Extended Gravity and Connections to Dark Energy

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Abstract

Lecture notes on extensions of gravity and their connections to dark energy

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1 Introduction

The cosmological standard model contains a cosmological constant, Λ. The energy density associated with this component makes up almost 70% of the energy density in the universe today. In these lectures we will see how difficult it is to naturally encorporate this into our standard models of gravity and quantum mechanics. Dark energy is the name we give to the substance, fields or modifications of standard physics that solves this puzzle (although some authors prefer to use the name dark energy specifically for a field which reproduces all of the observed phenomenology).

In 2011 the Nobel prize was awarded to Saul Perlmutter, Brian Schmidt and Adam Riess for their observations of Type 1a supernovae. These exploding stars have approximately the same luminosity wherever and whenever they occur in the universe, and therefore can be used as standard candels to form a cosmological distance ladder and help us reconstruct the recent expansion history of the universe. These observations [1, 2] (combined with observations of the CMB [3]) conclusively proved that the universe could not only be filled with matter and radiation. It needs, in addition, a significant component of something that looks like a cosmological constant.[1] Before these observations there were already a number of indicators for a significant cosmological constant, including a tension between the ages of the oldest stars and the calculated age of the universe (in the absence of a cosmological constant), the number densities of galaxies, and the observed flatness of the universe [5].

There are no widely accepted solutions to the cosmological constant problem. In this way the problem of dark energy is different to dark matter where a number of different possible candidates have been identified, and the challenge is in experimentally/observationally identifying which option the universe prefers. Solutions to the cosmological constant problem [5–8] have been suggested within string theory using hierarchies between extra dimensions, modifications of gravity (giving the graviton a mass would have been an elegant solution, but unfortunately is in tension with other observations) and modifications to our fundamental physical principles such as locality. It has also been suggested that we should just accept that physics is fine tuned.

In these lectures I will focus in particular on the possibility that the explanation for how the universe is evolving is due to a modification of gravity. Gravity has been extremely well tested in the laboratory, and in the solar system [9]. But it is an extrapolation over a vast range of scales to assume the same theory applies on cosmological scales. It is important to note though, that the distinction between what is a modification of gravity and what is the introduction of novel matter is rather arbitrary. Rather than debating classifications, we should focus on what degrees of freedom are present, what couples to what at what scale, and what the observational consequences are.

In these lectures I aim to demonstrate the problems associated with the cosmological constant and its solutions. Rather than trying to be comprehensive I will use illustrative scalar field models through-out. We will see that we may want to modify gravity on long distance scales. New physics often means new particles - and scalars are the simplest option (especially if we don't have a reason to introduce direction or spin dependence). Examples that introduce new scalars include $f(R)$ modified gravity, massive gravity and quintessence models of dark energy. Because we are looking for new physics on longer distance scales in the universe these scalar fields are typically light.

I use the $(-, +, +, +)$ metric convention.

Key references used in putting together these notes are:

[1] I noted that there is debate about whether these observations conclusively show the acceleration of the expansion, see for example [4].

[5] Dynamics of dark energy. *Edmund J. Copeland, M. Sami, and Shinji Tsujikawa.* https://arxiv.org/hep-th/0603057
[6] Modified Gravity and Cosmology. *Timothy Clifton, Pedro G. Ferreira, Antonio Padilla, Constantinos Skordis.* https://arxiv.org/1106.2476
[7] Everything You Always Wanted To Know About The Cosmological Constant Problem (But Were Afraid To Ask). *Jérôme Martin.* https://arxiv.org/1205.3365
[8] Beyond the Cosmological Standard Model. *Austin Joyce, Bhuvnesh Jain, Justin Khoury and Mark Trodden.* https://arxiv.org/1407.0059

2 General Relativity is Special

General relativity is our current best theory of gravity, it can be equivalently thought of as the theory of a curved space time manifold, or as the theory of a massless spin two field.

Lovelock's theorem is one way of expressing the uniqueness of General Relativity. It states that in a four dimensional space-time the only second order equations of motion obtained from an action of the form

$$S = \int d^4x \mathcal{L}(g_{\mu\nu}) \tag{1}$$

is

$$\alpha\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) + \lambda\sqrt{-g}g^{\mu\nu} = 0 \ , \tag{2}$$

where α and λ are constants. The simplest choice of action that gives such an equation of motion is

$$S = \int d^4x\sqrt{-g}\left(\alpha\frac{R}{2} - \lambda\right) \ . \tag{3}$$

If we include additional matter fields into this action, it must be done in a coordinate independent way, which means it must have the form

$$S = \int d^4x\sqrt{-g}\left(\alpha\frac{R}{2} - \lambda + \mathcal{L}_m(g_{\mu\nu}, \psi_i)\right) \ . \tag{4}$$

Variation of equation (4), and appropriate choice of the constants α and λ, gives the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = M_p^{-2}T_{\mu\nu} \ . \tag{5}$$

Note that the contracted Bianchi identities give the continuity equation

$$\nabla_\mu T^{\mu\nu} = 0 \ . \tag{6}$$

3 Friedmann and Conservation Equations

Imposing the cosmological principles of homogeneity and isotropy, the Einstein equations become the Friedmann equations. If the scale factor of the universe is a, so that the Hubble 'constant' is $H = \dot{a}/a$. The Friedmann equations are then

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3} \ , \tag{7}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \ . \tag{8}$$

From now on we are going to set K, the curvature of the universe, to be zero. We are treating the matter in the universe as a perfect fluid, and the corresponding conservation equation is

$$\dot{\rho} + 3H(\rho + p) = 0 \ . \tag{9}$$

If the universe is dominated by a cosmological constant then

$$H = \sqrt{\frac{\Lambda}{3}} \tag{10}$$

and

$$a \propto e^{\sqrt{\Lambda/3}t} \tag{11}$$

so we see that the cosmological constant drives an accelerated expansion.

We can also ask what properties matter would have to have to mimic this expansion. This means imposing the unusual requirements

$$\rho = \frac{\Lambda}{8\pi G} \tag{12}$$

and $p = -\rho$ (equivalently $w = -1$).

4 The Problems of the Cosmological Constant

The stress-energy tensor of matter is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial S_m}{\partial g^{\mu\nu}} \ . \tag{13}$$

The stress energy of matter in vacuum has to be of the form

$$\langle 0|T_{\mu\nu}|0\rangle = -\rho_{\rm vac} g_{\mu\nu} \ , \tag{14}$$

where the terms on the right hand side are $g_{\mu\nu}$ so that the vacuum is Lorentz invariant and $\rho_{\rm vac}$ is a constant so that the stress energy is conserved.

For a scalar field

$$S = -\int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right) \ . \tag{15}$$

In vacuum we find

$$\langle T_{\mu\nu}\rangle = -V(\phi_{\rm min})g_{\mu\nu} \ . \tag{16}$$

The classical cosmological constant problem is whether we can fix Λ to be zero (or small). We can see now that it's possible to do this before or after a phase transition, but not both. We also know that the universe has gone through at least two phase transitions, the electroweak, and the QCD, during its history.

4.1 Quantum Zero Point Energy

The cosmological constant problem gets worse when we consider that the universe is not only described by classical physics. Still thinking about our scalar field, we choose its potential to be

$$V(\phi) = \frac{1}{2}m^2\phi^2 \ . \tag{17}$$

As ϕ is a free field we can Fourier expand it

$$\phi(t,x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3\vec{k}}{\sqrt{2\omega}}\left(c_k e^{-i\omega t+i\vec{k}\cdot\vec{x}} + c_k^\dagger e^{i\omega t-i\vec{k}\cdot\vec{x}}\right) \ , \tag{18}$$

where $\omega^2 = k^2 + m^2$.

Substituting into the expression for the energy momentum tensor we find that

$$\langle\rho\rangle = \frac{1}{(2\pi)^3}\times\frac{1}{2}\times\int d^3k\omega(k) \ , \tag{19}$$

$$\langle p\rangle = \frac{1}{(2\pi)^3}\times\frac{1}{6}\times\int d^3k\frac{k^2}{\omega(k)} \ , \tag{20}$$

but both of these integrals blow up!

We clearly need to regulate these divergences, but we need to be careful. If we just impose a hard cut off at the energy scale M we find

$$\langle\rho\rangle = \frac{M^4}{16\pi^2} \ , \tag{21}$$

$$\langle p\rangle = \frac{1}{3}\times\frac{M^4}{16\pi^2} \ , \tag{22}$$

which gives an equation of state of $w = 1/3$. Is this radiation?!

In fact the problem here is that our regulation scheme breaks Lorentz invariance. To see what happens if we use a scheme which respects Lorentz invariance we instead try using dimensional regularization. This gives

$$\langle\rho\rangle = \frac{\mu^4}{2(4\pi)^{(d-1)/2}}\frac{\Gamma(-d/2)}{\Gamma(-1/2)}\left(\frac{m}{\mu}\right)^d \ , \tag{23}$$

$$\langle p\rangle = -\frac{\mu^4}{2(4\pi)^{(d-1)/2}}\frac{\Gamma(-d/2)}{\Gamma(-1/2)}\left(\frac{m}{\mu}\right)^d \ , \tag{24}$$

where μ is the regularization scale. So this time we find an equation of state with $w = -1$. If we subtract the pole in $\Gamma(-d/2)$ then we find

$$\langle\rho\rangle = \frac{m^4}{64\pi^2}\ln\left(\frac{m^2}{\mu^2}\right) \ . \tag{25}$$

So the amount of vacuum energy scales with the mass of the heaviest particle in our theory. This is a problem when the observed energy scale is $\sim 10^{-3}$ eV!

5 Why Extend Gravity?

- **Why not?** There could be lots of interesting new phenomenology to study!

- **Dark energy and the cosmological constant problem.** As we have seen, standard physics does not explain the observed acceleration of the expansion of the universe (absent a massive fine tuning).
- **UV completion.** We can write a low energy effective field theory for gravity, however this theory is not UV complete. One way of addressing this could be through modifications of gravity.

6 How to Extend Gravity

There are many (many, many) ways to extend gravity. What we will discuss here is retaining Lorentz invariance and universal coupling, but adding in additional fields, specifically an additional scalar field. We have already seen how to add a scalar field to the matter sector, what does it mean to add a scalar in the gravitational sector?

The way we will introduce our scalar modification here is to couple it non-minimally to gravity, so that the gravitational action is

$$S = \int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}A^2(\phi)R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right) \ , \tag{26}$$

we can think of this as making Newton's constant (or equivalently the Planck mass) dependent on the scalar field.

There is an equivalent description of this theory, known as the Einstein frame, which we find if we do the field redefinitions $\tilde{g}_{\mu\nu} = A^2(\phi)g_{\mu\nu}$ and

$$\left(\frac{d\tilde{\phi}}{d\phi}\right)^2 = \frac{1}{A^2}\left(1 + 6M_P^2\left(\frac{dA}{d\phi}\right)^2\right) \ , \tag{27}$$

which results in the action

$$S = \int d^4x\sqrt{-\tilde{g}}\left(\frac{M_P^2}{2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi} - \tilde{V}(\tilde{\phi})\right) \ , \tag{28}$$

where $\tilde{V} = \sqrt{\tilde{g}/g}V$.

For specific choices of $A(\phi)$ and $V(\phi)$ there is a third way of framing these theories as $f(R)$ theories of gravity. The scalar mode appears because higher powers of R in the action lead to higher derivative terms in the equations of motion. The gravitational instability of general relativity means that the new scalar mode does not introduce a ghost instability.

7 How to Drive Accelerated Expansion

7.1 Quintessence

As hinted at above, an unusual form of matter with an equation of state $w = -1$ can mimic a cosmological constant and drive an accelerated expansion. This can be achieved with a scalar field with the action

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}(\nabla\phi)^2 - V(\phi)\right) \ . \tag{29}$$

The background cosmological evolution is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \ , \tag{30}$$

and the components of the energy momentum tensor are

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi) \ , \tag{31}$$

$$p = \frac{\dot{\phi}^2}{2} - V(\phi) \ . \tag{32}$$

We see that the equation of state of this fluid can approach $w = -1$ if we are in a 'slow roll' regime where $\dot{\phi}^2 \ll V(\phi)$ (note we get accelerated expansion as long as $w < -1/3$).

We can get this slow roll behaviour in a few different ways, particularly if the field is settling into the minimum of its potential, or if Hubble friction stops the field from rolling down its potential. A common choice of potential which allows Hubble friction to stop the field at late times, known as 'freezing', is an inverse power law $V(\phi) = \Lambda^5/\phi$.

A quintessence field has the advantages that there can exist tracking solutions which can help solve the coincidence problem - why does the cosmological constant term come to dominate the evolution of the universe around the time of the formation of the solar system. The existence of scaling solutions can also remove dependence on initial conditions. However it doesn't help to answer the question of why the cosmological constant is not huge, and arguably quintessence models contain a 'hidden' cosmological constant so all of the cosmological constant problems remain.

7.2 Self-Acceleration

For scalar field theories there is an alternative way to drive an accelerated expansion known as self-acceleration. Thinking about the scalar tensor theory we introduced earlier we had two descriptions, a 'Jordan frame' where the scalar couples explicitly to the metric, and an 'Einstein frame' where the scalar couples explicitly to matter, which are related by field redefinitions. Self-acceleration is the idea that the scale factor in the Jordan frame will accelerate but the expansion in the Einstein frame will not accelerate. If the calculations are done carefully observables are the same which ever frame we calculate in (they should not be changed by field redefinitions!) however, implicitly, we normally do cosmological analysis in the Jordan frame (as we assume that particle masses are constants). The Jordan frame accelerated expansion comes entirely from the conformal transformation between the metrics and the dynamics of the scalar field.

The Jordan and Einstein scale factors are related by $a_J = Aa_E$. Comparing Friedman equations (this discussion follows [10]) we can show that

$$a_J\ddot{a}_J - a_E\ddot{a}_E = \left(\frac{A'}{A}\right)' \ , \tag{33}$$

where a dot is a derivative with respect to proper time, and a prime is a derivative with respect to conformal time. If the Einstein frame scale factor is not accelerating then we must have

$$a_J\ddot{a}_J \leq \left(\frac{A'}{A}\right)' \ , \tag{34}$$

implying that $1 \lesssim \Delta A/A$ over a (Jordan frame) Hubble time. Therefore the scalar field has to evolve significantly to drive self-acceleration.

8 Scalar Forces and Screening Mechanisms

If a scalar field couples to Standard Model matter (and without a good reason to forbid these interaction we should include them) it will mediate a new force. The force will be long range if the scalar is light. Experiments constrain long range forces to have couplings $\sim 10^5$ times weaker than gravity. This means introducing an energy scale five orders of magnitude above the Planck scale. If we don't want to introduce another fine tuning - what can we do?

8.1 Scalar Forces

We will now compute the tree level 2-2 particle scattering interaction by exchange of a light scalar (This section follows the discussion in the textbook by Peskin and Schroeder). We start from a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + g\bar{\psi}\psi\phi \ . \tag{35}$$

The scalar propagator is

$$\frac{i}{q^2 - m_\phi^2 + i\epsilon} \ . \tag{36}$$

The fermion propagator is

$$\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \ , \tag{37}$$

and the vertex contributes

$$-ig \ . \tag{38}$$

We want to work out this interaction in the non-relativistic limit $p = (m, \vec{p})$ and $k = (m, \vec{k})$, where the three-momenta are small, and $(p' - p)^2 = -|\vec{p}' - \vec{p}|^2 + \mathcal{O}(\vec{p}^4)$. The external fermion is

$$u^s(p_0) = \sqrt{m}\begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \ , \tag{39}$$

where ξ is a 2 component spinor, and the factor of $\sqrt{m}$ is a convenient normalization such that $\bar{u}^r u^s = 2m\delta^{rs}$.

Now we can compute the scattering amplitude of our Feynman diagram

$$i\mathcal{M} = (-ig^2)\left(\bar{u}(p')u(p)\frac{i}{(p'-p)^2 - m_\phi^2}\bar{u}(k')u(k)\right) \ , \tag{40}$$

$$\approx -g^2\left(2m\frac{i}{-|\vec{p}' - \vec{p}|^2 - m_\phi^2}2m\right) \ , \tag{41}$$

$$i\mathcal{M} \approx \frac{4im^2g^2}{|\vec{p}' - \vec{p}|^2 - m_\phi^2} \ , \tag{42}$$

where we have used the non-relativistic approximation in the second line.

We can compare this with non-relativistic quantum-mechanics governed by the Schrodinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = 0 \ . \tag{43}$$

If a particle with average momentum $\hbar\vec{k}$ is incident on a potential V, the scattering amplitude is defined as the coefficient of the outgoing wave in the asymptotic solution.

If we assume that scattering is weak and the total wavefunction is approximately the incident wave function

$$\langle p'|iT|p\rangle = -i\tilde{V}(q)(2\pi)\delta(E_{\vec{p}'} - E_{\vec{p}}) \; , \tag{44}$$

where $\vec{q} = \vec{p}' - \vec{p}$ and $\tilde{V}$ is the Fourier transformed potential.

In field theory

$$\langle \mathrm{in}|iT|\mathrm{out}\rangle = (2\pi)^4\delta^{(4)}(k_{\mathrm{in}} - k_{\mathrm{out}})i\mathcal{M} \; , \tag{45}$$

so we identify

$$\tilde{V}(\vec{q}) = -\frac{g^2}{|q|^2 + m_\phi^2} \; , \tag{46}$$

where we have had to divide by $1/(2m)^2$ to convert from relativistic to non-relativistic normalizations.

Inverting this Fourier transform (close the integration contour with a semi-circle in the upper half of the complex plane) we find

$$V(r) = -\frac{g^2}{4\pi}\frac{1}{r}e^{-m_\phi r} \; . \tag{47}$$

8.2 Universally Coupled Scalars

We now return to the universally coupled scalar field introduced earlier. Matter fields move on geodesics of the rescaled metric

$$\tilde{g}_{\mu\nu} = A(\phi)g_{\mu\nu} \; , \tag{48}$$

where $g_{\mu\nu}$ is the metric that determines the geometry of spacetime.

To understand how the scalar field affects matter we work with a simplified situation, assuming that spacetime is flat, $g_{\mu\nu} = \eta_{\mu\nu}$ and that the 'coupling function' is $A(\phi) \approx (1 + \phi/M)$. The motion of a matter particle, with position $\tilde{X}$, is governed by the geodesic equation

$$\frac{\partial^2\tilde{X}^\nu}{\partial\lambda^2} + \tilde{\Gamma}^\nu_{\mu\rho}\frac{\partial\tilde{X}^\mu}{\partial\lambda}\frac{\partial\tilde{X}^\rho}{\partial\lambda} = 0 \; . \tag{49}$$

Now we write a new four velocity u^μ such that

$$\eta_{\mu\nu}u^\mu u^\nu = -1 \; , \tag{50}$$

and acceleration

$$a^\mu = u^\nu\nabla_\nu u^\mu \; . \tag{51}$$

By transforming the quantities in the geodesic equation we find

$$a^\nu = u^\mu\partial_\mu u^\nu = -\frac{\phi_{,\nu}/M}{1 + \phi/M}(3u^\mu u^\nu + \eta^{\mu\nu}) \; . \tag{52}$$

If we consider a static, spherically symmetric situation such that $u^\mu = (1, \vec{0})$ then we find that

$$a_r = -\frac{\phi_{,\nu}}{M} \tag{53}$$

to first order in ϕ/M.

8.3 Scalar Field Around a Source

We take the following Lagrangian

$$\mathcal{L}_\phi = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \mathcal{L}_m(\psi_i, (1+\phi/M)g_{\mu\nu}) \;, \tag{54}$$

and the overall energy-momentum tensor is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}}\left(1-\frac{\phi}{M}\right)\frac{\delta\mathcal{L}_m}{\delta\tilde{g}^{\mu\nu}} \;. \tag{55}$$

The equation of motion for ϕ is then

$$\Box\phi - m^2\phi - \frac{1}{M\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta\tilde{g}^{\mu\nu}}g^{\mu\nu} = 0 \;, \tag{56}$$

$$\Box\phi - m^2\phi - \frac{1}{M}g^{\mu\nu}\left(-\frac{T_{\mu\nu}}{2(1-\phi/M)}\right) = 0 \;, \tag{57}$$

$$\Box\phi - m^2\phi - \frac{1}{2M}T^\mu_\mu = 0 \;. \tag{58}$$

If matter is a static, non-relativistic and spherically symmetric we can write the energy momentum tensor as $T^\mu_\mu = \mathrm{diag}(-\rho(r), \vec{0})$, so that the equation of motion is

$$\Box\phi = m^2\phi + \frac{1}{2M}\rho(r) \;. \tag{59}$$

Now if we assume that the source has mass M_s, constant density ρ and radius R then

$$\phi'' + \frac{2\phi'}{r} - m^2\phi = \frac{1}{2M}\rho\Theta(R-r) \;, \tag{60}$$

we can solve this by finding solutions for $r < R$ and $r > R$ and then imposing that ϕ and ϕ' are continuous at the surface of the source. We also impose that the field is regular at the origin and decays to zero at infinity. This becomes

$$\phi = \frac{\rho}{2Mm^3}\left(\frac{\sinh mr}{r} - m\right), \quad r < R \;, \tag{61}$$

$$\phi = \frac{1}{2M}\frac{M_s}{8\pi}\frac{e^{-m(R-r)}}{r}, \quad r > R \;, \tag{62}$$

and again we recover the Yukawa potential.

The force experienced by a test particle is $F_\phi = \nabla\phi/M$, and inside the Compton wavelength of the scalar field we find

$$F_\phi = -\frac{1}{2M}\frac{M_s}{8\pi r^2} \;. \tag{63}$$

8.4 Screening Around a Source

8.4.1 The Chameleon Model

We take the following Lagrangian, where we have chosen an inverse power law potential inspired by quintessence models

$$\mathcal{L}_\phi = -\frac{1}{2}(\partial\phi)^2 - \frac{\Lambda^5}{\phi} + \mathcal{L}_m(\psi_i, (1+\phi/M)g_{\mu\nu}) \;. \tag{64}$$

In the presence of a non-relativistic background matter density ρ the field now moves in an effective potential

$$V_{\rm eff}(\phi) = \frac{\Lambda^5}{\phi} + \frac{\phi\rho}{M} \ . \tag{65}$$

For a given ρ the minimum of the effective potential is

$$\phi_{\rm min} = \left(\frac{\Lambda^5 M}{\rho}\right)^{1/2} \ , \tag{66}$$

and the mass of small fluctuations around this minimum is

$$m^2_{\rm min} = 2\Lambda^5 \left(\frac{\rho}{\Lambda^5 M}\right)^{3/2} \ . \tag{67}$$

It is therefore possible that the field behaves very differently inside and outside a compact source. Screening of the fifth force occurs if the field is so massive inside the source that there is a region inside the source where the field is essentially constant, and so no gradients of the field are built up. We call the radius of this region R_*. For $r < R_*$ we will assume that the field is constant and at the minimum of the effective potential. For $R_* < r < R$ we assume the potential is well approximated by $V_{\rm eff} \approx \rho\phi/M$. Then outside the source we assume $V_{\rm eff} \approx (1/2)m^2_\infty(\phi - \phi_\infty)^2$ where ϕ_∞ and m_∞ are the minimum of the potential and the mass of small fluctuations in the background.

Constructing the field profile as before and imposing continuity of the field and its first derivative at R_* and R we find

$$\phi = \phi_{\rm in}, \qquad r < R_* \ , \tag{68}$$

$$\phi = \phi_{\rm in} + \frac{\rho_{\rm in} r^2}{6M}\left(1 - \frac{3R_*^2}{r^2} + \frac{2R_*^3}{r^3}\right), \qquad R_* < r < R \ , \tag{69}$$

$$\phi = \phi_\infty - \frac{\rho_{\rm in} R^3}{3M}\left(1 - \frac{R_*^3}{R^3}\right)\frac{e^{-m_\infty(R-r)}}{r}, \qquad r > R \ , \tag{70}$$

and the position of the surface R_* is determined by

$$1 - \frac{R_*^2}{R^2} = \frac{2M}{\rho_{\rm in}} R^2(\phi_\infty - \phi_{\rm in}) \ , \tag{71}$$

we see that if R_* is close to R the field in the exterior of the source is suppressed.

We can take the ratio of the chameleon screened force to the unscreened Yukawa force with the same mass in the background to find

$$\frac{F_{\rm cham}}{F_{\rm Yuk}} = 2\left(1 - \frac{R_*^3}{R^3}\right) \approx \frac{3M}{\rho R^3}(\phi_\infty - \phi_{\rm in}) \ . \tag{72}$$

8.4.2 Cubic Galileon

In this section we consider an example where screening of the fifth force arrises from a modifcation of the kinetic term for the scalar field

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{c_3}{\Lambda^3}\Box\phi(\partial\phi)^2 \ , \tag{73}$$

again we couple to matter through a linear ϕ/M coupling. Despite higher order derivative terms in the Lagrangian, we find that the equations of motion are at most second order in derivatives

$$\Box\phi + \frac{c_3}{\Lambda^3}[(\Box\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi] = \frac{\rho}{M} \ . \tag{74}$$

Taking the source to be spherically symmetric and of constant density as before we find

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\left[\left(\frac{\phi'}{r}\right)+\frac{c_3}{\Lambda^3}\left(\frac{\phi'}{r}\right)^2\right]\right)=\frac{\rho}{M}\Theta(R-r)\ , \tag{75}$$

which has solutions

$$\phi'=\frac{\Lambda^3 r}{2c_3}\left(-1+\sqrt{1+\frac{4c_3\rho}{3M\Lambda^3}}\right)\ ,\quad r<R\ , \tag{76}$$

$$\phi'=\frac{\Lambda^3 r}{2c_3}\left(-1+\sqrt{1+\frac{4c_3\rho}{3M\Lambda^3}\frac{R^3}{r^3}}\right)\ ,\quad r>R\ . \tag{77}$$

When $R<r\ll R_V$ where $R_V^3=c_3M_s/\pi M\Lambda^3$ we find that the ratio of the screened to unscreened scalar forces is

$$\frac{F_{\rm gal}}{F_{\rm unscreen}}=2\left(\frac{r}{R_V}\right)^{3/2}\ . \tag{78}$$

9 Summary

The main message of these lectures has been that solving the cosmological constant problem is hard. Even if we assume that some, as yet unknown, mechanism sets the observed cosmological constant to zero, it is still a challenge to explain the observed accelerated expansion without coming into conflict with other measurements. What makes this problem even more interesting is that the energy scale associated with it is a very accessible one, being roughly that of neutrino masses, and a distance scale of roughly 0.1 mm. This is a very well tested experimental regime, explaining why it is so difficult to construct new theories which pass all existing tests. Perhaps this is another indication that the solution must be non-linear, and that what we observe is a reprocessing of other more fundamental scales.

There are many topics in this area that we have not touched on in these lectures. One significant one is the constraints that come from cosmological observations and also observations of gravitational waves. In the context of scalar tensor theories of gravity, a very nice review of these constraints can be found in this Reference by Johannes Noller [11].

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