}^{BG})\,(F_D - Z_D)\,F_{EFG}\\
&\quad - \hat{\mathcal{H}}_E^{[A}\,(F_D - Z_D)\,F^{B]DE} + \tfrac{1}{2}\,(\eta^{CE}\,\eta^{DF} - \hat{\mathcal{H}}^{CE}\,\hat{\mathcal{H}}^{DF})\,\hat{\mathcal{H}}^{G[A}\,F_{CD}{}^{B]}\,F_{EFG}\,. \qquad (3.20)
\end{aligned}$$

Then the equations of motion, namely $R = 0$ and $G^{AB} = 0$, are manifestly covariant under the $\mathrm{O}(D,D)$ rotation

$$\begin{aligned}
F_{ABC} &\to C_A{}^D\,C_B{}^E\,C_C{}^F\,F_{DEF}\,, \qquad & Z_A &\to C_A{}^B\,Z_B\,,\\
\hat{\mathcal{H}}_{AB} &\to C_A{}^C\,C_B{}^D\,\hat{\mathcal{H}}_{CD}\,, & F_A &\to C_A{}^B\,F_B\,.
\end{aligned} \tag{3.21}$$

The transformations in the first line are equivalent to a redefinition of generators

$$T_A \to C_A{}^B\,T_B\,, \tag{3.22}$$

while those in the second line determine the transformation rules of $\hat{\mathcal{H}}_{AB}$ and φ . This $\mathrm{O}(D,D)$ symmetry is the Jacobi–Lie T-plurality and is a manifest symmetry of DFT. Since we are assuming $Z^a = 0$, the constant $\mathrm{O}(D,D)$ matrix $C_A{}^B$ needs to satisfy

$$C^{ab}\,Z_b = 0\,. \tag{3.23}$$

For later convenience, let us also find the transformation rule of the generalized Ricci tensor $\mathcal{S}_{MN}$. We define the double vielbein $V_{\mathcal{A}}{}^B \equiv (V_a{}^B,\,V_{\bar{a}}{}^B) \in \mathrm{O}(D,D)$ and its inverse $V_A{}^{\mathcal{B}}$ through

$$\hat{\mathcal{H}}_{AB} = V_A{}^{\mathcal{A}}\,V_B{}^{\mathcal{B}}\,\hat{\mathcal{H}}_{\mathcal{A}\mathcal{B}}\,, \qquad \eta_{AB} = V_A{}^{\mathcal{A}}\,V_B{}^{\mathcal{B}}\,\eta_{\mathcal{A}\mathcal{B}}\,, \qquad V_A{}^{\mathcal{C}}\,V_{\mathcal{C}}{}^B = \delta_A^B\,, \tag{3.24}$$

where

$$(\hat{\mathcal{H}}_{\mathcal{A}\mathcal{B}}) \equiv \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix}, \qquad (\eta_{\mathcal{A}\mathcal{B}}) \equiv \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix}, \tag{3.25}$$

and $\eta_{ab} \equiv \eta_{\bar{a}\bar{b}} \equiv \mathrm{diag}(-1, 1, \dots, 1)$. We suppose that the double vielbein is transformed as

$$V_A{}^{\mathcal{B}} \to C_A{}^C\, V_C{}^{\mathcal{B}}\,, \tag{3.26}$$

under the Jacobi–Lie T-duality, and then the transformation rule for

$$\mathcal{G}^{\mathcal{A}\mathcal{B}} \equiv V_A{}^{\mathcal{A}}\, V_B{}^{\mathcal{B}}\, \mathcal{G}^{AB}\,, \tag{3.27}$$

is found as

$$\mathrm{e}^{-2\Delta'}\, \mathcal{G}'^{\mathcal{A}\mathcal{B}} = \mathrm{e}^{-2\Delta}\, \mathcal{G}^{\mathcal{A}\mathcal{B}}\,. \tag{3.28}$$

We find that the only non-vanishing components of $\mathcal{G}^{\mathcal{A}\mathcal{B}}$ are $\mathcal{G}^{a\bar{b}}$, and using these, we can express the generalized Ricci tensor as

$$\mathcal{S}_{MN} = \bigl(\mathcal{E}_{MA}\, \mathcal{E}_{NB} + \mathcal{E}_{NA}\, \mathcal{E}_{MB}\bigr)\, V_c{}^A\, V_{\bar{d}}{}^B\, \mathcal{G}^{c\bar{d}}\,. \tag{3.29}$$

Then, using (3.28), we find the transformation rule of the generalized Ricci tensor $\mathcal{S}_{MN}$ as

$$\mathrm{e}^{-2\Delta'}\, \mathcal{E}'_A{}^M\, \mathcal{E}'_B{}^N\, \mathcal{S}'_{MN} = \mathrm{e}^{-2\Delta}\, C_A{}^C\, C_B{}^D\, \mathcal{E}_C{}^M\, \mathcal{E}_D{}^N\, \mathcal{S}_{MN}\,. \tag{3.30}$$

Namely, under the Jacobi–Lie T-plurality, or a local $\mathrm{O}(D,D)$ rotation of the generalized metric,

$$\mathcal{H}_{MN}(x) \to \mathcal{H}'_{MN}(x') = \bigl[h\, \mathcal{H}(x)\, h^{\mathrm{t}}\bigr]_{MN}\,, \qquad h_M{}^N \equiv \mathcal{E}'_M{}^A(x')\, C_A{}^B\, \mathcal{E}_B{}^N(x)\,, \tag{3.31}$$

the generalized Ricci tensor transforms as

$$\mathcal{S}_{MN}(x) \to \mathcal{S}'_{MN}(x') = \mathrm{e}^{2\,(\Delta'-\Delta)}\bigl[h\, \mathcal{S}(x)\, h^{\mathrm{t}}\bigr]_{MN}\,. \tag{3.32}$$

Unlike the case of the Poisson–Lie T-duality, the generalized Ricci tensor is transformed by a local $\mathrm{O}(D,D) \times \mathbb{R}^+$ rotation. As we discuss below, this additional $\mathbb{R}^+$ transformation makes the transformation rule of the R–R fields slightly non-trivial. Before considering the R–R fields, let us provide several examples.

3.1 An example without Ramond–Ramond flux

Let us consider a eight-dimensional Leibniz algebra with

$$f_{12}{}^2 = -1\,, \quad f_{12}{}^3 = 1\,, \quad f_{13}{}^3 = -1\,, \quad Z_1 = -2\,, \tag{3.33}$$

which is a direct sum of the six-dimensional Leibniz algebra $((\mathrm{IV}, -4\tilde{X}^1), (\mathrm{I}, 0))$ of [25] and a two-dimensional Abelian algebra. Using a parameterization $g = \mathrm{e}^{x\,T_1}\,\mathrm{e}^{y\,T_2}\,\mathrm{e}^{z\,T_3}\,\mathrm{e}^{w\,T_4}$, we find

$$\begin{aligned} &v_1 = \partial_x + y\,\partial_y + (z-y)\,\partial_z\,, \quad v_2 = \partial_y\,, \quad v_3 = \partial_z\,, \quad v_4 = \partial_w\,, \\ &e_1 = \partial_x\,, \quad e_2 = \mathrm{e}^x(\partial_y - x\,\partial_z)\,, \quad e_3 = \mathrm{e}^x\,\partial_z\,, \quad e_4 = \partial_w\,. \end{aligned} \tag{3.34}$$

Computing the matrix $M_A{}^B$, we find

$$\pi^{ab} = 0\,, \qquad \Delta = -2\,x\,. \tag{3.35}$$

Then, using the constant matrices

$$\hat{g}_{ab} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \hat{\beta}^{ab} = 0\,, \tag{3.36}$$

we obtain a 4D metric

$$\mathrm{d}s^2 = 2\,\mathrm{e}^{3x}\,\mathrm{d}x\,(\mathrm{d}z + x\,\mathrm{d}y) + \mathrm{e}^{2x}\,\mathrm{d}y^2 + \mathrm{e}^{4x}\,\mathrm{d}w^2\,. \tag{3.37}$$

In order to find a solution of DFT, we choose the function φ as

$$\varphi = -\tfrac{4}{3}\,x\,, \tag{3.38}$$

which yields

$$F_A = \bigl(-\tfrac{8}{3}, 0, 0, 0, 0, 0, 0, 0\bigr)\,. \tag{3.39}$$

Then the DFT dilaton and the standard dilaton become

$$\mathrm{e}^{-2\,d} = \mathrm{e}^{\frac{14x}{3}}\,, \qquad \mathrm{e}^{-2\,\Phi} = \mathrm{e}^{-\frac{4x}{3}}\,. \tag{3.40}$$

We can check that this dilaton and the metric (3.37) satisfy the equations of motion. In the following, we consider the Jacobi–Lie T-pluralities of this solution.

3.1.1 Generalized Yang–Baxter deformation

Let us perform an $\mathrm{O}(4,4)$ rotation $T_A \to C_A{}^B\,T_B$ with

$$C_A{}^B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 1 & 0 & 0 \\ 0 & -c & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.41}$$

which corresponds to a generalized Yang–Baxter deformation. After this rotation, the structure constants becomes

$$f_{12}{}^2 = -1\,,\quad f_{12}{}^3 = 1\,,\quad f_{13}{}^3 = -1\,,\quad f_1{}^{23} = 2\,c\,,\quad Z_1 = -2\,, \tag{3.42}$$

and this corresponds $((\mathrm{IV}, -4\tilde{X}^1), (\mathrm{II}, 0))$ or $((\mathrm{IV.iii}, 4\tilde{X}^1), (\mathrm{II}, 0))$ of [25] (accompanied by the two-dimensional Abelian algebra), for $c = 1/2$ or $c = -1/2$, respectively.[4]

Again we employ the same parametrization of the group element and the left-/right-invariant vector fields (3.34). Here, we find the Jacobi–Lie structure as

$$\pi = c\,(1 - \mathrm{e}^{-2x})\,\partial_y \wedge \partial_z\,, \tag{3.43}$$

and φ is not changed because $\mathcal{F}_A$ is not deformed under this $\mathrm{O}(4,4)$ rotation: $\mathcal{F}_A = C_A{}^B\,\mathcal{F}_B$. Then, we find the deformed supergravity fields as

$$\begin{aligned} \mathrm{d}s^2 &= 2\,\mathrm{e}^{3x}\,\mathrm{d}x\,(\mathrm{d}z + x\,\mathrm{d}y) + \mathrm{e}^{2x}\,\mathrm{d}y^2 + \mathrm{e}^{4x}\,\mathrm{d}w^2 + \mathrm{e}^{2x}\,c^2\,\mathrm{d}z^2\,, \\ B_2 &= c\,\mathrm{e}^{5x}\,\mathrm{d}x \wedge \mathrm{d}y\,, \qquad \mathrm{e}^{-2\,\Phi} = \mathrm{e}^{-\frac{4x}{3}}\,. \end{aligned} \tag{3.44}$$

This is again a supergravity solution for an arbitrary value of c .

3.1.2 Another Jacobi–Lie T-plurality

Here we consider another $\mathrm{O}(4,4)$ transformation

$$C_A{}^B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.45}$$

We then obtain the algebra with

$$f_{12}{}^2 = -2\,,\quad f_{12}{}^3 = -1\,,\quad f_{13}{}^2 = -1\,,\quad f_{13}{}^3 = -2\,,\quad f_1{}^{23} = 1\,,\quad Z_1 = -2\,. \tag{3.46}$$

The six-dimensional part of this algebra is known as $((\mathrm{VI}_2, -4\,\tilde{X}^1), (\mathrm{II}, 0))$. Using the parameterization, $g = \mathrm{e}^{x\,T_1}\,\mathrm{e}^{y\,T_2}\,\mathrm{e}^{z\,T_3}\,\mathrm{e}^{w\,T_4}$, we obtain

$$\begin{aligned} &v_1 = \partial_x + (2y + z)\,(\partial_y + \partial_z)\,,\quad v_2 = \partial_y\,,\quad v_3 = \partial_z\,,\quad v_4 = e_4 = \partial_w\,, \\ &e_1 = \partial_x\,,\quad e_2 = \tfrac{\mathrm{e}^x}{2}\big[(\mathrm{e}^{2x} + 1)\,\partial_y + (\mathrm{e}^{2x} - 1)\,\partial_z\big]\,,\quad e_3 = \tfrac{\mathrm{e}^x}{2}\big[(\mathrm{e}^{2x} - 1)\,\partial_y + (\mathrm{e}^{2x} + 1)\,\partial_z\big]\,. \end{aligned} \tag{3.47}$$

[4]The algebra with $c > 0$ or $c < 0$ can be mapped to to the one with $c = 1/2$ or $c = -1/2$, respectively.

We can compute several quantities as

$$\pi = x\,\partial_y \wedge \partial_z\,, \qquad \Delta = -2\,x\,, \qquad \varphi = -\tfrac{4}{3}\,x\,. \tag{3.48}$$

The associated supergravity fields are found as

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{e}^{4x}(\mathrm{d}w^2 - x^2\,\mathrm{d}x^2) - 2\,\mathrm{e}^{3x}\,\mathrm{d}x\,(\mathrm{d}y - \mathrm{d}z) + \tfrac{1}{4}\,\mathrm{e}^{-2x}(\mathrm{d}y + \mathrm{d}z)^2\,, \\ B_2 &= \tfrac{1}{2}\,\mathrm{e}^{x}\,x\,\mathrm{d}x \wedge (\mathrm{d}y + \mathrm{d}z)\,, \qquad \mathrm{e}^{-2\,\Phi} = \mathrm{e}^{\frac{2x}{3}}\,, \end{aligned} \tag{3.49}$$

and this is a solution of the supergravity.

3.2 Ramond–Ramond fields

We here introduce the R–R fields by considering the case $D = 10$. In the presence of the R–R fields, the equations of motion for the generalized metric and the DFT dilaton become

$$\mathcal{R} = 0\,, \qquad \mathcal{S}_{MN} = \mathcal{E}_{MN}\,, \tag{3.50}$$

where $\mathcal{E}_{MN}$ denotes the energy-momentum tensor of the R–R fields. Obviously, if we transform the energy-momentum tensor as

$$\mathrm{e}^{-2\Delta}\,\mathcal{E}_A{}^M\,\mathcal{E}_B{}^N\,\mathcal{E}_{MN} = \mathrm{e}^{-2\Delta'}\,\mathcal{E}'_A{}^M\,\mathcal{E}'_B{}^N\,\mathcal{E}'_{MN}\,, \tag{3.51}$$

the equations of motion for the generalized metric transform covariantly as

$$\mathrm{e}^{-2\Delta}\,\mathcal{E}_A{}^M\,\mathcal{E}_B{}^N\,\big(\mathcal{S}_{MN} - \mathcal{E}_{MN}\big) = \mathrm{e}^{-2\Delta'}\,\mathcal{E}'_A{}^M\,\mathcal{E}'_B{}^N\,\big(\mathcal{S}'_{MN} - \mathcal{E}'_{MN}\big)\,. \tag{3.52}$$

By using the results of the Poisson–Lie T-duality [20–22], we can easily see that the transformation rule (3.51) can be realized by using the ansatz

$$|F\rangle = \mathrm{e}^{-d(x)}\,\mathrm{e}^{\Delta(x)}\,S_U|\hat{\mathcal{F}}\rangle\,, \tag{3.53}$$

where $U \equiv (\mathcal{E}_M{}^A)$ and S_U is a matrix representation of U in the spinor representation (see [22] for our convention). The presence of $\mathrm{e}^{\Delta(x)}$ is the only difference from the Poisson–Lie T-duality. The $\mathrm{O}(10,10)$ spinor $|\hat{\mathcal{F}}\rangle$ is constant, and in type IIA/IIB theory, it can be expanded as

$$|\hat{\mathcal{F}}\rangle = \sum_{p\,:\,\text{even/odd}} \frac{1}{p!}\,\hat{\mathcal{F}}_{a_1\cdots a_p}\,\Gamma^{a_1\cdots a_p}|0\rangle\,, \tag{3.54}$$

where $|0\rangle$ is the Clifford vacuum satisfying $\Gamma_a|0\rangle = 0$. Under the ansatz (3.53), the equations of motion of the R–R fields become the algebraic relation

$$\big(\tfrac{1}{3!}\,\Gamma^{ABC}\,F_{ABC} - \tfrac{1}{2}\,\Gamma^A\,F_A + \Gamma^A\,Z_A\big)\,|\hat{\mathcal{F}}\rangle = 0\,. \tag{3.55}$$

When we consider an $\mathrm{O}(D,D)$ rotation (3.21), by rotating the constant spinor $|\hat{\mathcal{F}}\rangle$ also as

$$|\hat{\mathcal{F}}\rangle \to S_C\,|\hat{\mathcal{F}}\rangle \qquad \big(\Gamma^B\,C_B{}^A = S_C\,\Gamma^A\,S_C^{-1}\big)\,, \tag{3.56}$$

the equations of motion are manifestly covariant. This shows that the whole DFT equations of motion are covariant under the Jacobi–Lie T-plurality. When the supergravity fields have the form (3.2) and (3.53), we call the background the Jacobi–Lie symmetric.

For convenience, let us also express (3.53) in terms of the differential form. By using a polyform

$$F \equiv \sum_{p\,:\,\text{even/odd}} \frac{1}{p!}\, F_{m_1\cdots m_p}\, \mathrm{d}x^{m_1} \wedge \cdots \wedge \mathrm{d}x^{m_p}\,, \tag{3.57}$$

in type IIA/IIB theory, we have

$$F = \mathrm{e}^{-\varphi(x)}\, \mathrm{e}^{-(p-\frac{D+1}{2})\,\Delta(x)} |\det a_a{}^b|^{\frac{1}{2}}\, \mathrm{e}^{\frac{1}{2}\pi^{ab}\iota_a\iota_b} \left[\sum_{p\,:\,\text{even/odd}} \tfrac{1}{p!}\, F_{a_1\cdots a_p}\, r^{a_1} \wedge \cdots \wedge r^{a_p}\right]. \tag{3.58}$$

Note that here we are using the field strength in the A-basis (which satisfies $\mathrm{d}F = 0$) and this is related to the one in C-basis as

$$G = \mathrm{e}^{-B_2\wedge}\, F\,, \tag{3.59}$$

that satisfies the Bianchi identity

$$\mathrm{d}G + H_3 \wedge G = 0\,. \tag{3.60}$$

3.3 An example with Ramond–Ramond fluxes

Let us consider a 20-dimensional DD$^+$ with the structure constants

$$f_{12}{}^2 = -1\,, \quad f_{12}{}^3 = -1\,, \quad f_{13}{}^2 = -1\,, \quad f_{13}{}^3 = -1\,, \quad Z_1 = -2\,. \tag{3.61}$$

The non-trivial subalgebra generated by $\{T_1,\, T_2,\, T_3\}$ are known as $((\mathrm{III}, -4\tilde{X}^1),\, (\mathrm{I}, 0))$. Using the parameterization $g = \mathrm{e}^{x\,T_1}\, \mathrm{e}^{y\,T_2}\, \mathrm{e}^{z\,T_3}\, \mathrm{e}^{w_4\,T_4} \cdots \mathrm{e}^{w_{10}\,T_{10}}$, the non-trivial part of v_a^m and e_a^m are found as (the other components are just $v_a = e_a = \partial_a$)

$$\begin{aligned} &v_1 = \partial_x + (y+z)\,(\partial_y + \partial_z)\,, \quad v_2 = \partial_y\,, \quad v_3 = \partial_z\,, \\ &e_1 = \partial_x\,, \quad e_2 = \tfrac{1}{2}\left[(\mathrm{e}^{2x}+1)\,\partial_y + (\mathrm{e}^{2x}-1)\,\partial_z\right], \quad e_3 = \tfrac{1}{2}\left[(\mathrm{e}^{2x}-1)\,\partial_y + (\mathrm{e}^{2x}+1)\,\partial_z\right]. \end{aligned} \tag{3.62}$$

We introduce constants

$$\hat{g}_{ab} = \begin{pmatrix} 1 & 1 & 0 & 0 & & 0 \\ 1 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & & 1 \end{pmatrix}, \quad \hat{\beta}^{ab} = 0\,, \quad |\hat{\mathcal{F}}\rangle = 6\sqrt{2}\,\Gamma^1\left[(\Gamma^2+\Gamma^3)\Gamma^{4\cdots10} - 1\right]|0\rangle\,, \tag{3.63}$$

and then, by using $\Delta = -2x$ and supposing $\varphi = 0$, the supergravity fields are found as

$$
\begin{aligned}
&\mathrm{d}s^2 = \mathrm{e}^{4x}\big[\mathrm{d}x^2 + \mathrm{d}x\,(\mathrm{d}y - \mathrm{d}z) + \mathrm{d}s^2_{T^7}\big] + \mathrm{e}^{2x}\,\mathrm{d}x\,(\mathrm{d}y + \mathrm{d}z) + (\mathrm{d}y + \mathrm{d}z)^2\,,\\
&B_2 = 0\,,\qquad \mathrm{e}^{-2\,\Phi} = \mathrm{e}^{-16\,x},\qquad F_1 = -6\sqrt{2}\,\mathrm{e}^{-8\,x}\,\mathrm{d}x\,,
\end{aligned}
\tag{3.64}
$$

where $\mathrm{d}s^2_{T^7} \equiv \mathrm{d}w_4^2 + \cdots + \mathrm{d}w_{10}^2$ is a seven-dimensional flat metric. This is a solution of type IIB* supergravity.

Now we consider a generalized Yang–Baxter deformation with

$$
r^{23} = \frac{\eta}{2}\,. \tag{3.65}
$$

The resulting DD$^+$ has the structure constants

$$
f_{12}{}^2 = -1\,,\quad f_{12}{}^3 = -1\,,\quad f_{13}{}^2 = -1\,,\quad f_{13}{}^3 = -1\,,\quad f_1{}^{23} = \eta\,,\quad Z_1 = -2\,. \tag{3.66}
$$

The structure constants $f_1{}^{23}$ produces the Jacobi–Lie structure $\pi = \frac{\eta}{2}\,(1 - \mathrm{e}^{-2x})\,\partial_y \wedge \partial_u$ and the supergravity fields are

$$
\begin{aligned}
&\mathrm{d}s^2 = \mathrm{e}^{4x}\big[\mathrm{d}x^2 + \mathrm{d}x\,(\mathrm{d}y - \mathrm{d}z) + \mathrm{d}s^2_{T^7}\big] + \mathrm{e}^{2x}\,\mathrm{d}x\,(\mathrm{d}y + \mathrm{d}z) + (\mathrm{d}y + \mathrm{d}z)^2 - \tfrac{\eta^2}{4}\,\mathrm{e}^{4x}\,\mathrm{d}x^2\,,\\
&B_2 = -\tfrac{\eta}{2}\,\mathrm{e}^{2x}\,\mathrm{d}x \wedge (\mathrm{d}y + \mathrm{d}z)\,,\qquad \mathrm{e}^{-2\,\Phi} = \mathrm{e}^{-16\,x},\qquad F_1 = -6\sqrt{2}\,\mathrm{e}^{-8\,x}\,\mathrm{d}x\,.
\end{aligned}
\tag{3.67}
$$

This is again a solution of type IIB* supergravity and the Jacobi–Lie T-duality indeed works as a solution generating technique.

3.4 Jacobi–Lie T-plurality with spectator fields

The inclusion of the spectator fields is straightforwardly similar to the case of the Poisson–Lie T-duality/T-plurality (see Appendix B of [22]). Here, instead of repeating the presentation of [22], we only comment on some non-trivialities that are specific to the Jacobi–Lie T-plurality.

We consider a ten-dimensional spacetime with the "internal coordinates" x^m $(m = 1, \dots, D)$ and the "external coordinates" y^μ $(\mu = D + 1, \dots, 10)$. In the string sigma model, the scalar fields $y^\mu(\sigma)$ are called the spectator fields because they are invariant under the non-Abelian duality. We formally double all of the directions, and the generalized coordinates are given by $x^M = (x^m, \tilde{x}_m, y^\mu, \tilde{y}_\mu)$. The "flat" indices A, B and $\mathcal{A}, \mathcal{B}$ also run over the 20 directions. The underlying algebra DD$^+$ is associated with the $2D$-dimensional doubled coordinates $\{x^m, \tilde{x}_m\}$, and for example, the generalized frame fields constructed in the previous sections are embedded into the first $2D \times 2D$-block of the 20×20 matrix $\mathcal{E}_A{}^M$. We assume that $\mathcal{E}_A{}^M$ and the double vielbein $V_{\mathcal{A}}{}^{\mathcal{B}}$ have block-diagonal forms, i.e., they are given by direct sums of the $2D \times 2D$-block associated with the internal directions and $(20 - 2D) \times (20 - 2D)$-blocks associated with the external directions. In particular, we suppose that the external block of $\mathcal{E}_A{}^M$

is an identity matrix. With such understanding, the conditions for the Jacobi–Lie symmetry in the presence of the spectator fields, but without the R–R fields, are given by

$$\mathcal{H}_{MN} = \mathcal{E}_M{}^A(x)\,\mathcal{E}_N{}^B(x)\,\hat{\mathcal{H}}_{AB}(y)\,, \qquad \hat{\mathcal{H}}_{AB}(y) \equiv V_A{}^{\mathcal{A}}(y)\,V_B{}^{\mathcal{B}}(y)\,\hat{\mathcal{H}}_{\mathcal{A}\mathcal{B}}\,, \tag{3.68}$$

$$\mathrm{e}^{-2d} = \mathrm{e}^{-2\hat{d}(y)}\,\mathrm{e}^{-2\mathsf{d}(x)}, \qquad \mathrm{e}^{-2\mathsf{d}(x)} \equiv \mathrm{e}^{-2\varphi(x)}\,\mathrm{e}^{-\Delta(x)}|\det \ell_m^a|\,. \tag{3.69}$$

The difference is that $V_A{}^{\mathcal{A}}(y)$ is no longer constant and that the dilaton also acquires the y-dependence $\hat{d}(y)$. By following the same discussion as [22], we can show that the $\mathrm{O}(D,D)$ transformation which rotates the internal indices is a symmetry of the equation of motion.

When the R–R fields are also present, the symmetry becomes slightly subtle. In the presence of the spectator fields, the tensor $\mathcal{G}^{\mathcal{A}\mathcal{B}}$ becomes

$$\begin{aligned}\mathcal{G}^{\mathcal{A}\mathcal{B}} &\equiv 2\,\hat{\mathcal{H}}^{\mathcal{D}[\mathcal{A}}\,\mathcal{D}^{\mathcal{B}]}\mathcal{F}_{\mathcal{D}} - \tfrac{1}{2}\,\hat{\mathcal{H}}^{\mathcal{D}\mathcal{E}}\,\big(\eta^{\mathcal{A}\mathcal{F}}\,\eta^{\mathcal{B}\mathcal{G}} - \hat{\mathcal{H}}^{\mathcal{A}\mathcal{F}}\,\hat{\mathcal{H}}^{\mathcal{B}\mathcal{G}}\big)\,\big(\mathcal{F}_{\mathcal{D}} - \mathcal{D}_{\mathcal{D}}\big)\,\mathcal{F}_{\mathcal{E}\mathcal{F}\mathcal{G}} \\ &\quad - \hat{\mathcal{H}}_{\mathcal{E}}{}^{[\mathcal{A}}\,\big(\mathcal{F}_{\mathcal{D}} - \mathcal{D}_{\mathcal{D}}\big)\,\mathcal{F}^{\mathcal{B}]\mathcal{D}\mathcal{E}} + \tfrac{1}{2}\,\big(\eta^{\mathcal{C}\mathcal{E}}\,\eta^{\mathcal{D}\mathcal{F}} - \hat{\mathcal{H}}^{\mathcal{C}\mathcal{E}}\,\hat{\mathcal{H}}^{\mathcal{D}\mathcal{F}}\big)\,\hat{\mathcal{H}}^{\mathcal{G}[\mathcal{A}}\,\mathcal{F}_{\mathcal{C}\mathcal{D}}{}^{\mathcal{B}]}\,\mathcal{F}_{\mathcal{E}\mathcal{F}\mathcal{G}}\,,\end{aligned} \tag{3.70}$$

where $\mathcal{D}_{\mathcal{A}} \equiv V_{\mathcal{A}}{}^B\,\mathcal{E}_B{}^M\,\partial_M$ and the fluxes contain both the external and internal parts:

$$\mathcal{F}_{\mathcal{A}} = \hat{\mathcal{F}}(y) + \mathrm{e}^{\Delta(x)}\,V_{\mathcal{A}}{}^B(y)\,\mathcal{F}_B\,, \tag{3.71}$$

$$\mathcal{F}_{\mathcal{A}\mathcal{B}\mathcal{C}} = \hat{\mathcal{F}}_{\mathcal{A}\mathcal{B}\mathcal{C}}(y) + \mathrm{e}^{\Delta(x)}\,V_{\mathcal{A}}{}^D(y)\,V_{\mathcal{B}}{}^E(y)\,V_{\mathcal{C}}{}^F(y)\,F_{DEF}\,. \tag{3.72}$$

The internal/external parts contribute to the internal/external components of the matrix $\mathcal{G}^{\mathcal{A}\mathcal{B}}$, respectively. Then, the internal components of $\mathcal{G}^{\mathcal{A}\mathcal{B}}$ (or $\mathcal{S}_{MN}$) scale as $\mathrm{e}^{2\Delta}$ while the external components are independent of Δ. In order to realize the equations of motion $\mathcal{S}_{MN} = \mathcal{E}_{MN}$, the energy-momentum tensor $\mathcal{E}_{MN}$ also should scale in the same way, but it is non-trivial.

Then we can consider two possibilities: (i) the external components of $\mathcal{S}_{MN}$ vanish, or (ii) the internal components of $\mathcal{S}_{MN}$ vanish by themselves. The former is the case studied in the previous sections. In that case, we choose the R–R fields as

$$|F\rangle = \mathrm{e}^{-\mathsf{d}(x)}\,\mathrm{e}^{\Delta(x)}\,S_U|\hat{\mathcal{F}}(y)\rangle\,, \tag{3.73}$$

which is a natural extension of (3.53) including the y-dependence into $|\hat{\mathcal{F}}\rangle$. In the latter case, the scale factor $\mathrm{e}^{\Delta(x)}$ is not necessary and we consider

$$|F\rangle = \mathrm{e}^{-\mathsf{d}(x)}\,S_U|\hat{\mathcal{F}}(y)\rangle\,. \tag{3.74}$$

In terms of the differential form, this can be expressed as

$$F = \mathrm{e}^{-\varphi(x)}\,\mathrm{e}^{\frac{D-1}{2}\,\Delta(x)}|\det a_a{}^b|^{\frac{1}{2}}\,\mathrm{e}^{\frac{1}{2}\pi^{ab}\iota_a\iota_b}\left[\sum_{p:\text{even/odd}}\tfrac{1}{p!}\,F_{\hat{a}_1\cdots\hat{a}_p}(y)\,\mathcal{E}^{\hat{a}_1}\wedge\cdots\wedge\mathcal{E}^{\hat{a}_p}\right], \tag{3.75}$$

where we have defined $\mathcal{E}^{\hat{a}} \equiv \mathcal{E}^{\hat{a}}{}_{\hat{m}}\,\mathrm{d}x^{\hat{m}}$ with $x^{\hat{m}} \equiv (x^m,\, y^\mu)$ and $\{\hat{a}\} = \{a,\, \dot{\mu}\}$. Here, the dotted indices $\{\dot{\mu}\}$ denote the "flat" indices associated with $\{\mu\}$ and $\mathcal{E}^{\hat{a}}{}_{\hat{m}}$ is a component of $\mathcal{E}_A{}^M$.

The existence of the two options are specific to the Jacobi–Lie T-plurality, and these two are degenerate in the case of the Poisson–Lie T-duality (where $\Delta = 0$). In the next subsection, we present an example using the latter option (3.74).

3.5 An example with spectator fields

We consider an eight-dimensional DD^+ $(D=4)$ with the structure constants given in Eq. (3.61). We introduce the ten-dimensional coordinates

$$\{x^m;\, y^\mu\} = \{x, y, u, v\,;\, z, r, \xi, \phi_1, \phi_2, \phi_3\}\,, \tag{3.76}$$

and y^μ are the spectator fields. Using the parameterization $g = \mathrm{e}^{x\,T_1}\,\mathrm{e}^{y\,T_2}\,\mathrm{e}^{u\,T_3}\,\mathrm{e}^{v\,T_4}$, we obtain the left-/right-invariant vector fields as given in Eq. (3.62). We choose the metric $\hat{g}_{ab}(y)$, dilaton $\hat{d}(y)$, the R–R field $|\hat{\mathcal{F}}(y)\rangle$, and $\varphi(x)$ as

$$\begin{gathered}
\hat{g}_{ab} = \left(\begin{array}{ccccc|c}
\frac{1}{z^2} & \frac{1}{z^2} & 0 & 0 & 0 & \\
\frac{1}{z^2} & \frac{1}{z^2} & -\frac{1}{z^2} & 0 & 0 & \\
0 & -\frac{1}{z^2} & \frac{1}{z^2} & 0 & 0 & 0_{5\times5} \\
0 & 0 & 0 & \frac{1}{z^2} & 0 & \\
0 & 0 & 0 & 0 & \frac{1}{z^2} & \\ \hline
 & & 0_{5\times5} & & & g_{\mathrm{S}^5}
\end{array}\right), \quad \hat{\beta}^{ab} = 0\,, \quad \mathrm{e}^{-2\hat{d}} = \frac{\cos r\cos\xi\sin^3 r\sin\xi}{z^5}\,, \\
|\hat{\mathcal{F}}\rangle = 4\left(-z^{-5}\,\Gamma^{\dot{u}\dot{v}\dot{x}\dot{y}\dot{z}} + \sin^3 r\cos r\sin\xi\cos\xi\,\Gamma^{\dot{r}\dot{\xi}\dot{\phi}_1\dot{\phi}_2\dot{\phi}_3}\right), \qquad \varphi = -2\,x\,,
\end{gathered} \tag{3.77}$$

where the metric g_{S^5} on S^5 corresponds to the line element

$$\mathrm{d}s^2_{\mathrm{S}^5} \equiv \mathrm{d}r^2 + \sin^2 r\left(\mathrm{d}\xi^2 + \cos^2\xi\,\mathrm{d}\phi_1^2 + \sin^2\xi\,\mathrm{d}\phi_2^2\right) + \cos^2 r\,\mathrm{d}\phi_3^2\,. \tag{3.78}$$

Using $\pi^{ab} = 0$ and $\Delta = -2\,x$, the generalized frame fields become

$$\mathcal{E}_A{}^M(x) = \left(\begin{array}{cccccccc|c}
\mathrm{e}^{-2x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & \mathrm{e}^{-x}\cosh x & \mathrm{e}^{-x}\sinh x & 0 & 0 & 0 & 0 & 0 & \\
0 & \mathrm{e}^{-x}\sinh x & \mathrm{e}^{-x}\cosh x & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & \mathrm{e}^{-2x} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathrm{e}^{2x} & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \mathrm{e}^{x}\cosh x & -\mathrm{e}^{x}\sinh x & 0 & \\
0 & 0 & 0 & 0 & 0 & -\mathrm{e}^{x}\sinh x & \mathrm{e}^{x}\cosh x & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{2x} & \\ \hline
 & & & 0 & & & & & \mathbf{1}_{12\times12}
\end{array}\right). \tag{3.79}$$

By acting the twist, we find that this is the $\mathrm{AdS}_5\times\mathrm{S}^5$ solution of type IIB supergravity,

$$\begin{aligned}
&\mathrm{d}s^2_{\mathrm{AdS}_5\times\mathrm{S}^5} = z^{-2}\left(\mathrm{d}s^2_{4\mathrm{D}} + \mathrm{d}z^2\right) + \mathrm{d}s^2_{\mathrm{S}^5}\,, \qquad B_2 = 0\,, \qquad \Phi = 0\,, \\
&\mathrm{d}s^2_{4\mathrm{D}} \equiv \mathrm{e}^{4x}\left[\mathrm{d}x^2 + \mathrm{d}x\,\mathrm{d}y + \mathrm{d}y^2 + \mathrm{d}u^2 - \mathrm{d}u\,(\mathrm{d}x + 2\,\mathrm{d}y) + \mathrm{d}v^2\right] + \mathrm{e}^{2x}\,\mathrm{d}x\,(\mathrm{d}u + \mathrm{d}y)\,, \\
&F = 4\left[-\frac{\mathrm{e}^{6x}\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}u\wedge\mathrm{d}v\wedge\mathrm{d}z}{z^5} + \sin^3 r\cos r\sin\xi\cos\xi\,\mathrm{d}r\wedge\mathrm{d}\xi\wedge\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2\wedge\mathrm{d}\phi_3\right].
\end{aligned} \tag{3.80}$$

Here we have used $\mathrm{e}^{-\varphi(x)}\,\mathrm{e}^{\frac{D-1}{2}\,\Delta(x)}|\det a_a{}^b|^{\frac{1}{2}} = 1$ (where $D=4$), and

$$\mathcal{E}^1\wedge\cdots\wedge\mathcal{E}^4\wedge\mathcal{E}^{\dot{z}} = \mathrm{e}^{6x}\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}u\wedge\mathrm{d}v\wedge\mathrm{d}z\,. \tag{3.81}$$

Again we perform a generalized Yang–Baxter deformation (3.65) and obtain the DD^+ given in Eq. (3.66). The Δ is not changed and the Jacobi–Lie structure is $\pi = \frac{\eta}{2}\,(1-\mathrm{e}^{-2x})\,\partial_y\wedge\partial_u$.

The deformed geometry is

$$
\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}s^2_{\mathrm{AdS}_5\times\mathrm{S}^5} - \frac{\eta^2\,\mathrm{e}^{4x}(2\,\mathrm{e}^{2x}-1)^2\mathrm{d}x^2}{4\,z^6}\,, \qquad B_2 = \frac{\eta\,\big(\mathrm{e}^{6x}-\frac{1}{2}\,\mathrm{e}^{4x}\big)}{z^4}\,\mathrm{d}x\wedge(\mathrm{d}y-\mathrm{d}u)\,,\\
\Phi &= 0\,, \qquad G_3 = \frac{2\,\eta\,\mathrm{e}^{5x}\,(\cosh x + 3\sinh x)\,\mathrm{d}x\wedge\mathrm{d}v\wedge\mathrm{d}z}{z^5}\,,\\
G_5 &= 4\,\Big[-\frac{\mathrm{e}^{6x}\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}u\wedge\mathrm{d}v\wedge\mathrm{d}z}{z^5} + \sin^3 r\cos r\sin\xi\cos\xi\,\mathrm{d}r\wedge\mathrm{d}\xi\wedge\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2\wedge\mathrm{d}\phi_3\Big]\,.
\end{aligned}
\tag{3.82}
$$

This also satisfies the type IIB supergravity equations of motion.

In order to perform more interesting Jacobi–Lie T-plurality, the classification of the six-dimensional DD^+ will be useful. The classification of the Jacobi–Lie bialgebra has been done in [25] but which bialgebras are in the same orbit $\mathrm{O}(D,D)$ rotations have not been studied. If such a classification is worked out, we may find more dual geometries from the $\mathrm{AdS}_5\times\mathrm{S}^5$ solution (3.80).

4 Jacobi–Lie T-plurality in string theory

In the string sigma model, we can clearly see the symmetry of the Poisson–Lie T-duality by using a formulation called the $\mathcal{E}$-model [38]. The $\mathcal{E}$-model is defined by a Hamiltonian

$$
H = \frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\,\hat{\mathcal{H}}^{AB}\,j_A(\sigma)\,j_B(\sigma)\,,
\tag{4.1}
$$

and the current algebra

$$
\{j_A(\sigma)\,,j_B(\sigma')\} = F_{AB}{}^{C}\,j_C(\sigma) + \eta_{AB}\,\delta'(\sigma-\sigma')\,,
\tag{4.2}
$$

where $\hat{\mathcal{H}}^{AB}$ is a constant $\mathrm{O}(D,D)$ matrix, $F_{AB}{}^{C}$ is a certain structure constant, and η_{AB} is the $\mathrm{O}(D,D)$-invariant metric. The dynamics is governed by the $\mathrm{O}(D,D)$-manifest equations (4.1) and (4.2), and the time evolution of the currents can be determined by $\partial_\tau j_A = \{j_A,\,H\}$. If we consider string theory on a target space with the generalized metric

$$
\mathcal{H}_{MN} = E_M{}^{A}\,E_N{}^{B}\,\hat{\mathcal{H}}_{AB}\,,
\tag{4.3}
$$

where $\hat{\mathcal{H}}_{AB}\in\mathrm{O}(D,D)$ are certain constants and $E_A{}^{M}$ are the generalized frame fields satisfying $\hat{\mathcal{L}}_{E_A}E_B = -F_{AB}{}^{C}\,E_C$ with $F_{AB}{}^{C}$ the structure constants of a Drinfel'd double, the string equations of motion can be expressed as Eqs. (4.1) and (4.2). Here, the current is given by

$$
j_A(\sigma) = E_A{}^{M}(x(\sigma))\,Z_M(\sigma)\,, \qquad Z_M(\sigma) \equiv \begin{pmatrix} p_m(\sigma) \\ \partial_\sigma x^m(\sigma) \end{pmatrix}\,,
\tag{4.4}
$$

where p_m are the canonical momenta associated with x^m. Then we can see the covariance of the string equations of motion under the Poisson–Lie T-duality/T-plurality.

Now let us consider the case of the Jacobi–Lie T-plurality by assuming $Z^a = 0$. Here, the generalized metric is expressed as

$$\mathcal{H}_{MN} = \mathcal{E}_M{}^A\, \mathcal{E}_N{}^B\, \hat{\mathcal{H}}_{AB}\,, \tag{4.5}$$

where $\mathcal{E}_M{}^A$ satisfies

$$\hat{\pounds}_{\mathcal{E}_A} \mathcal{E}_B{}^M = -\,\mathrm{e}^{\Delta}\big(X_{AB}{}^C - 2\, Z_{[A}\, \delta^C_{B]} - \eta_{AB}\, Z^C\big)\, \mathcal{E}_C{}^M = -\,\mathrm{e}^{\Delta}\, F_{AB}{}^C\, \mathcal{E}_C{}^M\,, \tag{4.6}$$

and $F_{AB}{}^C$ is the one given in (2.5) with $Z^a = 0$. Then introducing the currents

$$\mathcal{J}_A(\sigma) \equiv \mathcal{E}_A{}^M(x(\sigma))\, Z_M(\sigma)\,, \tag{4.7}$$

we obtain the Hamiltonian and the current algebra as

$$H = \frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma\, \hat{\mathcal{H}}^{AB}\, \mathcal{J}_A(\sigma)\, \mathcal{J}_B(\sigma)\,, \tag{4.8}$$

$$\{\mathcal{J}_A(\sigma)\,, \mathcal{J}_B(\sigma')\} = \mathrm{e}^{\Delta(x(\sigma))}\, F_{AB}{}^C\, \mathcal{J}_C(\sigma) + \eta_{AB}\, \delta(\sigma - \sigma')\,. \tag{4.9}$$

We find that the explicit x-dependence in $\mathrm{e}^{\Delta(x(\sigma))}$ complicates the right-hand side of the equation of motion $\partial_\tau \mathcal{J}_A = \{\mathcal{J}_A,\, H\}$, and accordingly the covariance under the Jacobi–Lie T-plurality is not manifest.

Let us also discuss the covariance from another perspective. If we start with the action

$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma\, \sqrt{-\gamma}\, \big(\gamma^{\alpha\beta} - \varepsilon^{\alpha\beta}\big)\, \big(g_{mn} + B_{mn}\big)\, \partial_\alpha x^m\, \partial_\beta x^n\,, \tag{4.10}$$

the equations of motion can be expressed as

$$\mathrm{d}J_a = \frac{1}{2}\, \big(\pounds_{v_a} g_{mn}\, \mathrm{d}x^m \wedge *\mathrm{d}x^n + \pounds_{v_a} B_{mn}\, \mathrm{d}x^m \wedge \mathrm{d}x^n\big)\,, \tag{4.11}$$

where

$$J_a \equiv v_a^m\, \big(g_{mn} *\mathrm{d}x^n + B_{mn}\, \mathrm{d}x^n\big)\,. \tag{4.12}$$

If we identify the metric and the B-field as $g_{mn} + B_{mn} = E_{mn}$, by using Eq. (3.10), the equations of motion can be rewritten in a suggestive form [17]

$$\mathrm{d}J_a = \tfrac{1}{2}\, \mathrm{e}^{-2\Delta} (f_a{}^{bc} + 2\, \delta_a^b\, Z^c - 2\, \delta_a^c\, Z^b)\, J_b \wedge J_c\,. \tag{4.13}$$

However we cannot say anything more from this relation.

In the case of the Poisson–Lie T-duality, where $\Delta = 0$ and $Z^a = 0$, we can regard the relation (4.13) as a Maurer–Cartan equation and identify the current J_a as the right-invariant 1-form

$$\mathrm{d}\tilde{g}\, \tilde{g}^{-1} = J_a\, T^a\,, \qquad \tilde{g} \equiv \mathrm{e}^{\tilde{x}_a\, T^a}\,. \tag{4.14}$$

Then, we can rewrite the equations of motion in a manifestly $\mathrm{O}(D,D)$-covariant form as (see section 6.1 of [22] for the details)

$$\hat{\mathcal{P}}^A = \hat{\mathcal{H}}^A{}_B * \hat{\mathcal{P}}^B\,, \tag{4.15}$$

where $\hat{\mathcal{P}}^A$ is constructed by using an element of the Drinfel'd double $l \equiv g\,\tilde{g}$ as

$$\hat{\mathcal{P}} \equiv \hat{\mathcal{P}}^A\,T_A \equiv \mathrm{d}l\,l^{-1}\,. \tag{4.16}$$

The equations of motion can be also expressed as the $\mathrm{O}(D,D)$ covariant Maurer–Cartan equation for the Drinfel'd double

$$\mathrm{d}\hat{\mathcal{P}}^A + \frac{1}{2}\,F_{BC}{}^A\,\hat{\mathcal{P}}^B \wedge \hat{\mathcal{P}}^C = 0\,. \tag{4.17}$$

In the case of the Jacobi–Lie T-duality of [17], due to the presence of Δ in Eq. (4.13), J_a cannot be expressed by using $\tilde{g}$ and it is not clear how to construct a covariant or geometric object similar to $\hat{\mathcal{P}}^A$. If we instead identify the metric and the B-field as $g_{mn} + B_{mn} = \mathcal{E}_{mn}$ as in the case of the Jacobi–Lie T-plurality, Eq. (3.8) leads to

$$\mathrm{d}J_a = \tfrac{1}{2}\,(f_a{}^{bc} + 2\,\delta_a^b\,Z^c - 2\,\delta_a^c\,Z^b)\,J_b \wedge J_c + 2\,Z_b\,r^b \wedge J_a\,. \tag{4.18}$$

In this case, there is no scale factor, but due to the presence of the last term, this again cannot be regarded as a Maurer–Cartan equation. According to the above considerations, we suspect that the Jacobi–Lie T-plurality is not a symmetry of the string sigma model.

One of the reasons for the issue may be that the DD^+ is a Leibniz algebra instead of a Lie algebra. In the case of the Poisson–Lie T-duality, a string is fluctuating on the Drinfel'd double and the position of the string is described by a map, $l : \Sigma \to \mathcal{D}$, from the worldsheet to a Drinfel'd double $\mathcal{D}$. However, in the case of the Leibniz algebra, a group-like global structure is complicated and it is not clear how to describe the position of the string on the doubled geometry similar to the case of the Drinfel'd double. A recent study [39] may be useful in clarifying this point.

5 Conclusions

In this paper, we proposed a Leibniz algebra DD^+ and showed that this provides an alternative description of the Jacobi–Lie bialgebra. Extending the standard procedure developed in the Poisson–Lie T-duality, we showed that a DD^+ systematically constructs a Jacobi–Lie structures and the generalized frame fields satisfying $\hat{\mathcal{L}}_{E_A} E_B = -X_{AB}{}^C\,E_C$. Using the generalized frame fields, we proposed a natural extension of the Poisson–Lie T-duality, which we call the Jacobi–Lie T-plurality. We then showed that the Jacobi–Lie T-plurality (with the

R–R fields and the spectator fields) is a symmetry of the equations of motion of DFT. As a demonstration, we provided several examples of the Jacobi–Lie T-plurality. At the level of the string sigma model, we were faced with a difficulty in the realization of the Jacobi–Lie T-plurality, and this may indicate that the scale symmetry $\mathbb{R}^+$ is not a (classical) symmetry of string theory. To clarify the status of this scale symmetry, it is important to check whether the Jacobi–Lie T-plurality remains as a symmetry of α'-corrected supergravity by extending recent works on the Poisson–Lie T-duality [40–42].

In M-theory, the exceptional Drinfel'd algebra (associated with the SL(5) duality group) has been found as

$$\begin{aligned}
T_a \circ T_b &= f_{ab}{}^c\, T_c\,, \qquad T^{a_1a_2} \circ T^{b_1b_2} = -2\, f_c{}^{a_1a_2[b_1}\, T^{b_2]c}\,,\\
T_a \circ T^{b_1b_2} &= f_a{}^{b_1b_2c}\, T_c + 2\, f_{ac}{}^{[b_1}\, T^{b_2]c} + 3\, Z_a\, T^{b_1b_2}\,,\\
T^{a_1a_2} \circ T_b &= -f_b{}^{a_1a_2c}\, T_c + 3\, f_{[c_1c_2}{}^{[a_1}\, \delta^{a_2]}_{b]}\, T^{c_1c_2} - 9\, Z_c\, \delta^{[c}_b\, T^{a_1a_2]}\,.
\end{aligned} \tag{5.1}$$

If we decompose the index as $a = \{\dot{a}, \sharp\}$ and assume $f_{\dot{a}\dot{b}}{}^{\sharp} = 0$, we find that the generators $\{T_{\dot{a}},\, T^{\dot{a}} \equiv T^{\dot{a}\sharp}\}$ satisfy the subalgebra

$$\begin{aligned}
T_{\dot{a}} \circ T_{\dot{b}} &= f_{\dot{a}\dot{b}}{}^{\dot{c}}\, T_{\dot{c}}\,, \qquad T^{\dot{a}} \circ T^{\dot{b}} = -f_{\dot{c}}{}^{\dot{a}\dot{b}\sharp}\, T^{\dot{c}}\,,\\
T_{\dot{a}} \circ T^{\dot{b}} &= -f_{\dot{a}}{}^{\dot{b}\dot{c}\sharp}\, T_{\dot{c}} - f_{\dot{a}\dot{c}}{}^{\dot{b}}\, T^{\dot{c}} + (3\, Z_{\dot{a}} - f_{\dot{a}\sharp}{}^{\sharp})\, T^{\dot{b}}\,,\\
T^{\dot{a}} \circ T_{\dot{b}} &= f_{\dot{b}}{}^{\dot{a}\dot{c}\sharp}\, T_{\dot{c}} + f_{\dot{b}\dot{c}}{}^{\dot{a}}\, T^{\dot{c}} - (3\, Z_{\dot{b}} - f_{\dot{b}\sharp}{}^{\sharp})\, T^{\dot{a}} + (3\, Z_c - f_{\dot{c}\sharp}{}^{\sharp})\, \delta^{\dot{a}}_{\dot{b}}\, T^{\dot{c}}\,.
\end{aligned} \tag{5.2}$$

This is noting but the DD^+ under the identifications, $f_{\dot{a}}{}^{\dot{b}\dot{c}} = -f_{\dot{a}}{}^{\dot{b}\dot{c}\sharp}$, $Z^{\dot{a}} = 0$, and $2\, Z_{\dot{a}} = 3\, Z_{\dot{a}} - f_{\dot{a}\sharp}{}^{\sharp}$. Similarly, the extended Drinfel'd algebra in the type IIB picture also contains the DD^+ as a subalgebra. Thus, the Jacobi–Lie T-plurality is a subset of the proposed Nambu–Lie U-duality.[5] An issue in the Nambu–Lie U-duality is that the equations of motion of the exceptional field theory are complicated and the covariance under the Nambu–Lie U-duality cannot be easily proven. The results of this paper show that the non-Abelian duality works as a solution generating transformation even when the Z_a is present. Further steps towards the proof of Nambu–Lie U-duality will be taken in future work.

Another future direction is to study an extension of the Jacobi–Lie structure. An extension of the Poisson structure is known as the Nambu–Poisson structure, and its further extension is known as a Nambu–Jacobi structure [43]. In the context of the non-Abelian U-duality, some generalized Nambu–Lie structures have been introduced and it is interesting to study the extension by introducing a bi-vector $E^{(2)}$ that corresponds to the vector E introduced in (2.32). In the case of the Jacobi–Lie structure, the vector fields are constructed as $E \propto Z^a\, e_a$ but in the case of the extended Drinfel'd algebras (in the M-theory picture), the bi-vector will

[5] We note that some DD^+ cannot be embedded into the extended Drinfel'd algebra (see [4, 6]), and accordingly, some Jacobi–Lie T-plurality cannot be realized as a Nambu–Lie U-duality.

be given by $E^{(2)} \propto Z^{ab}\, e_a\, e_b$ by using a bi-vector $Z^{a_1a_2}$ which appears in the decomposition of the gauging $Z_A = (Z_a, \frac{Z^{a_1a_2}}{\sqrt{2!}}, \cdots)$. In the Nambu–Lie U-duality, we usually restrict ourselves to keep only the first components Z_a, but as we discussed in section 2, this restriction is not necessary to construct the Jacobi–Lie structures. It will be an interesting future work to keep $Z^{a_1a_2}$ or higher multi-vectors to formulate a certain bialgebra associated with generalized Nambu–Jacobi structures. It is also interesting to study the associated generalized Yang–Baxter equations.

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