New τ-based evaluation of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment

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Abstract

We revisit the radiative corrections to the $\tau^- \to \pi^-\pi^0\nu_\tau(\gamma)$ decays using Resonance Chiral Theory for the structure-dependent part. This channel is the most important one to understand the hadronic vacuum polarization piece of the muon g-2, which dominates the uncertainty of its data-driven prediction in the Standard Model. Using our results, the discrepancy between theory and experiment for this observable is reduced to $\sim 2.2\sigma$.

Contents

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1 Introduction

The combination of the first measurement from the new muon g-2 experiment at FNAL [1] with the final average from the muon g-2 measurements at BNL [2] yields

$$a_\mu^{\text{Exp}} = 116592061(41) \times 10^{-11}\,. \tag{1}$$

The Muon g-2 Theory Initiative quotes [3]

$$a_\mu^{\text{SM}} = 116591810(43) \times 10^{-11} \tag{2}$$

as the Standard Model prediction for a_μ [1], which deviates 4.2σ from $a_\mu^{\rm Exp}$. However, the very precise lattice evaluation of the (leading order) hadronic vacuum polarization contribution to the muon g-2 ($a_\mu^{HVP,LO}$) by the BMW Coll. reduces this difference to only 1.6σ [39]. While this result is corroborated or refuted by another lattice calculation with commensurate accuracy (see Aida El-Khadra and Gilberto Colangelo, these proceedings), it is timely to reexamine the dominant contribution to $a_\mu^{HVP,LO}$ using hadronic tau decay data (instead of $\sigma(e^+e^-\to \text{hadrons})$, as in [3]). We will present here our results [40], for an updated evaluation of the radiative corrections needed for such purpose and our value of $a_\mu^{HVP,LO}$, based on tau data (see also ref. [41]). Our study extends Refs. [42, 43], which first used Resonance Chiral Theory ($R\chi T$) [44, 45] in this setting [2], by going one order further in the chiral expansion.

In section 2 we introduce $a_\mu^{HVP,LO}$. In section 3 we summarize the computation of the radiative corrections to di-pion tau decays and in section 4 we present our results. Finally, we conclude in section 5.

2 Hadronic vacuum polarization

Based on analyticity and unitarity, loop integrals containing HVP insertions in photon propagators can be expressed as dispersive integrals over the cross-section of a virtual photon decaying into hadrons

$$a_\mu^{HVP,LO} = \frac{\alpha^2}{3\pi^2}\int_{m_\pi^2}^{\infty} \mathrm{d}s\frac{K(s)}{s}R(s)\,, \tag{3}$$

where $R(s) = \sigma(e^-e^-\to \text{hadrons}(\gamma))/\sigma(e^+e^-\to\mu^+\mu^-)|_{LO}$ and $K(s)$ is a kernel function behaving as $\sim 1/s$, which enhances the weight of the low-energy contributions. Indeed, about 73% of these (and 58% of their uncertainty) comes from the $\pi^+\pi^-(\gamma)$ final states for $4m_\pi^2 \le s \le 0.8$ GeV^2, in which we will focus here. Use of hadronic tau decay data in $a_\mu^{HVP,LO}$ requires to account for isosping-breaking (IB) corrections. At LO in them [42]

$$\sigma(e^+e^-\to\pi^+\pi^-(\gamma))(s) = \frac{K_\sigma(s)}{K_\Gamma(s)}\frac{d\Gamma(\tau^-\to\pi^-\pi^0\nu_\tau)}{ds}\frac{R_{IB}(s)}{S_{EW}}\,, \tag{4}$$

where $s=(p_\pi+p_\pi')^2$, and

$$\begin{aligned} R_{IB}(s) &= \frac{FSR(s)}{G_{EM}(s)}\frac{\beta^3_{\pi^+\pi^-}(s)}{\beta^3_{\pi^-\pi^0}(s)}\frac{F_V^{\pi^+\pi^-}(s)}{f_+^{\pi^-\pi^0}(s)}\,, \quad K_\sigma(s) = \frac{\pi\alpha^2}{3s}\,,\\ K_\Gamma(s) &= \frac{G_F^2|V_{ud}|^2M_\tau^3}{384\pi^3}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+\frac{2s}{M_\tau^2}\right). \end{aligned} \tag{5}$$

FSR stands for final-state radiation, the ratio of β functions corresponds to the slightly different phase space (PS) for $\pi^{0/+}$, and that of the form factors accounts for $\rho^-,\ldots$ or $\rho^0/\omega,\ldots$ resonances exchanged in the charged/neutral channel. G_{EM} includes the long-distance (real and virtual) radiative corrections and S_{EW} the universal short-distance ones. $K_{\sigma/\Gamma}$ collect the remaining differing factors between these processes. $\pi\pi$ upper/lower indices will be omitted in what follows.

For years, evaluations of $a_\mu^{HVP,LO}$ using tau data [29, 42, 43, 48–53] for the $\pi\pi$ cut have been closer to the world average for $a_\mu^{\rm exp}$ than those employing e^+e^- data. We must also note that

[1]This result is based on refs. [4–38].

[2]$R\chi T$ has also been employed in calculating the lightest pseudoscalar's pole contribution to a_μ [46, 47], in agreement with ref. [3].

the $\rho^0 - \gamma$ mixing was proposed [54] to bring both sets of data into agreement (although the ρ meson was treated as an elementary boson). These differences have been interpreted in the context of effective Lagrangians for non-standard interactions [55–57], yielding constraints on new physics at the TeV level (see P. Roig, these proceedings).

3 $\tau^- \to \pi^-\pi^0\nu_\tau(\gamma)$ decays

These decays were studied within $R\chi T$ [42, 43] including the contributions saturating the next-to-leading (NLO) order chiral low-energy constants (LECs) [3]. Here, we go [40] one order further in this expansion (NNLO). Refs. [60–62] analyzed these processes within a vector meson dominance model and put forward the importance of the $\rho - \omega - \pi$ couplings (these arise at chiral NNLO and thus were absent in refs. [42,43]) on the $G_{EM}(s)$. Our work [40] (see also [63] and Z. H. Guo, these proceedings) wanted to verify the importance of the $\rho - \omega - \pi$ vertices on $G_{EM}(s)$ and, further, to estimate an uncertainty for the original $R\chi T$ computation [42, 43] (and for ours, at the next chiral order), which is mandatory to assess the e^+e^- vs. τ data discrepancy in the two-pion channels, and its impact on $a_\mu^{HVP,LO}$.

The matrix element for the process with a real photon reads [64]

$$T = eG_F V_{ud}^* \epsilon^\mu(K)^* \Big\{ F_\nu \bar{u}(q)\gamma^\nu(1-\gamma_5)(M_\tau + \not{P} - \not{k})\gamma_\mu u(P) + (V_{\mu\nu} - A_{\mu\nu})\bar{u}(q)\gamma^\nu(1-\gamma_5)u(P) \Big\}, \quad (6)$$

where $F_\nu = (p_0 - p_-)_\nu f_+(s)/(2p \cdot k)$ [4]. The $V_{\mu\nu} - A_{\mu\nu}$ term can be split into structure-independent (SI) and dependent (SD) parts:

$$V^{\mu\nu} = V_{SI}^{\mu\nu} + V_{SD}^{\mu\nu}, \quad A^{\mu\nu} = A_{SD}^{\mu\nu}, \quad (7)$$

with the SI pieces fulfilling low-energy theorems [67, 68]. The evaluation of the SD part in $R\chi T$ up to NLO in the chiral expansion [43] only depends -keeping just the lightest spin-one resonance multiplets- on three resonance couplings (F_V, G_V, F_A) and the $U(3)$ $M_{V,A}$ masses. All of them can be determined employing a consistent set of short-distance constraints on the two-point Green functions, according to the operator product expansion of QCD, leading to [44, 45, 69]

$$F_V = \sqrt{2}F, F_A = F, G_V = \frac{F}{\sqrt{2}}, M_A = \sqrt{2}M_V, \quad (8)$$

with $M_V \sim M_\rho(770)$ and $F \sim F_\pi \sim 92$ MeV. These were employed in Ref. [43], without estimating any uncertainty associated to the previous high-energy restrictions or to the missing higher-order contributions in the chiral expansion.

Including the $R\chi T$ Lagrangian which saturates the chiral LECs at NNLO [70, 71] increases the number of couplings constants (now also in the odd-intrinsic parity sector) entering $(V/A)_{SD}^{\mu\nu}$ to 55. Short-distance QCD constraints on both sectors [70–72] reduce them to 41 undetermined couplings [5], challenging predictive power. Different phenomenological determinations have bounded six of them [58, 74–80]. To estimate the remaining 35, we rely on chiral counting [81–83]. For the indeed most important $\rho - \omega - \pi$ couplings we also fit the corresponding LECs [84]. This yields results compatible with the chiral counting and allows to reduce the error on these couplings by a half, approximately. Still, our uncertainty estimation based on this procedure appears to be extremely conservative when we float Gaussianly all parameters

[3] Radiative corrections for the one-meson tau decays have also been computed within $R\chi T$ [58, 59].

[4] For our numerical evaluations we use the dispersive $f_+(s)$ form factor from refs. [65, 66].

[5] These give rise to one-, two- and three-resonance mediated contributions, in both intrinsic parity sectors, to $\tau^- \to \pi^-\pi^0\nu_\tau\gamma$ decays, see figs. 1-7 in ref. [40]. Among these, the strength of the $\rho - \omega - \pi$ vertices is encoded in the κ_i^V couplings [71], see also ref. [73].

within their corresponding range (with so many of them, it is likely that a few are close to their maxima, producing thus an unrealistically big error). Lacking a better way to take this uncertainty into account, we report the errors obtained in this way [40]. We have estimated the errors of ref. [43], stemming from uncomputed higher-order chiral corrections, by adding, at NNLO, only couplings which are predicted using short-distance QCD constraints (this should give a lower bound on their uncertainty).

For different photon energy cuts, we predict [40] the di-pion invariant mass distribution, the branching ratio and the photon energy distribution in $\tau^- \to \pi^-\pi^0\nu_\tau$ decays (see figs. 10-14 and tables 2 and 3 in this reference). For definiteness, when cutting $E_\gamma < 100$ MeV, we obtain a branching ratio of $(1.9\pm0.3)\cdot10^{-3}$, while this is $(9.5^{+3.5}_{-0.5})\cdot10^{-4}$ for ref. [43] input. We emphasize the importance of measuring these observables at Belle-II. This will immediately shrink the errors of $a_\mu^{HVP,LO}$ using tau data and increase the new physics reach of other tests using them (see P. Roig, these proceedings).

The $G_{EM}(s)$ function is defined via

$$\frac{d\Gamma(\tau^-\to\pi^-\pi^0\nu_\tau(\gamma))}{ds} = \frac{G_F^2|V_{ud}|^2M_\tau^3S_{EW}}{384\pi^3}|f_+(s)|^2\left(1-\frac{s}{M_\tau^2}\right)^2\left(1-\frac{4m_\pi^2}{s}\right)^{3/2}\left(1+\frac{2s}{M_\tau^2}\right)G_{EM}(s). \tag{9}$$

In fig. 1 the solid line shows the $G_{EM}(s)$ function neglecting the structure-dependent part (only SI), the dashed and dotted lines are the NLO $G_{EM}(s)$ function (with either the constraints (8), or with the ones applying to two- and three-point Green functions, respectively). The blue shaded region is the full NNLO contribution, including (overestimated) uncertainties [6].

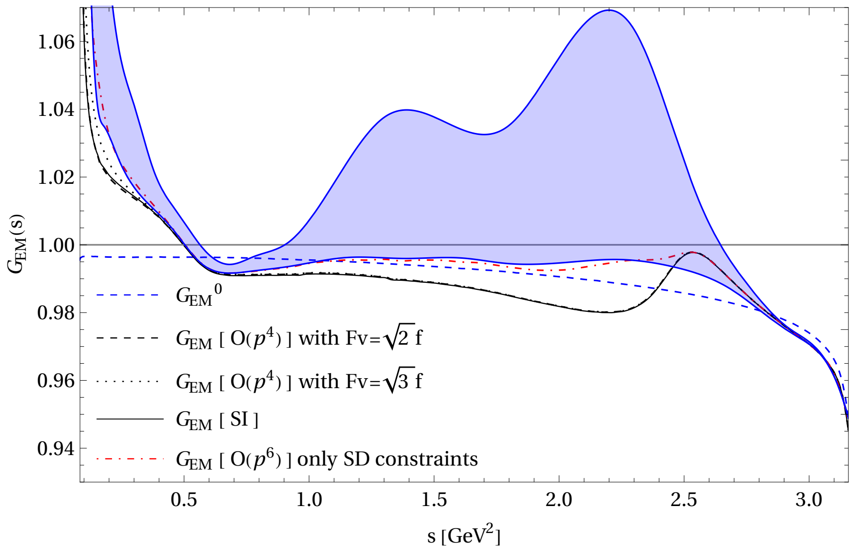

Figure 1: $G_{EM}(s)$ function (see main text for details).

4 Isospin-breaking corrections for the τ-based evaluation of $a_\mu^{HVP,LO}$

The effect of each IB correction can be computed using

$$\Delta a_\mu^{HVP,LO} = \frac{1}{4\pi^3}\int_{4m_\pi^2}^{s_{cut}} ds\, K(s)\frac{K_\sigma(s)}{K_\Gamma(s)}\frac{d\Gamma(\tau^-\to\pi^-\pi^0\nu_\tau(\gamma))}{ds}\left(\frac{R_{IB}(s)}{S_{EW}}-1\right), \tag{10}$$

[6] The $G_{EM}^0(s)$ function was introduced in ref. [42], and corresponds to the leading contribution in a low-energy photon expansion [67].

and is summarized in table 1. Di-pion tau decay data from ALEPH [85], Belle [86], CLEO [87] and OPAL [88] was used.

Source	$\Delta a_\mu^{HVP,LO}(\times 10^{11})$
S_{EW}	-103.1
PS	-74.5
FSR	-45.6 ± 4.6
FF	$+40.9 \pm 48.9$ $(+77.6 \pm 24.0)$
EM	$-15.9^{+5.7}_{-16.0}$
Total	$-107.1^{+49.4}_{-51.7}$ $\left(-70.4^{+25.1}_{-29.2}\right)$

Table 1: Different IB contributions to $a_\mu^{HVP,LO}$, obtained using tau data. For the form factors effect (FF) we follow the two approaches proposed in refs. [43] ([50]) (see ref. [40] for details).

As a test of these radiative corrections, the branching ratio ($\mathcal{B}$) of di-pion tau decays can be estimated from the measured $\sigma(e^+e^- \to \pi^+\pi^-(\gamma))$, by means of

$$\mathcal{B}(\tau^- \to \pi^-\pi^0\nu_\tau(\gamma)) = \mathcal{B}(\tau^- \to e^-\bar{\nu}_e\nu_\tau(\gamma)) \int_{4m_\pi^2}^{M_\tau^2} ds\,\sigma(e^+e^- \to \pi^+\pi^-(\gamma))\mathcal{N}(s)\frac{S_{EW}}{R_{IB}(s)}\,, \quad (11)$$

where $\mathcal{N}(s) = \frac{3|V_{ud}|^2}{2\pi\alpha^2 M_\tau^3} s\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+\frac{2s}{M_\tau^2}\right)$. Using BaBar $e^+e^- \to \pi^+\pi^-(\gamma)$ data [89] and the required IB corrections, we get

$$\mathcal{B}(\tau^- \to \pi^-\pi^0\nu_\tau(\gamma)) = (24.68 \pm 0.15)\%,\ \left(24.70^{+0.26}_{-0.15}\right)\% \quad (12)$$

where the first (second) result includes NLO (NNLO) chiral contributions to $G_{EM}(s)$ [7]. NNLO results agree at the one σ level with the ALEPH [85] and Belle [86] measurements: (25.24±0.39)% and (25.47 ± 0.13)%, respectively (a 4σ tension appears at NLO with respect to ALEPH). A comparison between the BaBar [89] and KLOE [90] $e^+e^- \to \pi^+\pi^-(\gamma)$ data and the prediction obtained from $\tau^- \to \pi^-\pi^0\nu_\tau(\gamma)$ including IB corrections is seen in Fig. 2. We are only showing it for the difference between the charged and neutral di-pion vector form factor called 'FF1' in ref. [40]. Results for 'FF2' are very similar (although the comparison is slightly worse). Our prediction is in better agreement with BaBar data. The dashed and solid lines represent the contributions at NLO using (8) or its generalization, respectively. The dotted line is the SI contribution. The red line depicts the envelope of $G_{EM}(s)$ at NNLO, with un overestimated uncertainty. The blue dotdashed line is the NNLO prediction using only those couplings which are determined by asymptotic QCD.

Accounting for all di-pion tau decay data [85–88], we get

$$10^{10} \times a_\mu^{HVP,LO} = 519.6 \pm 2.8_{\text{spectra}+\mathcal{B}}{}^{+1.9}_{-2.1\,\text{IB}}\,, \text{ at NLO,} \quad (13)$$

and

$$10^{10} \times a_\mu^{HVP,LO} = 514.6 \pm 2.8_{\text{spectra}+\mathcal{B}}{}^{+5.0}_{-3.9\,\text{IB}}\,, \text{ at NNLO.} \quad (14)$$

When these results are supplemented with the four-pion tau decay measurements and with e^+e^- data for the other channels and energies higher than the tau mass [8], we find the overall contribution

$$10^{10} \times a_\mu^{HVP,LO|_{\tau\,\text{data}}} = 705.7^{+4.0}_{-4.1}\,, \text{ at NLO}\,, \quad (15)$$

[7] Contributions to the uncertainty are given in ref. [40]. At NLO, errors are dominated by statistical uncertainty of BaBar data and by the error on $\Gamma_{\rho^+} - \Gamma_{\rho^0}$. At NNLO, these are overcome by the (asymmetric) error on $G_{EM}(s)$.

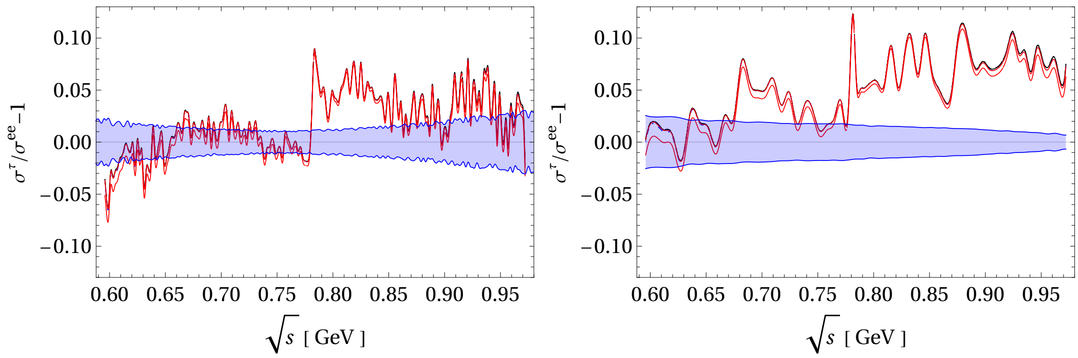

Figure 2: Comparison between the different data sets from BaBar (left) and KLOE (right) with our prediction using tau data and IB.

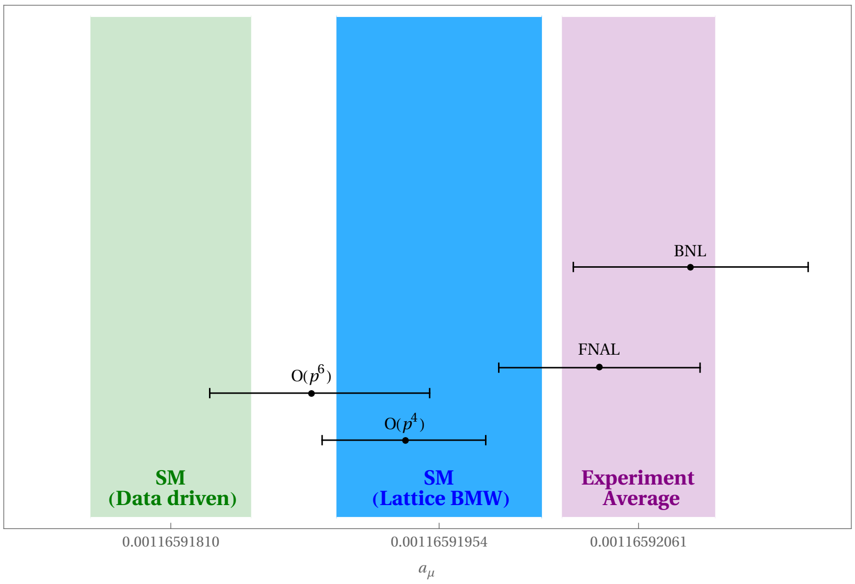

Figure 3: SM prediction [3], BMW value [39] and our results [40] compared to the a_μ measurements [1,2].

and

$$10^{10}\times a_\mu^{HVP,LO|_{\tau\ \mathrm{data}}} = 700.7^{+6.1}_{-5.2}\,,\ \text{at NNLO}\,. \tag{16}$$

When all other (QED, EW and subleading hadronic) contributions are added, the 4.2σ deficit of the SM prediction with respect to the FNAL+BNL average is reduced to

$$\Delta a_\mu = a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{th}} = (12.5\pm 6.0)\times 10^{-10}\,,\ \text{at NLO}\,, \tag{17}$$

and

$$\Delta a_\mu = \left(17.5^{+6.8}_{-7.5}\right)\times 10^{-10}\,,\ \text{at NNLO}\,, \tag{18}$$

2.1 and 2.3σ deviations, respectively. Our results [40] are compared to the BNL [2] and FNAL [1] measurements, and to the White Paper SM prediction [3] and the BMW lattice result [39] in fig. 3.

5 Conclusion

There is a global effort towards improving the Standard Model predictions of the hadronic contributions to a_μ. Specifically, there are dedicated studies to ameliorate lattice evaluations, dispersive data-driven computations, measurements of hadronic e^+e^- cross-section and τ decays, and related Monte Carlos.

The measurement of several observables in $\tau^- \to \pi^-\pi^0\nu_\tau\gamma$ decays will reduce drastically the model-dependent errors of the isospin-breaking corrections needed to use tau input in $a_\mu^{HVP,LO}$. Our computation of these improves agreement (also in spectra) between the measurements of e^+e^- and tau di-pion channels.

Our tau-based evaluation yields $\Delta a_\mu = a_\mu^{\rm exp} - a_\mu^{\rm th} = (12.5 \pm 6.0) \times 10^{-10}$, at NLO, and $\Delta a_\mu = \left(17.5^{+6.8}_{-7.5}\right) \times 10^{-10}$ at NNLO, corresponding to 2.1 and 2.3σ deviations, respectively.

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