# Exact dynamical correlations of nonlocal operators in quadratic open Fermion systems: a characteristic function approach

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## Abstract

**The dynamical correlations of nonlocal operators in general quadratic open fermion systems is still a challenging problem. Here we tackle this problem by developing a new formulation of open fermion many-body systems, namely, the characteristic function approach. Illustrating the technique, we analyze a finite Kitaev chain with boundary dissipation and consider anyon-type nonlocal excitations. We give explicit formula for the Green's functions, demonstrating an asymmetric light cone induced by the statistical angle $\phi$ and an increasing relaxation rate with $\phi$. We also analyze some other types of nonlocal operator correlations such as the full counting statistics of the charge number and the Loschmidt echo in a quench from the vacuum state. The former shows clear signature of a nonequilibrium quantum phase transition, while the later exhibits cusps at some critical times and hence demonstrates dynamical quantum phase transitions.**

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## Contents

# 1 Introduction

The interaction of a quantum system with its environment [1–3] can lead to various dissipation behaviors and the emergence of new collective phenomena, such as nonequilibrium phases and phase transitions driven by dissipation [4–9], universality and dynamic scaling behaviors at quantum transitions [10–16]. Understanding and controlling the behavior of quantum dissipative systems is also fundamental to the development of quantum-enhanced cutting-edge technologies such as quantum computing [17], quantum metrology [18], quantum state preparation or quantum reservoir engineering [19–28]. Although significant experimental advancements have been made in this context [29–32], dissipative quantum many-body problems are still quite challenging in theory. Within the so-called Markovian approximation, the open systems' Liouvillian dynamics is described by the Lindblad master equation [33, 34] for the time-dependent density matrix. A standard way of analyzing the master equation is by means of perturbation methods [35,36]. In addition, some exact solutions of the nonequilibrium steady states and the full spectrum of the Liouvillian have been obtained in some specific representative cases [37–46].

One specific instance that has attracted many interests is the open fermionic systems with *quadratic Lindbladian* [47–57], which can be solved exactly. However, even for such simple solvable systems, the dynamics of nonlocal operators is still challenging and desires efficient computation methods. Here we use *nonlocal operators* to refer to those operators containing a string operator of the form $\hat{O}_j = \exp[i\phi \sum_{l\leq j} \hat{c}_j^\dagger \hat{c}_j]$ (or more generally, an exponential function of bilinear fermion operators). Such operators appear in many important physical problems. For example, string order parameters have been used to characterize topological properties of quantum systems [58–61]. They also emerge in the studies of the Tonks-Girardeau gas [62, 63], the impenetrable anyons [64, 65], the XY Heisenberg chain [66], and the full counting statistics of quantum transport [67, 68]. The dynamical correlation functions of nonlocal operators in dissipative systems have not been investigated systematically, even in quadratic open systems. It represents a challenging and highly nontrivial theoretical problem.

Motivated by such challenges, here we put forward a new theoretical approach to open fermion systems by applying the idea of mappings between the Liouville-Fock space $\mathcal{K}$ and a Grassmann algebra $\mathcal{G}$, which can map operators to analytic functions of Grassmann variables and vice versa. The quantum master equation is transformed to a partial differential equation of the characteristic function of the density matrix, and all physical observables can be expressed in terms of this function. We name this new approach as *characteristic function approach* since the $\mathcal{K}$-$\mathcal{G}$ mappings and the characteristic function are essential concepts. This method could be seen as a fermion analogue of the phase-space method widely used in quantum optics [69, 70].

Our method, which can be useful for generic open fermion systems, is then applied to general quadratic fermion systems with linear Lindblad operators. We give exact solutions of the master equation, the steady state, the single-particle Green's function, the dynamical response function, and most importantly, the dynamical correlations of nonlocal operators. These general results are then applied to the Kitaev chain with boundary dissipation [50, 71, 72]. We obtain the spectrum of the matrix that determines the dissipative dynamics of the system, finding an excited state quantum phase transition (ESQPT) and its relationship with the nonequilibrium quantum phase transition (NQPT). We also compute the Green's functions of nonlocal excitations, namely, the hard-core anyons with statistical angle $\phi$, and find that the propagation of the excitations displays an asymmetric light-cone for $\phi \neq 0, \pi$, and the relaxation rate increases with the statistical angle. In addition, other types of nonlocal operator correlations such as the full counting statistics (FCS) of the charge number in a subsystem and the Loschmidt echo in quench dynamics can also be analyzed easily in our new approach and explicit formulas can be obtained. The FCS shows clear signature of the NQPT mentioned above, while the Loschmidt echo rate function exhibits cusps at some critical times in the quench from the vacuum state, giving evidence of dynamical quantum phase transitions (DQPT) in this dissipative system. These analyses demonstrate the feasibility and powerfulness of the characteristic function approach.

This paper is organized as follows. In Sec.2, we present the general formalism of the characteristic function approach and use it to give the exact solutions of various physical properties of the open fermion systems with quadratic Lindbladian, with emphasis on the dynamical correlations of nonlocal operators. In Sec.3 we analyze the boundary-driven Kitaev chain as an example, focusing on the Green's function of the hard-core anyons, the full counting statistics of the charge number in a subsystem, and the Loschmidt echo rate in a quench dynamics from the vacuum state. We conclude in Sec.4 with a summary of our main results and some discussions.

# 2 The characteristic function approach

## 2.1 Basic Formalism

We first develop a new general approach to solve quantum master equations of fermion systems. The basic idea is quite simple: the Liouville-Fock space $\mathcal{K}$ generated by fermion creation and annihilation operators $\{\hat{c}_1^\dagger, \hat{c}_1, \ldots, \hat{c}_N^\dagger, \hat{c}_N\}$ and the Grassmann algebra $\mathcal{G}$ generated by Grassmann variables $\{\bar{\xi}_1, \xi_1, \cdots, \bar{\xi}_N, \xi_N\}$ have the same dimension $2^{2N}$ and hence we can construct one-to-one mappings between these two spaces. In analogy to the phase-space functions and characteristic functions widely used in quantum optics [69], we define the mapping $\Theta$ from $\mathcal{K}$ to $\mathcal{G}$ as the *characteristic function* of the operators in $\mathcal{K}$:

$$\Theta : \hat{A} \in \mathcal{K} \to A_C(\bar{\xi}, \xi) \equiv \mathrm{Tr}[\hat{D}(\xi)\hat{A}], \tag{1}$$

where $\hat{D}(\xi) \equiv e^{\hat{c}^\dagger \xi - \bar{\xi}\hat{c}}$ is the fermion analogue of the boson displacement operator. Inversely, we have

$$\Omega : A_C(\bar{\xi}, \xi) \in \mathcal{G} \to \hat{A} = \int d\bar{\xi} d\xi \, A_C(\bar{\xi}, \xi) \left[ \frac{e^{i\pi\hat{N}} + \mathbb{1}}{2} \hat{D}^\dagger(\xi) + \frac{e^{i\pi\hat{N}} - \mathbb{1}}{2} \hat{D}(\xi) \right], \tag{2}$$

where $\hat{N} = \sum_i \hat{c}_i^\dagger \hat{c}_i$ is the total fermion number operator. It's straightforward to prove that $\Theta$ and $\Omega$ are reciprocal linear mappings. To do this, it's enough to show that for any analytic function $f(\bar{\eta},\eta) \in \mathcal{G}$, we have $f = \Theta[\Omega(f)]$.

$$\begin{aligned}
\Theta[\Omega(f)] &= \int d\bar{\alpha}d\alpha\, f(\bar{\alpha},\alpha)\,\mathrm{Tr}\left[e^{i\pi\hat{N}}\hat{D}^{\dagger}(\alpha)\hat{D}(\eta)\right] \\
&= \int d\bar{\alpha}d\alpha\, f(\bar{\alpha},\alpha)\,\mathrm{Tr}\left[e^{i\pi\hat{N}}\hat{D}(\eta-\alpha)\right] D(\alpha|\eta/2) \\
&= \int d\bar{\alpha}d\alpha\, f(\bar{\alpha},\alpha)\,\prod_k\left[(\alpha_k-\eta_k)(\bar{\alpha}_k-\bar{\eta}_k)\right] D(\alpha|\eta/2) \\
&= f(\bar{\eta},\eta).
\end{aligned}$$

We should note that the parity of the operators in $\mathcal{K}$ and the functions in $\mathcal{G}$ has significance in making these mappings. See Appendix.A for some details and useful formulas.

Using these mappings we can transform problems in the Liouville-Fock space, for example, the quantum master equation, to problems in the Grassmann algebra, and transform back if necessary. The advantage is that for functions in the Grassmann algebra we have rich analytic and algebraic tools.

Now consider an open system of $N$ sites with spinless fermions, whose dynamics is described by the quantum Lindblad master equation [33, 34] with Lindbladian $\mathcal{L}$ (we set $\hbar = 1$)

$$\partial_t\rho = \mathcal{L}(\rho) = -i[\hat{H},\rho] + \sum_\mu\left(2\hat{L}_\mu\rho\hat{L}_\mu^\dagger - \{\hat{L}_\mu^\dagger\hat{L}_\mu,\rho\}\right) \tag{3}$$

where $\hat{L}_\mu$ are the so-called Lindblad or jump operators. Although the *characteristic function approach* is a quite general theory for treating open fermion systems, here, for simplicity and as a starting point, we focus on general quadratic Hamiltonians

$$\hat{H} = \frac{1}{2}(\hat{c}^\dagger,\hat{c})\mathbb{H}\begin{pmatrix}\hat{c}\\ \hat{c}^\dagger\end{pmatrix}, \tag{4}$$

and linear Lindbaldian operators

$$\hat{L}_\mu = L_\mu^\dagger\begin{pmatrix}\hat{c}\\ \hat{c}^\dagger\end{pmatrix}, \quad \hat{L}_\mu^\dagger = (\hat{c}^\dagger,\hat{c})L_\mu, \tag{5}$$

where $(\hat{c}^\dagger,\hat{c}) = (\hat{c}_1^\dagger,\hat{c}_2^\dagger,\ldots,\hat{c}_N^\dagger,\hat{c}_1,\ldots,\hat{c}_N)$, $L_\mu(L_\mu^\dagger)$ are $2N$-dimensional column (row) vectors, while $\mathbb{H}$ is a $2N\times 2N$ matrix satisfying the symmetry requirement

$$\mathbb{H} + \tau_x\mathbb{H}^T\tau_x = 0, \tag{6}$$

where $\tau_{x,y,z}$ denote the Pauli matrices in the particle-hole subspace. Although such a *quadratic Lindbaldian* can be solved exactly by various methods [48–56], the computation of dynamical correlations of nonlocal operators is still a nontrivial and challenging problem. In the characteristic function approach we transform the quantum master equation of the density matrix into an equation for its characteristic function $F(\bar{\xi},\xi) \equiv \mathrm{Tr}[\hat{D}(\xi)\rho]$,

$$\partial_t F + (\bar{\xi},\xi)\left[i\mathbb{H} + \mathbb{X}_+\right]\begin{pmatrix}\bar{\partial}\\ \partial\end{pmatrix}F = -\frac{1}{2}(\bar{\xi},\xi)\mathbb{X}_-\begin{pmatrix}\xi\\ \bar{\xi}\end{pmatrix}F, \tag{7}$$

where

$$\mathbb{X}_\pm = \sum_\mu\left[L_\mu L_\mu^\dagger \pm \tau_x(L_\mu L_\mu^\dagger)^*\tau_x\right], \tag{8}$$

and $(\bar{\partial},\partial) = (\partial/\partial\bar{\xi}_1,\ldots,\partial/\partial\bar{\xi}_N,\partial/\partial\xi_1,\ldots,\partial/\partial\xi_N)$. The solution with initial condition $F(\bar{\xi},\xi;t=0) = F_0(\bar{\xi},\xi)$ is

$$F = F_0\left[(\bar{\xi},\xi)\mathbb{Q}(t)\right]\exp\left[-\frac{1}{2}(\bar{\xi},\xi)\mathbb{M}(t)\begin{pmatrix}\xi\\ \bar{\xi}\end{pmatrix}\right], \tag{9}$$

where the arguments of $F(\bar{\xi},\xi;t)$ have not been written explicitly for brevity, and

$$\mathbb{Q}(t) = e^{-(\mathbb{X}_+ + i\mathbb{H})t},\quad \bar{\mathbb{Q}}(t) = e^{-(\mathbb{X}_+ - i\mathbb{H})t},\quad \mathbb{M}(t) = \int_0^t dt'\ \mathbb{Q}(t')\ \mathbb{X}_-\ \bar{\mathbb{Q}}(t'). \tag{10}$$

The solution of Eq.(9) is a linear mapping from $F_0(\bar{\xi},\xi)$ to $F(\bar{\xi},\xi;t)$, which will be denoted as $F(\bar{\xi},\xi;t) = \mathcal{U}_t[F_0(\bar{\xi},\xi)]$. Obviously, $F(\bar{\xi},\xi;t) = \Theta[\rho(t)] = \Theta[e^{\mathcal{L}t}(\rho_0)] = \mathcal{U}_t[\Theta(\rho_0)]$, or more generally,

$$\Theta\star e^{\mathcal{L}t} = \mathcal{U}_t\star\Theta, \tag{11}$$

where $\star$ denotes the composition of two linear mappings.

### 2.2 Physical observables

Now let's discuss some physical properties of the open fermion system based on the solution given by Eq.(9). We remark that the results in this subsection could also be obtained by other methods, however, here we briefly present these results to show the completeness of our new method.

(i) The steady state can be obtained by taking the limit $t\to\infty$. If all the eigenvalues $\lambda_\alpha$ of $(\mathbb{X}_+ + i\mathbb{H})$ have positive real parts, i.e., $\mathrm{Re}\lambda_\alpha > 0$, then $\mathbb{Q}(t)\to 0$ while $\mathbb{M}(t)\to\mathbb{M}_\infty$ as $t\to\infty$, and the characteristic function approaches to

$$F_\infty = \exp\left[-\frac{1}{2}(\bar{\xi},\xi)\mathbb{M}_\infty\begin{pmatrix}\xi\\ \bar{\xi}\end{pmatrix}\right]. \tag{12}$$

This is a Gaussian state determined solely by the Hamiltonian and the dissipators, independent of the initial state. On the contrary, if some eigenvalues $\lambda_\alpha$ have zero real parts, $\mathbb{Q}(t)$ may not approach to zero and the system would have no unique steady state.

(ii) The covariance (or equal-time correlation) matrix can be expressed in terms of the characteristic function:

$$\mathbb{C}\equiv\left\langle\begin{pmatrix}\hat{c}\\ \hat{c}^\dagger\end{pmatrix}(\hat{c}^\dagger,\hat{c})\right\rangle = \frac{1}{2}\mathbb{1} + \begin{pmatrix}\bar{\partial}\\ \partial\end{pmatrix}(\partial,\bar{\partial})F(\bar{\xi},\xi)\Big|_0 \tag{13}$$

where $f(\bar{\xi},\xi)|_0$ means taking $\xi=\bar{\xi}=0$ at last. For the steady state described by Eq.(12), we have

$$\mathbb{C}_\infty = \frac{1}{2}\left(\mathbb{1} + \mathbb{M}_\infty - \tau_x\mathbb{M}_\infty^T\tau_x\right) = \frac{1}{2}\mathbb{1} + \mathbb{M}_\infty. \tag{14}$$

(iii) The nonequilibrium Green's functions, which describe the excitations in the steady state, can also be expressed in terms of the characteristic function. For example, the retarded Green function can be obtained through

$$G^{\mathrm{R}}(t)\equiv -i\theta(t)\left\langle\left\{\begin{pmatrix}\hat{c}(t)\\ \hat{c}^\dagger(t)\end{pmatrix},(\hat{c}^\dagger,\hat{c})\right\}\right\rangle_s = -i\theta(t)\begin{pmatrix}\bar{\partial}\\ \partial\end{pmatrix}\mathcal{U}_t\left[(\bar{\xi},\xi)\ F_s(\bar{\xi},\xi)\right]\Big|_0, \tag{15}$$

where $F_s$ is the characteristic function of the steady state $\rho_s$. For the Gaussian state given by Eq.(12) the retarded Green function simply reads $G^{\mathrm{R}}(t) = -i\theta(t)\mathbb{Q}(t)$.

(iv) Furthermore, the dynamical response function or the density-density correlation function can be defined as

$$D_{ij}(t) \equiv -i\theta(t)\langle[\hat{n}_i(t), \hat{n}_j]\rangle, \tag{16}$$

where $\hat{n}_j = \hat{c}_j^\dagger \hat{c}_j$. Using the same technique as that for the Green's functions we can obtain its expression in the steady state given by Eq.(12):

$$\begin{aligned} D_{ij}(t) &= -i\theta(t)\left\{[\mathbb{Q}\mathbb{M}_\infty]_{ij}[\bar{\mathbb{Q}}]_{ji} - [\mathbb{Q}]_{ij}[\mathbb{M}_\infty\bar{\mathbb{Q}}]_{ji} \right. \\ &\qquad \left. -[\mathbb{Q}\mathbb{M}_\infty]_{i+N,j}[\bar{\mathbb{Q}}]_{j,i+N} + [\mathbb{Q}]_{i+N,j}[\mathbb{M}_\infty\bar{\mathbb{Q}}]_{j,i+N}\right\}, \end{aligned} \tag{17}$$

where the time dependence of $\mathbb{Q}(t)$ and $\bar{\mathbb{Q}}(t)$ have not been written explicitly for brevity. In the same manner all dynamical correlation functions of local operators can be obtained by taking derivatives of the characteristic function, just as in Eq.(15).

### 2.3 Dynamical correlations of nonlocal operators

Now we turn to our main problem: the dynamical correlations of nonlocal operators. We would call the exponential of a general bilinear form of fermion creation and annihilation operators as *Gaussian operators*, and denote them as

$$\hat{\Gamma}_2(\mathbb{K}) \equiv \exp\left[\frac{1}{2}(\hat{c}^\dagger, \hat{c})\mathbb{K}\begin{pmatrix}\hat{c}\\ \hat{c}^\dagger\end{pmatrix}\right], \tag{18}$$

where $\mathbb{K}$ is a $2N \times 2N$ matrix satisfying $\mathbb{K} + \tau_x\mathbb{K}^T\tau_x = 0$. String operators can be treated as a special kind of Gaussian operators.

We would consider two types of dynamical correlations of nonlocal operators, namely,

$$\text{Type-I:} \qquad \mathrm{Tr}\left\{\hat{\Gamma}_2(\mathbb{K}_1)e^{\mathcal{L}t}\left[\hat{\Gamma}_2(\mathbb{K}_2)\hat{\Gamma}_2(\mathbb{K}_0)\right]\right\}, \tag{19}$$

which is a scalar and

$$\text{Type-II:} \quad \mathrm{Tr}\left\{\begin{pmatrix}\hat{c}\\ \hat{c}^\dagger\end{pmatrix}\hat{\Gamma}_2(\mathbb{K}_1)e^{\mathcal{L}_f t}\left[\hat{\Gamma}_2(\mathbb{K}_2)(\hat{c}^\dagger, \hat{c})\hat{\Gamma}_2(\mathbb{K}_0)\right]\right\}, \tag{20}$$

which is a $2N \times 2N$ matrix. Note that the Lindbladian superoperators in the above two equations are different, and this difference is explained in Appendix.B. We will give explicit formulas for these correlation functions. Before that, it's convenient to define the following matrices: $\mathbb{B}_0 \equiv \left[\mathbb{1} + e^{\mathbb{K}_0}\right]^{-1}$, $\mathbb{W}_{20} \equiv e^{\mathbb{K}_2}e^{\mathbb{K}_0}$, $\mathbb{W}_{02} \equiv e^{\mathbb{K}_0}e^{\mathbb{K}_2}$,

$$\begin{aligned} \mathbb{B}_{20} &\equiv \frac{1}{2}\mathbb{1} + \frac{1}{2}\mathbb{Q}(t)\frac{\mathbb{1} - \mathbb{W}_{20}}{\mathbb{1} + \mathbb{W}_{20}}\bar{\mathbb{Q}}(t) + \mathbb{M}(t), \\ \mathbb{B}_{02} &\equiv \frac{1}{2}\mathbb{1} + \frac{1}{2}\mathbb{Q}(t)\frac{\mathbb{1} - \mathbb{W}_{02}}{\mathbb{1} + \mathbb{W}_{02}}\bar{\mathbb{Q}}(t) + \mathbb{M}(t), \end{aligned}$$

and $\mathbb{R}_{20} \equiv \mathbb{B}_0 + e^{\mathbb{K}_2}(\mathbb{1} - \mathbb{B}_0)$, $\mathbb{R}_{02} \equiv \mathbb{B}_0 + (\mathbb{1} - \mathbb{B}_0)e^{\mathbb{K}_2}$, $\mathbb{S}_{20} \equiv \mathbb{B}_{20} + (\mathbb{1} - \mathbb{B}_{20})e^{\mathbb{K}_1}$, $\mathbb{S}_{02} \equiv \mathbb{B}_{02} + (\mathbb{1} - \mathbb{B}_{02})e^{\mathbb{K}_1}$.

Using the three linear mappings $\Omega, \Theta$ and $\mathcal{U}_t$, we have

$$\mathrm{Tr}\left\{\hat{\Gamma}_2(\mathbb{K}_1)e^{\mathcal{L}t}\left[\hat{\Gamma}_2(\mathbb{K}_2)\hat{\Gamma}_2(\mathbb{K}_0)\right]\right\} = \mathrm{Tr}\left\{\hat{\Gamma}_2(\mathbb{K}_1)\Omega \star \mathcal{U}_t \star \Theta\left[\hat{\Gamma}_2(\mathbb{K}_2)\hat{\Gamma}_2(\mathbb{K}_0)\right]\right\}.$$

Now we compute the three mappings one by one:

$$\begin{aligned}
&\text{(i).} && \Theta\left[\hat{\Gamma}_2(\mathbb{K}_2)\hat{\Gamma}_2(\mathbb{K}_0)\right] = \sqrt{\det(\mathbb{1}+\mathbb{W}_{20})}\exp\left[-\frac{1}{2}(\bar{\xi},\xi)\frac{1}{\mathbb{1}+\mathbb{W}_{20}}\begin{pmatrix}\xi\\ \bar{\xi}\end{pmatrix}\right],\\
&\text{(ii).} && \mathcal{U}_t\star\Theta\left[\hat{\Gamma}_2(\mathbb{K}_2)\hat{\Gamma}_2(\mathbb{K}_0)\right]\\
&= && \sqrt{\det(\mathbb{1}+\mathbb{W}_{20})}\exp\left[-\frac{1}{2}(\bar{\xi},\xi)\left(\mathbb{Q}(t)\frac{1}{\mathbb{1}+\mathbb{W}_{20}}\bar{\mathbb{Q}}(t)+\mathbb{M}(t)\right)\begin{pmatrix}\xi\\ \bar{\xi}\end{pmatrix}\right]\\
&= && \sqrt{\det(\mathbb{1}+\mathbb{W}_2)}\exp\left[-\frac{1}{2}(\bar{\xi},\xi)\mathbb{B}_{20}\begin{pmatrix}\xi\\ \bar{\xi}\end{pmatrix}\right],\\
&\text{(iii).} && \Omega\star\mathcal{U}_t\star\Theta\left[\hat{\Gamma}_2(\mathbb{K}_2)\hat{\Gamma}_2(\mathbb{K}_0)\right] = \sqrt{\det(\mathbb{1}+\mathbb{W}_{20})}\sqrt{\det\mathbb{B}_{20}}\,\hat{\Gamma}_2(\mathbb{K}_{B_{20}}),
\end{aligned}$$

where $\mathbb{K}_{B_{20}}$ is defined through $\mathbb{B}_{20}(\mathbb{1}+e^{\mathbb{K}_{B_{20}}})=\mathbb{1}$. Note that in (ii) we have changed the matrix in the exponential to $\mathbb{B}_{20}$ to satisfy the requirement $\mathbb{B}_{20}+\tau_x\mathbb{B}_{20}^T\tau_x=\mathbb{1}$. Finally, taking the trace gives the result:

$$\mathrm{Tr}\left\{\hat{\Gamma}_2(\mathbb{K}_1)e^{\mathcal{L}t}\left[\hat{\Gamma}_2(\mathbb{K}_2)\hat{\Gamma}_2(\mathbb{K}_0)\right]\right\} = \sqrt{\det(\mathbb{1}+\mathbb{W}_{20})\det\mathbb{S}_{20}}. \tag{21}$$

When $t=0$, $\mathbb{Q}=\mathbb{1},\mathbb{M}=0$, and $\mathbb{B}_{20}=[1+\mathbb{W}_{20}]^{-1}$, then we can obtain the static correlation function $\mathrm{Tr}\left\{\hat{\Gamma}_2(\mathbb{K}_1)\hat{\Gamma}_2(\mathbb{K}_2)\hat{\Gamma}_2(\mathbb{K}_0)\right\}=\sqrt{\det\left[\mathbb{1}+e^{\mathbb{K}_1}e^{\mathbb{K}_2}e^{\mathbb{K}_0}\right]}$.

Two remarks should be added here. (1) An issue of the determinant formulas is that the sign of the square root of the determinant has to be determined. In some simple cases the square root of a determinant can be rewritten as a Pfaffian [81]. However, this is difficult for general cases, especially for products of several Gaussian operators. In practical calculations the sign can be determined as follows. For $Z(\mathbb{A})=\sqrt{\det[\mathbb{1}+e^{\mathbb{A}}]}$, we consider $Z(\lambda\mathbb{A})$, which should be an analytic function of $\lambda$. This determines the correct way of taking the sign of the square root: the sign has to be taken so that $Z(\lambda\mathbb{A})$ is everywhere analytic and at $\lambda=0$ one has $Z(0)=2^N$. (2) Some matrices used in these formulas should satisfy certain symmetry requirements, namely, $\mathbb{A}+\tau_x\mathbb{A}^T\tau_x=0$ for $\mathbb{A}=\mathbb{H},\mathbb{M}(t),\mathbb{K}_{0,1,2}$, while $\mathbb{A}+\tau_x\mathbb{A}^T\tau_x=\mathbb{1}$ for $\mathbb{A}=\mathbb{B}_0,\mathbb{B}_{20}$ and $\mathbb{B}_{02}$.

Now consider the dynamical correlations of nonlocal single-particle operators, which takes the type-II form of Eq.(20). Even for quadratic Lindbladian these correlations are difficult to compute. Here we use the characteristic function approach to solve this problem. The correlation can be rewritten as

$$\begin{aligned}
&\mathrm{Tr}\left\{\begin{pmatrix}\hat{c}\\ \hat{c}^\dagger\end{pmatrix}\hat{\Gamma}_2(\mathbb{K}_1)e^{\mathcal{L}_f t}\left[\hat{\Gamma}_2(\mathbb{K}_2)(\hat{c}^\dagger,\hat{c})\hat{\Gamma}_2(\mathbb{K}_0)\right]\right\}\\
=\ &\mathrm{Tr}\left\{\begin{pmatrix}\hat{c}\\ \hat{c}^\dagger\end{pmatrix}\hat{\Gamma}_2(\mathbb{K}_1)e^{i\pi\hat{N}}e^{\mathcal{L}t}\left[e^{i\pi\hat{N}}\hat{\Gamma}_2(\mathbb{K}_2)(\hat{c}^\dagger,\hat{c})\hat{\Gamma}_2(\mathbb{K}_0)\right]\right\}\\
=\ &\mathrm{Tr}\left\{\begin{pmatrix}\hat{c}\\ \hat{c}^\dagger\end{pmatrix}\hat{\Gamma}_2(\mathbb{K}_1)e^{i\pi\hat{N}}\Omega\star\mathcal{U}_t\star\Theta\left[e^{i\pi\hat{N}}\hat{\Gamma}_2(\mathbb{K}_2)(\hat{c}^\dagger,\hat{c})\hat{\Gamma}_2(\mathbb{K}_0)\right]\right\}.
\end{aligned}$$

Then we can do the three mappings $\Omega,\mathcal{U}_t$ and $\Theta$ one by one, and make the trace to obtain the final result:

$$\begin{aligned}
&\mathrm{Tr}\left\{\begin{pmatrix}\hat{c}\\ \hat{c}^\dagger\end{pmatrix}\hat{\Gamma}_2(\mathbb{K}_1)e^{\mathcal{L}_f t}\left[\hat{\Gamma}_2(\mathbb{K}_2)(\hat{c}^\dagger,\hat{c})\hat{\Gamma}_2(\mathbb{K}_0)\right]\right\}\\
=\ &\frac{\sqrt{\det[\mathbb{R}_{20}]\det[\mathbb{S}_{20}]}}{\sqrt{\det[\mathbb{B}_0]}}e^{\mathbb{K}_1}[\mathbb{S}_{20}]^{-1}\mathbb{Q}(t)\mathbb{B}_0[\mathbb{R}_{20}]^{-1}e^{\mathbb{K}_2}.
\end{aligned} \tag{22}$$

By exchanging $\mathbb{K}_2$ and $\mathbb{K}_0$, we have another form

$$\begin{aligned}&\mathrm{Tr}\left\{\left(\begin{array}{c}\hat{c}\\ \hat{c}^\dagger\end{array}\right)\hat{\Gamma}_2(\mathbb{K}_1)e^{\mathcal{L}_f t}\left[\hat{\Gamma}_2(\mathbb{K}_0)(\hat{c}^\dagger,\hat{c})\hat{\Gamma}_2(\mathbb{K}_2)\right]\right\}\\ =&\frac{\sqrt{\det[\mathbb{R}_{02}]\det[\mathbb{S}_{02}]}}{\sqrt{\det[\mathbb{B}_0]}}e^{\mathbb{K}_1}[\mathbb{S}_{02}]^{-1}\mathbb{Q}(t)[\mathbb{R}_{02}]^{-1}(\mathbb{1}-\mathbb{B}_0).\end{aligned} \tag{23}$$

We would not give the technical details here since the procedure is lengthy but straightforward. We just give three remarks.

(i) If $\mathbb{K}_1=\mathbb{K}_2=0$, then $\mathbb{R}_{20}=\mathbb{S}_{20}=\mathbb{1}$, and the correlations would reduce to that of local operators:

$$\mathrm{Tr}\left\{\left(\begin{array}{c}\hat{c}\\ \hat{c}^\dagger\end{array}\right)e^{\mathcal{L}_f t}\left[(\hat{c}^\dagger,\hat{c})\hat{\Gamma}_2(\mathbb{K}_0)\right]\right\}=\mathbb{Q}(t)\frac{\sqrt{\det\left[\mathbb{1}+e^{\mathbb{K}_0}\right]}}{\mathbb{1}+e^{\mathbb{K}_0}}.$$

(ii)If $t=0$, then $\mathbb{Q}=\mathbb{1},\mathbb{M}=0$ and $\mathbb{B}_{20}=(\mathbb{1}+\mathbb{W}_{20})^{-1}$, and the result would reduce to the static correlations:

$$\mathrm{Tr}\left\{\left(\begin{array}{c}\hat{c}\\ \hat{c}^\dagger\end{array}\right)\hat{\Gamma}_2(\mathbb{K}_1)\hat{\Gamma}_2(\mathbb{K}_2)(\hat{c}^\dagger,\hat{c})\hat{\Gamma}_2(\mathbb{K}_0)\right]=\frac{\sqrt{\det\left[\mathbb{1}+e^{\mathbb{K}_1}e^{\mathbb{K}_2}e^{\mathbb{K}_0}\right]}}{\mathbb{1}+e^{\mathbb{K}_1}e^{\mathbb{K}_2}e^{\mathbb{K}_0}}e^{\mathbb{K}_1}e^{\mathbb{K}_2}, \tag{24}$$

(iii) If we consider the correlations in the steady state given by Eq.(12), we should note that the corresponding density matrix is

$$\rho_s=\sqrt{\det\left(\frac{1}{2}\mathbb{1}+\mathbb{M}_\infty\right)}\,\hat{\Gamma}_2(\mathbb{K}_0), \tag{25}$$

where $\mathbb{K}_0$ is determined by $\left(\frac{1}{2}\mathbb{1}+\mathbb{M}_\infty\right)\left(\mathbb{1}+e^{\mathbb{K}_0}\right)=\mathbb{1}$, and the corresponding $\mathbb{B}_0=\frac{1}{2}\mathbb{1}+\mathbb{M}_\infty$.

## 3 Kitaev chain with boundary dissipation

In this section we take the Kitaev chain with boundary dissipation as an example to illustrate the general techniques developed above.

### 3.1 The Model and the spectrum

The Hamiltonian is

$$\hat{H}_K=\sum_{l=1}^{N-1}\left[(J\hat{c}_l^\dagger\hat{c}_{l+1}+\Delta\hat{c}_l\hat{c}_{l+1})+\mathrm{h.c.}\right]-\mu\sum_{l=1}^{N}\hat{c}_l^\dagger\hat{c}_l, \tag{26}$$

which can be rewritten as a bilinear form of Eq.(4). We consider single-particle gain and loss dissipators,

$$\hat{L}_{j+}=\sqrt{\gamma_{j+}}\,\hat{c}_j^\dagger,\quad \hat{L}_{j-}=\sqrt{\gamma_{j-}}\,\hat{c}_j, \tag{27}$$

The simplest nontrivial dissipations act only on the first and last site, i.e., $\gamma_{1\pm}=\gamma_{N\pm}=\gamma_\pm$ and all other dissipators vanish. With this setting the model is essentially equivalent to the boundary-driven XY spin chain [47–50]. Therefore we can immediately infer that there is an NQPT [47] in the $\Delta$-$\mu$ space at the critical lines $\pm\mu_c/J=\pm2[1-(\Delta/J)^2]$. Namely, there is the so called long-range magnetic correlation (LRMC) phase for $|\mu|<\mu_c$ and the non-LRMC phase for $|\mu|>\mu_c$.

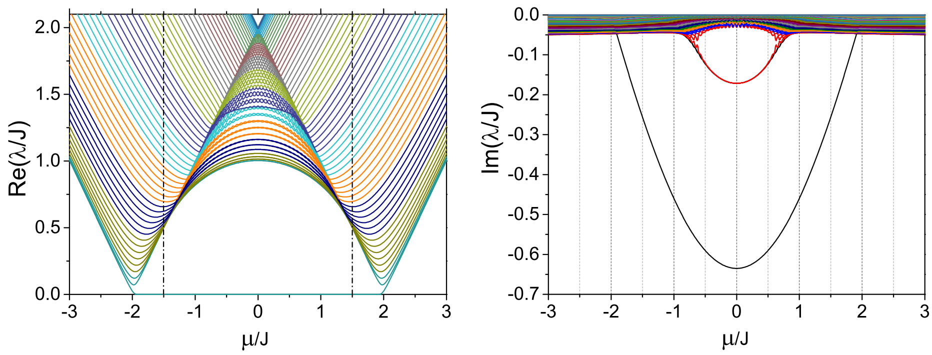


Figure 1: The real and imaginary part of the eigenvalues $\lambda_\alpha$ of $\mathbb{H} - i\mathbb{X}_+$. Since the real part is symmetric about the origin, only the positive half has been shown. The parameters are chosen as: $\Delta/J = 0.5, \gamma_-/J = 0.5, \gamma_+/J = 0.2$ and $N = 64$. The dashed lines in the left plot denote the critical chemical potential $\pm\mu_c/J = \pm 2[1 - (\Delta/J)^2] = \pm 1.5$. Between the two dashed lines there is a region where the energy levels have high degeneracy. In the right plot the lowest black line corresponds to the edge modes with zero real energy.

As seen from the solution of the quadratic Lindbladian, the dynamics is completely determined by three matrices: $\mathbb{H}$ and $\mathbb{X}_\pm$. In fact, the matrix $\mathbb{H} - i\mathbb{X}_+$ determines the dissipative dynamics and the Liouvillian spectrum. In Fig.1 we plot the real and imaginary parts of the eigenvalues $\lambda_\alpha, \alpha = 1, 2, \ldots, 2N$ of the matrix $\mathbb{H} - i\mathbb{X}_+$. Two features can be observed: (i) There are two degenerate modes with $\mathrm{Re}\lambda = 0$ when $|\mu/J| \leq 2$. The corresponding left and right eigenvectors are localized at the edges, similar to the Majorana zero modes in the closed system. However, in the steady state phase diagram there is no corresponding topological phase transition at $\mu/J = \pm 2$. This is because these edge modes do not contribute to the steady state as a result of the particle-hole symmetry of the edge modes and the matrix $\mathbb{X}_-$. Furthermore, the imaginary part of the eigenvalues of the edge modes is negative and has large absolute value. This means that the edge modes decay very rapidly in the dissipative dynamics.

(ii) In the left plot of Fig.1 we also observe that there is a region where the energy levels have many crossings. This abrupt change of level degeneracy is a characteristic signature of the so-called ESQPT [73]. In fact the level structure is similar to (but different from) that of the nonlinear Kerr oscillator where the ESQPT has been investigated systematically in a recent paper [74]. In the thermodynamic limit $N \to \infty$ the bulk spectrum is insensitive to the boundary dissipation and is given by the spectrum of $\mathbb{H}$,

$$\mathrm{Re}\lambda = \pm 2J\sqrt{\left(\cos q - \frac{\mu}{2J}\right)^2 + \frac{\Delta^2}{J^2}\sin^2 q} \tag{28}$$

with $q \in (-\pi, \pi]$ (see, e.g., [75, 76]). The structure of this dispersion relation qualitatively changes as the chemical potential crosses the critical values, $\pm\mu_c/J = \pm 2[1 - (\Delta/J)^2]$. These critical values determine phase boundaries of both the ESQPT and the NQPT. This coincidence suggests us a close relationship between ESQPT and NQPT: in the weak dissipation limit ($\gamma_\pm \to 0$) a NQPT would correspond to an ESQPT, but not the ground-state quantum phase transition. This relationship is an interesting issue that deserves further investigations.

### 3.2 The Green's function

Now we compute the dynamics of nonlocal excitations, namely, the Green's functions of the hard-core anyons. In one dimension it's well-known that the hard-core anyons satisfy the exchange statistics

$$\hat{f}_l\hat{f}_m^\dagger + e^{-i\phi\,\mathrm{sgn}(l-m)}\hat{f}_m^\dagger\hat{f}_l = \delta_{lm}, \qquad \hat{f}_l\hat{f}_m + e^{i\phi\,\mathrm{sgn}(l-m)}\hat{f}_m\hat{f}_l = 0, \tag{29}$$

where

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases}$$

They can be transformed to spinless fermions multiplied by a string operator,

$$\hat{f}_l^\dagger \equiv \hat{c}_l^\dagger e^{i\phi\sum_{m\le l}\hat{n}_l}, \quad \hat{f}_l \equiv e^{-i\phi\sum_{m\le l}\hat{n}_l}\hat{c}_l. \tag{30}$$

Our motivation of studying such excitations is twofold. First, if the fermionic Hamiltonian is obtained from a hard-core anyon or hard-core boson (Tonks-Girardeau gas or XY spin chain) model, correlations of such nonlocal operators would have physical importance in the original system. Second, even in this fermion model, string order parameters may be useful to characterize topological properties [58–61].

Here we express the Green's functions explicitly. For that purpose we define the following matrices:

$$\begin{aligned}
&\mathbb{R}_\pm^{j0} \equiv \mathbb{B}_0 + e^{\pm i\phi\tau_z\mathbb{D}_j}(\mathbb{1}-\mathbb{B}_0), \quad \mathbb{R}_\pm^{0j} \equiv \mathbb{B}_0 + (\mathbb{1}-\mathbb{B}_0)e^{\pm i\phi\tau_z\mathbb{D}_j},\\
&\mathbb{B}_\pm^{j0} \equiv \frac{1}{2}\mathbb{1} + \frac{1}{2}\mathbb{Q}(t)\frac{\mathbb{1}-e^{\pm i\phi\tau_z\mathbb{D}_j}e^{\mathbb{K}_0}}{\mathbb{1}+e^{\pm i\phi\tau_z\mathbb{D}_j}e^{\mathbb{K}_0}}\bar{\mathbb{Q}}(t) + \mathbb{M}(t),\\
&\mathbb{B}_\pm^{0j} \equiv \frac{1}{2}\mathbb{1} + \frac{1}{2}\mathbb{Q}(t)\frac{\mathbb{1}-e^{\mathbb{K}_0}e^{\pm i\phi\tau_z\mathbb{D}_j}}{\mathbb{1}+e^{\mathbb{K}_0}e^{\pm i\phi\tau_z\mathbb{D}_j}}\bar{\mathbb{Q}}(t) + \mathbb{M}(t),\\
&\mathbb{S}_{ab}^{j0l} \equiv \mathbb{B}_a^{j0} + (\mathbb{1}-\mathbb{B}_a^{j0})e^{bi\phi\tau_z\mathbb{D}_l}, \quad \mathbb{S}_{ab}^{0jl} \equiv \mathbb{B}_a^{0j} + (\mathbb{1}-\mathbb{B}_a^{0j})e^{bi\phi\tau_z\mathbb{D}_l},
\end{aligned}$$

where $a, b = \pm$, $\mathbb{B}_0 = \frac{1}{2}\mathbb{1} + \mathbb{M}_\infty$, $\tau_z\mathbb{D}_j$ means $\tau_z \otimes \mathbb{D}_j$, and $\mathbb{D}_j$ is a diagonal $N \times N$ matrix with diagonal elements $(\mathbb{D}_j)_{mm} = 1$ if $m \le j$ and 0 otherwise.

First, the greater Green's function for $t > 0$ reads

$$\begin{aligned}
iG_{lj}^>(t) &= \langle \hat{f}_l(t)\hat{f}_j^\dagger\rangle = \mathrm{Tr}\left\{e^{-i\phi\hat{Q}_l}\hat{c}_l e^{\mathcal{L}_f t}\left[\hat{c}_j^\dagger e^{i\phi\hat{Q}_j}\rho_s\right]\right\}\\
&= e^{i\phi(j-l)/2}\sqrt{\det\mathbb{B}_0}\,\mathrm{Tr}\left\{\hat{c}_l\hat{\Gamma}_2(-i\phi\tau_z\mathbb{D}_l)e^{\mathcal{L}_f t}\left[\hat{\Gamma}_2(i\phi\tau_z\mathbb{D}_j)\hat{c}_j^\dagger\hat{\Gamma}_2(\mathbb{K}_0)\right]\right\}.
\end{aligned}$$

Using Eq.(22) and setting $\mathbb{K}_1 = -i\phi\tau_z\mathbb{D}_l$, $\mathbb{K}_2 = i\phi\tau_z\mathbb{D}_j$, we obtain

$$iG_{lj}^>(t) = e^{i\phi(j-l)/2}\sqrt{\det\mathbb{R}_+^{j0}\det\mathbb{S}_{+-}^{j0l}}\left\{\left[\mathbb{S}_{+-}^{j0l}\right]^{-1}\mathbb{Q}\mathbb{B}_0\left[\mathbb{R}_+^{j0}\right]^{-1}\right\}_{lj}. \tag{31}$$

Similarly we can obtain

$$iG_{lj}^>(-t) = e^{i\phi(j-l)/2}\sqrt{\det\mathbb{R}_-^{0l}\det\mathbb{S}_{-+}^{0lj}}\left\{\left[\mathbb{S}_{-+}^{0lj}\right]^{-1}\mathbb{Q}\left[\mathbb{R}_-^{0l}\right]^{-1}(\mathbb{1}-\mathbb{B}_0)\right\}_{N+j,N+l}. \tag{32}$$

We can prove that they satisfy the relation, $iG_{jl}^>(-t) = \left[iG_{lj}^>(t)\right]^*$.

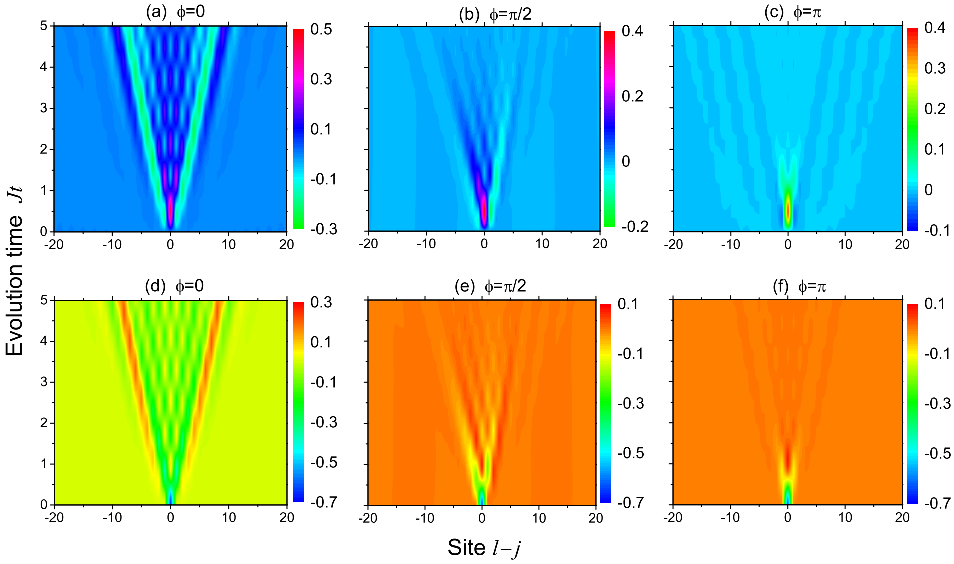


Figure 2: The real (top panel) and imaginary (bottom panel) part of the greater Green's function $G^>_{lj}(t)$ in a chain with $N = 65$ sites for three different statistical angles $\phi = 0, \pi/2$ and $\pi$. The site $j$ is fixed at the center of the chain, $j = 33$, and $\mu/J = 2.0, \Delta/J = 0.1, \gamma_-/J = 0.1, \gamma_+/J = 0.05$.

Second, the lesser Green's function $iG^<_{lj}(t) = \langle \hat{f}^\dagger_j \hat{f}_l(t)\rangle$ for $t > 0$ can be obtained in a similar manner:

$$iG^<_{lj}(t) = e^{i\phi(j-l)/2}\sqrt{\det \mathbb{R}^{0j}_+ \det \mathbb{S}^{0jl}_{+-}}\left\{\left[\mathbb{S}^{0jl}_{+-}\right]^{-1}\mathbb{Q}\left[\mathbb{R}^{0j}_+\right]^{-1}(\mathbb{1}-\mathbb{B}_0)\right\}_{lj}, \quad (33)$$

$$iG^<_{lj}(-t) = e^{i\phi(j-l)/2}\sqrt{\det \mathbb{R}^{l0}_- \det \mathbb{S}^{l0j}_{-+}}\left\{\left[\mathbb{S}^{l0j}_{-+}\right]^{-1}\mathbb{Q}\mathbb{B}_0\left[\mathbb{R}^{l0}_-\right]^{-1}\right\}_{N+j,N+l}. \quad (34)$$

When $t = 0$, the lesser Green's function would reduce to the steady-state one-particle density matrix, which is studied in Appendix.C.

In Fig.2 we plot the real and imaginary part the greater Green's function $G^>_{lj}(t)$ in a chain with $N = 65$ sites for three different statistical angles $\phi = 0, \pi/2$ and $\pi$. The site $j$ is fixed at the center of the chain and the figure displays the propagation of the excitation in space-time. For $\phi = 0$, i.e., spinless fermions, the propagation shows a clear symmetric light cone. However, for $0 < \phi < \pi$, the light-cone becomes asymmetric, as shown in Fig.2(b) and Fig.2(e) for $\phi = \pi/2$. This asymmetric propagation is induced by the statistical angle. To show this, we label the Green's function $G^>_{lj}(t)$ with the angle $\phi$. Then we have

$$G^>_{lj}(t;\phi) = G^>_{l'j'}(t;-\phi), \quad (35)$$

where $l'(j')$ is the site that $l$ is mapped to under reflection about the center of the chain. So the light-cones in Fig.2 should be symmetric only for $\phi = 0, \pi$.

We also observe that the greater Green's function decay rapidly for large statistical angles. This behavior could be seen clearly in Fig.3, where the local Green's function $G^>_{jj}(t)$ at the center of the chain is plotted as a function of time for $\phi = 0, \pi/5, \pi/2$ and $\pi$. The relaxation rate increases with the statistical angle $\phi$. Physically, the hard-core anyons have strong interactions due to the hard-core constraint, leading to scattering processes and finite relaxation rate. The boundary dissipation also lead to relaxation of excitations, however, it's weak for the system parameters chosen in Fig.2 and Fig.3.

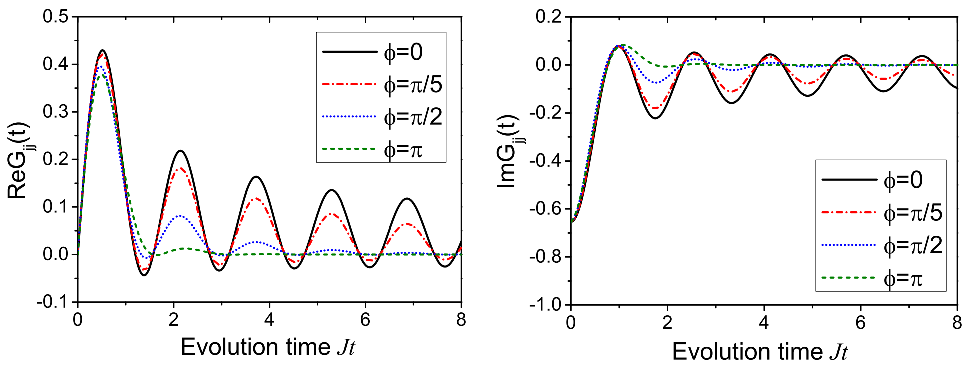


Figure 3: The real and imaginary part of the local greater Green's function $G^{>}_{jj}(t)$ at the center $j = 33$ in a chain with $N = 65$ sites for $\phi = 0, \pi/5, \pi/2$ and $\pi$. The other parameters are the same as that in Fig.2.

### 3.3 Full counting statistics of charge number

The charge number fluctuations in a subsystem is an important quantity in quantum many-body systems. It has been demonstrated that fluctuations and the full counting statistics (FCS) of charge or other conserved quantities (such as the block magnetization in certain spin chains) may contain information about the full entanglement scaling of a system split into two parts [77–80]. Here we consider the FCS of the charge distribution of a subsystem $A$ in the chain. For this purpose, we define the number operator $\hat{Q}_A$ as $\hat{Q}_A = \sum_{j\in A} \hat{c}_j^\dagger \hat{c}_j$, and a diagonal $N \times N$ matrix $\mathbb{D}_A$ with diagonal elements

$$(\mathbb{D}_A)_{ll} = \begin{cases} 1 & \text{if } j \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then $e^{\lambda \hat{Q}_A} = \hat{\Gamma}_1(\lambda \mathbb{D}_A) = \hat{\Gamma}_2(\lambda \tau_z \mathbb{D}_A) e^{\lambda \mathrm{Tr}(\mathbb{D}_A)/2}$, which can be taken as a special Gaussian operator. Suppose that the initial state is a Gaussian state with the density matrix

$$\rho(0) = \frac{e^{-\beta \hat{H}_0}}{\mathrm{Tr} e^{-\beta \hat{H}_0}}, \quad \hat{H}_0 = \frac{1}{2}(\hat{c}^\dagger, \hat{c}) \mathbb{H}_0 \begin{pmatrix} \hat{c} \\ \hat{c}^\dagger \end{pmatrix}.$$

Then the counting statistic function at time $t$ is

$$\chi(\lambda, t) = \sum_n P_n(t) e^{\lambda n} = \frac{1}{\mathrm{Tr}[e^{-\beta \hat{H}_0}]} \mathrm{Tr}\left\{ e^{\lambda \hat{Q}_A} e^{\mathcal{L}t}[e^{-\beta \hat{H}_0}] \right\}, \tag{36}$$

which could be taken as a special case of Eq.(21), and hence the result can be obtained immediately,

$$\chi(\lambda, t) = e^{\lambda \mathrm{Tr}(\mathbb{D}_A)/2} \sqrt{\det\left[\mathbb{B}(t) + e^{\lambda \tau_z \mathbb{D}_A}(\mathbb{1} - \mathbb{B}(t))\right]}, \tag{37}$$

where $\mathbb{B}(t) = \frac{1}{2}\mathbb{1} + \mathbb{Q}(t)\left(\mathbb{B}_0 - \frac{1}{2}\mathbb{1}\right)\bar{\mathbb{Q}}(t) + \mathbb{M}(t)$, and $\mathbb{B}_0 = [\mathbb{1} + e^{-\beta \mathbb{H}_0}]^{-1}$. This expression generalizes the result obtained by Klich [81] to dissipative systems. As $t \to \infty$, the state would approaches to the steady state with the density matrix $\rho_s = \sqrt{\det\left(\frac{1}{2}\mathbb{1} + \mathbb{M}_\infty\right)}\, \hat{\Gamma}_2(\mathbb{K}_0)$,

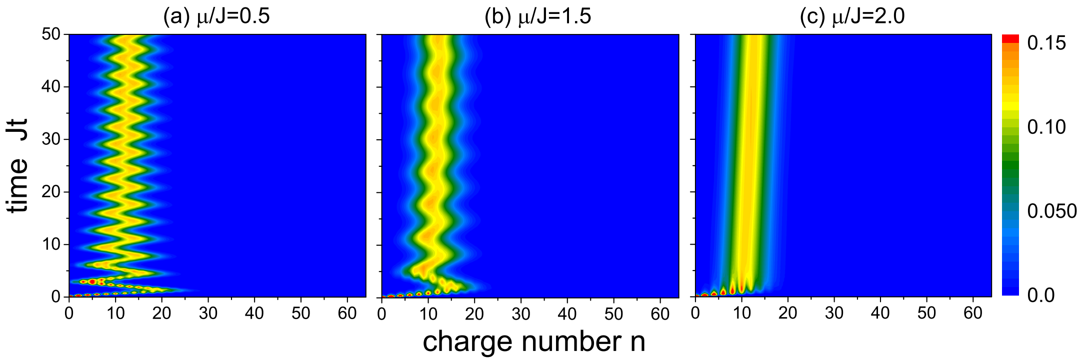


Figure 4: The dynamical evolution of the FCS $P_n(t)$ of the charge number in half of the chain from an initial vacuum state. The parameters are: $\Delta/J = 0.5, \gamma_-/J = 0.1, \gamma_+/J = 0.05$ and $N = 128$.

and the counting statistic function approaches to its steady value

$$\chi_s(\lambda) = e^{\lambda \mathrm{Tr}(\mathbb{D}_A)/2} \sqrt{\det\left[\left(\frac{1}{2}\mathbb{1} + \mathbb{M}_\infty\right) + e^{\lambda\tau_z \mathbb{D}_A}\left(\frac{1}{2}\mathbb{1} - \mathbb{M}_\infty\right)\right]}. \tag{38}$$

From this expression of the counting statistic function we can derive the probability distribution $P_n$ of the charge number $\hat{Q}_A$.

In Fig.4 we plot the dynamical evolution of the FCS of the charge number in half of the chain with $N = 128$ sites. The initial state is chosen as the vacuum state, $\rho_0 = |0\rangle\langle 0|$, and hence at $t = 0$ we have $P_0 = 1, P_{n\neq 0} = 0$. As the system evolves, the distribution $P_n(t)$ changes with time. For $\mu = 0.5J < \mu_c$, the distribution $P_n(t)$ oscillates rapidly, while for $\mu = 2.0J > \mu_c$, the distribution almost does not oscillate and monotonically approaches to its steady-state value. This could be taken as a dynamical signature of the NQPT occurring at $\mu = \mu_c$. For the parameters chosen in Fig.4, the relaxation time is very long and hence we plot the steady-state value in Fig.5. The left plot shows the distribution $P_n$ as a function of $\mu$ while the right plot shows the distribution for three representative chemical potentials, $\mu = 0, \mu = 1.5J$ and $\mu = 3.0J$. We see that there are obvious singularities at $\mu = \pm\mu_c$ and $\mu = 0$, where NQPT occurs. So we conclude that both the dynamical evolution and the steady-state value of the FCS of the charge number could reveal the NQPT.

### 3.4 Loschmidt Echo and Dynamical Quantum Phase Transitions

One particularly interesting phenomenon in real-time dynamics of quantum many-body systems are DQPTs in the sense that an observable changes nonsmoothly at a critical time after a quench [82, 83]. Since in many experiments the physical systems are subject to dissipation, it is important to consider the fate of DQPTs in nonunitary dynamics. It has been shown that for simple Fermionic models the DQPTs may persist in the presence of dissipation [84–88]. Here we consider the possibility of DQPTs in the boundary-driven Kitatev chain. To characterize the quench dynamics we need a generalization of the Loschmidt echo $L(t)$ for mixed states. Following a recent Letter [88] we use the definition $L(t) = \mathrm{Tr}[\rho(0)\rho(t)]$, and the rate function $r(t) = -(1/N)\ln L(t)$. As initial state we

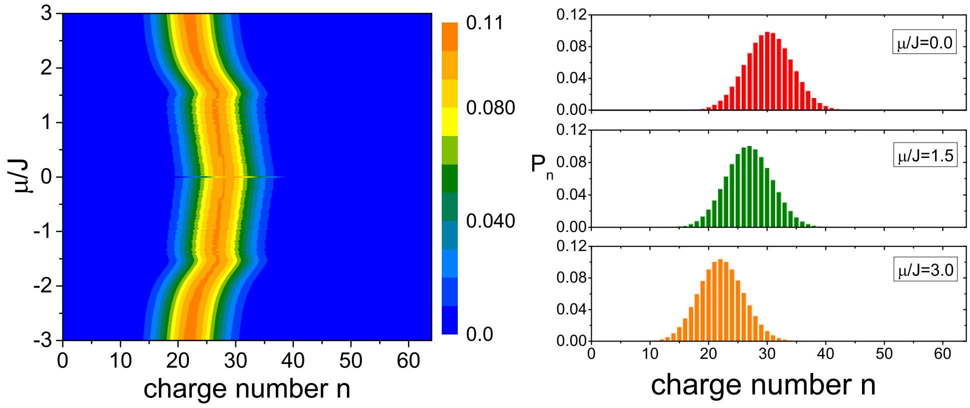


Figure 5: The steady-state FCS $P_n$ of the charge number in half of the chain. The parameters are the same as that in Fig.4. The left plot shows singularities at $\mu = 0$ and $\mu = \pm\mu_c = \pm 1.5J$.

choose the vacuum state, which corresponds to the fully polarized ferromagnetic state in the context of the XY spin chain. This state can be taken as a Gaussian state with the density matrix $\rho = e^{-\beta \hat{H}_0}/\mathrm{Tr}[e^{-\beta \hat{H}_0}]$, where $\hat{H}_0 = -\mu \sum_l \hat{c}_l^\dagger \hat{c}_l$ and $\beta\mu \to -\infty$. Then the Loschmidt echo $L(t)$ takes the form of Eq.(21) and can be simplified as

$$L(t) = \sqrt{\det\left[\mathbb{B}_0\mathbb{B} + (\mathbb{1} - \mathbb{B}_0)(\mathbb{1} - \mathbb{B})\right]}, \tag{39}$$

and the rate function

$$r(t) = -\frac{1}{2N}\mathrm{Tr}\ln\left[\mathbb{B}_0\mathbb{B} + (\mathbb{1} - \mathbb{B}_0)(\mathbb{1} - \mathbb{B})\right], \tag{40}$$

where $\mathbb{B} = \frac{1}{2}\mathbb{1} + \mathbb{Q}(t)\left(\mathbb{B}_0 - \frac{1}{2}\mathbb{1}\right)\bar{\mathbb{Q}}(t) + \mathbb{M}(t)$ and $\mathbb{B}_0 = \left[\mathbb{1} + e^{-\beta\mathbb{H}_0}\right]^{-1}$.

In Fig.6 we show this rate function for several different dissipation rates and system sizes. We see that for the chosen parameters DQPTs occur, i.e., the rate function develops cusps at critical times. In the left plot we fix the dissipation rates $\gamma_{1\pm} = \gamma_{N\pm} = \gamma_{\pm}$. and increase the system size $N$. We see that the cusps are smoothed for small system sizes, but becomes sharper and sharper as the size increases. In the right plot we fix the system size $N = 100$ and increase the dissipation rates. It's obvious that the dissipations lead to a damping of the peaks but the cusps still persist. Even more interestingly, for the chosen parameters, a new cusp emerges near $Jt = 5$, where the unitary dynamics shows a plateau. The persistence of DQPTs and the emergence of new cusps in dissipative dynamics is generic and does not require fine turning of parameters. This can be easily verified numerically by using our theoretical approach.

## 4 Conclusion and discussion

In summary, we have developed a general theoretical approach to solve open fermion systems and apply it to systems with quadratic Lindbladian. We focus on the dynamical correlations of nonlocal operators and give exact explicit formulas based on our characteristic function approach. We then take the boundary-driven Kitaev chain as an example to illustrate the general ideas and formulas. We compute the Green's functions of hard-core

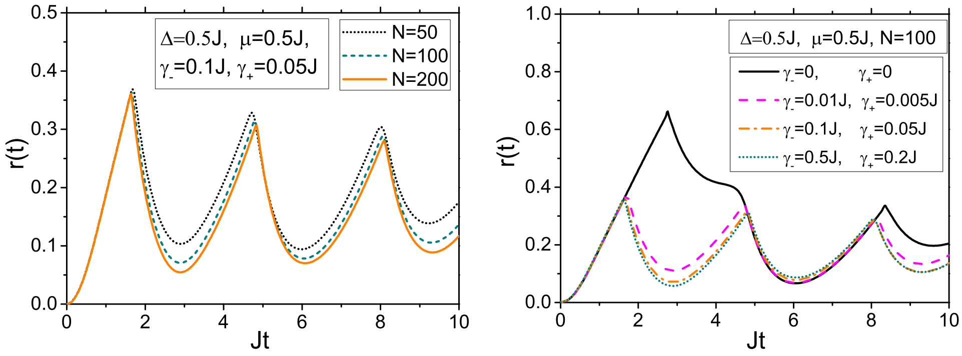


Figure 6: Loschmidt rate function $r(t)$ of the boundary-driven Kitaev chain. The dissipation rates are chosen to be $\gamma_{1\pm} = \gamma_{N\pm} = \gamma_{\pm}$. The left plot shows the rate function for fixed dissipation and different system sizes $N$. The right plot shows the rate function for fixed $N = 100$ and increasing dissipation rates.

anyons with statistical angle $\phi$, and find that the propagation of the nonlocal excitations displays an asymmetric light-cone for $0 < \phi < \pi$, and the relaxation rate increases with $\phi$. In addition, two other types of nonlocal operator correlations such as the FCS of the charge number and the Loschmidt echo in quench dynamics are also analyzed and explicit formulas are obtained. The FCS shows clear signature of the steady-state NQPT, while the Loschmidt echo rate function exhibits cusps at some critical times in the quench from the vacuum state, demonstrating DQPTs in this dissipative system.

The characteristic function approach is a new and general theoretical method to treat open fermion systems. We would apply and extend this method to solve some other physical problems. For example, in the presence of dephasing, the Liouvillian is no longer quadratic and has no simple solutions like the quadratic Lindbladian. However, we find that the dynamical correlation functions can be obtained by making Taylor expansions of the characteristic function. Another important application is the full counting statistics in dissipative transport. Introduction of a counting field brings nonlocal operators naturally, which can be treated by using the techniques given in this paper. Results in these directions would be presented in future works.

## Acknowledgements

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## A Some useful formulas

In this appendix we give some concepts and formulas that are useful in deriving the results in the main text.

(1) The parity operator $\hat{P}_F$ in $\mathcal{K}$ can be defined by the transformation $\hat{P}_F(\hat{c}, \hat{c}^\dagger)\hat{P}_F =$

$(-\hat{c}, -\hat{c}^\dagger)$. Obviously, one representation of the parity operator is $\hat{P}_F = e^{i\pi\hat{N}}$. Similarly, the parity operator $P_g$ in $\mathcal{G}$ can be defined as $P_g f(\bar{\xi}, \xi) = f(-\bar{\xi}, -\xi)$, and one representation of $P_g$ is

$$P_g = \exp\left[i\pi \sum_k (\xi_k \partial_k + \bar{\xi}_k \bar{\partial}_k)\right]. \tag{41}$$

(2) The displacement operator $\hat{D}(\xi) \equiv e^{\hat{c}^\dagger \xi - \bar{\xi}\hat{c}}$ has the properties:

$$\mathrm{Tr}\hat{D}(\xi) = 2^N, \quad \mathrm{Tr}\left[e^{i\pi\hat{N}}\hat{D}(\xi)\right] = \prod_{k=1}^{N} \xi_k \bar{\xi}_k, \tag{42}$$

and the integration is

$$\int d\bar{\xi} d\xi \, \hat{D}(\xi) = \frac{1}{2^N} e^{i\pi\hat{N}}. \tag{43}$$

(3) A mixed operator involves both fermion operators and Grassmann variables, i.e., it's an element of the direct product space $\mathcal{K} \bigotimes \mathcal{G}$. Since fermion creation/annihilation operators anticommute with Grassmann variables, we should be careful in computing traces of such operators. We can use the following rules: (i) If $f(\bar{\eta}, \eta)$ has even parity, i.e., $f(\bar{\eta}, \eta) = f(-\bar{\eta}, -\eta)$, then $\mathrm{Tr}[\hat{A} f(\bar{\eta}, \eta)] = \mathrm{Tr}[\hat{A}] f(\bar{\eta}, \eta)$; (ii) If $f(\bar{\eta}, \eta)$ has odd parity, i.e., $f(\bar{\eta}, \eta) = -f(-\bar{\eta}, -\eta)$, then $\mathrm{Tr}[\hat{A} f(\bar{\eta}, \eta)] = \mathrm{Tr}[\hat{A} e^{i\pi\hat{N}}] f(\bar{\eta}, \eta)$.

(4) The $\Theta$ mapping of basic Gaussian operators:

$$\mathrm{Tr}\left[\hat{\Gamma}_2(\mathbb{K})\hat{D}(\xi)\right] = \sqrt{\det(\mathbb{1} + e^{\mathbb{K}})} \exp\left[-\frac{1}{2}(\bar{\xi}, \xi)\frac{1}{\mathbb{1} + e^{\mathbb{K}}}\begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}\right], \tag{44}$$

$$\mathrm{Tr}\left[(\hat{c}^\dagger, \hat{c})\hat{\Gamma}_2(\mathbb{K}) e^{i\pi\hat{N}}\hat{D}(\xi)\right] = -\left\{(\bar{\xi}, \xi)\frac{1}{\mathbb{1} + e^{\mathbb{K}}}\right\} \mathrm{Tr}\left[\hat{\Gamma}_2(\mathbb{K})\hat{D}(\xi)\right]. \tag{45}$$

(5) The $\Omega$ mapping of basic Gaussian functions:

$$\Omega\left\{\exp\left[-\frac{1}{2}(\bar{\xi}, \xi)\mathbb{B}\begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}\right]\right\} = \sqrt{\det \mathbb{B}}\, \hat{\Gamma}_2(\mathbb{K}), \tag{46}$$

$$\Omega\left\{(\bar{\xi}, \xi)\exp\left[-\frac{1}{2}(\bar{\xi}, \xi)\mathbb{B}\begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}\right]\right\} = -(\hat{c}^\dagger, \hat{c})\frac{\sqrt{\det \mathbb{B}}}{\mathbb{B}} \hat{\Gamma}_2(\mathbb{K}) e^{i\pi\hat{N}}, \tag{47}$$

where $\mathbb{B}(\mathbb{1} + e^{\mathbb{K}}) = \mathbb{1}$ satisfies the relation $\mathbb{B} + \tau_x \mathbb{B}^T \tau_x = \mathbb{1}$, while the matrix $\mathbb{K}$ satisfies $\mathbb{K} + \tau_x \mathbb{K}^T \tau_x = 0$.

# B The sign problem of the Green's function

The conventional dissipation superoperator $\mathcal{D}$ with Lindblad operator $\hat{L}, \hat{L}^\dagger$ reads

$$\mathcal{D}[\circ] = 2\hat{L} \circ \hat{L}^\dagger - \left\{\hat{L}^\dagger \hat{L}, \circ\right\}. \tag{48}$$

However, if both the operator $\circ$ and the Lindblad operator $\hat{L}^{(\dagger)}$ are fermionic operators, i.e., they have odd Fermion number parity, then the dissipation superoperator should differ from the above one by having a minus sign in front of the $2\hat{L} \circ \hat{L}^\dagger$ term, leading to a new superoperator [89]:

$$\mathcal{D}_f[\circ] = -2\hat{L} \circ \hat{L}^\dagger - \left\{\hat{L}^\dagger \hat{L}, \circ\right\}. \tag{49}$$

We should note that these two superoperators are intimately connected: If $\hat{P}_F \hat{L} \hat{P}_F = -\hat{L}$, then

$$\hat{P}_F \mathcal{D}_f[\hat{P}_F \circ] = \mathcal{D}[\circ], \quad \hat{P}_F e^{\mathcal{D}_f t}[\hat{P}_F \circ] = e^{\mathcal{D}t}[\circ]. \tag{50}$$

Similarly,

$$\mathcal{D}_f[\circ \hat{P}_F]\hat{P}_F = \mathcal{D}[\circ], \quad e^{\mathcal{D}_f t}[\circ \hat{P}_F]\hat{P}_F = e^{\mathcal{D}t}[\circ]. \tag{51}$$

The proof is straightforward:
(1)

$$\begin{aligned} \hat{P}_F \mathcal{D}_f[\hat{P}_F \circ] &= -2\hat{P}_F L \hat{P}_F \circ L^\dagger - \hat{P}_F \left\{ L^\dagger L, \hat{P}_F \circ \right\} \\ &= 2L \circ L^\dagger - \left\{ L^\dagger L, \circ \right\} = \mathcal{D}[\circ]. \end{aligned}$$

(2) Define $\tilde{A}(t) = \hat{P}_F e^{\mathcal{D}_f t}[\hat{P}_F A]$, and $A(t) = e^{\mathcal{D}t}[A]$, then

$$\frac{\partial}{\partial t}\tilde{A}(t) = \hat{P}_F \mathcal{D}_f \left\{ e^{\mathcal{D}_f t}[\hat{P}_F A] \right\} = \hat{P}_F \mathcal{D}_f \left\{ \hat{P}_F \, \hat{P}_F \, e^{\mathcal{D}_f t}[\hat{P}_F A] \right\} = \mathcal{D}[\tilde{A}(t)],$$

with the initial condition $\tilde{A}(t=0) = A$. On the other hand, $A(t)$ satisfies the equation

$$\frac{\partial}{\partial t} A(t) = \mathcal{D}[A(t)],$$

with the initial condition $A(t=0) = A$. So we see that $\tilde{A}(t)$ and $A(t)$ satisfy the same equation of motion and the same initial condition, and hence $\tilde{A}(t) = A(t)$, i.e.,

$$\hat{P}_F e^{\mathcal{D}_f t}[\hat{P}_F \circ] = e^{\mathcal{D}t}[\circ].$$

Similarly we can prove the other equations.

# C Steady State and Static Correlations

Suppose that the non-Hermitian matrix $\mathbb{X}_+ + i\mathbb{H}$ has the spectral decomposition

$$\mathbb{X}_+ + i\mathbb{H} = \sum_{k=1}^{2N} \lambda_k |\varphi_k^R\rangle\langle\varphi_k^L|,$$

where $\{\lambda_k\}$ are the eigenvalues and $\{|\varphi_k^{R(L)}\rangle\}$ the right (left) eigenvectors of $\mathbb{X}_+ + i\mathbb{H}$, satisfying the biorthonormal condition $\langle\varphi_k^L|\varphi_q^R\rangle = \delta_{k,q}$. We can prove that $\text{Re}\lambda_k \geq 0$ for all $k$. For the boundary-driven Kitaev chain with a finite size $N$, we can numerically verify that $\text{Re}\lambda_k > 0$ for all $k$. Then the steady state characteristic function is given by Eq.(12) with

$$\mathbb{M}_\infty = \sum_{m,n} \frac{\langle\varphi_m^L|\mathbb{X}_-|\varphi_n^L\rangle}{\lambda_m + \lambda_n^*} |\varphi_m^R\rangle\langle\varphi_n^R|. \tag{52}$$

Here we focus on the momentum distribution of anyons defined as [90]

$$n(k) \equiv \frac{1}{N} \sum_{j,l=1}^{N} e^{ik(j-l)} \langle \hat{f}_j^\dagger \hat{f}_l \rangle.$$

Such correlation functions of nonlocal operators can be computed by takeing the $t = 0$ limit of the lesser Green's function. In Fig.7 we plot this distribution for two statistical parameters $\phi = 0$ and $\phi = \pi$. We see that the behavior of $n(k)$ is qualitatively the same for different statistical parameters. When $|\mu| < |\mu_c|$, the $k$-distribution shows two maximums at $k \neq 0, \pi$, otherwise it shows only one maximum at $k = 0$ or $\pi$. So the NQPT occurring at $\mu_c$ can be clearly characterized by the $k$-distribution function.

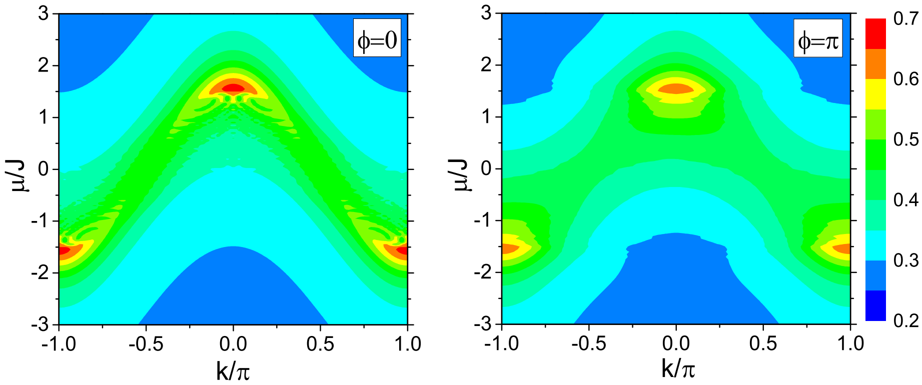


Figure 7: The $k$-distribution $n(k)$ in the steady state with the statistical parameter $\phi = 0$ (left) and $\phi = \pi$ (right). The other parameters $\Delta, \gamma_{\pm}$ and $N$ are the same as in Fig.1. The critical chemical potential is $\mu_c/J = \pm 1.5$.

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