Beyond $\mathcal{PT}$-symmetry: Towards a symmetry-metric relation for time-dependent non-Hermitian Hamiltonians. I linear amplification

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(Dated:)

Abstract

In this work we first propose a method for the derivation of a general continuous antilinear time-dependent (TD) symmetry operator $I(t)$ for a TD non-Hermitian Hamiltonian $H(t)$. Assuming $H(t)$ to be simultaneously $\rho(t)$-pseudo-Hermitian and $\Xi(t)$-anti-pseudo-Hermitian, we also derive the antilinear symmetry $I(t) = \Xi^{-1}(t)\rho(t)$, which retrieves an important result obtained by Mostafazadeh [J. Math, Phys. 43, 3944 (2002)] for the time-independent (TI) scenario. We apply our method for the derivaton of the symmetry associated with a TD non-Hermitian linear Hamiltonian: a cavity field under linear amplification. The computed TD symmetry operator is then particularized for the equivalent TI linear Hamiltonian and its $\mathcal{PT}$-symmetric restriction. In this TI scenario we retrieve the well-known Bender-Berry-Mandilara result for the symmetry operator: $I^{2k} = 1$ with k odd [J. Phys. A 35, L467 (2002)]. The results here derived together with those in the sequel, where we extend our analysis for a TD non-Hermitian quadratic Hamiltonian, enables us to propose a useful symmetry-metric relation for TD non-Hermitian Hamiltonians.

I. INTRODUCTION

In the last two decades, since the seminal contributions of Bender and Boettcher [1] and Mostafazadeh [2, 3], $\mathcal{PT}$-symmetric Hamiltonians —invariant under parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) symmetries— have been investigated in practically all domains of physics, from low to high energies, revealing to be an increasingly autonomous and thought-provoking field. The $\mathcal{PT}$-symmetry condition, much less demanding than that of Hermiticity, greatly expands the possibility of the Hamiltonian description of physical systems (with real eigenvalues [1] and conservation of the norm [2]), which is one of the strong calls for the field. And much has been done recently, such as the experimental realizations of Floquet $\mathcal{PT}$-symmetric systems [4] and $\mathcal{PT}$-symmetric flat bands [5], besides enhanced sensing based on $\mathcal{PT}$-symmetric electronic circuits [6] and $\mathcal{PT}$-symmetric topological edge-gain effect [7]. The linear response theory for a pseudo-Hermitian system-reservoir interaction was developed [8], as well as a protocol to approach non-Hermitian non-commutative quantum mechanics [9].

In this work we propose a method to derive a general time-dependent (TD) continuous symmetry operator for a TD non-Hermitian Hamiltonian. This will be done in the broader scenario of non-autonomous Hamiltonians, and to this reason we revisit the TD non-Hermitian Hamiltonians of a cavity field under linear [10] and parametric [11] amplifications, the latter in Part II of this sequel. These Hamiltonians have been considered for approaching TD non-Hermitian Hamiltonians under TD Dyson maps, thus extending the method proposed by Mostafazadeh [2]. This extension was also undertaken in Refs. [12, 13].

Many interesting contributions to the subject of TD non-Hermitian Hamiltonians have been presented [14]. We mention, in particular, a method that adds to the achievements of Ref. [10], enabling the unitarity of the time-evolution and the observability of non-Hermitian Hamiltonians through particular TD Dyson maps that define time-independent (TI) metric operators [15]. Moreover, we stress the introduction of the all-creation and all-annihilation TD pseudo-Hermitian bosonic Hamiltonians [16], able to generate an infinite squeezing degree at a finite time. A TD pseudo-Hermitian Hamiltonian for a cavity mode with complex frequency is also able to generate an infinite squeezing at a finite time [17]. Finally, we mention the enhancement of photon creation through the pseudo-Hermitian dynamical Casimir effect [18].

Our subject, pseudo-Hermiticity beyond $\mathcal{PT}$-symmetry, is in fact at the foundations of pseudo-Hermitian quantum mechanics. A theorem by Mostafazadeh [3] —formulated for TI non-Hermitian Hamiltonians, symmetries and metric operators— asserts that a *diagonalizable (non-Hermitian) Hamiltonian is pseudo-Hermitian if an only if it has an antilinear symmetry, i.e., a symmetry generated by an invertible antilinear operator.* Moreover, Bender, Berry and Mandilara [19] have shown that a non-Hermitian Hamiltonian presents a real spectrum not only when invariant under $\mathcal{PT}$-symmetry, but also under any antiunitary operator I satisfying $I^{2k} = 1$ with k odd. We also mention the demonstration that supersymmetry gives rise to non-$\mathcal{PT}$-symmetric families of complex potentials with entirely real spectra [20], and also the proposition of chiral metamaterials with $\mathcal{PT}$ symmetry and beyond [21]. Despite the generality of the pseudo-Hermitian requirement, the particular case of $\mathcal{PT}$-symmetric Hamiltonians gained prominence due to Bender and Boettcher's seminal work and certainly due to the strong physical appeal of parity and time-reversal invariance.

Our objective is precisely to explore more general symmetries than $\mathcal{PT}$ starting from the general scenario of TD non-Hermitian Hamiltonians. The method we propose for the derivation of TD symmetries for TD non-Hermitian Hamiltonians applies indistinctly to *linear* or *antilinear*, *unitary* or *nonunitary symmetries*. However, we assume the symmetry to be an *antilinear* operator aiming to retrieve the results by Mostafazadeh [3] and Bender-Berry-Mandilara [19] in the particular case of a TI scenario, i.e., TI non-Hermitian Hamiltonians, metrics and symmetries. The above mentioned theorem by Mostafazadeh [3] is retrieved when considering an *antilinear symmetry* while the result by Bender-Berry-Mandilara is retrieved when considering a *unitary antilinear* or *antiunitary symmetry*.

In addition, guided by the results in Refs. [2, 19], here we explore the connection between *antilinear symmetries* and *metrics*. We derive a relation between the TD symmetry and a pair of TD metrics operators, one linear and the other antilinear, which is annalogous to the Mostafazadeh's relation [3] for the TI scenario. In the sequel, this connection between symmetry and metric is explored a little further, after the results derived here, for the non-Hermitian linear amplification, and there for the non-Hermitian quadratic amplification.

After presenting our method to derive the symmetry operator, we then apply it for a TD non-Hermitian linear Hamiltonian, leaving the quadratic Hamiltonian to Part II of this sequel. As expected, we have derived TD continuous antilinear symmetries far more complex than the spatial reflection and time reversal. These TD symmetries are then particularized to the equivalent TI non-Hermitian linear and quadratic Hamiltonians and their $\mathcal{PT}$-symmetric restrictions. Then, in the TI scenario, the results in Refs. [2, 19] are perfectly retrieved.

Our analysis corroborate the fact that the $\mathcal{PT}$-symmetry is a particular case of more general symmetries in which spatial reflection is generalized to continuous rotations followed by additional displacement and/or squeezing in phase space. This in a way gives us physical support to extend the scope of pseudo-Hermitian Hamiltonians beyond those invariants under $\mathcal{PT}$ operation.

Our paper is organized as follows. In Section II we briefly revisit the foundations of pseudo-Hermitian quantum mechanics for TI and TD Hamiltonians. In Section III we present a method for the construction of a general TD symmetry operator for a TD non-Hermitian Hamiltonian. In Section IV we assume that the TD non-Hermitian Hamiltonian is simultaneously $\rho(t)$-pseudo-Hermitian and $\Xi(t)$-anti-pseudo-Hermitian, to derive the relation $I(t) = \Xi^{-1}(t)\rho(t)$ for our TD antilinear symmetry operator. From this relations, which must be better explored in paper II, we retrieve the Mostafazadeh's theorems for the TI scenario [3]. The TD non-Hermitian Hamiltonian describing a cavity field under linear amplification is introduced in Section V. We then compute the TD symmetry operator for this non-Hermitian linear Hamiltonian using the method presented in Section III. We demonstrate that this TD symmetry reduces to the $\mathcal{PT}$ operator when the non-Hermitian linear Hamiltonian is assumed to be $\mathcal{PT}$ symmetric. An ansatz for the Dyson map is then proposed for the construction of the pseudo-Hermiticity relation. In Section VI we address the TI equivalent of the TD non-Hermitian Hamiltonian introduced in Section V. In this TI scenario we retrieves the Bender-Berry-Mandilara [19] result, and when considering a $\mathcal{PT}$-symmetric TI non-Hermitian Hamiltonian, we verify that the TI symmetry again reduces to the $\mathcal{PT}$ operator. In Section VII we present our conclusions.

II. PSEUDO-HERMITICITY FOR TD AND TI NON-HERMITIAN HAMILTONIANS

We start our review following the Ref. [10], where a method is presented for approaching the quantum mechanics of TD non-Hermitian and non-observable Hamiltonians with TD metric operators. Alternative developments for the TD scenario are also given in Refs. [12, 13]. From the particularization of these results for TI non-Hermitian Hamiltonians and metric operators, we then derive the results presented by Mostafazadeh in Ref. [2].

Considering a TD non-Hermitian Hamiltonian $H(t)$ and a nonunitary TD Dyson map $\eta(t)$, the Schrödinger equation $i\partial_t |\Psi(t)\rangle = H(t) |\Psi(t)\rangle$ ($\hbar = 1$) is transformed to $i\partial_t |\psi(t)\rangle = h(t) |\psi(t)\rangle$, with

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\left(\frac{\partial}{\partial t}\eta(t)\right)\eta^{-1}(t), \tag{1}$$

and $|\psi(t)\rangle = \eta(t) |\Psi(t)\rangle$. This transformed Hamiltonian $h(t)$ becomes Hermitian as long as the TD pseudo-Hermiticity relation

$$H^\dagger(t)\rho(t) - \rho(t)H(t) = i\partial_t\rho(t), \tag{2}$$

is satisfied, where $\rho(t) = \eta^\dagger(t)\eta(t)$ is the TD metric operator ensuring the norm-conservation:

$$\left\langle \Psi(t) \middle| \tilde{\Psi}(t) \right\rangle_{\rho(t)} = \left\langle \Psi(t) \left| \rho(t)\ \tilde{\Psi}(t) \right. \right\rangle = \left\langle \psi(t) \middle| \tilde{\psi}(t) \right\rangle. \tag{3}$$

In the same way that Eq. (2) ensures the norm-conservation —through the time derivative of Eq. (3)—, it also ensures the similarity transformation

$$O(t) = \eta^{-1}(t)o(t)\eta(t), \tag{4}$$

between the observables $O(t)$ and $o(t)$ in the pseudo-Hermitian and Hermitian systems, respectively, thus enabling the computation of the matrix elements

$$\left\langle \Psi(t) \left| O(t) \right| \tilde{\Psi}(t) \right\rangle_{\rho(t)} = \left\langle \Psi(t) \left| \rho(t)O(t) \right| \tilde{\Psi}(t) \right\rangle = \left\langle \psi(t) \left| o(t) \right| \tilde{\psi}(t) \right\rangle. \tag{5}$$

The reason why a TD Dyson map is required for the construction of the Hermitian counterpart $h(t)$ of an equally TD non-Hermitian $H(t)$, is to avoid unwanted constraints between the parameters defining $H(t)$. When considering a TI non-Hermitian H so that an equally TI Dyson map η can be considered, as in Ref. [2], the TD Dyson relation (1) simplifies to the similarity transformation

$$h = \eta H \eta^{-1}, \tag{6}$$

whereas the Eq. (2) simplifies to the well-known pseudo-Hermiticity relation

$$H^\dagger\rho = \rho H. \tag{7}$$

We now analyze the consequences of the TD extension of the Mostafazadeh's method based on the last two equations (6) and (7). First, within the TD extension and then the

TD Dyson relation (1), we lose the similarity transformation (6) which ensures the observability of the Hamiltonian. However, the similarity transformation (4) remains valid for all operators $O(t)$ other than the Hamiltonian, making the TD pseudo-Hermitian Hamiltonians as pertinent as their TI partners. It is worth noting that the observability of TD Hamiltonians is a sensitive point even in Hermitian quantum mechanics, where, as discussed in Ref. [15], the Hamiltonian acts essentially as the generator of the model's dynamics.

We end this brief review noting that in Ref. [15] a method is proposed for the derivation of particular TD Dyson maps which ensures the observability of TD pseudo-Hermitian Hamiltonians —as much as in Hermitian quantum mechanics— by restoring the similarity transformation between $H(t)$ and $h(t)$. In the treatment we have developed bellow, however, we are not considering the method in [15], and the Hamiltonians $H(t)$ and $h(t)$ must be transformed through the TD pseudo-Hermiticity relation (2).

III. A METHOD FOR THE CONSTRUCTION OF A GENERAL TD SYMMETRY OPERATOR

In order to explore more general symmetries than $\mathcal{PT}$ for a TD non-Hermitian Hamiltonian $H(t)$, we first propose a method to derive this symmetry $I(t)$ which applies indistinctly to linear or antilinear, unitary or nonunitary symmetries. However, as anticipated above, from now on we assume this symmetry to be antilinear so that we can retrieve the results in Refs. [3, 19] for the particular case of TI Hamiltonians and symmetries. Moreover, as well as the Hamiltonian, we assume the symmetry to be a TD operator. Starting from the Schrödinger equation for $H(t)$, we apply the antilinear operator $I(t)$ on both its left-hand sides and then replace t by $-t$, to obtain

$$i\frac{\partial}{\partial t}I(-t)\left|\psi(-t)\right\rangle = \left(I(-t)H(-t)I^{-1}(-t) + i\frac{\partial I(-t)}{\partial t}I^{-1}(-t)\right)I(-t)\left|\psi(-t)\right\rangle . \tag{8}$$

Therefore, for the transformation $I(t)$ to be a symmetry of the system modeled by the Hamiltonian $H(t)$, thus producing an independent solution $I(-t)\left|\psi(-t)\right\rangle$ of the Schrodinger equation from a given solution $\left|\psi(t)\right\rangle$, we ends up with the equation

$$i\frac{\partial I(t)}{\partial t} + H(-t)I(t) - I(t)H(t) = 0. \tag{9}$$

If we had considered a linear instead of antilinear transformation $I(t)$, we would have obtained the well-known equation of an invariant operator [22–24]

$$i\frac{\partial I(t)}{\partial t}+[I(t),H(t)]=0, \tag{10}$$

associated, however, with a non-Hermitian Hamiltonian $H(t)$. Therefore, an TD invariant is a linear TD symmetry of the system.

For a TI symmetry I, the Eq. (9) simplifies to the form

$$IH(t)I^{-1}=H(-t), \tag{11}$$

and for the case where both the symmetry and the Hamiltonian are TI operators, the condition (11) is further simplified to the commutation

$$[I,H]=0. \tag{12}$$

Regarding the TI $\mathcal{PT}$ operation, the condition for a TD Hamiltonian to be $\mathcal{PT}$-symmetric is given by

$$\mathcal{PT}H(t)\left(\mathcal{PT}\right)^{-1}=H(-t), \tag{13}$$

which reduces, for TI Hamiltonians, to the commutation relation $[\mathcal{PT},H]=0$.

We thus verify that the condition for the TD operator $I(t)$ to be the symmetry associated with a TD Hamiltonian $H(t)$, given by the differential equation (9), simplifies to the algebraic equation (11) for a TI symmetry operator. This represents a major reduction in the generality of the symmetry operator, which becomes even greater for a TI Hamiltonian.

Following the reasonings in Ref. [23], where a method for the construction of nonlinear Lewis-Riesenfeld TD invariants is presented, we define the general symmetry operator as the product $I(t)=\Lambda(t)\mathcal{U}(t)$, with $\Lambda(t)$ being either a unitary or non-unitary operator. Regarding $\mathcal{U}(t)$, from now on we assume it to be antilinear in accordance with the condition imposed in references [3, 19], whose results we want to rescue in the scenario of TI Hamiltonians, symmetry and metric operators.

Considering the product $I(t)=\Lambda(t)\mathcal{U}(t)$, Eq. (9) can be rewritten in the form

$$\left(i\frac{\partial\Lambda(t)}{\partial t}+H(-t)\Lambda(t)-\Lambda(t)H(t)\right)\mathcal{U}(t)+\Lambda(t)\left(i\frac{\partial\mathcal{U}(t)}{\partial t}+[H(t),\mathcal{U}(t)]\right)=0. \tag{14}$$

By also rewriting the Hamiltonian as $H(t) = H_0(t)+V(t)$, with $H_0(t)$ being either a diagonal or nondiagonal operator with known eigenstates, we propose the ansatz $\mathcal{U}(t) = \mathcal{R}(t)\mathcal{T}$, with $\mathcal{T}$ being the time-reversal operator and $\mathcal{R}(t) = e^{i\phi(t)H_0(t)}$, with a TD complex parameter $\phi(t)$. For an Hermitian $H_0(t)$, $\mathcal{U}(t)$ then becomes an antiunitary operator. We thus define the TD operator

$$\Theta(t) = \left(i\frac{\partial\mathcal{U}(t)}{\partial t} + [H(t),\mathcal{U}(t)]\right)\mathcal{U}^{-1}(t), \tag{15}$$

such that Eq. (14) becomes

$$i\frac{\partial\Lambda(t)}{\partial t} + H(-t)\Lambda(t) - \Lambda(t)H(t) = -\Lambda(t)\Theta(t). \tag{16}$$

In summary, to obtain $I(t) = \Lambda(t)\mathcal{U}(t)$, we first compute the TD operator $\Theta(t)$ from Eq. (15), by taking the advantage of the known eigenstate basis of $H_0(t)$ which defines $\mathcal{R}(t)$. Next, starting from an ansatz for $\Lambda(t)$, based on the symmetry group of $V(t)$, we then compute this operator from Eq. (16), what finally gives us the symmetry $I(t)$. It is evidently straightforward to derive the equivalent of Eq. (9) for a linear symmetry operator, with the same ansatz $I(t) = \Lambda(t)\mathcal{U}(t)$ applying for its solution.

IV. THE ANTILINEAR SYMMETRY DESCRIBED BY A COUPLE OF LINEAR AND ANTILINEAR METRIC OPERATORS

Let us consider a TD non-Hermitian Hamiltonian $H(t)$ which obeys the TD pseudo-Hermiticiy relation given by Eq. (2): $H^\dagger(t)\rho(t) - \rho(t)H(t) = i\partial_t\rho(t)$. Starting with the Schrödinger equation for $H(t)$, $i\partial_t|\psi(t)\rangle = H(t)|\psi(t)\rangle$, applying the linear metric operator $\rho(t)$ on its l.h.s., and assuming the relation in Eq. (2), we obtain

$$i\frac{\partial}{\partial t}|\chi(t)\rangle = H^\dagger(t)|\chi(t)\rangle. \tag{17}$$

where we have defined $|\chi(t)\rangle = \rho(t)|\psi(t)\rangle$. Next, assuming that $H(t)$ also obeys a TD anti-pseudo-Hermiticiy relation

$$H^\dagger(t)\Xi(t) - \Xi(t)H(-t) = i\dot{\Xi}(t), \tag{18}$$

for the TD antilinear metric operator $\Xi(t)$, the application of the operator $\Xi(-t)$ on the l.h.s. of the Schrödinger equation for $H(t)$, leads us again to the Eq. (17) once we define $|\chi(t)\rangle = \Xi(t)|\psi(-t)\rangle$.

Considering both the TD pseudo-Hermiticity relations, in Eqs. (2) and (18), we derive $H^{\dagger}(t) = \rho(t)H(t)\rho^{-1}(t) + i\dot{\rho}(t)\rho^{-1}(t)$ from the former and then substitute this adjoint Hamiltonian into the latter to obtain

$$i\frac{\partial}{\partial t}\left[\Xi^{-1}(t)\rho(t)\right] = \left[\Xi^{-1}(t)\rho(t)\right]H(t) - H(-t)\left[\Xi^{-1}(t)\rho(t)\right]. \tag{19}$$

It is straightforward and remarkable to verify that the above expression recovers the Eq. (9) for the TD antilinear symmetry operator defined as

$$I(t) = \Xi^{-1}(t)\rho(t). \tag{20}$$

In fact, for the case where only the Hamiltonian $H(t)$ is a TD operator, we then obtain the simplified linear and antilinear pseudo-Hermiticity relations $H^{\dagger}(t)\rho = \rho H(t)$ and $H^{\dagger}(t)\Xi = \Xi H(-t)$, with the TI antilinear symmetry $I = \Xi^{-1}\rho$. When the Hamiltonian is also a TI operator, we then retrieve from our assumption of a TD anti-pseudo-Hermitian relation (18), the results proved by Mostafazadeh in Ref. [3], that *every (non-Hermitian) diagonalizable Hamiltonian is anti-pseudo-Hermitian and that the pseudo-Hermiticity of the Hamiltonian implies the presence of an antilinear symmetry.* In fact, in this case we have $\rho H\rho^{-1} = H^{\dagger} = \Xi H\Xi^{-1}$, and hence $[H, \Xi^{-1}\rho] = 0$.

We have thus verified that, for TD Hamiltonians, symmetry and metric operators, we have derived the TD counterpart of the important Mostafazadeh's relation for the symmetry operator, $I = \Xi^{-1}\rho$. We do not, of course, have a counterpart to the theorem proved by Mostafazadeh in the TI scenario, but verifying that the symmetry operator $I(t) = \Lambda(t)\mathcal{U}(t)$ we have derive through Eq. (9) can also be written in the form that generalizes Mostafazadeh's expression to the TD scenario, is significant and will be explored in the sequel.

V. THE TD NON-HERMITIAN HAMILTONIAN OF A CAVITY FIELD UNDER LINEAR AMPLIFICATION

The TD non-Hermitian Hamiltonian modeling a cavity field under linear amplification is given by

$$H(t) = \omega(t)a^{\dagger}a + \alpha(t)a + \beta(t)a^{\dagger}, \tag{21}$$

with the TD parameters $\omega(t)$, $\alpha(t)$, and $\beta(t)$ being complex functions. Here we just demand that $H^{\dagger}(t) \neq H(t)$, such that $\omega^{*}(t) \neq \omega(t)$ and/or $\alpha^{*}(t) \neq \beta(t)$. The usual requirement for the Hamiltonian (21) to be $\mathcal{PT}$-symmetric, given by Eq. (13), imposes the more restrictive conditions $\omega^*(-t) = \omega(t)$, $\alpha^*(-t) = -\alpha(t)$, and $\beta^*(-t) = -\beta(t)$. From Eq. (13) we also verify that the Hamiltonian (21) is $\mathcal{PT}$-symmetric under spatial reflection about both $x_0 = 0$ and

$$x_0 = -\sqrt{\frac{1}{2m\omega(t)}}\frac{\alpha(t)+\beta(t)}{\omega(t)} = \text{real constant}. \tag{22}$$

For the case of a TI Hamiltonian, $x_0 \neq 0$ implies a Hermitian Hamiltonian, whereas for a TD Hamiltonian, $x_0 \neq 0$ imposes constraints on the Hamiltonian parameters which do not occur for $x = 0$.

A. The TD antilinear symmetry operator

Considering the method proposed for deriving the symmetry operator, we rewrite the Hamiltonian (21) in the form $H(t) = H_0(t) + V(t)$, with $H_0(t) = \omega(t)a^\dagger a$ and $V(t) = \alpha(t)a + \beta(t)a^\dagger$. We then define the operator $\mathcal{R}(t) = e^{-i\phi(t)a^\dagger a}$, such that $\mathcal{U}(t) = e^{-i\phi(t)a^\dagger a}\mathcal{T}$. Consequently, using Eq. (15) we obtain

$$\Theta(t) = \left[\omega(t) - \omega^*(t) + \dot{\phi}(t)\right] a^\dagger a + \left[\alpha(t) - \alpha^*(t)e^{i\phi(t)}\right] a + \left[\beta(t) - \beta^*(t)e^{-i\phi(t)}\right] a^\dagger. \tag{23}$$

where the dot indicates a time derivative. Next, we consider, as an ansatz, the generalized displacement operator

$$\Lambda(t) = e^{\nu(t)a^\dagger + \lambda(t)a + \mu(t)}, \tag{24}$$

which becomes a unitary operator for $\lambda(t) = -\nu^*(t)$, and an Hermitian operator for $\lambda(t) = \nu^*(t)$. We thus obtain from Eqs. (24) and (16):

$$i\frac{\partial\Lambda(t)}{\partial t} + H(-t)\Lambda(t) - \Lambda(t)H(t) = -\left[A(t)a^\dagger a + B(t)a + C(t)a^\dagger + D(t)\right]\Lambda(t) \tag{25}$$

where

$$A(t) = \omega(t) - \omega(-t), \tag{26a}$$

$$B(t) = -\dot{\lambda}(t) + \omega(t)\lambda(t) + \alpha(t) - \alpha(-t), \tag{26b}$$

$$C(t) = -\dot{\nu}(t) - \omega(t)\nu(t) + \beta(t) - \beta(-t), \tag{26c}$$

$$\begin{aligned} D(t) = {} & -\dot{\mu}(t) - \frac{1}{2}\left[\dot{\nu}(t)\lambda(t) - \nu(t)\dot{\lambda}(t)\right] \\ & - \omega(t)\lambda(t)\nu(t) - \alpha(t)\nu(t) + \beta(t)\lambda(t). \end{aligned} \tag{26d}$$

From the r.h.s. of Eqs. (16) and (25), it follows that

$$\Lambda(t)\Theta(t)\Lambda^{-1}(t) = A(t)a^\dagger a + B(t)a + C(t)a^\dagger + D(t), \tag{27}$$

and by substituting Eqs. (23) and (24) in Eq. (27), we obtain

$$\phi(t) = \phi_0 + \int_0^t \left[\omega^*(\tau) - \omega(-\tau)\right] d\tau, \tag{28a}$$

$$\dot{\lambda}(t) = \omega(-t)\lambda(t) + \alpha^*(t)e^{i\phi(t)} - \alpha(-t), \tag{28b}$$

$$\dot{\nu}(t) = -\omega(-t)\nu(t) + \beta^*(t)e^{-i\phi} - \beta(-t), \tag{28c}$$

$$\begin{aligned}\mu(t) = \mu_0 - \frac{1}{2}\int_0^t &\left\{\left[\alpha^*(\tau)e^{i\phi(\tau)} + \alpha(-\tau)\right]\nu(\tau)\right.\\ &\left.- \left[\beta^*(\tau)e^{-i\phi(\tau)} + \beta(-\tau)\right]\lambda(\tau)\right\} d\tau.\end{aligned} \tag{28d}$$

Note from Eq. (28d) that the parameter $\mu(t)$ is added to the generalized displacement operator to avoid undesirable constraints in the Hamiltonian's parameters. For the particular case of a unitary operator $\Lambda(t)$, where $\lambda(t) = -\nu^*(t)$, we use Eqs. (28b) and (28c) to obtain

$$\nu(t) = \frac{\alpha(t) + \beta^*(t)}{\omega(-t) + \omega^*(-t)}e^{-i\phi} - \frac{\alpha^*(-t) + \beta(-t)}{\omega(-t) + \omega^*(-t)}. \tag{29}$$

Therefore, from Eqs. (28) we obtain the parameters defining the TD antilinear symmetry operator

$$I(t) = \mathcal{D}(t)\mathcal{R}(t)\mathcal{T}, \tag{30}$$

where we have replaced Λ for $\mathcal{D}$, which, for a unitary Λ becomes the displacement operator. This symmetry operator describes the successive actions of a time-reversal operator $\mathcal{T}$, a TD global rotation in phase space $\mathcal{R}(t) = e^{-i\phi(t)a^\dagger a}$ and, finally, let us say, a TD generalized displacement in phase space $\mathcal{D}(t) = e^{\nu(t)a^\dagger + \lambda(t)a + \mu(t)}$. For a unitary Λ, this TD symmetry $I(t) = \mathcal{D}(t)\mathcal{R}(t)\mathcal{T}$ resembles the evolution operator for the Hermitized counterpart of the TD Hamiltonian in Eq. (21), except, of course, for the time-reversal operation. Such evolution operator can be derived following the reasonings in Refs. [11, 16, 23]. Therefore, if applied to a given state of the Hermitized counterpart of our Hamiltonian, this peculiar TD symmetry operator $I(t) = \mathcal{D}(t)\mathcal{R}(t)\mathcal{T}$ causes the probability distribution to trace an upward spiral in phase space.

B. The Dyson map and pseudo-Hermiticiy relation

For treating a TD non-Hermitian Hamiltonian we consider a TD Dyson map η which results, in general [15], in a TD metric operator $\rho = \eta^{\dagger}\eta$. Otherwise, the TD pseudo-Hermiticity relation (2) imposes undesirable constraints on the TD parameters of the Hamiltonian. For the TD Dyson map we consider the ansatz

$$\eta = e^{\epsilon a^{\dagger}a+\gamma a+\gamma^{*}a^{\dagger}}, \tag{31}$$

with $\epsilon(t)$ being a real function. To determine its time derivative we use the method of parameter differentiation [25], by which

$$\frac{\partial}{\partial t}e^{Z} = \int_{0}^{1} e^{xZ}\frac{\partial Z}{\partial t}e^{-xZ}dxe^{Z}, \tag{32}$$

where $Z = \epsilon a^{\dagger}a + \gamma a + \gamma^{*}a^{\dagger}$. We thus obtain the Hamiltonian

$$h = Wa^{\dagger}a + Ua + Va^{\dagger} + F, \tag{33}$$

where

$$W = i\dot{\epsilon} + \omega, \tag{34a}$$

$$U = i\dot{\gamma} + i\left(1 - \frac{1-e^{-\epsilon}}{\epsilon}\right)\left(\frac{\gamma}{\epsilon}\dot{\epsilon} - \dot{\gamma}\right) + \omega\gamma\frac{1-e^{-\epsilon}}{\epsilon} + \alpha e^{-\epsilon}, \tag{34b}$$

$$V = i\dot{\gamma}^{*} + i\left(1 + \frac{1-e^{\epsilon}}{\epsilon}\right)\left(\frac{\gamma^{*}}{\epsilon}\dot{\epsilon} - \dot{\gamma}^{*}\right) + \omega\gamma^{*}\frac{1-e^{\epsilon}}{\epsilon} + \beta e^{\epsilon}, \tag{34c}$$

$$\begin{aligned} F = {} & 2\left|\gamma\right|^{2}\left(i\frac{\dot{\epsilon}}{\epsilon} + \omega\right)\frac{1-\cosh\epsilon}{\epsilon^{2}} - \frac{i}{\epsilon}\left(1 - \frac{1-e^{-\epsilon}}{\epsilon}\right)\gamma^{*}\dot{\gamma} \\ & - \frac{i}{\epsilon}\left(1 + \frac{1-e^{\epsilon}}{\epsilon}\right)\gamma\dot{\gamma}^{*} - \frac{\alpha\gamma^{*}}{\epsilon}\left(1-e^{-\epsilon}\right) - \frac{\beta\gamma}{\epsilon}\left(1-e^{\epsilon}\right). \end{aligned} \tag{34d}$$

To ensure the Hermiticity of $h(t)$ we impose a complex TD frequency $\omega(t) = \omega_R(t) - i\dot{\epsilon}(t)$, with $\omega_R(t)$ being a real function, in addition to $U = V^{*}$ and $F \in \mathbb{R}$, what demands that

$$\dot{\gamma} + \left(\frac{\epsilon\coth\epsilon - 1}{\epsilon}\dot{\epsilon} - i\omega_R\right)\gamma - i\frac{\epsilon}{2\sinh\epsilon}\left(\alpha e^{-\epsilon} - \beta^{*}e^{\epsilon}\right) = 0, \tag{35a}$$

$$\begin{aligned} & 2i\left(\frac{\sinh\epsilon}{\epsilon} - 1\right)\left(\gamma\dot{\gamma}^{*} + \gamma^{*}\dot{\gamma}\right) + 2\left(\omega - \omega^{*} + 2i\frac{\dot{\epsilon}}{\epsilon}\right)\left|\gamma\right|^{2}\frac{1-\cosh\epsilon}{\epsilon} \\ & - \left(\alpha\gamma^{*} - \alpha^{*}\gamma\right)\left(1-e^{-\epsilon}\right) - \left(\beta\gamma - \beta^{*}\gamma^{*}\right)\left(1-e^{\epsilon}\right) = 0. \end{aligned} \tag{35b}$$

From (35a) we obtain

$$\gamma = e^{-\chi}\left(\gamma_0 + i\int_0^t \frac{\epsilon e^{\chi}}{2\sinh\epsilon}\left(\alpha e^{-\epsilon} - \beta^* e^{\epsilon}\right) d\tau\right), \tag{36}$$

with

$$\chi = \int_0^t \left(\frac{(\epsilon\coth\epsilon - 1)}{\epsilon}\dot{\epsilon} - i\omega_R\right) d\tau. \tag{37}$$

Now, by substituting Eq. (35a) into Eq. (35b), and admitting momentarily the approximation $\epsilon \ll 1$, we obtain

$$\epsilon \simeq \exp\left(-i\int_0^t \frac{(\alpha^* + \beta)\,\gamma - (\alpha + \beta^*)\,\gamma^*}{2\left|\gamma\right|^2} d\tau\right), \tag{38}$$

showing that the Hermiticity requirements in Eqs. (35) imposes no additional constraints on the Hamiltonian parameters, apart from the complex TD frequency $\omega(t) = \omega_R(t) - i\dot{\epsilon}(t)$ coming from Eq. (34a). Otherwise, when we assume that $\omega(t)$ is real from Eq. (21), it follows that ϵ must be constant, Which leads to a new Hamiltonian h in Eq. (33), and consequently to a new system in Eq. (34) and a new hermitization condition in Eq. (35). When the simplified Dyson map $\eta = e^{\gamma a + \gamma^* a^\dagger}$ is considered, as in Ref. [10], with $\epsilon = 0$, the pseudo-Hermiticity requirement of a complex frequency simplifies to that of a real one, $\omega(t) = \omega_R(t)$, still with no constraints on $\alpha(t)$ and $\beta(t)$.

Therefore, when considering the symmetry operator for the TD pseudo-Hermitian Hamiltonian (33), with γ and ϵ following from Eqs. (36) and (38), we must necessarily assume the function $\omega(t)$ appearing in Eqs. (28) to be of the form $\omega(t) = \omega_R(t) - i\dot{\epsilon}(t)$ (or $\omega = \omega_R$ for the particular Dyson map $\eta = e^{\gamma a + \gamma^* a^\dagger}$). Despite of the frequency constraint, the symmetry operator in Eq. (30), have no restrictions for the amplification parameters $\alpha(t)$ and $\beta(t)$.

C. From $I(t)$ in Eq. (30) to $\mathcal{PT}$

The TI $\mathcal{PT}$ operator can be directly recovered from Eq. (30), starting with the constraints under which the Hamiltonian (21) is $\mathcal{PT}$-symmetric: $\omega^*(-t) = \omega(t)$, $\alpha^*(-t) = -\alpha(t)$, and $\beta^*(-t) = -\beta(t)$. Assuming also a unitary $\Lambda(t)$, we verify from Eqs. (28a) and (28b) that the rotation is reduced to a TI operator with $\phi(t) = \phi_0$, while the parameter $\nu(t)$ of the displacement operator is simplified to

$$\nu(t) = \frac{\alpha(t) + \beta^*(t)}{\omega^*(t) + \omega(t)}\left(1 + e^{i\phi_0}\right). \tag{39}$$

For a TI symmetry operator, where $I = \mathcal{DU}$ must be a TI parameter, we must then assume, to avoid undesirable constraints on the Hamiltonian parameters, that $\phi_0 = (2n+1)\,\pi$, with $n \in \mathbb{Z}$. From this assumption we obtain $\nu(t) = \mu(t) = 0$, and noting that the parity operator can be written in the form $e^{-i(2n+1)\pi a^\dagger a}$, we finally recover the TI operator $\mathcal{PT}$ from Eq. (30), i.e.,:

$$I(t) \to I = e^{-i(2n+1)\pi a^\dagger a}\mathcal{T} = \mathcal{PT}. \tag{40}$$

VI. THE TI NON-HERMITIAN HAMILTONIAN OF A CAVITY FIELD UNDER LINEAR AMPLIFICATION

Now we consider the particular case of a TI non-Hermitian Hamiltonian

$$H = \omega a^\dagger a + \alpha a + \beta a^\dagger, \tag{41}$$

with $\omega^* \neq \omega$ and/or $\alpha^* \neq \beta$. The $\mathcal{PT}$-symmetry of H, now following from the commutation $[\mathcal{PT}, H] = 0$, imposes here the more restrictive conditions $\omega^* = \omega$, $\alpha^* = -\alpha$, and $\beta^* = -\beta$, and enables the spatial reflection only about $x_0 = 0$, for

$$\omega \in \mathbb{R},\ \ \alpha = |\alpha|\, e^{i(n+1/2)\pi},\ \beta = |\beta|\, e^{i(m+1/2)\pi},\ \text{with } n, m \in \mathbb{Z}. \tag{42}$$

The case $x_0 \neq 0$ implies a Hermitian Hamiltonian as anticipated above.

A. The TI antilinear symmetry operator

The condition for the TI Hamiltonian (41) to be invariant under a TI antilinear operator I is given by the commutation relation $[I, H] = 0$. From the knowledge of the TD symmetry operator in Eq. (30), it is natural to assume for its TI equivalent the form $I = \mathcal{DRT}$, with a TI global rotation $\mathcal{R} = e^{-i\phi a^\dagger a}$ and a TI $\mathcal{D} = e^{\nu a^\dagger + \lambda a}$. We have neglected the parameter μ added to the Eq. (24) since it is insensitive to the commutation relation $[I, H] = 0$, which imposes the equations

$$\omega^* = \omega, \tag{43a}$$

$$\omega\lambda - \alpha + \alpha^* e^{i\phi} = 0, \tag{43b}$$

$$\omega\nu + \beta - \beta^* e^{-i\phi} = 0, \tag{43c}$$

$$\omega\lambda\nu + \alpha^*\nu e^{i\phi} - \beta^*\lambda e^{-i\phi} = 0. \tag{43d}$$

For a unitary Λ ($\lambda = -\nu^*$), and using the polar forms $\alpha = |\alpha|\, e^{i\varphi_\alpha}$ and $\beta = |\beta|\, e^{i\varphi_\beta}$, it follows from Eqs. (43) that

$$\phi = i \ln \frac{\alpha^* - \beta}{\alpha - \beta^*}, \tag{44a}$$

$$|\nu| = \frac{1}{\omega}\left[|\alpha| \sin\left(\varphi_\alpha - \frac{\phi}{2}\right) - |\beta| \sin\left(\varphi_\beta + \frac{\phi}{2}\right)\right], \tag{44b}$$

$$\varphi_\nu = \left(n + \frac{1}{2}\right)\pi - \frac{\phi}{2}. \tag{44c}$$

Substituting Eqs. (44b) and (44c) into Eq. (43d) (with $\lambda = -\nu^*$), we obtain the expression

$$|\nu| \left[|\alpha| \cos\left(\varphi_\alpha - \frac{\phi}{2}\right) - |\beta| \cos\left(\varphi_\beta + \frac{\phi}{2}\right)\right] = 0, \tag{45}$$

which result in two different solutions, one for $|\nu| = 0$ and the other for $|\nu| \neq 0$. For $|\nu| = 0$, the Eq. (45) is automatically satisfied and using Eq. (44b) we obtain the constraints $\varphi_\alpha = \phi/2 + n\pi$ and $\varphi_\beta = -\phi/2 + m\pi$, with $n, m \in \mathbb{Z}$, which satisfy Eq. (44a) for $n = 0$, such that $\phi = 2\varphi_\alpha$ and the Hamiltonian's parameters become

$$\omega \in \mathbb{R}, \quad \alpha = |\alpha|\, e^{i\varphi_\alpha}, \quad \beta = |\beta|\, e^{i[m\pi - \varphi_\alpha]}. \tag{46}$$

and the TI antiunitary symmetry operator

$$I = e^{-2i\varphi_\alpha a^\dagger a}\mathcal{T}. \tag{47}$$

For $|\nu| \neq 0$, we get the constraint

$$\frac{\cos(\varphi_\alpha - \phi/2)}{\cos(\varphi_\beta + \phi/2)} = \frac{|\beta|}{|\alpha|} = \mathfrak{p}. \tag{48}$$

which, considering $\varphi_\alpha + \varphi_\beta = \varphi$, leads to the relation

$$\phi = 2\varphi_\alpha - 2\tan^{-1}\frac{1 - \mathfrak{p}\cos\varphi}{\mathfrak{p}\sin\varphi}, \tag{49}$$

in agreement with Eq. (44a). From the above results, we obtain from Eq. (44b) the expression

$$|\nu| = \frac{|\alpha|}{\omega} \frac{1-\mathfrak{p}^2}{1+\mathfrak{p}^2 - 2\mathfrak{p}\cos\varphi}, \tag{50}$$

which imposes

$$\mathfrak{f} = \mathfrak{p}^2 - \frac{2\omega\cos\varphi}{\omega+|\alpha|}\mathfrak{p} + \frac{\omega-|\alpha|}{\omega+|\alpha|} < 0, \tag{51}$$

together with $|\cos\varphi| > \sqrt{\omega^2 - |\alpha|^2}/2\omega$, and then $\mathfrak{p}_- < \mathfrak{f} < \mathfrak{p}_+$, with

$$\mathfrak{p}_\pm = \frac{\omega}{\omega+|\alpha|}\left(1 \pm \sqrt{1 - \frac{\omega^2-|\alpha|^2}{4\omega^2\cos^2\varphi}}\right). \tag{52}$$

The parameters in Eqs. (49) and (50), under the above constraints for φ and $\mathfrak{f}$, define the TI antiunitary symmetry operator

$$I = e^{\nu a^\dagger - \nu^* a} e^{-i\phi a^\dagger a}\mathcal{T}. \tag{53}$$

Although the TI continuous symmetry operators in Eqs. (47) and (53) are particular cases of the TD symmetry operator in Eq. (30), they are generalizations of the discrete parity and time-reversal transformation. Differently from the operator $I(t)$ in Eq. (30), whose TD parameters depends on the Hermiticity conditions only through the frequency requirement $\omega = \omega_R - i\dot{\epsilon}$ (or $\omega = \omega_R$ for the particular Dyson map $\eta = e^{\gamma a + \gamma^* a^\dagger}$), the operators in Eqs. (47) and (53) takes into account the constraints imposed on ω, α and β. The TI non-Hermitian Hamiltonians and consequently the associated symmetry operators are more vulnerable than their general TD equivalents to the constraints imposed by the pseudo-Hermiticity relation. This vulnerability to the constraints follows from the more stringent condition for the invariance of a TI Hamiltonian: $[I, H] = 0$.

1. *Bender-Berry-Mandilara*

Although it is straightforward to verify the validity of the relation $I^{2k} = 1$, with k odd [19], for the symmetry operator in Eq. (47), its validity for the operator in Eq. (53) demands a little algebra. In fact, for the operator in Eq. (53) we obtain

$$\begin{aligned} I^2 &= e^{\nu a^\dagger - \nu^* a} e^{-i\phi a^\dagger a} e^{\nu^* a^\dagger - \nu a} e^{i\phi a^\dagger a}\mathcal{T}^2 \\ &= \exp\left(\nu a^\dagger - \nu^* a\right)\exp\left(\nu^* e^{-i\phi} a^\dagger - \nu e^{i\phi} a\right) = 1, \end{aligned} \tag{54}$$

since it follows from Eqs. (44c) and (50) that $\nu e^{i\phi} = -\nu^*$.

It is important to note that the TD symmetry operator $I(t)$ does not obey the Bender-Berry-Mandilara relation. Although we still have the relation

$$I^2(t) = \exp\left[\nu(t)a^\dagger - \nu^*(t)a\right] \exp\left[\nu^*(t)e^{-i\phi(t)}a^\dagger - \nu(t)e^{i\phi(t)}a\right], \tag{55}$$

the equality $\nu(t)e^{i\phi(t)} = -\nu^*(t)$ is no longer satisfied in the TD scenario.

B. Dyson map and pseudo-Hermiticiy

For the TI Hamiltonian (41), we consider the TI Dyson map $\eta = e^{\epsilon a^\dagger a + \gamma a + \gamma^* a^\dagger}$, with $\epsilon \in \mathbb{R}$, leading to the Dyson relation (6)

$$h = \eta H \eta^{-1} = \omega a^\dagger a + ua + va^\dagger + f, \tag{56}$$

where u, v and f follow directly from U, V and F in Eqs. (34). The Hermiticity condition $h = h^\dagger$ imposes $\omega, f \in \mathbb{R}$ and $u = v^*$, such that

$$\gamma = \frac{\epsilon}{\omega} \frac{\alpha e^{-\epsilon} - \beta^* e^{\epsilon}}{e^{-\epsilon} - e^{\epsilon}}, \tag{57a}$$

$$\frac{\gamma^*}{\gamma} = \frac{\alpha^* \left(1 - e^{-\epsilon}\right) + \beta \left(e^{\epsilon} - 1\right)}{\alpha \left(1 - e^{-\epsilon}\right) + \beta^* \left(e^{\epsilon} - 1\right)}. \tag{57b}$$

By substituting Eq. (57a) into Eq. (57b) we obtain $\beta\alpha \in \mathbb{R}$. With the polar forms $\alpha = |\alpha| e^{i\varphi_\alpha}$ and $\beta = |\beta| e^{i\varphi_\beta}$, it follows that $\varphi_\beta = n\pi - \varphi_\alpha$, with $n \in \mathbb{Z}$. Therefore, for the chosen TI Dyson map η, the Hamiltonian (41) becomes pseudo-Hermitian, with γ given by Eq. (57a) and ϵ being a free real parameter, under the constraints

$$\omega \in \mathbb{R}, \;\; \alpha = |\alpha| e^{i\varphi_\alpha}, \;\; \beta = |\beta| e^{i(n\pi - \varphi_\alpha)}, \tag{58}$$

exactly those in Eq. (46). Therefore, for the case $|\nu| = 0$, the pseudo-Hermiticity does not impose additional constraints on the symmetry operator (47) beyond those already following from the commutation relation $[I, H] = 0$. The same does not apply to the case $|\nu| \neq 0$ which leads to the much more complex symmetry operator (53).

C. From I in Eq. (47) to $\mathcal{PT}$

Under the requirement for the $\mathcal{PT}$-symmetry invariance of the TI Hamiltonian (41), which imposes $\varphi_\alpha = \varphi_\beta = (n + 1/2)\,\pi$, the TI symmetry in Eq. (47) automatically reduces to

$$I = e^{-i(2n+1)\pi a^\dagger a}\mathcal{T} = \mathcal{PT}. \tag{59}$$

VII. CONCLUSIONS

In this work we have proposed a method for the derivation of general TD continuous symmetry operators for TD non-Hermitian Hamiltonians. Although our method applies indistinctly to linear or antilinear, unitary or nonunitary symmetries, we then assume an antilinear symmetry to retrieve the results by Mostafazadeh [3] and Bender-Berry-Mandilara [19] for the case of TI Hamiltonian and symmetry operators. In fact, assuming that the TD non-Hermitian Hamiltonian is simultaneously ρ-pseudo-Hermitian and Ξ-anti-pseudo-Hermitian, we then derive the relation $I(t) = \Xi^{-1}(t)\rho(t)$ for our TD antilinear symmetry operator. From this relation we recover the Mostafazadeh's theorem, for TI Hamiltonian and symmetry operators, asserting that the pseudo-Hermiticity of a Hamiltonian implies the existence of an antilinear symmetry of the form $I = \Xi^{-1}\rho$. We also retrieves the Bender-Berry-Mandilara result that a non-Hermitian Hamiltonian presents a real spectrum when invariant under any antiunitary operator I satisfying $I^{2k} = 1$ with k odd.

Our method is also based on a proposal in Ref. [23], for the construction of Lewis & Riesenfeld TD nonlinear invariants, and we have applied it for the case of a TD non-Hermitian linear Hamiltonian modelling a cavity field under linear amplification. We have thus derived a TD continuous symmetry operator, given in Eq. (30), which describes the successive actions of a time-reversal operator $\mathcal{T}$ and TD rotation and (unitary or nonunitary) displacement in phase space, respectively. This TD continuous symmetry automatically reduces to the TI discrete $\mathcal{PT}$ operator when we restrict our TD Hamiltonian to be $\mathcal{PT}$-symmetrical.

After computing the symmetry operator we then consider the pseudo-Hermitization of our TD linear Hamiltonian. For the case of the TI equivalent of our non-Hermitian linear Hamiltonian, the stringent invariance requirement $[I, H] = 0$, imposes a TI continuous symmetry operator which is a very particular case of the TD symmetry operator in Eq. (30), even though it is a generalization of the discrete parity and time-reversal transformation. The TI non-Hermitian Hamiltonians and the associated symmetry operators are more vulnerable, as expected, than their general TD equivalents to the constraints imposed by the pseudo-

Hermiticity relation.

Back to the TD general symmetry in Eq. (30), when considering the case of a TD antiunitary symmetry, it strongly resembles the evolution operator for the Hermitized counterpar of our TD non-Hermitian Hamiltonian, except for the time-reversal operation [16, 17, 24]. If applied to a given state of this Hermitized counterpart of our Hamiltonian, this peculiar symmetry operator causes the probability distribution to trace an upward spiral in phase space, with TD rotation and translation rates.

The results we have derived here together with those in the sequel for the TD non-Hermitian quadratic Hamiltonian, allow us to infer a relation between symmetry and metric that can be useful in the exploration of pseudo-Hermitian quantum mechanics beyond $\mathcal{PT}$-symmetry.

Acknowledgements

The authors would like to thank CAPES, CNPq, and INCT-IQ, for support..

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