Universal finite-size amplitude and anomalous entangment entropy of $z = 2$ quantum Lifshitz criticalities in topological chains

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Abstract

We consider Lifshitz criticalities with dynamical exponent $z = 2$ that emerge in a class of topological chains. There, such a criticality plays a fundamental role in describing transitions between symmetry-enriched conformal field theories (CFTs). We report that, at such critical points in one spatial dimension, the finite-size correction to the energy scales with system size, L, as $\sim L^{-2}$, with universal and anomalously large coefficient. The behavior originates from the specific dispersion around the Fermi surface, $\epsilon \propto \pm k^2$. We also show that the entanglement entropy exhibits at the criticality a non-logarithmic dependence on l/L, where l is the length of the sub-system. In the limit of $l \ll L$, the maximally-entangled ground state has the entropy, $S(l/L) = S_0 + (l/L)\log(l/L)$. Here S_0 is some non-universal entropy originating from short-range correlations. We show that the novel entanglement originates from the long-range correlation mediated by a zero mode in the low energy sector. The work paves the way to study finite-size effects and entanglement entropy around Lifshitz criticalities and offers an insight into transitions between symmetry-enriched criticalities.

Contents

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1 Introduction

A class of criticalities separate gapped symmetry protected phases [1–3] (SPTs) and topologically trivial ones. At these criticalities usually the system disperses linearly, $\epsilon = \pm v_F k$, around the Fermi surface, and the low-energy effective physics is described by conformal field theories [4–6] (CFTs). Several universal features characterize conformal critical points. One notable feature for quantum one-dimensional (1D) systems is the universal finite-size amplitude [7] together with the emergence of the universal characteristic of CFTs, the central charge, c. Namely, the finite-size correction to the ground state energy $E(L)$, e.g., in case of open boundary condition (b.c), always contains a universal term $c\pi/24L$. The other universal feature is the logarithmic entanglement entropy [8], e.g. , $S \sim c\ln(L)/6$ in the case of periodic b.c.

Topologically distinct and gapped phases are reached by adding the mass to the CFT criticalities [9]. A simple example is the hamiltonian $h(i\partial_x) = v_F\sigma_y i\partial_x + m\sigma_x$ and sign(m) is an integer to distinguish phases. Here $\sigma_{x,y}$ are Pauli matrices. Universal features also appear around the topological phase transitions [10], e.g., the finite-size correction emerges as a universal function of scale, $\omega = mL$.

Recently, it has been observed that CFT critical phases can have non-trivial topology and host boundary modes. Such criticalities are dubbed symmetry-enriched criticalities [11–13] or called gapless SPTs [14, 15]. At the transition between two symmetry-enriched CFTs, non-CFT criticalities can emerge [16]. The simplest case is the Lifshitz criticality [17–22] with dynamical exponent $z = 2$. Its role as a criticality between gapless SPTs is similar to CFT critical points separating gapped SPTs. Namely, one can reach topologically distinct gapless phases by adding velocity term v to $z = 2$ critical point. A simple Hamiltonian illustrates this fact,

$$h(-i\partial_x) = v\sigma_y(i\partial_x) + u\sigma_x\partial_x^2. \tag{1}$$

Here v is the velocity, and u is the curvature of the spectrum. The case with $v = 0$ corresponds to a non-CFT criticality, referred to as Π throughout this paper. With appropriate boundary conditions, one can find the eigenstate, $\psi(x)$, of the Hamiltonian Eq. 1, exhibiting boundary modes at sign(v) > 0 which however disappear at sign(v) < 0. Thus, adding velocity perturbations to the $z = 2$ criticality generates two gapless phases: one topologically trivial and another non-trivial.

In spite of its fundamental role of describing transitions between symmetry-enriched CFTs, the understanding of universal features of $z = 2$ critical points (with the dispersion $\epsilon \sim \pm k^2$) is still lacking. In this letter, we aim to understand the universal properties of Π criticality from two aspects: the study of the energy and entanglement entropy of the ground state. To this end, we consider two concrete lattice models and develop the low energy field theoretical description of the criticality. Lattice models considered below are Majorana/Kitaev chains [16, 23] with next-nearest neighbor terms from BDI symmetry class [23–26] and the generalized Su-Schrieffer–Heeger (SSH) model [27, 28] with next-next-nearest neighbor terms belonging to the AIII symmetry class.

The first result of the present letter corresponds to the ground state energy $E(L)$ as a function of the system size, L. At open boundary condition, the finite-size corrections [29–31] to $E(L)$ exhibit a universal behavior and read

$$E(L) = L\epsilon + b - nu\frac{A}{L^2} + O(L^{-3}), \tag{2}$$

Here ϵ is average bulk energy, b is the boundary energy, and $n \in \mathbb{Z}^+/2$ depends on degrees of freedom of the underlying field theory: $n = 1/2$ for the Majorana chain and $n = 1$ for the SSH model under consideration. The amplitude A is $A \simeq 0.887984$, which is universal for two lattice models and the low-energy field theory giving the same value. This indicates a possible set of rich phenomena of finite-size scaling functions around this criticality [10, 32–34]. We have checked that velocity perturbations modify A into a universal scaling function of $\omega = Lv$, and the function is sensitive to the topological nature of CFTs.

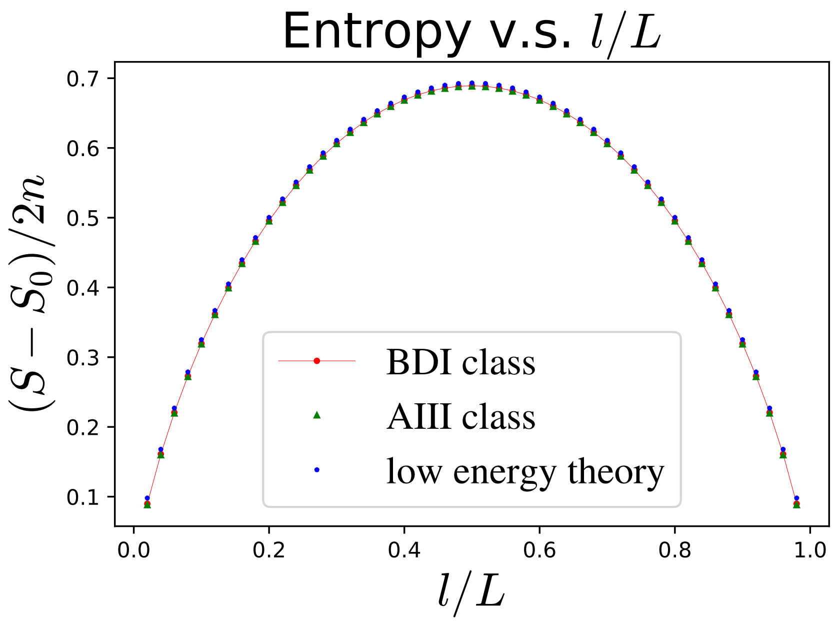

Figure 1: (Color online) Entanglement entropy $(S - S_0)/2n$ is plotted versus l/L. Here S_0 is the non-universal constant entropy, l is the size of subsystem, $n = 1/2$ for the Majorana chain in BDI class and $n = 1$ for the SSH model in AIII class. Three sets of data, including entropy of the Majorana chain, SSH model, and low energy theory, all fall into the same *universal* curve. The function, representing the plotted curve, is exactly the l/L-dependent term in Eq. 3.

We also find that the entanglement entropy [8] exhibits an interesting dependence on l/L. At periodic b.c, the von Neumann entropy of the maximally-entangled ground state is given by

$$S \simeq S_0 + 2n \cdot \left[\frac{l}{L} \ln\left(\frac{L}{l} - 1 \right) - \ln\left(1 - \frac{l}{L} \right) \right]. \tag{3}$$

Here l is the length of the subsystem, and S_0 is a non-universal constant. At the limit $l/L \ll 1$, S has a simple asymptote $\sim (l/L)\log(l/L)$, which is non-logarithmic. The l/L-dependent term is found to be universal, plotted in Fig. 1. Below we start with a definition of lattice models and observe the emergence of Π criticality.

The remainder of the paper is organized as follows. In Section II, we start with a definition of lattice models and observe the emergence of Π criticality. In Section III, the analytical study of the universal finite-size amplitude at the criticality is presented. Section IV presents the derivation of the anomalous entanglement entropy corresponding to the Π criticality. Finally, the conclusions are presented in Section V.

2 Lattice models and Criticality

We consider two concrete lattice models. One is the Majorana chain, containing both nearest site and next-nearest site hoppings and pairings. The Hamiltonian is given by

$$H_{\text{Majorana}} = \sum_n t_0 \tilde{\gamma}_n \gamma_n + t_1 \tilde{\gamma}_n \gamma_{n+1} + t_2 \tilde{\gamma}_n \gamma_{n+2}. \tag{4}$$

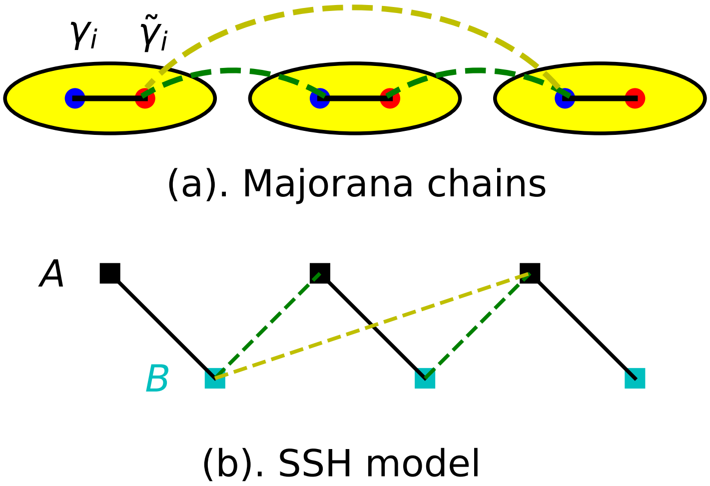

Figure 2: (Color online) Models with three unit cells are plotted to illustrate the hoppings and pairings. (a) Majorana chain. A single fermion is decomposed into two Majorana fermions, shown as blue and red dots. Black lines represent t_0 and dashed green/yellow lines represents t_1/t_2 hoppings (and pairings) in Eq. 4. (b) SSH model. Black/cyan rectangular dots represent A/B sublattices. Black, green and yellow lines represent u_0, u_1 and u_2 hoppings in Eq. 6.

Here $\{\gamma_n, \tilde{\gamma}_n\}$ are two Majorana fermions at the same physical site, and constants $t_i \in \mathbb{R}$, with $i = 0, 1, 2$. The model is schematically shown in Fig. 2a. Note that the model belongs to the BDI class of Cartan's classification of symmetric spaces. A critical line of the model, where the gap closes, corresponds to the case $t_2 + t_0 = t_1$. One can observe three distinct critical behaviors in this situation: (1) when $0 < t_2/t_1 < 1/2$, the low-energy sector is described by Majorana CFT and two localized Majorana modes. (2) When $1 > t_2/t_1 > 1/2$, the low-energy description is a single Majorana CFT. (3) At $t_2 = t_0 = t_1/2$, the Π criticality emerges around $k = \pi$ in the Brillouin zone. The Hamilotnian around the Fermi surface, in Bogoliubov-de-Gennes (BdG) formalism, can be written as

$$H_{\mathrm{FS}} = u \int dx \Psi^\dagger(x) \sigma_x \partial_x^2 \Psi(x). \tag{5}$$

Here $\Psi(x) = (\psi(x), \psi^\dagger(x))^T$ and $\psi(x)$ is the spinless fermion operator in the continuous space.

The second model under consideration is the generalized SSH model from AIII class. The model is schematically shown in Fig. 2b. The Hamiltonian includes nearest-neighbor and next-next-nearest neighbor hoppings of fermions $c^{(\dagger)}$ and is given by

$$H_{\mathrm{ssh}} = \sum_n u_0 c_{n,A}^\dagger c_{n,B} + \sum_{i=1,2} u_i c_{n,B}^\dagger c_{n+i,A} + h.c. \tag{6}$$

The model is defined on a bipartite lattice with A and B sublattices and real hopping parameters. It has a similar phase diagram with Majorana chains. Here the criticality Π emerges around $k = \pi$ when $u_0 = u_2 = u_1/2$. Now the Hamilotnian around the Fermi surface is described by Eq. 5 but with $\Psi(x) = (\psi_A(x), \psi_B(x))$.

3 Universal finite-size amplitude

This section starts with studying the finite-size correction to the ground state energy $E(L)$ at open boundary condition at criticality Π.

In the lattice models under consideration, the computation of the finite-size amplitude of the ground state is similar to the method used in references [10,32], that was applied to CFT criticalities. The method [35] has an error bar, $\sim L^{-1}$. Here we report the results for the SSH model and Majorana chain: we pick $L=500$ and A is found to be 0.88441 for the SSH model and 0.88440 for the Majorana chain. Compared to the value of A below Eq. 2, errors are at the expected order, $\sim 10^{-3}$.

The amplitude A is universal because it originates from the long-wavelength degrees of freedom around the Fermi surface. Below, we will validate this point by deriving the amplitude A from the low energy theory.

Consider the Hamiltonian Eq. 1 at $v=0$. One special property of the operator $\sigma_x\partial_x^2$ is that the free wave and the bound states can belong to the same subspace. Namely, the eigenstates are $\psi_k(x)=\exp(ikx)\cdot\chi_-$ and $\psi_{ik}(x)=\exp(-kx)\cdot\chi_+$ lie in the same energy level $\epsilon_k=uk^2$. Here $\chi_\pm$ satisfy $\sigma_x\chi_\pm=\pm\chi_\pm$ and $k\in(-\pi,\pi)$.

Now assume $h(-i\partial_x)$ acts on coordinate dependent wavefunctions with $x\in(0,L)$ and we impose open boundary conditions on wavefunctions, the left end with $\psi_1(0)=\partial_x\psi_1(0)=0$, and the right end with $\psi_2(L)=\partial_x\psi_2(L)=0$. Note that the wavefunction with the energy ϵ_k can be generally written as $\varphi_k(x)=\sum_{s=\pm}a_s\psi_{sk}(x)+b_s\psi_{isk}(x)$. Upon searching for solutions $\varphi_k(x)$, which obey the open b. c., one arrives at the quantization condition (QC) of the momentum,

$$\cos kL+1/\cosh kL=0,\quad 0<k<\pi, \tag{7}$$

different from conventional QC of Ising CFTs ($\cos kL=0$). When $kL\gg 1$, Eq. 7, the difference between the abovementioned QCs is exponentially small. However, when $kL\sim 1$, the difference is not negligible anymore. This difference indicates that there could be non-trivial finite-size effects. Solutions to Eq. 7 are shown in Fig. 3.

To compute the ground state energy, one must sum all the quasi-particle energies below the Fermi surface. Namely, $E(L)=-u\sum_{k\in\mathrm{QC}}k^2$. Note that all quantizations of k in Eq. 7 are invovled in the ground state energy. Summation in $E(L)$ can be written as a contour integral. Defining $z=kL$, $f(z)=\cos z+1/\cosh z$ and taking the analytical continuation of $f(z)$, one finds

$$E(L)=-\frac{1}{2L^2}\oint_C\frac{dz}{2\pi i}z^2\partial_z\ln f(z). \tag{8}$$

Here C is the contour in complex plane $z=x+iy$, shown in Fig. 4. One can decompose $\ln f(z)=\ln\exp(iz)+\ln\exp(-iz)f(z)$. The first term of this decomposition plugged into Eq. (8) gives the bulk energy $L\epsilon$ of Eq. 2 while the second term gives the leading finite-size correction, $\propto A/L^2$. Further, one can deform the contour to obtain a regular integral over a single real variable. Namely, the C is deformed to be contour D at the cost of exponentially small error, shown in Fig. 4. We find,

$$A=-\mathrm{Re}\int_0^{+\infty}\frac{x^2dx}{2\pi}\partial_x\ln\left(\frac{e^{(i-1)x}+1}{2}+\frac{2}{e^x+e^{-ix}}\right). \tag{9}$$

The above integral is evaluated numerically, yielding $A=0.887984$. This analytically found constant matches the value of A presented below Eq. 2. The value of n can also be argued from the low-energy sector: $n=1/2$ for BDI class is due to the property that the operator

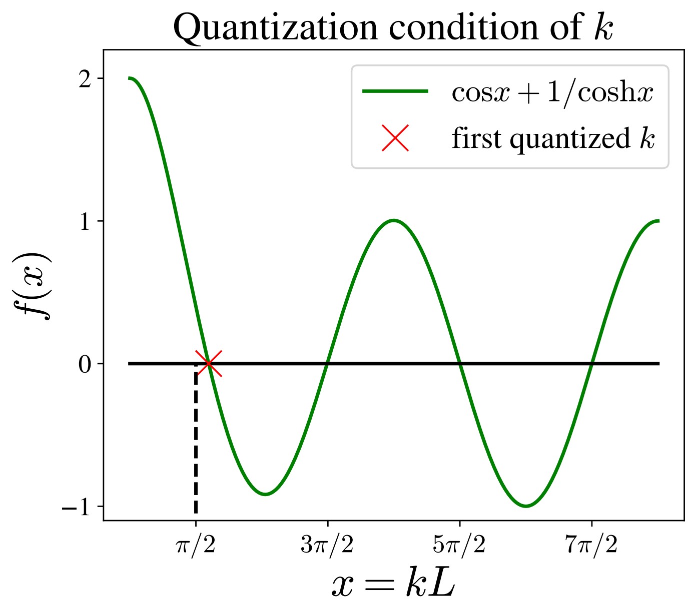

Figure 3: (Color online) Plot of the quantization condition in Eq. 7. The green curve plots the function $f(x) = \cos x + 1/\cosh x$ with $x = kL$. Intersections between $f(x)$ and x-axis determine quantized values of k. The first quantized value, located around $x_0 \simeq 1.875$, is marked by a red cross. This value deviate from the first quantized value in Ising CFT, $\pi/2$. Quanlitatively, the order of the amplitude A can be aruged from this deviation: the deviation in the spectrum level is given by $x_0^2 - (\pi/2)^2 \sim 1$, which is the order of A.

$\Psi(x) = (\psi(x), \psi^\dagger(x))^T$ is counted as 1/2 degree of freedom, while $n = 1$ for AIII class is due to the fact that $\Psi(x) = (\psi_A(x), \psi_B(x))^T$ can be counted as 1 degree of freedom. In this way, we proved the Eq. 2.

4 Anomalous Entanglement entropy

The other universal data, which can be extracted from the Hamiltonian, is the entanglement entropy S. Below we take the Majorana chain as an example to illustrate the emergence of the anomalous entanglement. The consideration for the SSH model is similar.

At first glance, one may observe the eigenstates of Eq. 5 are not different from ones of the gapped Hamiltonian ($k^2 \to m$ and $m \neq 0$). Thus one expects short-ranged correlations and constant (non-universal) entanglement entropy, known as features of a gapped 1D quantum system. However, the presence of zero-modes at the Fermi surface changes the scenario. With periodic boundary conditions, the $k = 0$ eigenstate leads to the double degeneracy of the ground state. Tracing the maximally-entangled ground state, we find that the asymptotic correlation function is given by

$$\langle \gamma_x \tilde{\gamma}_y \rangle \simeq \frac{2i}{L} e^{ik_F(x-y)}, \text{ when } |x-y| \gg 1 \tag{10}$$

Here L is the size of the system, $k_F = \pi$ is the Fermi momentum, and a is the lattice space. So this L-dependent long-range correlation originates from $k = 0$ zero modes at the Fermi surface.

For free fermions, the correlation function encodes the information of entanglment spec-

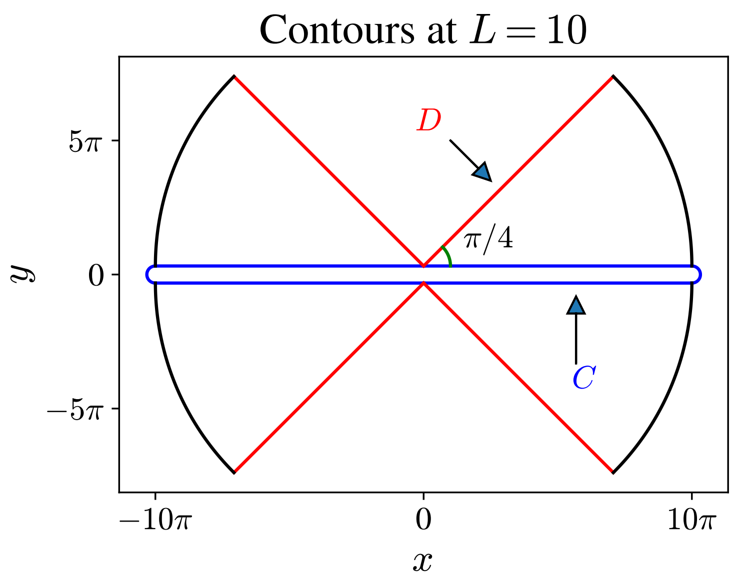

Figure 4: (Color online) Contours of integration. The blue contour C corresponds to Eq. 8. One can deform the contour C to D (the red lines), since the integrand on the arc (the black line) is exponentially small and the function $\ln f(z)$ is holomorphic when $\mathrm{Re}(z) \neq 0$ and $\mathrm{Im}(z) \neq 0$.

trum [36, 37]. It is reflected by a simple fact, $\langle \gamma_x \tilde{\gamma}_y \rangle = \mathrm{tr}(\gamma_x \tilde{\gamma}_y \rho_A)$. Here A is a subsystem and $x, y \in A$. Thus from the long-range correlation in Eq. 10, we find that entanglement spectrum contains a non-trivial value, $\epsilon_0 = \log(L/l - 1)$. Subsequently, the ϵ_0 results in the non-trivial entropy in Eq. 3. At the limit $l/L \ll 1$, the asymptotic expression of S is $\sim l/L \log(l/L)$. The form is highly-nontrivial, as it is a non-logarithmic function. But its magnitude is weaker than a pure logrithmic function [38].

Zero-modes are present and influencing entanglement entropy in other contexts [39–41], including CFTs [42]. But the effects of zero-modes are negligible in CFTs. On one hand, Eq. 10 is subleading relative to the $1/|x - y|$ decaying correlations in CFTs. On the other hand, the entanglement entropy in Eq. 3 is weaker than the logarithmic entropy. Thus the criticality Π is a better platform to observe the effect of zero-modes in field theory rather than CFTs.

5 Conclusion

For the criticality Π with quadratic dispersion, $\epsilon \sim \pm k^2$, we find a universal finite-size amplitude A as the coefficient in front of L^{-2} term in the ground state energy of the system. The magnitude of A is anomalously large as it is of the order of one. There exists rich phenomena in finite-size scaling functions around this criticality [10, 32–34]. For example, with Eq. 1 at $v \neq 0$, a universal finite-size scaling emerges as a function of the scale Lv, and the function has a peak at the topological side. In principle, the Lifshitz criticality can also be enriched by symmetries. In that case, the presence of boundary modes around Fermi surface and in case of breaking of some discrete symmetries (including chiral symmetry and time-reversal symmetry), one expects the emergence of a non-monotonic universal function of some scaling variable. This is an interesting and open problem.

The entanglement of the ground state is also found to be non-trivial, carrying a non-logarithmic entropy. This originates from the zero modes at the Fermi surface. Compared to CFTs, zero modes play a much more crucial role at the criticality Π. This offers an opportunity to observe the effects of zero modes in the fermionic field theory [39–41]. Similarly, one can also explore the behaviors of entanglement entropy and boundary entropy [32, 43–46] around Π.

Effects of interactions are not explored in the present work. The exciting question is establishing the interacting theory of the low energy sector of Π criticality. This question is beyond the scope of the Luttinger liquid [47, 48], where mostly the linear dispersion is considered.

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