Applying Quantum Tomography to Hadronic Interactions

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Abstract

A proper description of inclusive reactions is expressed with density matrices. Quantum tomography reconstructs density matrices from experimental observables. We review recent work that applies quantum tomography to practical experimental data analysis. Almost all field-theoretic formalism and modeling used in a traditional approach is circumvented with great efficiency. Tomographically-determined density matrices can express information about quantum systems which cannot in principle be expressed with distributions defined by classical probability. Topics such as entanglement and von Neumann entropy can be accessed using the same natural language where they are defined. A deep relation exists between *separability*, as defined in quantum information science, and *factorization*, as defined in high energy physics. Factorization acquires a non-perturbative definition when expressed in terms of a conditional form of separability. An example illustrates how to go from data for momentum 4-vectors to a density matrix while bypassing almost all the formalism of the Standard Model.

1 Introduction

A revolution is underway in how to understand and use quantum mechanics to describe experimental physics. History began with wave function-based descriptions of non-relativistic exclusive reactions. Applications of quantum field theory adopted much the same architecture. Yet the art of physics consists of identifying the minimal theoretical machinery that describes a system of interest. Density matrices are the universal language of inclusive processes, free and clear of unobservable theoretical superstructure. We will describe how density matrices can be deduced directly from experimental data, which is called *quantum tomography*. The rules of quantum probability with density matrices cannot generally be emulated with classical distributions, because quantum probability is a larger mathematical framework. Then we describe how data analysis using quantum probability can find features and regularities that analysis based on making distributions cannot generally reproduce. Among those features are consequences of *entanglement*, which is both conceptually fascinating and powerful for discovery of unexpected phenomena.

The thrust of reference [1] is practical data analysis by mapping experimental data into the density matrix that produced it. The work develops a practical, data-driven application which reconstructs the polarization density matrix of Z-bosons produced in high energy collisions. The entire superstructure of the traditional approach is bypassed. There is no field-theoretic

S-matrix, no structure functions, no parton model, and no dependence on coordinate frames. The numerical application leads to a convex optimization fitting algorithm guaranteed to have one global minimum of a χ^2 statistic. A supplemental Mathematica code goes straight from momentum 4-vectors to the hadronic density matrix in seconds. The density matrix makes available true quantum-mechanical observables such as the *entanglement entropy*, which cannot be found with classical distributions.

Entanglement in terms of density matrices provides natural classifications of observables. This casts a new light on hadronic physics and perturbative QCD. The perturbative framework called *factorization* involves subsystems that are called *separable* in quantum information science (QIS). Separability is exceptional for interacting systems. Dynamically interesting systems are not separable, and indeed separability and *entanglement* are mutually exclusive. While entanglement certainly occurs in hadronic reactions, there exists a freedom to restrict subprocess of interest to make it unobservable, and thereby *define* separable density matrices. This introduces a concept of *conditionally separable systems* which we believe is new.

In the next Section we review background. Much of it is at the undergraduate level, yet it does not appear in most undergraduate or graduate textbooks.

1.1 Background

A density matrix ρ is an operator with positive eigenvalues. Since having real eigenvalues makes an operator Hermitian, $\rho = \rho^\dagger$ follows without needing a separate postulate. The *observable* of an operator A is its expectation value:

$$< A >= \frac{tr(\rho A)}{tr(\rho)} \rightarrow tr(\rho A). \tag{1}$$

Here tr stands for the trace. The last term imposes a convention $tr(\rho) = 1$.

Linear combinations of all operators of a given dimension form a vector space. The Hilbert-Schmidt inner product of operators A, B is

$$< A|B >= tr(A^\dagger B) \rightarrow tr(AB),$$

the last assuming Hermiticity. Then $< A >=< \rho|A >$ measures the geometrical projection of the system ρ onto the observable A. Let G_ℓ be a set of orthonormal Hermitian operators, $< G_\ell|G_k >= tr(G_\ell G_k) = \delta_{\ell k}$. The set need not be complete, while it forms a basis for the subspace it spans. Expanding ρ in the basis gives

$$\begin{aligned} \rho &= \textstyle\sum_\ell |G_\ell >< G_\ell|\rho >, \\ \rho &= \textstyle\sum_\ell < G_\ell > G_\ell. \end{aligned} \tag{2}$$

Thus each independent observable $< G_\ell >$ represents one independent projection of the density matrix. We say that each *probe* G_ℓ discovers one *fact*. Probes are chosen from systems with known density matrices. Observing a sufficient number of probes reconstructs an arbitrarily density matrix. This is *quantum tomography.*

An infinite number of observables exists on the infinite dimensional spaces of quantum mechanics. We maintain that quantum mechanics is concerned with what is observable. We use the density matrix as representing *our description* of a system. It is exactly the density matrix that can be observed. That is what Eq. 2 with no limit on the sum over ℓ represents. This is very powerful. In one step it bypasses the tradition of classifying and modeling in advance infinitely many possible outcomes first, and projecting onto what is observed last. Quantum tomography never deals with what is not observed, and that is extremely efficient.

1.2 The Born Rule

Quantum mechanics textbooks have several versions of "the Born Rule." When a book says that $\psi^*(x)\psi(x)$ is the Born rule, it is wrong. Misunderstanding the Born Rule and misinterpreting the $\psi^*(x)\psi(x)$ as a classical probability distribution[1] leads to almost all the confusion over entanglement that has beset quantum physics.

Let $|e_\alpha>$ be the normalized eigenvectors of ρ, with eigenvalues ρ_α. The "spectral resolution" of the operator is

$$\rho = \sum_\alpha |e_\alpha > \rho_\alpha < e_\alpha|.$$

Consider the expectation value of an operator $E_\beta = |e_\beta >< e_\beta|$, for some value of β. It is

$$< E_\beta >= tr(|e_\beta >< e_\beta|\rho) =< e_\beta|\rho|e_\beta >= \rho_\beta,$$

barring degeneracies. With ρ being positive and $tr(\rho) = 1$, the diagonal elements of ρ in its eigenframe are positive numbers, summing to one, that can be interpreted as classical probabilities. But we are not finished. Suppose $|f>$ is not an eigenvector of ρ, and $F = |f >< f|$. Then

$$< F >= tr(|f >< f|\rho) = \sum_\alpha \rho_\alpha| < f|e_\alpha > |^2.$$

This is *not* an outcome of a classical distribution. It reproduces the specialized Born rule probability to find $|f>$, given $|e_\alpha>$, summed over with weights to find each $|e_\alpha>$ that *do not* come from a distribution.

In an exceptional case ρ has $rank$-1, namely one non-zero eigenvector, denoted $|\psi>$, which defines the *wave function* or "pure state". Eigenvectors have no definite phase or normalization, which *derives* why wave functions have these features as "ray representations". When $\rho \to |\psi >< \psi|$, then $< A >\to< \psi|A|\psi >$. This, plus the Born rule specialized to rank-1, reproduces the rules of elementary quantum mechanics, exposed to refer to the rare case called the "pure state". While they dominate the educational system, there is little experimental evidence for pure states, since interactions and entanglement do not preserve the concept. When $rank(\rho) \neq 1$ no wave function exists to replace the density matrix, which is simply the "generic state."

1.3 Information in a Density Matrix

A density matrix tomographically reconstructed from experiment is the ideal summary of what is known about a system. Theory and experiment share a well-defined framework that is as fundamental as quantum mechanics itself.

Moreover, true quantum mechanical observables *not directly observed* can be computed. Here are three examples:

- The quantum probability of a normalized state $|f>$ is $< f|\rho|f>$. The classical probability of a state is radically different. Suppose an engineer measures a time series of voltages V_t with an oscilloscope. There are 3000 discrete sample times t measured with 8-bit resolution. The vector space of the voltage data has dimension 256^{3000}. Spaces of such dimension cannot in principle be sampled using all the time and atoms in the universe. Moreover as the voltage resolution go to zero there are infinitely many mutually-exclusive states. In quantum probability only orthogonal states are mutually-exclusive.

[1] Notable examples of both mishaps are the elementary books by *Griffiths*.

The vector space of the oscilloscope data has only 3000 dimensions. We may imagine that 3000 dimensional data is complicated, but it is a shocking reduction of complexity from the classical case.

- The entropy S of a density matrix ρ is

$$S = -tr(\rho \log(\rho)).$$

The entropy is a measure of an effective dimension $D_{eff} = e^S$. When ρ is equivalent to a pure state, then $S = 0$, and $D_{eff} = 1$. When ρ has no information, it equals the unit matrix times a normalization. Then $S = \log(D)$, and $D_{eff} = D$ for a $D \times D$ matrix. When the entropy show a significant change from a baseline, it is a signal that something significant is underway. Every experimental data set we have examined has shown unexpected features in the entropy.

- Observables of a density matrix exhibit *outcome dependence*. An example is the measurement of a pure state $|f>$ which defines a projector $\pi_f = |f><f|$. In an ideal projective measurement $\rho \to \pi_f \rho \pi_f$. If after that a different projector π_g is measured, the sequence is represented by $\rho \to \pi_g \pi_f \rho \pi_f \pi_g$. In the reverse order $\rho \to \pi_f \pi_g \rho \pi_g \pi_f$, which is generally different. As a rule no fixed distribution of numbers (eigenvalues, etc.) associated with the state can emulate the variety of possible outcomes. This underlies many so-called paradoxes of quantum measurement. The paradoxes come from invoking classical particles treated with classical probability that is not faithful to how probability works in quantum mechanics.

1.4 Hadronic Reactions

Consider a generic inclusive reaction $p_1 + p_2 \to k_1 + k_2 + ... + X$. The conventional description is based on a S matrix elements $< p_1, p_2|_{in} S| k_1, k_2...X >_{out}$. Actually pure states are seldom involved: No pure states exist to describe unpolarized electrons, photons, or protons. The density matrix description of a cross section $d\sigma$ is

$$d\sigma \sim \sum_X M_{p_1 p_2 k_1 k_2 ... X} M^\dagger_{p_1 p_2 k_1 k_2 ... X} dLIPS. \tag{3}$$

Here X is not detected, and $dLIPS$ absorbs the flux and phase space factors. The notation does not imply any pure states, and simply indicates some multi-variate matrix of the form $MM^\dagger$. Any matrix product $MM^\dagger$ has positive eigenvalues, and is a density matrix.

Density matrices are at the heart of quantum mechanics, and always appear when making contact with experiment. The prototype inclusive reaction is deeply inelastic scattering. In is conventionally described with symbols $d\sigma = L^{\mu\nu} W_{\mu\nu}$, where $L^{\mu\nu}$ and $W_{\mu\nu}$ are certain "structure functions." Both tensors are positive, so they are density matrices. A more general description is

$$d\sigma = tr(\rho_{lep} \rho_X) = < \rho_{lep} | \rho_X >$$

where ρ_{lep} might be a more general lepton probe. Since this is an inner product, classifying the space of operators in the probe ρ_{lep} automatically classifies the space of ρ_X. This is the mirror trick, and it makes a separate classification of ρ_X redundant for experimental purposes. That is, *the scattered leptons* are observed, not the target. Notice there is no model here, other than the information in ρ_{lep}. There is no attempt to "predict" what is being described, and that is why it is very general. One may or may not invoke partons, factorization, and all that: The description simply encodes what is measured. To this day, theory predicts very little about strong interactions, compared to what *quantum tomography* has been doing, unrecognized, for more than 40 years.

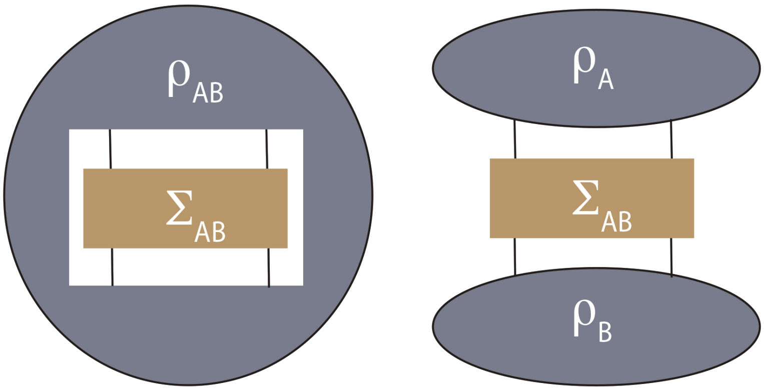
1.5 Density Matrix Entanglement and Reduction

The word *entanglement* was coined by Schrödinger coined to expose misunderstandings and misrepresentation of quantum probability. In elementary quantum mechanics two systems with variables $\vec{x}_1$, $\vec{x}_2$ are *not entangled* if a joint wave function $\psi(\vec{x}_1, \vec{x}_2) = \psi_1(\vec{x}_1)\psi_2(\vec{x}_2)$. Otherwise systems are entangled. Entanglement does not mean "interacting," and non-interacting systems (e. g. "identical particles") can be entangled.

The density matrix theory of entanglement is much deeper. Two systems A, B are *separable* if the following criterion holds:

$$\rho(A, B) = \sum_{\ell} P_\ell \rho_\ell(A) \otimes \rho_\ell(B). \tag{4}$$

Here $P_\ell > 0$, and each factor $\rho(\ell)$ is positive. Systems are *entangled* if they are not separable. It is easy to show that separable systems remain separable if they do not interact. Conversely, maintaining separability in interacting systems is not a natural thing to expect.



[ht]

Figure 1: Classifying composite systems. *Left:* An non-factorized, or entangled system, with a generic probe. Right: A factorized, or separable system *conditional upon* a special class of probe Σ_{AB}.

The fundamental role of the density matrix was suppressed for a long time. For example, in 1957 Fano [3] discovered quantum tomography and wrote Eq. 2. It was largely ignored until the modern era of quantum computing resurrected it. The criteria for separability and entanglement of density matrices was rather late to appear. It seems to have first been stated in QIS by Werner [2] in 1989, some 63 years after quantum mechanics was discovered by Schrödinger in 1926. Expressing Eq. 4 with diagrams (Figure 1) reveals that separability is a statement of *factorization*. Not only does recognition of factorization pre-date separability in QIS, but particle and nuclear physics has been subdivided into processes that are *conditionally separable*. Thus factorization holds for some probes and subprocesses and not others, and holds in some kinematic limit of leading twist, etc. This concept does not seem to be known in QIA.

Reduction is the process of removing quantum mechanical subspaces that will not be observed. If nothing about system B is observed, then all operators on system A are proportional to the unit matrix 1_B. Projecting onto this space of operators is done by taking the trace over the B labels, indicated by $\rho(A, B) \to \rho(A) = tr_B(\rho(A, B))$. In general both pure states and separable systems become mixed states, with increased entanglement upon reduction. Yet due to a psychological bias, there is an improper general default to describe systems of interest as if they are autonomous, ignoring reduction. The result is that theoretical analysis often starts off naively, and then takes years to come to the description that was needed from the start.

Feynman's partons were essentially pure states. Integrating their density matrix over gluon radiation (a process of reduction) led to the scale-dependent distributions of DGLAP. Restoring all the density matrix polarization features of quarks and gluons [**?**] led to a gold mine of information about hadronic spin physics which is still being explored and planned for the Electron Ion Collider. [5]

1.6 A Specific Example: Inclusive Lepton Pair Production

Reference [1] develops the example of inclusive lepton pair production from Z-bosons in complete detail. A lepton pair defines 6 independent momenta ℓ_μ, ℓ'_μ, which are expressed with the the total $Q^\mu = k_\mu + k'_\mu$ and difference $\ell_\mu = k_\mu - k'_\mu$ momenta. The difference momenta are orthogonal to Q and define polar and azimuthal angles θ, ϕ with respect to an event-by-event coordinate system that depends on Q and a beam 4-vector P. The coordinate system is defined with 4-vectors $X_\mu(Q, P)$, $Y_\mu(Q, P)$, $Z_\mu(Q, P)$ obeying $Q \cdot X = Q \cdot Y = Q \cdot Z = X \cdot Y = X \cdot Z = Y \cdot Z = 0$. As a result angular quantities are mapped into Lorentz scalars such as $\cos(\theta) = \ell \cdot Z / \sqrt{Z^2 \ell^2}$, which are evaluated directly with 4-momenta measured in the lab. Up to a normalization factor, the angular distribution $d\sigma / d \cos\theta d\phi = tr(\rho_{lep} \rho_X)$, which leads to quantum-tomographically determining the system density matrix ρ_X. Every element of Standard Model formalism is bypassed except for 2 parameters appearing in ρ_{lep}. A Cholesky representation of $\rho_X = M(m) M(m)^\dagger$ explicitly maintains positivity of the unknown density matrix, with parameters m being fit to the measured angular distribution by a maximum likelihood procedure guaranteed to minimize a χ^2 statistic which has one global minimum, and no isolated minima. Mathematica code is included that determines the system density matrix ρ_X directly from an imported file of lepton-pair 4-vectors.

2 Conclusion

Density matrices describe inclusive reactions. By quantum tomography, inclusive reactions describe density matrices. Quantum tomography circumvents unobservable formalism to describe quantum systems directly from how they are observed. A density matrix tomographically reconstructed from data has complete information to explore entanglement and all other quantum mechanical observables that are desired. A small part of the total information can be put into correspondence with classical probability and the traditional process of characterizing data with distributions. The rest *cannot* be described by distributions, and is waiting to be explored.

Acknowledgements

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