A continuum determination of the strong isospin-breaking contribution to the muon anomalous magnetic moment

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Abstract

Lattice determinations of the Standard Model expectation for the leading order hadronic vacuum polarization contribution to the anomalous magnetic moment of the muon are now sufficiently precise that further progress requires the inclusion of contributions from strong and electromagnetic isospin-breaking effects. We provide a continuum, $SU(3)$ chiral perturbation theory based estimate of the former, using flavor-breaking hadronic τ decay sum rules to determine a crucial input higher-order low-energy constant. Implications of the form of this result are also discussed.

Contents

1 Introduction

The $3-4\sigma$ discrepancy between the BNL E989 result [1–3] and Standard Model (SM) expectation for a_μ, the anomalous magnetic moment of the muon, has been the subject of much ongoing attention. Interest in this discrepancy was increased further by the release of the FNAL E989 result [4] which produces a new world average $> 4\sigma$ from the SM expectation. [1]

[1]For a detailed review of the current experimental and theoretical situation, see Ref. [5].

Uncertainties in the determinations of hadronic contributions (in particular the leading-order hadronic-vacuum-polarization (LO HVP) and hadronic light-by-light contributions) currently dominate the uncertainty on the SM expectation. This paper focuses on the LO HVP contribution, $a_\mu^{LO,HVP}$, and ongoing attempts to reduce the uncertainty on this quantity.

There are currently two approaches to determining $a_\mu^{LO,HVP}$ in the SM, one "dispersive", based on experimental $e^+e^- \to hadrons$ cross-sections, and one employing the lattice.

In the dispersive approach, $a_\mu^{LO,HVP}$ is obtained as a weighted integral over the inclusive hadro-production cross-section ratio $R(s)$. The weight in question is exactly known and decreases monotonically with hadronic invariant squared mass, s, strongly emphasizing the low-s region, with $\sim$ 73% of the full contributions coming from the $\pi\pi$ exclusive mode. A current practical complication is the long-standing discrepancy between BaBar [6, 7] and KLOE [8] results for the $e^+e^- \to \pi^+\pi^-$ cross sections, which subsequent determinations by CMD2 [9–11], BESIII [12], CLEO-c [13] and SND [14] have so far failed to resolve.

In the lattice approach, an alternate representation of $a_\mu^{LO,HVP}$, as a weighted integral with respect to $Q^2 = -s$, with exactly known weight, of the $Q^2 = 0$-subtracted version, $\hat{\Pi}_{EM}(Q^2) = \Pi_{EM}(Q^2) - \Pi_{EM}(0)$, of the vacuum polarization of the electromagnetic (EM) current two-point function, $\Pi_{EM}^{\mu\nu}(Q)$ [15, 16]. Since $\Pi_{EM}^{\mu\nu}(Q)$ can be measured on the lattice, lattice determinations of $\hat{\Pi}_{EM}(Q^2)$, and hence $a_\mu^{LO,HVP}$ are also possible [17]. Progress on this approach has been rapid, with recent updates from all of BMW [18, 19], ETMC [20–22], RBC/UKQCD [23–25], FNAL/HPQCD/MILC [26, 27], Mainz [28], PACS [29] and Aubin *et al.* [30]. The current best lattice result, from BMW [19], has a precision of 0.8%. Further improved, sub-% precision determinations are expected from various lattice groups in the near future. At sub-% precision, evaluations of strong and EM isospin-breaking (IB) contributions are mandatory. These receive both quark-line-connected and quark-line-disconnected contributions. The latter are considerably more numerically challenging on the lattice.

This paper focuses on a_μ^{SIB}, the strong IB (SIB) contribution to $a_\mu^{LO,HVP}$. A number of lattice results exist for the connected part [19, 21, 23, 25, 26], but only one, from BMW [19], for the disconnected part. The BMW results show a strong cancellation between connected and disconnected contributions, as anticipated from the partially quenched chiral perturbation theory study of the $\pi\pi$ contributions reported in Ref. [25].This cancellation, and the numerical effort involved in evaluating disconnected contributions, motivate looking for an alternate continuum determination. This paper provides such a determination, using the Euclidean integral representation of $a_\mu^{LO,HVP}$ and the $SU(3)$ chiral perturbation theory (ChPT) representation of the SIB contribution to $\hat{\Pi}_{EM}(Q^2)$. Sec. 2 provides relevant background and notation, and Sec. 3 details of the required ChPT representation and our final result.

2 Background and notation

With $J_\mu^a = \bar{q}\frac{\lambda^a}{2}\gamma_\mu q$ the flavor octet of light-quark vector currents, the light-quark (u, d, s) part of the EM current, J_μ^{EM}, has the standard decomposition into $I = 1$ and 0 ($a = 3$ and 8) parts, $J_\mu^{EM} = V_\mu^3 + V_\mu^8/sqrt3$. The subtracted EM vacuum polarization has the related decomposition,

$$\hat{\Pi}_{EM}(Q^2) = \hat{\Pi}^{33}(Q^2) + \frac{2}{\sqrt{3}}\hat{\Pi}^{38}(Q^2) + \frac{1}{3}\hat{\Pi}^{88}(Q^2) \tag{1}$$

into pure isovector ($ab = 33$), pure isoscalar ($ab = 88$), and mixed isospin ($ab = 38$) parts. SIB is induced by the $I = 1$, $O(m_d - m_u)$ part of the QCD mass operator, and hence, to

$O(m_d - m_u)$, occurs only in the 38 part of $\hat{\Pi}_{EM}$. The SIB contribution to $\hat{\Pi}_{EM}(Q^2)$ is thus

$$\hat{\Pi}^{SIB}(Q^2) = \frac{2}{\sqrt{3}}\hat{\Pi}^{38}_{QCD}(Q^2)\,. \tag{2}$$

The QCD subscript, which denotes the $O(m_d - m_u)$ QCD contribution, will be dropped below.

In the weighted Euclidean integral formulation of Refs. [15, 16],

$$a_\mu^{LO,HVP} = -4\alpha^2 \int_0^\infty dQ^2 f(Q^2)\hat{\Pi}_{EM}(Q^2)\,, \tag{3}$$

with α the EM fine structure constant and $f(Q^2)$ an exactly known kernel. Replacing $\hat{\Pi}_{EM}(Q^2)$ by $\hat{\Pi}^{SIB}(Q^2)$ on the RHS of Eq. (3) produces the analogous representation of a_μ^{SIB}. For use in what follows, we also define the auxiliary quantity

$$a_\mu^{SIB}[Q^2_{max}] \equiv -4\alpha^2 \int_0^{Q^2_{max}} dQ^2 f(Q^2)\hat{\Pi}^{SIB}(Q^2)\,. \tag{4}$$

and the analogous auxiliary quantity $a_\mu^{LO,HVP}[Q^2_{max}]$. $f(Q^2)$ diverges as $1/\sqrt{Q^2}$ as $Q^2 \to 0$ and falls rapidly with increasing Q^2, creating a peak in the integrand of the integral for $a_\mu^{LO,HVP}$ at very low $Q^2 \simeq m_\mu^2/4$ – so low that $\hat{\Pi}_{EM}(Q^2)$ is very well approximated as linear in Q^2 in the region up to and including the peak. The Q^2 dependence of $f(Q^2)$ is thus such that the location of the peak is essentially just that of the peak in the product $Q^2 f(Q^2)$. This will, for the same reason, be true of the location of the peak in the integrand for a_μ^{SIB}. This raises the possibility that a_μ^{SIB} might be estimated using the ChPT representation of $\Pi^{SIB}(Q^2)$.

The convergence of $a_\mu^{LO,HVP}[Q^2_{max}]$ to $a_\mu^{LO,HVP}$ with increasing Q^2_{max} was investigated for the dominant $I = 1$ contribution using a highly physical dispersive model for $\hat{\Pi}^{33}(Q^2)$ constructed from precision non-strange τ decay data in Refs. [32, 33]. $\sim 82\%$, $\sim 92\%$ and $\sim 94\%$ of the full contribution was found to arise from the regions $Q^2 < 0.10\ GeV^2$, $Q^2 < 0.2\ GeV^2$ and $Q^2 < 0.25\ GeV^2 \simeq m_K^2$, respectively. For reasons discussed in detail in Sec. IIB of Ref. [31], we expect a similarly rapid approach to the $Q^2_{max} \to \infty$ limit for a_μ^{SIB}. With the region $0 < Q^2 < 025\ GeV^2 \simeq m_K^2$ plausibly in the range of validity of $SU(3)_F$ ChPT, a determination of $a_\mu^{SIB}[0.25\ GeV^2]$ obtained using ChPT for $\hat{\Pi}^{SIB}(Q^2)$ is thus expected to miss only $\sim 6\%$ of the total contribution to a_μ^{SIB}, provided the ChPT representation employed is accurate in this integration region.

The dispersive model of Refs. [32, 33] was also employed to explore the accuracy of the use of ChPT in the low-Q^2 region for the analogous $I = 1$ (33) contribution to $a_\mu^{LO,HVP}$. Using the next-to-next-to-leading-order (NNLO) form so that important ρ-region spectral contributions first encoded in the NNLO low-energy constant, C_{93}, are included, one finds an estimate for $a_\mu^{33}[0.25\ GeV^2]$ which overshoots the $Q^2_{max} = 0.25\ GeV^2$-truncated dispersive model result by $\sim 4.8\%$ and is only $\sim 1.5\%$ below the full $Q^2_{max} \to \infty$ model result. The 4.8% overshooting is a result of the NNLO form missing small yet-higher-order contributions of the ρ peak to the curvature of $\hat{\Pi}_{EM}(Q^2)$ with respect to Q^2, and naturally works in the opposite direction to the undershooting produced by truncating the integral at $Q^2_{max} = 0.25\ GeV^2$. A qualitatively similar cancellation of chiral-order-truncation and $Q^2_{max} \simeq 0.25\ GeV^2$ truncation effects is expected for the SIB case [31]. We will thus take the ChPT-based result for $a_\mu^{SIB}[0.25\ GeV^2]$ as our estimate for a_μ^{SIB}, and assign what should be a conservative 10% error for the uncertainty produced by the combination of the truncation in chiral order and truncation of the integral for a_μ^{SIB} at $Q^2_{max} = 0.25\ GeV^2$.

3 Results

The NNLO representation of Π^{38}, worked out in Ref. [36], implies

$$\hat{\Pi}^{SIB}(Q^2) = \frac{1}{2}\left(m_{K^0}^2 - m_{K^+}^2\right)_{QCD}\left[\frac{2i\bar{B}(\bar{m}_K^2, Q^2)}{Q^2} - \frac{1}{48\pi^2\bar{m}_K^2} + \right.$$
$$\left.\frac{8i\bar{B}(\bar{m}_K^2, Q^2)}{f_\pi^2}\left(\frac{i}{2}\bar{B}_{21}(m_\pi^2, Q^2) + i\bar{B}_{21}(\bar{m}_K^2, Q^2) + \frac{\log\left(m_\pi^2\bar{m}_K^4/\mu^6\right)}{384\pi^2} - L_9^r(\mu)\right)\right], \quad (5)$$

where $(m_{K^0}^2 - m_{K^+}^2)_{QCD}$ is the non-EM contribution to the kaon mass-squared splitting, $\bar{m}_K^2$ is the non-EM part of the average kaon squared mass, $f_\pi \simeq 92\ MeV$ is the pion decay constant, L_9^r is the usual renormalized NLO LEC of Gasser and Leutwyler [35], μ is the chiral renormalization scale. $\bar{B}(m^2, Q^2)$ and $\bar{B}_{21}(m^2, Q^2)$ are standard subtracted, equal-mass, two-propagator loop functions, whose explicit forms are given in Ref. [31]. The first and second lines of Eq. (5) contain the sums of NLO and NNLO contributions, respectively.

In what follows, we take $L_9^r(\mu = 0.77\ GeV) = 0.00593(43)$ from Ref. [37], and evaluate $(m_{K^0}^2 - m_{K^+}^2)_{QCD}$ using the FLAG 2019 result [38], 0.79(7), for the parameter, ϵ_D, which characterizes the breaking of Dashen's theorem [39]. As is well known, there is an $O(\alpha(m_d + m_u))$ ambiguity in the separation of strong and EM effects. [2]. Our SIB result corresponds to the FLAG choice of separation scheme and can be directly compared to lattice SIB results in the literature. With the above input, we find, to NNLO, the contributions

$$\begin{aligned}\left[a_\mu^{SIB}(0.25\ GeV^2)\right]_{NLO} &= 0.073 \times 10^{-10}\\ \left[a_\mu^{SIB}(0.25\ GeV^2)\right]_{NNLO} &= 0.552(37) \times 10^{-10}\,.\end{aligned} \quad (6)$$

The absence of an NLO pion loop contribution and smallness of the integrated NLO contribution reflects the exact NLO-level cancellation between connected and disconnected contributions from $\pi\pi$ intermediate states noted in Ref. [25].

The sum of the NLO and NNLO contributions in Eqs. (6), $0.625(37) \times 10^{-10}$, is considerably smaller than estimates of the contribution from the ρ-ω interference region obtained by integrating the IB part of the $\pi\pi$ cross section obtained from fits to those cross-sections. This is no surprise, since the effects of resonances, integrated out in forming the effective chiral Lagrangian, show up first in the chiral expansion of $\hat{\Pi}^{SIB}(Q^2)$ as tree-level contributions proportional to Q^2 and one factor of $(m_d - m_u)$. Since two further momentum factors are required in the relevant effective operator to produce the transverse kinematic tensor factor in the SIB two-point function $\Pi_{\mu\nu}^{SIB}(Q)$, such an effective operator will involve four derivatives and one power of the quark mass matrix, and hence be NNNLO in the chiral counting. To incorporate numerically relevant ρ-ω interference (and higher) region contributions, one must thus include also tree-level NNNLO contributions.

Fortunately, (i) such tree-level NNNLO contributions involve only a single NNLO LEC combination (associated with NNNLO operators number 944 and 945 in the basis of Ref. [40]), and (ii) this same combination also determines the only NNNLO tree-level contribution to the ChPT representation of the flavor-breaking (FB) difference of non-strange (flavor ud) and strange (flavor us) vector current polarizations, the spectral functions of which can be extracted from inclusive differential hadronic τ decay distributions [41]. The relevant NNNLO LEC combination, denoted $\delta C_{93}^{(1)}$ in Ref. [34], can thus be determined by an inverse-moment finite-energy sum rule (IMFESR) analysis of hadronic τ decay data. This was done initially, with older input, in Ref. [34]. We have rerun this analysis, employing updated information on

[2] See, e.g., Secs. 3.1.1 and 3.1.2 of the 2019 FLAG report [38] for a discussion.

exclusive-mode strange τ branching fractions [42], updated OPE input [38, 44, 45], and the recently updated version of the $I=1$, vector spectral function detailed in Ref. [43]. Further detail is provided in Ref. [31]. The updated IMFESR analysis produces an updated result for the slope of the FB polarization combination with respect to Q^2 at $Q^2=0$, which in turn, assuming the NNNLO tree-level term dominates contributions beyond NNLO, produces the updated result

$$\delta C_{93}^{(1)}\left(m_K^2-m_\pi^2\right)=0.00534(37)\,GeV^{-2}\,, \tag{7}$$

to which we assign a 30% uncertainty to reflect possible yet-higher-chiral-order corrections.

The corresponding tree-level NNNLO contribution to $\hat{\Pi}^{SIB}(Q^2)$,

$$\left[\hat{\Pi}^{SIB}(Q^2)\right]_{NNNLO,LEC}=-\frac{8}{3}Q^2\left(m_{K^0}^2-m_{K^+}^2\right)_{QCD}\,\delta C_{93}^{(1)}\,, \tag{8}$$

produces an additional contribution to a_μ^{SIB} of

$$\left[a_\mu^{SIB}\right]_{NNNLO}\simeq\left[a_\mu^{SIB}(0.25\;GeV^2)\right]_{NNNLO}=2.69(18)\times10^{-10}\,. \tag{9}$$

As expected, since it encodes numerically important resonance region contributions, this is significantly larger than the sum of NLO and NNLO contributions. It is also similar in size to phenomenological fit based estimates of contributions from the $\pi\pi$ ρ-ω interference region, but has the advantage over such estimates of including all contributions, from this and other regions, up to the order considered in the chiral expansion.

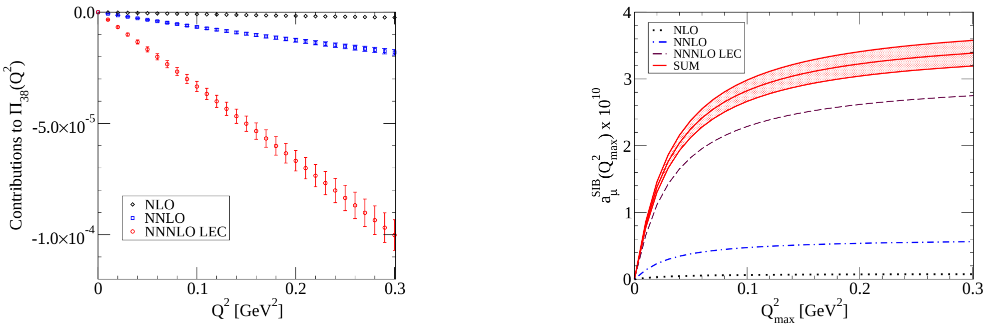

Figure 1: Left panel: NLO, NNLO and NNNLO LEC contributions to $\hat{\Pi}^{38}(Q^2)$, with errors as described in the text. Right panel: The Q^2_{max} dependence of the NLO, NNLO and NNNLO LEC contributions and NLO+NNLO+NNNLO LEC sum to a_μ^{SIB} as a function of Q^2_{max}, with error (as described in the text) shown only on the sum.

In the left panel of Fig. 1, we display the Q^2 dependence of the NLO, NNLO and NNNLO LEC contributions to $\hat{\Pi}^{38}(Q^2)$. It is relevant to note that, though the loop functions which determine the NLO and NNLO contributions are not strictly linear in Q^2, they are numerically very close to being so in the region of Q^2 of interest to the determination of a_μ^{SIB}. Adding the NLO, NNLO and NNNLO LEC contributions, we obtain as our final estimate for a_μ^{SIB},

$$a_\mu^{SIB}=3.32(4)(19)(33)(81)\times10^{-10}\,, \tag{10}$$

where the first error results from the uncertainty on the input for L_9^r, the second from the uncertainty on $\delta C_{93}^{(1)}$ quoted in Eq. (7) (dominated by uncertainties in the experimental strange hadronic τ decay distributions), the third from our 10% estimate for the combined uncertainty associated with the truncation of the integral for a_μ^{SIB} at $Q^2_{max}=0.25\;GeV^2$ and neglect of contributions beyond NNLO to the curvature of $\hat{\Pi}^{SIB}(Q^2)$ with respect to Q^2, and the fourth from our 30% estimate for the uncertainty in $\delta C_{93}^{(1)}$ induced by possible higher-chiral-order contributions to the slope of the FB polarization at $Q^2 = 0$ obtained from the updated version of the FB IMFESR analysis of Ref. [34].

The central values of the NLO, NNLO and NNNLO LEC contributions to $a_\mu^{SIB}[Q^2_{max}]$, together with the corresponding NLO+NNLO+NNNLO LEC sum, are shown as a function of Q^2_{max} in the right panel of Fig. 1. The error band on the total, which shows the quadrature sum of the LEC-uncertainty-induced NNLO and NNNLO LEC errors plotted in left panel, is dominated by the fourth of the uncertainties detailed above.

4 Conclusion

We have obtained a continuum, ChPT-based estimate of a_μ^{SIB}, the SIB contribution to the anomalous magnetic moment of the muon. A key ingredient in this analysis is the determination of the crucial NNNLO LEC, $\delta C_{93}^{(1)}$, from an FB IMFESR analysis of hadronic τ decay data. Our result, $a_\mu^{SIB} = 3.32(90) \times 10^{-10}$, agrees within errors with the only full lattice result, $1.93(1.20) \times 10^{-10}$, obtained by summing the connected and disconnected contributions reported in Ref. [19]. The dominance of our result by the contribution of the NNNLO LEC $\delta C_{93}^{(1)}$ makes clear that, as for $a_\mu^{LO,HVP}$, a_μ^{SIB} is dominated by resonance region contributions. As such, we expect small (few to several percent) finite volume (FV) effects on the lattice for the full connected+disconnected SIB sum. While FV effects for $a_\mu^{LO,HVP}$ at this level are not negligible on the scale of the current target a_μ precision, they are completely negligible, on this same precision scale, for the much smaller a_μ^{SIB} contribution. This is true only for the connected+disconnected SIB sum and not for the separate connected and disconnected parts, where significant FV effects have, e.g., been observed for the separate connected contribution.

Finally, given the dominance of the result for a_μ^{SIB} by the NNNLO contribution proportional to $\delta C_{93}^{(1)}$, and the linear-in-Q^2 behavior of this contribution, it would be of interest for future lattice studies to quote results for $d\hat{\Pi}^{SIB}(Q^2)/dQ^2$ at $Q^2 = 0$, a result which can be otained from the t^4 time moment of the lattice two-point function at zero spatial momentum [46].

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