Straightening Out the Frobenius-Schur Indicator

Steven H. Simon[1] and Joost K. Slingerland[2,3]

[1] *Rudolf Peierls Centre for Theoretical Physics, Clarendon Laboratory, Oxford, OX1 3PU, UK*
[2] *Department of Mathematical Physics, National University of Ireland, Maynooth, Ireland*
[3] *Dublin Institute for Advanced Studies, School of Theoretical Physics, 10 Burlington Rd, Dublin, Ireland.*

(Dated: September 12, 2022)

Amplitudes for processes in topological quantum field theory (TQFT) are calculated directly from spacetime diagrams depicting the motion, creation, annihilation, fusion and splitting of any particles involved. One might imagine these amplitudes to be invariant under any deformation of the spacetime diagram, this being almost the meaning of the "topological" in TQFT. However, this is not always the case and we explore this here, paying particular attention to the Frobenius-Schur indicators of particles and vertices. The Frobenius-Schur indicator is a parameter $\kappa_a = \pm 1$ assigned to each self-dual particle a in a TQFT, or more generally in any tensor category. If κ_a is negative then straightening out a timelike zig-zag in the worldline of a particle of type a can incur a minus sign and in this case the amplitude associated with the diagram is not invariant under deformation. Negative Frobenius-Schur indicators occur even in some of the simplest TQFTs such as the $SU(2)_1$ Chern-Simons theory, which describes semions. This has caused some confusion about the topological invariance of even such a simple theory to space-time deformations. In this paper, we clarify that, given a TQFT with negative Frobenius-Schur indicators, there are two distinct conventions commonly used to interpret a spacetime diagram as a physical amplitude, only one of which is isotopy invariant — the non-isotopy invariant interpretation is used more often in the physics literature. We clarify in what sense TQFTs based on Chern-Simons theory with negative Frobenius-Schur indicators are isotopy invariant, and we explain how the Frobenius-Schur indicator is intimately linked with the need to frame world-lines in Chern-Simons theory. Further, in the non-isotopy-invariant interpretation of the diagram algebra we show how a trick of bookkeeping can usually be invoked to push minus signs onto the diagrammatic value of a loop (the "loop weight"), such that most of the evaluation of a diagram does not incur minus signs from straightening zig-zags, and only at the last step minus signs are added. We explain the conditions required for this to be possible. This bookkeeping trick is particularly useful in the construction of string-net wavefunctions, where it can be interpreted as simply a well-chosen gauge transformation. We then further examine what is required in order for a theory to have full isotopy invariance of planar spacetime diagrams, and discover that, if we have successfully pushed the signs from zig-zags onto the loop weight, the only possible obstruction to this is given by an object related to vertices, known as the "third Frobenius-Schur indicator". We finally discuss the extent to which this gives us full isotopy invariance for braided theories.

PACS numbers: PACS

I. INTRODUCTION

Topology is one of the most prominent themes in modern quantum condensed matter physics[1–3]. Among the prevalent topological ideas is that of the topological quantum field theory (TQFT) which can be used to describe systems with exotic particles such as anyons. At the same time there have been substantial mathematical advances[4–7] in the field of topology based on the physical ideas of TQFTs.

While there are several, essentially equivalent, definitions of a TQFT[1,3,5,7], for our purposes we will think of a TQFT as a set of rules that takes as an input a space-time diagram of particles tracing out world-lines and gives a complex number, or amplitude, as an output. The input space-time diagrams can include simple space-time motion of particles; creation or annihilation of particle-antiparticle pairs; fusion of particles when two particles meet to form a combined "bound state" particle; and the reverse process, splitting, where one particle divides to form two others (See Fig. 1 or 20 for example). The

essence of the TQFT is nothing more than a diagrammatic calculus that maps from the input diagram to the output amplitude.

One of the key applications of TQFTs is in the construction of knot and link invariants. The input spacetime diagram can be a labeled knot or link (representing world lines of particle types) and the output amplitude is the knot invariant. Knot invariants of this type were famously constructed by Witten[4] using Chern-Simons TQFTs. For such TQFTs the diagrammatic rules are constructed so as to be (regular) "isotopy invariant" meaning that any smooth deformation of the diagram leaves the output amplitude unchanged as shown in Fig. 1 (the word "regular", here means we should treat strands as thickened ribbons so as to keep track of self twists).

However, in TQFTs which contain a particle which happens to be its own antiparticle (we say the particle is "self-dual"), the usually applied rules of diagrammatic evaluation can have an obstruction to isotopy invariance. Each self-dual particle has a property known as its Frobenius-Schur indicator. The Frobenius-Schur
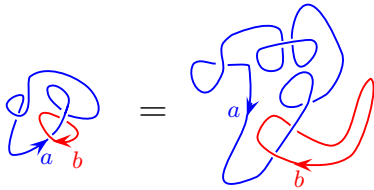
Figure 1. The two diagrams are (regular) isotopy equivalent to each other, i.e., they can be smoothly deformed into each other without cutting, treating each line as a thickened ribbon. In a TQFT we expect that these two diagrams should have the same amplitude.

indicator is simply a sign $\kappa = \pm 1$. Cases where $\kappa = -1$ are cases where there is an obstruction to isotopy invariance. In particular, as shown in Fig. 2, straightening out a "zig-zag" in the space-time path of a particle with $\kappa = -1$ incurs a minus sign in the amplitude corresponding to the space-time diagram compared to the diagram without the space-time zig-zag. (A zig-zag, as shown in Fig. 2 involves one particle pair creation event and one particle pair annihilation event.) This minus sign from this zig-zag cannot be removed by gauge choices and normalization choices, and must be treated very carefully. A good discussion of this issue is given, for example, in Ref. 8. Some other works which explore aspects of the Frobenius-Schur indicators and their generalizations include, for example, Refs. 9–16.

The signs associated with the Frobenius-Schur indicator have often been treated incorrectly or omitted in the physics literature. Such nontrivial signs occur even in theories, such as Chern-Simons theories, which we think of as being diffeomorphism invariant. The purpose of this article is to clarify the physics of the Frobenius-Schur indicator.

$$\mathord{\scalebox{1.5}{ζ}}_a = \kappa_a \ \Big|_a$$

Figure 2. A space-time path (time directed upwards) of a particle which is its own antiparticle (hence there is no directed arrow on the world-line). The Frobenius-Schur indicator $\kappa_a = \pm 1$ for particle type a is included for removing the "zig-zag" in space-time. (A zig-zag is defined to involve one particle pair creation event and one particle pair annihilation event.) If $\kappa_a = -1$ for a particle, then diagrams do not have isotopy invariance as the zig-zag on the left incurs a minus sign when it is straightened. Note, in section IV we will clarify that this diagram is to be interpreted via "Convention 1" where flags are assigned to all be pointing to the right.

Even if we are not considering fully-fledged 2+1 dimensional TQFTs, many tensor categories can be thought of as diagrammatic algebras — planar algebras without necessarily having a notion of over- and under-crossings. Self-dual objects in these theories can also be assigned

Frobenius-Schur indicators. One of the main applications of such planar diagrammatic algebra is in building toy model 2+1 dimensional Hamiltonians that have topological properties — so called string-net Hamiltonians[17,18]. We will comment briefly on how, in the context of string nets, the Frobenius-Schur indicator can be handled.

The realization that theories we think of as being isotopy (or diffeomorphism) invariant may incur signs when a zig-zag is straightened leads us to reconsider how invariant these theories actually are — and whether TQFTs have other constraints in deforming diagrams.

The outline of this paper is as follows:

In section II we introduce the basic definitions associated with the Frobenius-Schur indicator. In section III, to make this physics a bit more familiar, we draw an analogy with spin 1/2 particles.

In section IV we describe the use of so-called "flags" for keeping track of signs of Frobenius-Schur indicators — whereby we associate one of two possible flag directions with particle creation or annihilation. This technique is used commonly in the literature (See for example Refs. 3, 19, and 20). We then emphasize that there are two different conventions commonly used in how one assigns the direction of these flags to a diagram, and these different conventions have different physical interpretations. The first convention ("Convention 1") gives an output from any diagram that explicity incorporates signs associated with straightening zig-zags as shown in Fig. 2. However, the second convention ("Convention 2") is constructed so as to not give such signs and thereby give an output which is fully isotopy invariant.

In section V we explain how the knot invariants famously derived by Witten[4] from Chern-Simons theory correspond to our "Convention 2" for assigning these flags so as to obtain an isotopy invariant output from any input knot diagram. We explain how this convention arises from the need to give Wilson loops (or particle world lines) a framing.

In section VI we consider the very simplest anyon theory (or modular tensor category) having a nontrivial Frobenius-Schur indicator, the so-called semion theory or $SU(2)_1$. We consider in detail (in sections VI A and VI B) the above mentioned two conventions for assigning flags to a diagram and discuss the differing outputs from diagrammatic evaluations. We then introduce a third convention in section VI C which is physically equivalent to "Convention 1" but uses a bookkeeping trick called "Cap Counting" to take care of minus signs by pushing signs onto the diagrammatic value of a loop (the "loop weight"). As a result, the evaluation of the diagram *seems* isotopy invariant, until the last step where certain signs are added and the isotopy invariance is broken again.

We then turn to a very brief discussion of string nets in section VII. We explain in section VII B how the cap counting technique is particularly convenient in the case of building a string net from $SU(2)_1$ as it makes the ground state wavefunction look isotopy invariant. We

explain how the cap counting technique applied to a string net can be interpreted as simply a gauge transform. Further we discuss the effect of the interpretation of the Frobenius-Schur indicator in the TQFT that emerges from this string net. In section VII C we briefly outline how this cap counting would apply to another simple theory $SU(2)_2$, suggesting that the idea is more general.

In section VIII we then explain how the cap counting technique (and pushing minus signs onto the loop weight) can be generalized to many other theories so long as they admit a so-called "$\mathbb{Z}_2$ Frobenius-Schur grading". Existence of such a grading is extremely common, particularly among braided theories. It is fairly hard to find cases which do not admit such a grading — we give some examples in the appendices.

Having discussed isotopy invariance applied to evaluation of knots, in section IX we discuss whether there can be further impediments to isotopy invariance of diagram algebras coming from behavior at vertices (fusions and splittings). The particular worry is that there may be a phase incurred if we try to mutate a vertex with two legs going down and one going up to a vertex with two legs going up and one leg going down. Assuming that we have handled the signs from removing zig-zags using the cap-counting techniques, we find that there is only a single possible obstruction to turning legs up and down, and this is related to a closely related quantity known as the *third* Frobenius-Schur indicator. If this indicator is nontrivial, there is an obstruction to turning legs up and down (i.e., a phase that cannot be removed). This obstruction can occur, even for braided (or modular) theories — although such situations appear to be extremely unusual.

Assuming no such third Frobenius-Schur indicator obstruction occurs, and assuming we have a $\mathbb{Z}_2$ Frobenius-Schur grading, then for planar diagram algebra we have full isotopy — i.e., diagrams can be deformed smoothly within the plane and the value of the diagram remains unchanged. For 2+1 dimensional theories, i.e., theories with well defined braidings and which give rise to invariants of links and knots, we might wonder if we also have full isotopy invariance of diagrams with vertices. The answer is a bit subtle and we discuss this in more detail in section IX A. In particular one is allowed to deform diagrams — but only in limited ways. If we think of the diagrams as being made of ribbons, we are only allowed to consider configurations with one side of the ribbon facing forwards at both the beginning and end of the deformation. One way to describe this is to say that if we have a diagram in the shape of a tetrahedron, we are allowed to rotate the tetrahedron arbitrarily, but not invert it. There are some theories where inversion of the tetrahedron is also allowed, which we call "full tetrahedral symmetry", but these are not generic.

Finally in section X we briefly discuss more general transformations that might be possible beyond our "cap counting" scheme, and in section XI we give some brief

conclusions.

There are several major appendices to this work. Appendices A are devoted to exploring how common it is to have theories that have $\mathbb{Z}_2$ Frobenius-Schur gradings. We claim, particularly among braided theories, that it is extremely common and it is rather hard to find exceptions (although exceptions do exist), and we detail this claim in the appendix.

In Appendix B we detail the proof that the only obstruction to turning up and down legs of vertices is given by the third Frobenius-Schur indicator. We further prove that for braided (ribbon) theories, such obstruction cannot occur unless there are fusion multiplicities.

Finally Appendix C discusses a number of unusual examples. Section C 1 considers theories which do not admit a $\mathbb{Z}_2$ Frobenius-Schur grading. For braided theories, these fairly hard to find. Section C 2 gives an example of a modular (and therefore braided) theory which has a nontrivial third Frobenius-Schur indicator. Section C 3 discusses a modular (and therefore braided) theory which has isotopy invariance that allows rotation of tetrahedral diagrams, but not inversion.

II. SOME BASICS REGARDING THE DEFINITION OF THE FROBENIUS-SCHUR INDICATOR

A. Review of Diagrams, F-moves, Gauge Choice, and Isotopy Invariance

Let us first remind ourselves of the idea of F-moves and the structure of Hilbert space in topologically ordered systems. We will always assume we are describing a unitary topological system (a unitary TQFT) — i.e., a quantum mechanical system that might be physically realized. We draw diagrams of lines labeled with particle types $(a, b, c, \ldots)$ and arrows. Each particle has an antiparticle which we label with an overline such as $\bar{a}$. Reversing an arrow changes a particle to its anti-particle. If $a = \bar{a}$ we say the particle is self-dual and we do not draw an arrow on the corresponding line.

For a system with multiple anyons at a fixed moment in time, the Hilbert space dimension depends on the fusion rules of the theory. The (generically) multi-dimensional space can be described in several different bases, which we typically draw as trees as shown in, say, the left of Fig. 3. For example, the diagram on the left of the figure represents a particular state in the Hilbert space of three particles of types a, b and c. For this particular state (reading upwards) particle e splits into a and b showing that the quantum number of e is the same as that of a and b put together. We may also say that a and b fuse to e. Similarly the quantum number of all three particles a, b, c put together is the same as that of particle type d. It is important to note that diagrams having different values of e are orthogonal to each other.

On the other hand, we can also describe the same
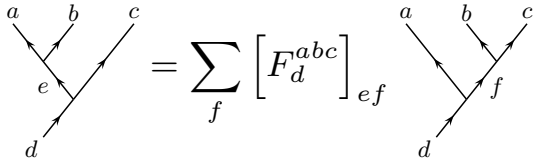
Figure 3. Fusion trees describing the structure of Hilbert space. The labels a, b, c are the physical particle types we are describing. Here, d, e and f are the results of fusing together these particles in various orders as shown in the diagrams. The basis of states shown in the left figure (a and b fusing to e) is described in terms of the basis of states described by the right figure (b and c fusing to f). The two different bases are related by the F-matrix as shown[21].

states via the tree on the right hand side, where f splits into b and c (and again, all three particles have the same quantum number as d). These two possible bases on the left and right of the figure are related by the F-matrix F_d^{abc} with elements $[F_d^{abc}]_{ef}$, the F-symbols, as shown in Fig. 3[21].

For each possible vertex (intersection of three lines) in a diagram, there is a choice of gauge. If we change this choice of gauge we multiply each vertex by an arbitrary phase factor[22] u_a^{bc} for a diagram where a splits into b and c like the one on the left of Fig. 4. Such gauge changes result in a change in the F-matrix via

$$[F_d^{abc}]_{ef}^{\text{new}} = \frac{u_f^{bc} u_d^{af}}{u_e^{ab} u_d^{ec}} [F_d^{abc}]_{ef}^{\text{old}} \tag{1}$$

So far we have described splitting diagrams, where, going upwards in the diagram (forward in time) a particle splits into others. We should think of such diagrams as being kets. The corresponding bras are obtained by turning the diagrams upside down and reversing all arrows – so for example, the diagram on the left of Fig. 4 is the ket corresponding to the bra on the right.
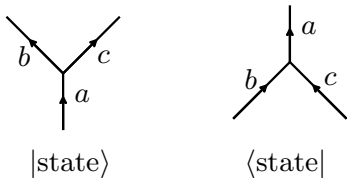
Figure 4. A ket (left) and its corresponding bra (right)[21].

Now consider the diagram shown in Fig. 5 which is the same as the left of Fig. 2, but with some identity lines drawn (dotted and labeled with 0). Here we view the up-down zig-zag as an inner product between a bra (upper half) and a ket (lower half) when cut horizontally at mid-height. In the middle of Fig. 5 we have evaluated the left diagram using the F-move from Fig. 3 (as well as using the fact that different values of the particle f in the ket on the right of Fig. 3 are orthogonal to each other — so we need only $[F]_{00}$ not $[F]_{0j}$).

We would like our diagrams to obey isotopy invariance, so that we can straighten out wiggly lines as in Fig. 2 or
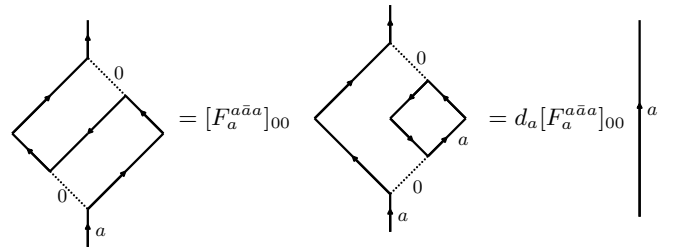
Figure 5. A zig-zag diagram. Here 0 represents the identity or vacuum particle.

the right of Fig. 5. Examining Fig. 5, we can achieve this invariance at least up to a phase factor by defining the diagrammatic loop weight d_a of particle a as

$$d_a = \left| [F_a^{a\bar{a}a}]_{00} \right|^{-1}.$$

where 0 indicates the identity, or vacuum particle. With this definition, this quantity, $d_a > 0$, is also known as the "quantum dimension" of the particle a which describes how the dimension of the Hilbert space increases asymptotically[19] with the number N_a of particles of type a, as N_a grows large, via $\text{Dim} \sim d_a^{N_a}$.

We assign this factor d_a to a closed loop of the particle a (i.e., creation of $\bar{a}$ and a followed by reannihilation of the same), as in Fig. 5. For consistency we must include factors of $d_a^{1/4}$ at any vertex including particle type a. In particular, a vertex like those shown in Fig. 4 should be associated with a factor

$$\text{vertex factor} = \left(\frac{d_b d_c}{d_a} \right)^{1/4} \tag{2}$$

This then gives us the modified orthonormality shown in Fig. 6 and the modified completeness relation shown in Fig. 7. In particular, in Fig. 6 with a being replaced by the identity, we must have $b = \bar{c}$, resulting in a closed loop of type b which is properly assigned the value d_b (since $d_0 = 1$ and $d_c = d_{\bar{c}}$). Note that since this closed particle loop is the inner product between a bra and its corresponding ket, we expect its inner product, and hence the loop weight d_b is positive definite.

$$\begin{array}{c} b \diamondsuit c \\ b \diamondsuit c \end{array} = \begin{array}{c} b \diamondsuit c \end{array} = \delta_{ad} \sqrt{\frac{d_b d_c}{d_a}} \qquad \Big\uparrow a$$
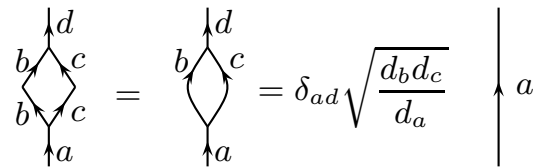
Figure 6. Modified Orthonormality (Bubble-Collapsing).

With the three diagrammatic rules (F-moves (Fig. 3), completeness (Fig. 7), and bubble-collapsing (Fig. 6) we can evaluate any planar diagram.

In order for this modified vertex normalization to allow us to freely straighten out a wiggly line as in Fig. 5 without incurring any phase, we only require that $[F_a^{a\bar{a}a}]_{00}$ be

Figure 7. Modified Completeness Relation

real and positive, so that the prefactor of the diagram on the right hand side of Fig. 5 cancels completely. Since we are free to choose the gauge of the two vertices which create a and $\bar{a}$ from the vacuum, we can use this freedom (See Eq. 1) to fix $[F_a^{a\bar{a}a}]_{00}$ to be positive if $a \neq \bar{a}$, by choosing the gauge transform constant $u_0^{a\bar{a}}/u_0^{\bar{a}a}$. This then achieves the desired isotopy invariance whenever $a \neq \bar{a}$.

B. Frobenius-Schur Indicator

Unfortunately, in the case where a particle is self-dual $a = \bar{a}$, the quantity $[F_a^{a\bar{a}a}]_{00} = [F_a^{aaa}]_{00}$ is gauge invariant, and gauge transformation is not possible. Hence there may be an obstruction to having a diagrammatic algebra that enjoys isotopy invariance.

For a self-dual particle a the quantity we are interested in is $[F_a^{aaa}]_{00}$. It is fairly easy to show that this quantity must be real[19]. We thus define a quantity known as the Frobenius-Schur indicator[23] for a self-dual particle type $(a = \bar{a})$ to be given by

$$\kappa_a = \text{sign}\,[F_a^{aaa}]_{00} \quad .$$

For a particle which is not its own antiparticle, there are various differing definitions of Frobenius-Schur indicator[19,24]. We will thus avoid making any statements about Frobenius-Schur indicators for non-self-dual particles.

In the case where $\kappa_a = -1$, one might think that we can make some redefinition to get rid of this sign. In particular we can declare that the loop weight d_a (the diagrammatic value of the closed loop) to be precisely $[F_a^{aaa}]_{00}^{-1}$ rather than its absolute value. However, this is problematic because then the inner product of a state with itself (the lower half of the loop with the upper half of the loop) would give a negative result — thus sacrificing positivity of norms, and resulting in a nonunitary theory that should not be used for quantum mechanical systems (although it may be perfectly good for defining knot invariants).

Thus when there is a negative Frobenius-Schur indicator, we find ourselves in a situation where we must either accept a negative quantum dimension (and negative normed states) or we must accept that removing zig-zags (as in Fig. 2 or Fig. 5) accumulates a minus sign.

This may seem quite disturbing. Particularly if we consider, for example, Chern-Simons theories, such as $SU(2)_1$, which are supposed to be diffeomorphism invariant and unitary. Yet, many particles in Chern-Simons theory (such as the only nontrivial particle in $SU(2)_1$) do indeed have a negative Frobenius-Schur indicator. In section V below we discuss how the idea of Frobenius-Schur indicators fits into Chern-Simons theory.

III. ANALOGY WITH SPIN-1/2 PARTICLES

While the physics of the Frobenius-Schur indicator might appear a bit unfamiliar it turns out that there is a familiar analog in angular momentum addition — where the particle type (the label a, b, c etc) corresponds to the eigenvalue of J^2.

Consider three spin-1/2 particles which all taken together are in an eigenstate of $J = 1/2$. We can describe the possible states of the system with fusion diagrams analogous to Fig. 3 — in this case where a, b, c and e are all labeled as individually having $J = 1/2$. In Fig. 3 we can (on the left of the figure) consider either the fusion of the leftmost two particles to some angular momentum $d = 0$ or $d = 1$, or we can (on the right of the figure) consider fusion of the rightmost two particles to either $f = 0$ or $f = 1$. (Here, 0 and 1 refer to singlet and triplet). The F-matrix that relates these two descriptions of the same space is given by $[F_{\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}\frac{1}{2}}]_{df}$ which is often known as a $6j$ symbol in the theory of angular momentum addition.

Given that the total spin is $1/2$ we can focus on the case where the total z-component of angular momentum is $J_z = 1/2$ as well. The state where the leftmost two particles fuse to the identity (or singlet $J = d = 0$) can then be written explicitly as

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle\right) \otimes |\uparrow_3\rangle \tag{3}$$

where the subscripts are the particle labels given in left to right order. This wavefunction is precisely analogous to the lower half (the "ket") of the left hand side of Fig. 5.

On the other hand, we could use a basis where we instead fuse the rightmost two particles together first, as in the righthand side of Fig. 3. We can write the state where the right two fuse to $J = f = 0$ analogously as

$$|\psi'\rangle = |\uparrow_1\rangle \otimes \left(|\uparrow_2\downarrow_3\rangle - |\downarrow_2\uparrow_3\rangle\right)\frac{1}{\sqrt{2}} \tag{4}$$

which is precisely analogous to (but the conjugate of) the top half (the "bra") of the left hand side of Fig. 5.

It is easy to check that the inner product of these two states $|\psi\rangle$ and $|\psi'\rangle$ (corresponding to the middle of Fig.5 is[25]

$$\langle\psi'|\psi\rangle = -1/2$$

By redefining the normalization of these states, we can arrange that this overlap have unit magnitude. However, the sign cannot be removed. The situation is the same for any two half-odd-integer spins fused to a singlet.

It is worth thinking for a moment about why the sign is hard to remove by any sort of redefinitions. The convention we have used above in Eq. 3 and 4 is that the fusion of two particles to the identity (to a singlet) should give $(|\uparrow_a\downarrow_b\rangle - |\downarrow_a\uparrow_b\rangle)/\sqrt{2}$ where a is always to the left of b. Whatever convention we choose should be something that we can evaluate locally — i.e., the result of fusing a with b should be the same whether we fuse the result with some c on its left later, or some d on its right later. This sort of locality requirement prevents us from finding a way to insert a minus sign into the definition of $|\psi\rangle$ or $|\psi'\rangle$.

IV. FLAGS

Let us now be a bit more precise with the rules for evaluating diagrams. Here we will use the so-called "flag" method. This is the method developed by Ref. 19 and also used in Ref. 20. It is equivalent to discussions in Refs 5, 6, and 26.

We first work in a gauge (as discussed above) where we can remove zig-zags freely for any non-self-dual particle, and also we can remove zig-zags freely for any self-dual particle with Frobenius-Schur indicator $\kappa = +1$. The difficult part is in keeping track of factors of the -1 for particles with the Frobenius-Schur indicator $\kappa = -1$. For these particles we introduce large arrows on the particle lines in the diagram, known as flags, whenever a pair of such self-dual particles is either created or destroyed[27] as shown in Fig. 8. We emphasize that these flag arrows are not the same arrows we put on lines to distinguish a from $\bar{a}$ (which we draw as smaller arrows as in Fig. 3 for example). Here the particles are self-dual, and for self-dual particles we do not draw the smaller arrows since $a = \bar{a}$.

We call the creation diagram a cup (right diagrams of top two lines of Fig. 8) and the annihilation diagrams are called caps (left diagrams of top two lines of Fig. 8). The reason we introduce the flags is because we wish to use two different cup states and two different cap states, their conjugates. The two cup (and cap) states differ by a factor κ_a, which allows us to remove any explicit factors of κ_a by choosing which of the cup and cap states to use in the diagrams. We then need the flags to distinguish the two cup (and cap) states. Hermitian conjugation of a cup gives a cap (and vice versa) but does not reverse the direction of the flag. As shown in the third line of Fig. 8, a cup and a cap may be assembled to form a positive definite inner product assuming that their flags are aligned.

When evaluating a diagram, a flag can be reversed at the price of a single factor of κ_a as shown in Fig. 9. Two flags in successive cup-cap pairs can be cancelled with each other to remove a zig-zag (as shown in Fig 10) if the arrows are oppositely directed.

Note that under twisting of lines, flags are rotated with the lines as in Fig. 12. This is because the Frobenius-

Schur indicator enters into the relationship between R matrix element R_0^{aa} for the self-dual particle a and the twist factor θ_a via

$$R_0^{aa} = \theta_a^* \kappa_a \qquad (a = \bar{a}),$$

as illustrated in Fig. 11,12,13.

It turns out the that the flags are not just an accounting trick but they represent a genuine degree of freedom of self-dual particles, connected to framing. We will return to discuss the meaning of this degree of freedom in section V.
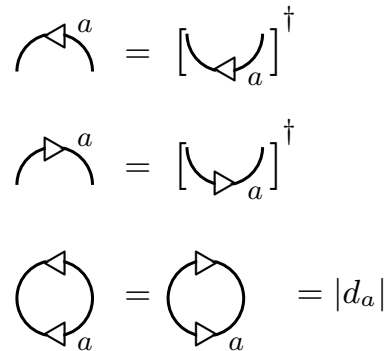
Figure 8. Some of the flag-bookkeeping rules for keeping track of signs given by Frobenius-Schur indicators used by Ref. 19. Large arrows represent flags on self-dual particle lines. The inner product of a cup and cap is positive definite if they have aligned flags.

Figure 9. Reversing a flag incurs a factor of the Frobenius-Schur indicator κ_a.
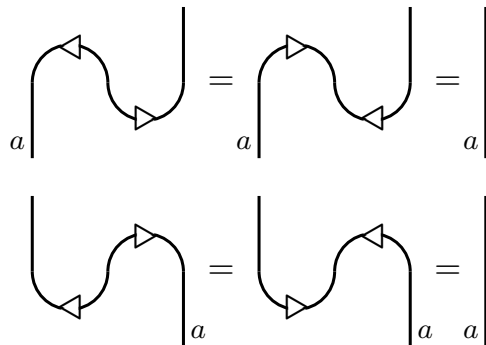
Figure 10. Opposite directed flags may be cancelled and zig-zags may be removed.

We note that the mathematical purist may want to decorate all cups and caps with flags whether or not the particle is self-dual and whether or not $\kappa_a = \pm 1$. While

$$= R_a^{bc}$$
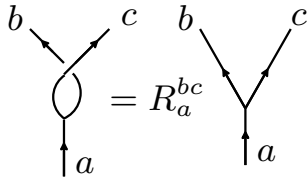
Figure 11. The general definition of the R matrix.[21]

$$= \theta_a^* \qquad = \kappa_a \theta_a^*$$
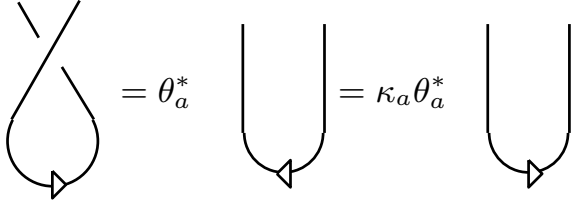
Figure 12. When untwisting, the direction of the flag follows the change in direction of the string (as if the flag is attached to the string). In order to get the flag pointing to the right again, one multiplies by the Frobenius-Schur indicator. Thus for a self-dual particle $R_0^{aa} = \theta_a^* \kappa_a$.

$$= \theta_a^*$$
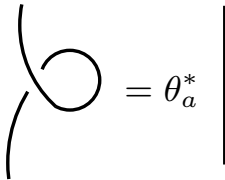
Figure 13. In comparison to the R-matrix, a twist of this sort does not accumulate a factor of the Frobenius-Schur indicator. This is true in either convention 1 or convention 2 for assigning flag directions.

there is a certain appeal to treating all particles on equal footing it is only for the self-dual particles with $\kappa_a = -1$ that we cannot live without the flags, and it is usually easier not to draw them in all other cases.

Suppose now we want to turn a labeled knot or link diagram into an output (our interpretation of a TQFT is a prescription for turning a diagram into an output). It is important to emphasize that if one is given such a diagram one must first decorate cups and caps with flags in order to fully define the value of the diagram. There are two conventions we may choose to use, so as to determine how flags are meant to point if they are not drawn (and the two different conventions give physically different results).

Convention 1: All Flags Point Right

In this convention we simply declare that at the beginning of a calculation all flags are right-pointing. Further, if we ever generate a left-pointing flag (say by the left equality of Fig. 12) we will immediately apply κ_a to re-orient the flag again to point right (as in the right of Fig. 12).

With this convention of all flags pointing right, note that straightening out a zig-zag necessarily incurs a factor of κ_a (this can be seen by combining Fig. 9 and Fig. 10).

As a result this means that our diagrams are not isotopy invariant.

Note that we have implicitly used this right-pointing convention in defining our F-matrices in Fig. 3. In cases where there is a cup (in either diagram) we assume the corresponding flag points right. Further, Fig. 2 has implicitly used this convention as well. Most diagrams drawn in the condensed-matter literature use this convention even if it is not stated.

This convention is most natural when one is describing physics in a Hamiltonian formalism. In such a case one might be presented with a particular (two-dimensional) wavefunction of a planar system. Give such a ket, one can label the positions and types of each particle that is found in the system. When two such particles (possibly self-dual) are created (annihilated), each such creation (annihilation) event looks identical to every other one. Thus it makes sense to label them all the same way and make sure the flags all point the same way.

With this convention, it is important to note that even though we do not have full isotopy invariance (straightening a zig-zag can incur a minus sign) there is still some degree of deformation of diagrams that is allowed. In particular any deformation of a diagram is allowed that does not change the time-direction of motion of any particles, and does not add or remove particle creation/annihilation events. See the related discussion in the Conclusions.

Convention 2: Alternating Flags

A different convention we might choose is that (starting with a knot or link diagram) we should decorate cups and caps with flags that alternate direction as we walk along the length of a strand (meaning that the first flag points in the direction of walking, the second points opposite the direction of walking and so forth). With such a convention, zig-zags can be freely straightened out (as in Fig. 10). The resulting diagrammatic evaluations are then (regular) isotopy invariant, thus giving us a way to construct true knot invariants.

The rule of decorating cups and caps with alternating flags may seem like a disturbingly nonlocal rule: if one locally examines a cup or a cap, one may have to walk a very long distance along the strand to find out whether this cup or cap should be decorated with a left-pointing or right-pointing flag. Nonetheless, as we shall see in section V, there is a sensible physical interpretation to this rule, and it is precisely this rule that needs to be applied for turning Chern-Simons theories into a knot invariant[4].

Note that flipping all of the flags in a knot or link leaves the final evaluation unchanged. This suggests that we have not actually distinguished between a and $\bar{a}$ which are meant to be the same.

This convention (or a convention equivalent to this in some language) is used commonly by mathematicians (category theorists, knot theorists, etc) and, as we will see in the next section, it arises naturally in an action

formalism such as Chern-Simons theory where the knots are Wilson loop operators. Here, it is much more natural to think in terms of space-time histories and the particular "state" of a system at some time-slice (which we discussed in "Convention 1" above) is not really a feature of the model.

V.   ISOTOPY INVARIANT CHERN-SIMONS THEORY

Let us review some basic notions of constructing knot invariants from Chern-Simons theory as pioneered by Witten[4]. We will be brief here, referring to the literature for further details[1,3,4].

For a given reference manifold $\mathcal{M}$ (often taken to be S^3) the Chern-Simons partition function for that manifold is given by the functional integral

$$Z(\mathcal{M}) = \int_{\mathcal{M}} \mathcal{D}a_\mu(x) \; e^{iS_{CS}[a_\mu(x)]}$$

where S_{CS} is the Chern-Simons action and $a_\mu(x)$ is the Chern-Simons vector potential, which is a Lie algebra valued vector for some given Lie algebra.

We now consider a Wilson loop operator $\hat{W}_L$ defined along a closed curve L

$$\hat{W}_L^R = \text{Tr}^R \left[P e^{\oint_L dl^\mu a_\mu} \right]$$

where P means that the integral should be path-ordered, and the superscript ^{R} means that the trace is taken in a representation R of the Lie algebra.

For simplicity, we assume that the closed curve L is embedded in a reference manifold $\mathcal{M} = S^3$. The famous result by Witten[4] is that one can define a knot invariant as the expectation of the Wilson loop operator

$$\text{Knot Invariant}_L^R = \langle \hat{W}_L^R \rangle \tag{5}$$
$$= \frac{\int_{S^3} \mathcal{D}a_\mu(x) \; \hat{W}_L^R \; e^{iS_{CS}[a_\mu(x)]}}{\int_{S^3} \mathcal{D}a_\mu(x) \; e^{iS_{CS}[a_\mu(x)]}}$$

These knot invariants correspond to a diagram with particle type R traveling along the closed loop L.

This prescription, while simple sounding, actually has a number of subtleties. The main subtlety we will focus on here is that the expression for the knot invariant Eq. 5 is actually not well defined as it is written. One must specify not only the oriented path of the Wilson line, but also a *framing* of the knot. By framing here we mean that we not only specify a loop L but we attach a narrow ribbon to L such that L is one of the boundaries of the ribbon. The reason for this requirement is that in order to evaluate integrals such as that in Eq. 5 one needs to regularize the Wilson loop integral by point-splitting, where the two edges of the ribbon correspond to the splitting of a single point into two very close points as shown in Fig. 14 (See the discussion in Ref. 4). Since our theory is topological we expect that any physical results
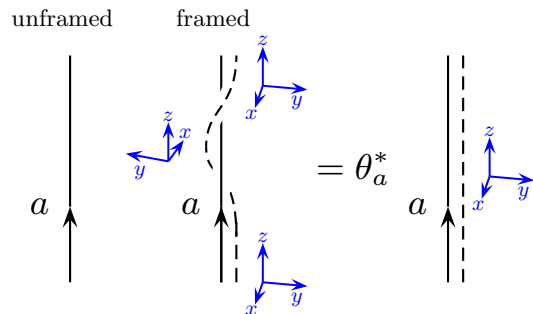
Figure 14. The unframed figure on the left is ambiguous in Chern-Simons theory. A framing is given for the diagrams on the right.

we may calculate should depend only on the topological properties of the framing.

If the ribbon twists around the particle path as shown in the middle of Fig. 14, the twist in framing can be removed at the cost of a factor of θ_a^* or θ_a depending on the chirality of the twist.

For simplicity of presentation, and in the spirit of developing a planar diagram algebra, one often uses so-called *blackboard framing* in drawing knots and links. In this representation the ribbon is assumed to lie flat on the page. A twist such as the middle of Fig. 14 is represented instead with a blackboard framed curl such as Fig. 13.

With this convention we can define a reference frame in the following way. The z-axis of our coordinate system points along the direction of the arrow. The y-axis points from the solid to the dashed line (the two edges of the ribbon), and the x-axis of our coordinate frame always points out of the plane of the page towards the reader as in Fig. 15.

If the dashed line lies to the right of the solid line while walking along the direction of the arrow, we have specified a right-handed coordinate system, whereas if the dashed line lies to the left of the solid line, then we have a left-handed coordinate system. Note that particle a being framed right handed is equivalent to particle $\bar{a}$ being framed left-handed as shown in the top of Fig. 15. This can be interpreted as the CPT theorem: charge, parity, and direction of motion are all reversed. A right-handed a can annihilate with a left-handed $\bar{a}$ as shown in the bottom of Fig. 15.

For each pair of particle types a and $\bar{a}$ we can establish a convention that one of them is always framed right-handed and the other is always framed left-handed so that the two can annihilate. However, we now see that we have an ambiguity if we try to establish a similar convention for self-dual particles. Since we do not have arrows along lines, we need to invoke flags in order to clarify how a line is meant to be framed. The flags are precisely what is needed to define a consistent framing convention. Let us use the dictionary shown in Fig. 16 which assigns a right-handed frame with the ribbon on the inside of
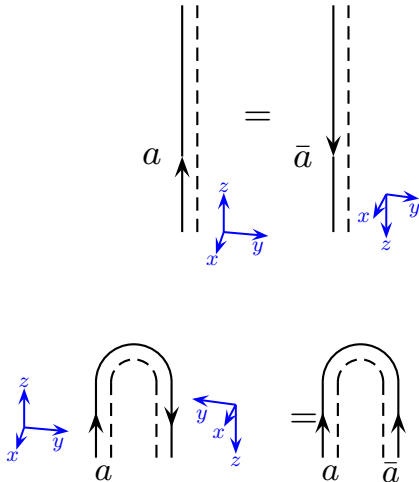
Figure 15. Top: Particle a being framed right-handed is equivalent to particle $\bar{a}$ being framed left-handed. Bottom: a right-handed particle a can annihilate a left-handed particle $\bar{a}$.
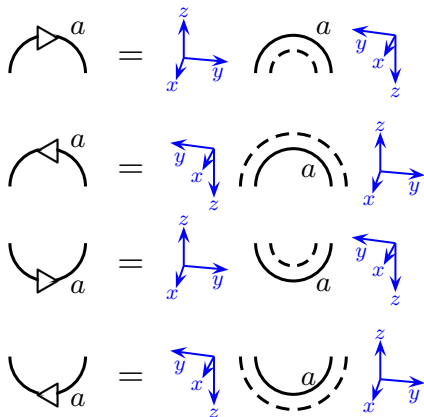
Figure 16. A dictionary between flags and framing for self-dual particles. The orientation of the local frames on the left and right of the ribbon is indicated. The y-axis always points from the solid edge to the dashed edge of the ribbon, the x-axis always points out of the page and the z-axis follows the solid edge in such a way that the frame is right handed.

the curved line to a right pointing cap/cup and assigns a right-handed frame with ribbon on the outside of the curved line to a left pointing cap/cup. Note this means that the z-axis follows the direction of the flag for caps and opposes the direction of the flag for cups. Other consistent conventions are also possible. Such a dictionary has a number of appealing features. First, inner products between a cup/cap and its Hermitian conjugate results in a properly framed positive definite diagram, as shown in Fig. 17. In addition the ribbons connect together so as to reproduce the zig-zag identity (Fig. 18). I.e., properly framed zig-zags can be pulled straight — meaning that properly framed zig-zags correspond to alternating flag directions.

Now that we understand that flags are essentially a
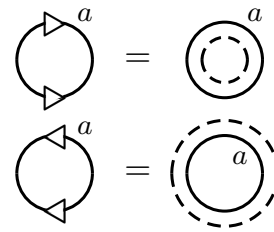
Figure 17. Taking the inner product of a ket and its Hermitian conjugate bra results in a properly framed diagram.
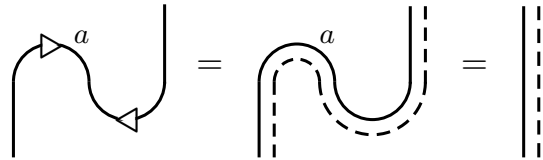
Figure 18. If flags orientations alternate as you walk along a string, then the corresponding framing is consistent with the ribbon extended on one side of the string. This allows the string to be smoothly deformed, thus giving the zig-zag identity of Fig. 10.

stand-in for framing of the strands, this sheds light on the alternating flag-assignment rule in "Convention 2" above. The demand that in any diagram representing an amplitude in Chern-Simons theory all curves must posses a consistent framing is a global requirement on the whole curves. This is why the flag assignment in "Convention 2" appears as a non-local procedure. Nevertheless, so long as we adhere to the rule of alternating flags, we will obtain a consistent framing of the curve, and Chern-Simons theory will give us an isotopy invariant result.

One should realize that Wilson lines in Chern-Simons theory *must* be assigned a framing corresponding to alternating flags. While one can twist framings as in Fig. 14, one cannot flip a flag as in Fig. 9. Thus the most general TQFT structure which allows arbitrary flag labels on diagrams is actually beyond what can be achieved with a Chern-Simons theory.

It is worth mentioning that the machinery of specifying a framing and putting flags on lines is not fundamentally special to self-dual particles. However, for non-self dual particles it is easiest to simply work with conventions that allow us to never think about flags.

Having now understood the meaning of the Frobenius-Schur indicator, it is useful to work through some examples. Further, we will introduce a third convention for evaluating diagrams which will make our "Convention 1" at least appear a bit more isotopy invariant.

VI.   SIMPLEST NONTRIVIAL FROBENIUS-SCHUR ANYON: $SU(2)_1$, THE SEMION MODEL

The simplest example of a theory with a nontrivial Frobenius-Schur indicator is $SU(2)_1$, the semion model. This model has the identity particle and a single nontrivial particle which we will call a which is self-dual with $\kappa_a = -1$. Diagrams are simple unlabeled linked loop diagrams (loops with no branching, with over and under-crossings allowed) with the lines representing the world lines of particle a. It is perhaps underappreciated that even this extremely simple anyon theory has a nontrivial Frobenius-Schur indicator.

We will focus on explaining the physics of this simple case. Once this much is understood, the more general case follows fairly straightforwardly.

A.   Isotopy Invariant Approach: Convention 2

If we want to construct an isotopy invariant (and unitary) theory, we must follows the rules of alternating flag directions as discussed in "Convention 2" above. For $SU(2)_1$ evaluating any *planar* diagram using this convention simply gives unity as an output. For diagrams with over and undercrossings, one can use two rules to turn diagrams into planar diagrams which then have the value unity.

- [**Rule A1**] Turning an overcrossing into an undercrossing (or vice-versa) multiplies a diagram by -1.

- [**Rule A2**] One can use the twist factor $\theta_a^* = -i$ to remove twists as in Fig. 13 or as in the left equality of Fig. 12 (Removing the mirror image twist gets the complex conjugate phase).

- [**Rule A3**] Any loop (unlinked to any other loop, and without self-twists) has value $+1$ independent of whether it has zig-zags.

This set of rules is now isotopy invariant. Note crucially in Rule A2, that if the twist is oriented as in the left of Fig. 12 one now does not flip the direction of the flag after untwisting (one does not implement the last equality of Fig. 12). The reason for this is our rule that flag directions should alternate as we walk along a strand. Once a planar diagram is achieved, the value of that planar diagram is always unity.

B.   Non-Isotopy Invariant: Convention 1

One may also choose to interpret a diagram using "Convention 1", which assumes all flags are right-pointing. In this convention, one must keep track of zig-zags. Each time a zig-zag is straightened, one incurs a

factor of $\kappa_a = -1$. A loop with no zig-zags has value of 1. For planar diagrams the value of the diagram is given by

$$\text{Diagram} = (-1)^{\text{number of zig-zags to be straightened}} \tag{6}$$

where here we count the number of zig-zags that must be straightened in order to obtain a set of loops in the plane each with no zig-zags.

For non-planar diagrams, in addition to this zig-zag rule, Rule A1 still applies. However, instead of using Rule A2 to remove twists, we now must consider how the twist is oriented on the page before deciding whether we need to reverse a flag after removing the twist. Thus our evaluation rules for non-planar diagrams are

- [**Rule B1**] Turning an overcrossing into an undercrossing (or vice-versa) multiplies a diagram by -1.

- [**Rule B2.1**] One can use the twist factor $\theta_a^* = -i$ to remove twists as in Fig. 13 (removing the mirror image twist gets the complex conjugate phase).

- [**Rule B2.2**] One can use the twist factor $\kappa_a \theta_a^* = +i$ to remove twists as in Fig. 12 (removing the mirror image twist gets the complex conjugate phase).

- [**Rule B3**] A simple loop (unlinked to other loops and with no zig-zags and no self-twists) has value $+1$. Removing a zig-zag multiplies the diagram by -1.

The two cases of Rule B2 make it crucial here to keep track of whether a twist is oriented vertically or horizontally, as the phase for removing the twist differs in the two cases (Fig. 13 versus Fig. 12). These rules are not isotopy invariant.

C.   Convention 3: Cap Counting as an Alternative to Convention 1

We now propose a different bookkeeping scheme which ends up equivalent to Convention 1, but is simpler in some respects. Here, we propose to move the minus sign onto the loop weight d_a, the diagrammatic value of a loop. However, to make up for this, and so that the value of a single loop is still positive, we evaluate a diagram and at the end, we multiply by (-1) to the power of the number of "caps" in the original diagram. A cap in this case, as shown in Fig. 19, is a place where a particle annihilates with another particle to form the vacuum (or equivalently it "turns over" in time). For example, a simple loop acquires $d = -1$ for being a loop, but since it has a cap, it gets an additional factor of -1. If we add a zig-zag to a line, as in the left of Fig. 2, then we count another cap, and hence we accumulate another factor of -1, thus accounting for the Frobenius-Schur indicator.
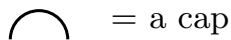
Figure 19. A cap is when a particle going upwards in time annihilates with another particle to form the vacuum.

The general evaluation of a planar diagram is given by

$$\text{Diagram} = (-1)^{\text{number of loops + number of caps}} \quad (7)$$

which can be easily seen to be equivalent to Eq. 6.

In evaluating a non-planar diagram, one must be cautious because removal of a twist as in Fig. 13 now also removes a cap. Thus the net twist factor incurred is $\tilde{\theta}_a^* = -\theta_a^*$. Note that this factor now matches the factor for removing a twist in Fig. 12, where the twist is simply oriented in a different direction on the page. So our rules for evaluating diagrams here are

- [**Rule C0**] Before evaluating a diagram count the number of caps, and call it n.

- [**Rule C1**] Turning an overcrossing into an undercrossing (or vice-versa) multiplies a diagram by -1.

- [**Rule C2**] One can use the twist factor $\tilde{\theta}_a^* = i$ to remove twists as in Fig. 13 or as in the left equality of Fig. 12 (Removing the mirror image twist gets the complex conjugate phase $\tilde{\theta}_a = -i$).

- [**Rule C3**] Any loop (unlinked to other loops and with no self-twists) has value -1 independent of whether it has zig-zags.

- [**Rule C4**] At the end of the evaluation, multiply the final result by $(-1)^n$.

The diagrammatic rules C1-C3 are (regular) isotopy invariant. We call these steps, the *nonunitary evaluation* of the diagram, since they correspond to an (auxiliary) nonunitary theory with negative loop weight ($d_a < 0$) which has a non-positive definite norm. (Note that if one is not interested in building physical models, that need to be unitary, one can certainly consider such non-unitary diagram rules for building knot invariants, for example).

While the nonunitary evaluation of the diagram is isotopy invariant, the application of rule Rule C0,C4 breaks this isotopy invariance, and in fact gives the same result as the evaluation described in section VI B (Convention 1). The advantage of this scheme is that for the main part of the evaluation (Rules C1,C2,C3) one works with isotopy invariant rules, and only at the beginning and the end does one break this invariance.

Note that for any planar diagram, one only applies Rules C0,C3,C4 and one recovers the result stated in Eq. 7.

This cap counting scheme has a significant advantage when the diagram algebra is used to build a string net model[17,28] which we discuss next.

VII. STRING NETS

A. String Nets in Brief

String nets[17,18,29] are a general construction which take as an input a planar diagram algebra (a spherical category). From this algebra one can build an explicit local Hamiltonian for a 2+1 dimensional system whose ground states are described by a 2+1 dimensional TQFT, and which correspondingly has anyon excitations. The TQFT which results is known as the quantum double (or Drinfel'd double) of the input spherical category (the input planar diagram algebra). We will give a very brief description of string nets here referring to the literature[17] for further details.

To build a string net, one usually starts with a 2D honeycomb lattice, though any type of planar trivalent lattice will do. One assigns a particle type of the planar diagram algebra to each directed edge of the lattice. These particle types then meet at the vertices and low energy configurations are required to satisfy the fusion rules there. Each such assignment is viewed as a fusion diagram in the sense of the diagram algebra. The string-net ground state wavefunction is given by

$$|\psi\rangle = \sum_{\text{all planar diagrams}} W(\text{diagram}) \, |\text{diagram}\rangle \quad (8)$$

where $W(\text{diagram})$ is the amplitude that results from the evaluation of the planar diagram. (We have ignored here possible ground state degeneracies, which can occur if the lattice models a surface of nontrivial topology rather than a plane, e.g. a torus.)

Interestingly, if the input planar algebra comes from a modular tensor category (a sufficiently well behaved anyon theory) then the output TQFT is just the product of the input anyon theory and its mirror image. Note that in the construction of the string net model, we never actually input any of the braiding properties associated with the planar algebra — the output braiding properties are emergent. Somehow in the system "knows" when the planar diagram algebra stems from a 2+1 dimensional theory and the emergent theory correctly reflects this.

One can also use input planar algebras that are not consistent with any braiding, and still the string net construction will produce a well defined 2+1 dimensional TQFT with nontrivial anyons as an output — although in such a case the output TQFT is very nontrivially related to the input.

If the input theory includes a negative Frobenius-Schur indicator, then the amplitudes $W(\text{diagram})$ in Eq. 8 should be evaluated using "Convention 1". This requires us to set some convention as to what we mean as "up" on the two dimensional plane. "Up" has the physical meaning of time in a 2+1 dimensional theory, but here "up" is just a reference direction in the plane. Note that the use of Convention 1 allows for the desired amplitudes to be generated by a local and Hermitian Hamiltonian,

and the resulting string net correctly generates a TQFT which is the quantum double of the input spherical category. One may be tempted to define a model by the requirement that the ground states look similar to Eq. 8 but with the amplitudes evaluated using Convention 2, but this does not correctly generate the quantum double. Since switching between the two conventions is a non-local procedure, the resulting topological orders will generally be different.

B. String Nets in the Semion Model

We now apply the string-net construction to the above discussed $SU(2)_1$ rules (Convention 1). Using the evaluation of diagrams from Eq. 6, we can write the string net wavefunction as

$$|\psi\rangle = \sum_{\text{all planar diagrams}} (-1)^{\text{number of zig-zags}} |\text{diagram}\rangle \tag{9}$$

where a diagram here is any planar diagrams of loops (i.e., a loop gas) now on a honeycomb lattice. Here we must know what direction is "up" in order to know what constitutes a zig-zag. The topological content of this string net is (i.e., the resulting output TQFT is) $SU(2)_1 \times \overline{SU(2)_1}$ with the overline meaning the mirror image theory.

Now we can try to instead use our cap counting scheme (Convention 3) for constructing a string net. Here we would equivalently write the signs as in Eq. 7. However, now we are not describing space-time world lines, but rather a component of a wavefunction. Thus, we can assign the -1 factors on every cap as being simply a gauge transformation on wavefunctions. To be more explicit, on a honeycomb lattice, we can choose a local basis for the Hilbert space of the direct neighborhood of each vertex in the spatial lattice which has two links pointing down and one link pointing up. (The links incident on these vertices also cover the neighborhoods of the remaining vertices). Such a basis would be labeled by (fusion respecting) particle type labelings around the vertex. Basis states for the Hilbert space of the entire lattice can be built as tensor products of these vertex states. We can then change the original choice of basis by multiplying any vertex basis vector representing a cap by -1. If we make such a transform on our in-plane wavefunctions

$$|\psi\rangle \to (-1)^{\text{number of of caps}}|\psi\rangle$$

where ψ is a particular in-plane loop configuration, we would completely remove the need to keep track of these factors in our bookkeeping. The remaining effect of the Frobenius-Schur indicator is that the value of a loop is $d = -1$. In particular, one obtains a fully isotropic (and fully isotopic) quantum loop gas on the honeycomb lattice where the ground state wavefunction is simply of the

form

$$|\psi\rangle = \sum_{\text{all planar diagrams}} (-1)^{\text{number of loops}} |\text{diagram}\rangle \tag{10}$$

The result of this construction is precisely the "double semion" wavefunction as it is described in Refs. 17, 28, and 30. An explicit Hamiltonian is written down by Ref. 17 that generates this wavefunction as a ground state.

The gauge transformation on the wave functions that we have performed here has an interesting relation to the gauge transformations u_c^{ab} in the tensor category that we discussed before Eq. 1. If the theory is unitary, then the phase factors will need to satisfy $u_{ab}^c = (u_c^{ab})^*$, so that the inner product is preserved and equations like Eq. 6 and Eq. 7 remain valid with the same, positive, coefficients. However, if unitarity is not required then u_{ab}^c and u_c^{ab} could be chosen independently and this would allow for instance $u_{a\bar{a}}^1 = -1$ and $u_1^{a\bar{a}} = 1$, which would change inner products and in particular the sign of the a-loop amplitude. We see then that the unitary gauge transform that we perform on the lattice achieves the same effect on lattice wave functions as such a non-unitary gauge at the level of the tensor category.

The interesting consequence here is that, although our string net is based on a unitary theory $SU(2)_1$ (so it gives a nice unitary theory as its output), it can be presented in a different gauge so that it appears to stem from a nonunitary theory (with $d = -1$). Nonetheless, the two string net wavefunctions Eqs. 9 and 10 are actually just unitary gauge transforms of each other!

As an aside we mention that, as noted, the tensor categories we deal with are all so called spherical categories and as such they are furnished with a spherical structure, which helps to define quantum traces (see e.g. Refs. 31 and 32 for details). In particular, if a category allows multiple spherical structures then changing the spherical structure can change the loop values (they are the traces of the trivial diagrams with a single line). The trick we use here appears to be related to spherical structures but we must remember that our trick is for diagram evaluation with the interpretation of Convention 1 and spherical structures are natural in the context of Convention 2. In particular, though different choices of spherical structure will give rise to different loop values in Convention 2, all choices should give rise to planar isotopy invariance of the diagrammatic calculus.

Note that, were we to build a semion string-net based on the Convention 2 rules listed in section VI A, we would obtain an equal weighting of all loop diagrams. This is the Toric Code ground state, rather than $SU(2)_1 \times \overline{SU(2)_1}$.
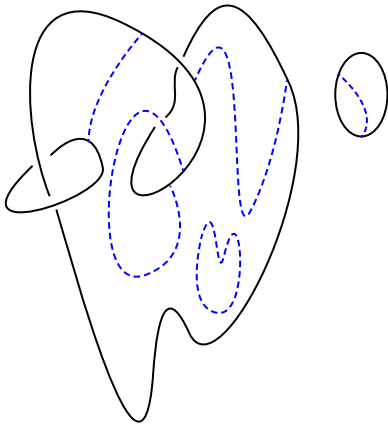
Figure 20. A space-time diagram for $SU(2)_2$. The solid lines represent σ and the dotted lines represent ψ.

C.   String Nets in $SU(2)_2$

As in the case of the doubled-semion model, the cap counting technique is useful in more general string-net models. As an example (and a preview of upcoming sections) let us consider a string net built from the commonly-discussed theory $SU(2)_2$. This theory also has a particle type with negative Frobenius-Schur indicator, whereas its close relative, the Ising theory does not. $SU(2)_2$ has two nontrivial particles, which we will call ψ and σ (another common notation is 1 and $1/2$, in analogy with $SU(2)$ spins). The nontrivial fusion rules are

$$\psi \times \psi = I$$
$$\psi \times \sigma = \sigma$$
$$\sigma \times \sigma = I + \psi$$

with I being the identity particle. The Frobenius-Schur indicators of σ is $\kappa_\sigma = -1$ whereas $\kappa_\psi = +1$. The fusion rules of the Ising theory are exactly the same as those of $SU(2)_2$ although all particles in the Ising theory have $+1$ Frobenius-Schur indicator.

A typical space-time diagram is shown in Fig. 20 where the solid lines represent σ and the dotted lines represent ψ. Notice that the σ lines, the particles with the nontrivial Frobenius-Schur indicator, form closed loops. Thus, we can handle the bookkeeping exactly the same as we did for the semion theory in section VI C above. We count a closed loop of σ as $-|d_\sigma|$ and each cap for a σ particle accumulates a -1.

When we build a string-net model, as in the case of the semion model, we pick out a preferred direction for "up" within the plane. Our fusion algebra is given in terms of the F-matrices of $SU(2)_2$ in "Convention 1". However, as in in section VII B we can simplify the theory by using instead "Convention 3" where we push the minus sign

onto the loop weight. Then, as in the case of the semion model we can gauge transform to remove a factor of -1 for each σ particle cap. The result is an isotropic string net model with isotopic weights in the plane and negative loop weight for the σ particle. Such a string net model has been studied in depth in Ref. 33.

The fact that we can apply the same scheme of generally moving minus signs from the Frobenius-Schur indicators onto the loop weight suggests how we can generally use cap counting for more general anyon theories.

VIII.   GENERALIZING CAP COUNTING TO OTHER ANYON THEORIES

It is useful to carefully extend the principle of cap-counting bookkeeping to other categories. This then gives us a convenient method of bookkeeping for complicated theories with nontrivial Frobenius-Schur indicators.

A. $\mathbb{Z}_2$ Frobenius-Schur Grading of the Fusion Algebra

In order to keep track of the particles with $\kappa = -1$, we introduce the notion of a $\mathbb{Z}_2$ Frobenius-Schur grading. We will use this grading below in our bookkeeping schemes. First, we explain what this grading is. In Appendix A we explain why such gradings usually exist, we give examples of the large families of theories where such gradings exist and we explain some reasons why exceptions are so rare.

We say that a theory can be given a $\mathbb{Z}_2$ Frobenius-Schur grading if we can define indices $\tilde{\kappa}_a = \pm 1$ for all particles a such that $\tilde{\kappa}_a = \kappa_a$ for any self-dual particle and

$$\tilde{\kappa}_a \tilde{\kappa}_b = \tilde{\kappa}_c \quad \text{when} \quad N_{ab}^c > 0 \tag{11}$$

for any a, b, c whether or not they are self-dual. Here, N_{ab}^c is the fusion multiplicity, so that $N_{ab}^c > 0$ means that c is among the possible fusion products of a and b.

The point of the $\mathbb{Z}_2$ grading is that the $\tilde{\kappa}$ indices multiply at any vertex. As a result, in any diagram, the union of all the paths of particles having $\tilde{\kappa} = -1$ is a collection of closed loops. An example of a diagram in a $\mathbb{Z}_2$ graded theory is shown in Fig. 21 where we can see the closed loops explicitly. In Convention 1, up-down zig-zags (Fig. 2) in these closed loops will incur a minus sign analogous to the case of the semion model. Here we introduce an analogous cap counting bookkeeping (Convention 3) which is equivalent to Convention 1.

Our bookkeeping system will be a fairly simply generalization of the bookkeeping for the semion model discussed above. As in that case we will arrange that all particles with $\tilde{\kappa}_a = -1$ have $d_a < 0$ with d_a the value of a loop. Full evaluation of a diagram will require (analogous to Rule C0) that we first count the number of caps
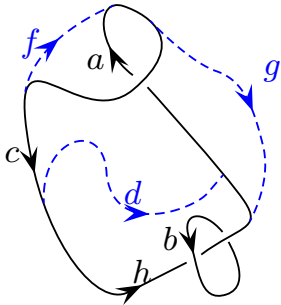
Figure 21. A spacetime diagram for a $\mathbb{Z}_2$ graded theory. Here particle types with $\tilde{\kappa} = -1$ are drawn black and solid. Particle types with $\tilde{\kappa} = +1$ are drawn blue dotted. Note that the $\tilde{\kappa} = -1$ lines form closed loops. There are three caps of $\tilde{\kappa} = -1$ particles in this figure.

of particles with $\tilde{\kappa}_a = -1$ and call this number n. (An example of such cap counting is shown in Fig. 21). We may then evaluate the diagram (making our "nonunitary evaluation") using isotopy invariant rules (analogous to Rules C1-C3) which allow us to straighten zig-zags and at the end we multiply by $(-1)^n$ (analogous to rule C4) which breaks the isotopy invariance again. As before, we can think of this procedure as simply working in a different gauge, either by applying a unitary basis transformation to a string net wave function, or by applying a non-unitary gauge transformation in our tensor category.

In counting the number of caps with negative $\tilde{\kappa}$, one has to be a bit cautious, because one can run into vertices of the type shown on the right of Fig. 4. Such a vertex should be counted as a cap if both $\tilde{\kappa}_b$ and $\tilde{\kappa}_c$ are negative. A simple rule that makes counting easy is just to erase all lines with positive $\tilde{\kappa}$ which leaves only loops, and then we count the caps of these loops.

In this scheme we arrange that the sign of d_a matches the sign of $\tilde{\kappa}_a$. Further since the $\tilde{\kappa}$ are multiplicative as in Eq. 11 we also have $\text{sign}(d_a)\text{sign}(d_b) = \text{sign}(d_c)$ for any nonzero a, b, c vertex (such as shown in Fig. 4). As a result, the argument of the square roots in Figs. 6 and 7 are always positive. However we still have the freedom to choose the sign of the square-root. It is easy to establish that *for consistency we must choose an overall negative sign to the square root if and only if d_a and d_b are both negative.* For example, in Fig. 6 if we consider the case of a and d being the vacuum, so that $b = \bar{c}$, the square root takes the sign of d_b, so that a loop of b has weight d_b rather than $|d_b|$. Another way to see that we should choose such a sign is that both the moves shown in Figs. 6 and 7 change the parity of the number of caps exactly when d_a and d_b are both negative.

This cap-counting technique allows us to work with diagrammatic rules that allow straightening of zig-zags, so long as we have a $\mathbb{Z}_2$ Frobenius-Schur grading. As mentioned above, we outline in Appendix A how many

(but not all) theories of interest do have such gradings. Such gradings can be subdivided into two types:

1. **Simply Graded Theories**

    For a particularly simple subclass of theories, all non-self-dual particles can be simply assigned $\kappa_a = +1$. For lack of a better word, we will call these theories "simply $\mathbb{Z}_2$ graded".

2. **Non-Simply Graded Theories**

    Although there are many theories that are simply graded, some are graded, but are not simply graded.

    The simplest example of a non-simply graded theory is $SU(6)_1$. This is an abelian theory with six particle types (called $\mathbb{Z}_6^{(3+1/2)}$ in the notation of Ref. 20). In this particular case, the particle types can be written as a^n for $n = 0 \ldots 5$ (with a^0 meaning the identity) with fusion rules $a^n \times a^m = a^{(n+m)\text{mod}6}$. The particle type a^3 is self dual with $\kappa_3 = -1$. Hence we must take the grading $\tilde{\kappa}_n = (-1)^n$. We see that this type of theory allows a $\mathbb{Z}_2$ grading so long as we allow some of the non-self-dual particles (a^1 and a^5) to be assigned negative values of $\tilde{\kappa}$.

    In a non-simply graded theory, we now have non-self-dual particles which we have assigned $\tilde{\kappa}_a = -1$. For our bookkeeping scheme to work, we want the corresponding value d_a of a loop to be negative. Fortunately, this can be arranged. To do so, we need only choose an appropriate gauge (See Eq. 1). In particular, for these particles we now choose $[F_a^{a\bar{a}a}]_{00}$ negative and choose d_a negative as well so that we have isotopy invariance in Fig. 5. Note that the gauge choice can have an effect on other F matrix elements as well.

In either case, we have a bookkeeping scheme that allows one to straighten zig-zags freely and minus signs are re-inserted at the last step. In particular, it assures us that, if we are considering knots (or links), the value of the diagram is unchanged under any isotopy of the knot, with again minus signs being only re-inserted at the end.

IX. ISOTOPY OF DIAGRAMS WITH VERTICES

In studying topological theories, we would like to be able to deform 3D diagrams in any way, and still have the diagram correspond to the same value. I.e., we want our theories to be "regular isotopic" (The word "regular" here meaning that we must be careful not to insert twists in strands. I.e, the diagrams should be thought of as being composed of ribbons rather than straight lines). As discussed above, negative Frobenius-Schur indicators present complications in obtaining such isotopy invariance.

So long as we are considering knots and links, using Convention 2 from section IV we are guaranteed to have a regular isotopic theory — i.e., a knot or link invariant. If we use Convention 1 (which is often used by physicists and particularly useful for string net models) we have to worry about Frobenius-Schur indicators, and straightening a zig-zag incurs a minus sign. However, so long as there is a $\mathbb{Z}_2$ Frobenius-Schur grading, our Convention 3 (cap counting) allows us to work with isotopy invariant rules. As described above, for particles with $\tilde{\kappa}_a = -1$ we set d_a, the value of a loop negative. Before a computation we count caps having $\tilde{\kappa} = -1$, and call this number n. We may then apply isotopy freely to our diagram to simplify it. After fully evaluating the diagram we multiply the final result by $(-1)^n$. This prescription is equivalent to Convention 1 where all flags are right-pointing and we incur a minus sign for straightening each zig-zag with $\tilde{\kappa} = -1$.

Despite these approaches to obtaining isotopy invariance for knots and links, there is still an issue to be cautious of. When we have diagrams involving vertices — fusions or splittings — we are not always guaranteed full isotopy invariance. We have so far obtained isotopy invariance for diagrams of knots and links, but not of graphs with fusion. This is somewhat curious: we have obtained isotopy invariance for the value of a knot or link, but in order to actually evaluate the knot or link we typically use F-moves to turn it into a fusion diagram, which may then break the isotopy invariance. Further we certainly had to consider fusions and F-moves in order to even define the Frobenius-Schur grading, just to obtain isotopy invariance of diagrams that have no fusions or splitting.

To give an example of how fusion diagrams may lose isotopy invariance we consider the diagrams shown in Fig. 22 where factors of F and d are incurred in diagrammatic transformation that would be allowed in a theory having full isotopy invariance.

$$\begin{aligned} & \bar{c}\ \Big|\ = \sqrt{\frac{d_a d_c}{d_b}}[F_a^{c\bar{c}a}]_{Ib} \end{aligned}$$

$$\begin{aligned} & \bar{c}\ \Big|\ = \sqrt{\frac{d_a d_c}{d_b}}[F_a^{a\bar{c}c}]_{bI}^* \end{aligned}$$

Figure 22. Turning-up and Turning-Down legs. The factors of $d^{1/2}$ in these equations are due to the vertex renormalization factors in Eq. 2. For theories with full isotopy invariance the total factor out front on the right is unity. Note the case $b = 1$ is just straightening out zigzags.

Let us assume that we are working with Convention 3 (cap counting) such that we can freely straighten zigzags. It turns out that very often we can fix a gauge such that all factors from turning-up and turning-down legs as in Fig. 22 are trivial. If we can set these factors to unity, then we have full isotopy invariance in the plane even with vertices.

The condition to have such full planar isotopy is a condition on a quantity known as the *third Frobenius-Schur indicator*[10,11,19,20,34]. This indicator can be defined if the fusion multiplicity $N_{aa}^{\bar{a}}$ is nonzero. I.e., if a and a fused together has $\bar{a}$ as one of the possible fusion products. If this is the case we define the operator C_a as in Fig. 23 to rotate a vertex by $2\pi/3$. This operator is an $N_{aa}^{\bar{a}}$ dimensional matrix (and in particular is just a scalar if there is no fusion multiplicity).

Figure 23. Rotating a vertex by $2\pi/3$. The indices μ, ν are vertex indices which must be included if there is a fusion multiplicity $N_{aa}^{\bar{a}} > 1$.

The third Frobenius-Schur indicator $\nu_3(a)$ is then defined as the trace of this matrix

$$\nu_3(a) = \text{Tr}[C_a] \tag{12}$$

We say that this indicator is trivial if $\nu_3(a) = N_{\bar{a}}^{aa}$, i.e., if all $N_{\bar{a}}^{aa}$ of the eigenvalues of C_a are unity. Otherwise we say that $\nu_3(a)$ is nontrivial. (A simple example of a theory with a nontrivial $\nu_3(a)$ is the generating cocycle of the group $\mathbb{Z}_3$.).

In appendix B, we prove the following important theorem:

Theorem: For a spherical tensor category with a $\mathbb{Z}_2$ Frobenius-Schur grading and trivial third Frobenius-Schur indicator, one can always choose a gauge which realizes full planar isotopy invariance, i.e., the prefactors in Fig. 22 are all unity. In particular this means that one can always obtain planar isotopy invariance if there is no particle such that $N_{aa}^{\bar{a}} > 0$.

Many theories we want to consider are also *ribbon*, meaning that in addition to having F matrices satisfying the pentagon, we have R matrices (see Fig. 11) satisfying the hexagon equation with these F's, and we have a consistent set of twist factors θ (See for example, Refs. 19 and 20). In fact every braided unitary theory has a unique ribbon structure (i.e., uniquely defined consistent twists)[35]. If we have such a ribbon theory then it is much harder to have nontrivial third Frobenius-Schur indicators. We study this case in detail in appendix B 2 with the following results:

1. In the case of a ribbon theory we cannot have such a nontrivial $\nu_3(a)$ unless $N_{aa}^{\bar{a}} > 1$ (i.e., unless there is a fusion multiplicity such that $\bar{a}$ occurs $N_{aa}^{\bar{a}} > 1$ times, in the fusion product of a with a).

2. If we do have such a fusion multiplicity, then we examine the $N_{aa}^{\bar{a}}$ dimensional matrix matrix $[R_{\bar{a}}^{aa}]_{\mu\nu}$ and the $N_{aa}^{\bar{a}}$ dimensional matrix $[(F_{\bar{a}}^{a\bar{a}\bar{a}})_{Ia}]_{\mu\nu}$. If $R_{\bar{a}}^{aa}$ commutes with $(F_{\bar{a}}^{a\bar{a}\bar{a}})_{Ia}$, then the third Frobenius-Schur indicator is trivial.

Obviously in the case where $N_{aa}^{\bar{a}} = 1$, these are scalars not matrices and therefore commute.

An example of a modular (therefore braided) theory with nontrivial third Frobenius-Schur indicator is given in Appendix C 2.

We note that there are other "higher" Frobenius-Schur indicators[10,11,36] which we write as $\nu_p(a)$ with $p > 3$. These are defined analogous to Fig. 23 except that the vertex has p lines all labeled a coming into a single point. However, our diagrammatic algebra is defined only for trivalent vertices so one can only define ν_p with $p > 3$ by resolving a p-valent vertex into multiple trivalent vertices. Then if the manipulation of the trivalent vertices is isotopy invariant, so will be the full diagram. Thus, ν_p for $p > 3$ cannot present a further obstruction to obtaining isotopy invariance for any theory which already has trivial third indicators and where all vertices in diagrams are defined to be trivalent only (which we generally assume).

A. How Much Isotopy?

Assuming we have a $\mathbb{Z}_2$ Frobenius-Schur grading, and we use Convention 3, and further we do not have any nontrivial third Frobenius-Schur indicators, then we have full isotopy for planar diagrams — i.e we can deform the diagrams in the plane, and turn up and down the legs of the vertices freely. However, for ribbon theories (i.e, those with R matrices satisfying the hexagon, and consistent twist factors θ) this does not necessarily translate into full 3D isotopy.

Let us consider ribbon theories with a $\mathbb{Z}_2$ Frobenius-Schur grading and with no nontrivial third Frobenius-Schur indicators. We use Convention 3 and consider the "nonunitary" evaluation of the diagram — i.e., the steps after the counting of caps and before the re-introduction of minus signs at the end. For this evaluation, when we start with a diagram, we think of it as being made of blackboard framed ribbons with branches of ribbons at vertices. We paint the front side (out of the blackboard) of the ribbon white and the backside (which we do not see in the blackboard framing) black. Any deformation (isotopy) of the ribbons leaves the value of the diagram unchanged if it again ends up with all the white sides facing forwards[37]. Any F-moves or R-moves performed are defined to start with white sides facing forwards, and end with white sides again facing forwards. However, it is generally not the case that the value of a diagram will
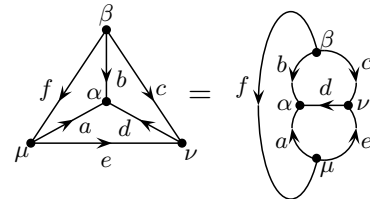
Figure 24. The Tetrahedral Diagram. Variables are attached to vertices for the general case with fusion multiplicity.

be unchanged (and in fact may not even be well defined) if the diagram is isotoped so that any of the black sides are forward. This appears to agree with the discussion of Refs. 5, 6, and 37.

One often considers theories with a higher degree of isotopy known as "tetrahedral symmetry". This is generally taken to mean that the tetrahedral diagram in Fig. 24 is invariant under all 24 symmetry operations of the tetrahedron — 12 rotations and 12 inversion-rotations. For ribbon theories with $\mathbb{Z}_2$ Frobenius-Schur grading, and with no nontrivial third Frobenius-Schur indicators, if we use Convention 3 as described in the previous paragraph, the tetrahedral diagram is invariant under all 12 rotations, but not necessarily under inversions. It is worth noting that, if we think of the tetrahedral diagram as being made out of ribbons, one can smoothly deform the diagram into the inverted tetrahedron, but if in the initial position if all the ribbons have the white side facing forwards, when the tetrahedron is inverted all of the black sides face forwards instead. Thus invariance under inversion is not something that our diagrammatic algebra generally guarantees. In appendix C 3 an example is given of a modular (therefore braided) theory where the tetrahedral diagram is invariant under all rotations, but not inversions.

We note in passing that there have been a number of attempts[38–40] to develop a diagrammatic calculus which is able to more generally describe ribbon diagrams such that one can properly give a value to half-twisting a ribbon — something that we cannot describe in the usual diagrammatic algebra.

Of course there certainly do exist many (ribbon) theories for which the tetrahedral diagram can be rotated and inverted without changing its value. For example, any Chern-Simons theory $SU(N)_k$ without fusion multiplicity[41] has this full tetrahedral symmetry including inversion. However, in cases where there are fusion multiplicities it is possible to find cases (indeed it may even be generic) where one can obtain tetrahedral rotational symmetry but cannot obtain inversion symmetry in any gauge. For theories which enjoy full tetrahedral invariance, including invariance under inversion, one does not need to keep track of the front and back of ribbons, and we believe diagrams for such theories are fully isotopy invariance in three-dimensional space.

X. MORE GENERAL TRANSFORMATIONS?

The cap-counting scheme described above allows us (usually) to work with isotopy invariant diagram algebras, at the price of (at least temporarily) working with a non-unitary theory (i.e., with negative loop-weights). The cap-counting signs added back in at the end fix the result so that it is completely equivalent to a unitary theory. In string-net models, as discussed briefly in section VII, the Convention 3 "nonunitary evaluation" is completely equivalent to making a well-chosen gauge transformation. It is clear that we can do this not only for the theories discussed in VII, but in fact for in any theory with a $\mathbb{Z}_2$ Frobenius-Schur grading.

Since we have accepted that we will be working with a non-unitary theory, we might wonder if we can do something more general than just pushing signs from the Frobenius-Schur indicators onto loop weights. For the case of constructing string-nets this possibility has been explored in detail by Refs. 18 and 29. There one considers independent gauge transforms for "bras" and "kets" for the planar diagram algebra, and these transformations can be more general than just involving signs. One might ask whether we can use a similar generalized strategy for describing 2+1 dimensional (braided) theories. Of course, we give up unitary to do this, and there may not be any easy trick (like cap counting) that will recover unitarity in the end. We leave it as an open question as to whether there are advantages to working with such more general non-unitary braided theories.

XI. CONCLUSION

The Frobenius-Schur indicator has been a source of a great deal of confusion in both the physics and mathematics literature. In this paper we have elucidated the meaning of this quantity. We showed that it is intimately related to the need to frame particle world lines. We show that different conventions for interpreting diagrams ends up treating signs associated with the Frobenius-Schur indicator differently, and it is this distinction that is the source of much of the confusion. One method of interpretation is isotopy invariant ("Convention 2") applicable to framed world lines in Chern-Simons theories, whereas the other convention ("Convention 1") is not isotopy invariant on account of signs associated with zig-zags. We discuss an alternate convention ("Convention 3") which, while equivalent to the non-isotopy invariant conventions, allows one to count caps of the diagram then work with an isotopy invariant theory thereafter, adding back in the non-isotopy invariant signs at the end. This method works so long as the theory has a $\mathbb{Z}_2$ Frobenius-Schur grading, which we argue is quite general. We point out that in the construction of string nets, the non-isotopy invariant signs can be removed by gauge transformation.

Assuming that we have a $\mathbb{Z}_2$ Frobenius-Schur grading, using Convention 3, we are able to straighten zig-zags

freely. One can obtain full isotopy of (fusion) diagrams in a plane, unless there is an obstruction caused by a nontrivial third Frobenius-Schur indicator. In the absence of this obstruction, we can freely deform diagrams in the plane. For ribbon theories (where nontrivial third Frobenius-Schur indicators are very rare) we can freely deform diagrams in three dimensions but we must treat the diagram as being made of ribbons and generally we must keep track of which side of the ribbon is facing forward.

Finally we would like to remark that any non-invariance of diagrams under isotopy that occurs in the topological formalism is always due to a change in the order of events. If we consider all fusion and splitting vertices in a diagram (including the creations and annihilations) to be the events of the spacetime history, then these events naturally form a partially ordered set, with the ordering induced by the flow of the particles from one vertex to another. Any deformation of the history which changes this partial order requires either bending the legs of one or more vertices from forward to backward in time or vice versa, and/or adding new creation and annihilation events (caps and cups). Isotopies which do not do this always leave the amplitudes invariant. We might call these "causal isotopies", since partially ordered sets essentially store the causal information in a history while forgetting, as much as possible, any distance information. Because of this, they have inspired a number of approaches to the study of quantum gravity[42,43], where the distance information must be somehow added back in to reproduce the classical limit. In describing systems of particles with only topological interactions, one might consider starting from the idea that amplitudes are allowed to depend on the partial order of the events as long as they are invariant under causal isotopy. Naively this could lead to richer theories. However the Hamiltonian approach to histories, essentially interpreting every timeslice as a state, seems to naturally allow only the limited non-invariance provided by the presence of nontrivial Frobenius-Schur indicators, at least in the planar case. With braiding, we gain non-invariance under inversion of the tetrahedron and one may wonder if there is more. Starting from the action formalism of Chern-Simons theory, one would have expected complete invariance, since the field theory is Euclidean and does not support any intrinsic order of events. It turns out the requirement of framing of particle world lines, which is a global requirement, not local in any "timeslice", actually allows for the expected full isotopy invariance at least for histories which are knots or links, resolving the non-invariance which is found in the Hamiltonian approach when there are nontrivial Frobenius-Schur indicators. With vertices, some non-invariance can actuallly remain. It would be interesting to know if there are any other global requirements on spacetimes histories, beyond framing, which can be imposed to enrich the symmetries of Hamiltonian models.

We hope that this paper goes a long way towards clar-

ifying the physics of the Frobenius-Schur indicators and will be useful to those studying topological models in both the physics and mathematics communities.

ACKNOWLEDGMENTS

The authors are grateful to Andre Henriques and Parsa Bonderson for helpful conversations. We are particularly grateful to Eddy Ardonne, both for giving comments on a draft of this work and also for helping us establish the $SU(3)_3$ example in Appendix C3. Statement of compliance with EPSRC policy framework on research data: This publication is theoretical work that does not require supporting research data. SHS has been supported by EPSRC Grants EP/I031014/1, EP/N01930X/1, and EP/S020527/1. JKS has been supported by Science Foundation Ireland through Principal Investigator Awards 12/IA/1697 and 16/IA/4524

Appendix A: The Many Theories With $\mathbb{Z}_2$ Frobenius-Schur Grading

The purpose of this appendix is to explore how common it is to be able to give a theory a $\mathbb{Z}_2$ Frobenius-Schur grading.

There appear to be a great many theories that admit a $\mathbb{Z}_2$ grading (simple or non-simple in the language of section VIII A) — and indeed while exceptions (non-gradeable theories) do exist, they are a bit unusual. To examine this further, let us consider a few common types of theories of interest which all have $\mathbb{Z}_2$ Frobenius-Schur gradings.

(i) **Products (and Possibly Condensations, and Cosets)**

In appendix A1 we discuss how, given theories that have a $\mathbb{Z}_2$ Frobenius-Schur grading, additional theories also having such a $\mathbb{Z}_2$ Frobenius-Schur grading may be constructed by several procedures. These procedures include taking the product of two theories, taking the quotient of two theories (i.e., forming a coset theory), and condensing a boson from a theory. It is easy to show that if two theories both admit a $\mathbb{Z}_2$ Frobenius-Schur grading then their product will also. The situation is less clear for condensations and cosets, but we conjecture that if we start with theories having $\mathbb{Z}_2$ Frobenius-Schur gradings, under fairly general conditions their condensations and cosets will too.

(ii) **Chern-Simons Theories**

Consider Chern-Simons theories G_k where G is a Lie-group and k is the level. Such Chern-Simons theories are examined in detail in Appendix A2. All of these theories admit a $\mathbb{Z}_2$ Frobenius-Schur grading. Almost all of these theories can obtain a $\mathbb{Z}_2$ Frobenius-Schur grading by assigning $\tilde{\kappa}_a = 1$ for any non-self-dual particles (i.e., they are "simply $\mathbb{Z}_2$ graded" in the language we introduced in section VIII A, item 1). The exception are theories of the form $SU(6+4n)_k$ where $n \geq 0$. These cases can also be given a $\mathbb{Z}_2$ grading by assigning some of the non-self-dual particles $\tilde{\kappa}_a = -1$. A detailed discussion of Chern-Simons theories is given in Appendix A2.

(iii) **All particles self-dual**

In many theories, all particles are self-dual (such as, $SU(2)_k$). As long as the so-called positivity conjecture holds (See appendix A3), then the Frobenius-Schur indicators immediately give the theory a $\mathbb{Z}_2$ grading. As mentioned in appendix A3, exceptions to the positivity conjecture are extremely rare[44,45], and cannot occur in a braided theory without fusion multiplicity $N_{ab}^c > 1$.

(iv) **Braided abelian Theories**

It is easy to show that any braided abelian theories will admit a $\mathbb{Z}_2$ Frobenius-Schur grading (See Appendix A4).

(v) **"Small" Discrete (twisted and untwisted) Gauge Theories**

As detailed in section C1 discrete gauge theories (twisted or untwisted) of groups of order 15 and less all admit a $\mathbb{Z}_2$ Frobenius-Schur grading.

In appendix C1 we give some theories which we know *do not* admit $\mathbb{Z}_2$ Frobenius-Schur gradings.

1. Products (and Possibly Condensations, and Cosets)

Given two theories G and H having a $\mathbb{Z}_2$ Frobenius-Schur grading, it is trivial to show that the product theory $G \times H$ will also have a $\mathbb{Z}_2$ Frobenius-Schur grading with $\tilde{\kappa}_{a_G \times b_H} = \tilde{\kappa}_{a_G} \tilde{\kappa}_{b_H}$ where a_G is a particle type from theory G, and b_H is a particle type from theory H.

What is more interesting is the possibility of taking condensations[46] of theories with $\mathbb{Z}_2$ Frobenius-Schur gradings. Bosons can only condense if they have trivial Frobenius-Schur indicators[47]. Further we can think of a condensed particle as being the fusion of a particle in the uncondensed theory with the vacuum — thus having the same $\mathbb{Z}_2$ Frobenius-Schur index as the uncondensed particle. It is also possible that under condensation a particle may split into multiple species. Physically we can think of this as the creation of new conserved quantities that can be assigned to a particle in the condensed phase. However, such splitting does not change the $\mathbb{Z}_2$ index. Under condensation, the fusion rules are preserved, and this is entirely consistent with the $\mathbb{Z}_2$ index being inherited from the uncondensed theory.

What is nontrvial here is the possibility that a new self-dual particle may emerge in the condensed theory

that was not there in the uncondensed theory. In particular, if we condense a particle b, if there is a particle a in the uncondensed theory such that $a \times a = b + \ldots$, then while a is not self-dual in the uncondensed theory, it becomes self-dual in the condensed theory and its Frobenius-Schur indicator seems as if it could be arbitrary, and this may not match the value of its Frobenius-Schur grading when it was uncondensed, thus breaking the idea of the condensed theory inheriting its grading from the uncondensed theory.

That said, if the uncondensed theory has a $\mathbb{Z}_2$ Frobenius-Schur grading and if it also has no nontrivial third Frobenius-Schur indicators, then the uncondensed theory can be put in a form with isotopy invariance for its diagrammatic algebra. Since every process in the condensed theory can be described as a process in the uncondensed theory, along with creation and annihilation of bosons, it then seems as if the diagrammatic algebra for the condensed theory must also have isotopy invariance, suggesting that it also has a $\mathbb{Z}_2$ Frobenius-Schur grading and no nontrivial third Frobenius-Schur indicators. However, this is a conjecture, not a proof.

Finally we consider the possibility of cosets of two theories. This case is simply a special case of condensations. Here we will use the statement that we can write a coset G/H as $G \times \bar{H}$ (with $\bar{H}$ being the mirror image theory of H) where in the product theory we condense all bosons[46] (or fully extend the chiral algebra, in another language). We can then invoke the discussion above regarding condensations.

2. Chern-Simons Theories

Here we discuss simple Chern-Simons theories for compact Lie group G at level k and we show that all can be given a $\mathbb{Z}_2$ Frobenius-Schur grading.

First we note that since the Frobenius-Schur indicator of a self-dual object can only take the values ± 1, it should be constant as a function of a parameter q when the representation theory is q-deformed even when we choose q to be root of unity despite the fact that the tensor product becomes truncated. Thus we expect that the (second) Frobenius-Schur indicators are the same for all levels k, and are the same as the corresponding classical Lie groups as well. It is crucial here that the truncation does not affect the channel where a and a fuse to the identity, so long as the particle a exists at the given level. As a result, we can focus only on the corresponding classical group.

We thus want to find a $\mathbb{Z}_2$ Frobenius-Schur grading for the classical Lie groups $(A_l, B_l, C_l, D_l, F_4, G_2, E_6, E_7, E_8)$. A tremendously elegant way to do this is to invoke results given in Ref. 48 (See also the discussion of Ref. 49). Given a representation a, let χ_a be the so-called central character of a (i.e., the rep a restricted to the center of the group

is χ_a times the identity). For any irrep we set

$$\tilde{\kappa}_a = \chi_a(\exp(2\pi i \rho^\vee))$$

where $\rho^\vee$ is half the sum of the positive coroots. Here $\exp(2\pi i \rho^\vee)$ is necessarily an element of order 2 in the center of the group. Due to the multiplicative property of characters of Abelian groups, this means that the $\tilde{\kappa}$'s form a $\mathbb{Z}_2$ grading. Further, as shown in Ref. 48, $\tilde{\kappa}_a$ is in fact κ_a, the Frobenius-Schur indicator when a is self-dual. This then confirms the existence of a $\mathbb{Z}_2$ Frobenius-Schur grading.

For completeness we mention some properties of the $\mathbb{Z}_2$ Frobenius-Schur gradings for the classical Lie groups.

For the Lie groups E_8, F_4, G_2, and D_n (or $SO(2n)$) with $n = 4m$, and B_n (or $SO(2n+1)$) with $n = 4m$ or $4m+3$ all irreps are self-dual, and all Frobenius-Schur indicators are $+1$, so these cases have trivial $\mathbb{Z}_2$ Frobenius-Schur gradings.

For E_6 and D_n (or $SO(2n)$) with n odd, and A_n ($SU(n+1)$) with $n \neq 4m+1$ not all irreps are self-dual, but all self-dual irreps have Frobenius-Schur indicator $+1$, so again we can trivially assign $\tilde{\kappa} = +1$ for all irreps.

For E_7, A_1 (or $SU(2)$), C_n (or $Sp(2n)$), B_n (or $SO(2n+1)$) with $n = 4m+1$ or $4m+2$, and D_n (or $SO(2n)$) with $n = 4m+2$, all irreps are self-dual and the Frobenius-Schur indicator is multiplicative under fusion as required.

The most interesting case is A_n or $SU(n+1)$ with $n = 4m+1$ and $m \geq 1$. In this case not all irreps are self-dual and for those irreps which are self-dual not all have positive Frobenius-Schur indicators. For this case we must nontrivially assign $\tilde{\kappa} = \pm 1$ to the non-self-dual irreps. I.e., this case is "non-simply graded". In this case it is well known that one can assign an index to each irrep which is conserved under fusion modulo $(n+1)$ (this is sometimes known as $(n+1)$-ality or the congruence class[50]). Further, for self-dual irreps this index is even for $\kappa_a = 1$ and odd for $\kappa_a = -1$. Assigning $\tilde{\kappa}_a$ to be the parity of this index then gives the Frobenius-Schur grading.

3. Positivity Conjecture

In 2003, Bantay proposed[24,44] a positivity conjecture that states that for any category with fusion rules

$$a \times b = \sum_c N_{ab}^c \, c$$

and corresponding Frobenius-Schur indicators κ_a for particle of type a, we should have

$$\kappa_a \kappa_b = \kappa_c \quad \text{when} \quad N_{ab}^c > 0 \quad \text{and } a, b, c \text{ all self-dual} \tag{A1}$$

Note that this conjecture applies to cases where a, b and c are all self-dual (in fact we do not even define κ for non-self dual particles here).

While it turns out that this conjecture is not actually true in all cases, it is quite challenging to find cases where the conjecture fails, the first one being published only in 2017 by Mason[44] being based on a group of order 128 (Another exception based on a group of order 64 was informally discussed on a website[45] in 2011). Further, there does exist a proof that the conjecture must be true whenever N_{ab}^c is odd (for any braided theory and indeed, many other theories that do not actually require a braiding, see Refs 44 and 51).

4. Braided Abelian Theories

Here we briefly show that any braided abelian theory, can be given a $\mathbb{Z}_2$ Frobenius-Schur grading and further cannot have a nontrivial third Frobenius-Schur indicator.

A beautiful theorem by Galindo and Jaramillo[52,53] reduces any modular abelian category to a product of so-called *prime* modular abelian categories. Of the prime categories only the right- and left-handed semion theories have a nontrivial Frobenius-Schur indicator (!). Since we know these two admit a $\mathbb{Z}_2$ Frobenius-Schur grading, and taking products of theories with gradings gives a theory that allows a grading (see section A 1), this implies that all modular abelian categories admit a $\mathbb{Z}_2$ Frobenius-Schur grading.

This result can also be extended to non-modular but braided abelian categories. To do so, we simply use a result from Ref. 54 that any non-modular but braided abelian theory can be written as a product of a modular abelian theory along with some number of fermions which also have trivial Frobenius-Schur indicators. Thus our result applies to all braided abelian theories.

Further, abelian theories have no fusion multiplicities, and as mentioned in the main text (and shown in appendix B 2) there can be no nontrivial third Frobenius-Schur indicator without fusion multiplicity. This means that it is always possible to put abelian theories into isotopy invariant form.

Appendix B: Turning-Up/Turning-Down and Gauge Transformation

In this appendix we consider the transformations of "turning-up/down" edges from vertices (as in Fig. 22). We will assume a $\mathbb{Z}_2$ Frobenius-Schur grading. We will assume we have handled the minus signs from Frobenius-Schur indicators using Convention 3. We now want to know whether we can further choose a gauge such that the turning-up/down transformations are trivial — i.e., in the diagrammatic algebra, one can turn up and down edges for free. While it is not always possible to choose such a gauge, we will isolate a single possible "obstruction" which is the third Frobenius-Schur indicator[10,11,19,20].

We will not assume any braiding for now. We introduce some notation that we will use to simplify the discussion later.

$$\tikz = V_{c;\mu}^{ab} \tag{B1}$$

The index μ is included when there is a fusion multiplicity $N_{ab}^c > 1$. For simplicity of notation we may supress these vertex indices except when they become important. Similarly we have

$$\tikz = V_{ab;\mu}^{c} \tag{B2}$$

We will assume we are working with "Convention 3" so that we can straighten zig-zags freely. Hermitian conjugation of a diagram is achieved by reflecting the diagram across a horizontal line and reversing all arrows, for example as in Eq. B3

$$\left[\tikz \right]^{\dagger} = \tikz \tag{B3}$$

or equivalently $[V_c^{ab}]^{\dagger} = V_{ab}^c$. Note however, we will generally have a non-positive-definite inner product because we may have chosen $d_a < 0$ for some particle types.

We will make use of the so-called pivotal property, which we write diagrammatically as shown in Fig. 25. This identity is proven very generally in Ref. 19.
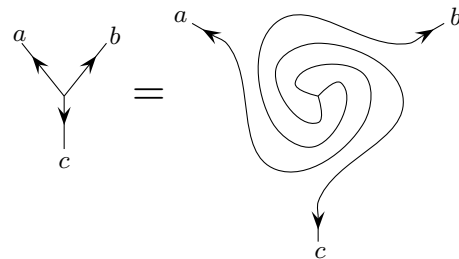
Figure 25. The pivotal property is rotation of a vertex by 2π

This property is equivalent to Fig. 26[19].
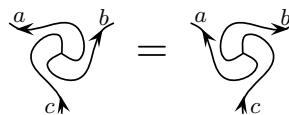
Figure 26. Another version of the pivotal property. Clockwise rotation of a vertex by π is equal to counterclockwise rotation by π.

Since the pivotal property is derived in Ref. 19 using Convention 1, one might worry that a sign could be introduced when moving to Convention 3. However, since

there are three caps on the right hand side of Fig. 25 (one of each type) the assumed $\mathbb{Z}_2$ grading assures that signs cancel and the pivotal property holds in this Convention 3 as well.

We now define the turning-up and turning-down operators

$$T_c^{ab} \left[\vcenter{\hbox{\includegraphics{y1}}} \right] = \vcenter{\hbox{\includegraphics{y2}}} \tag{B4}$$

$$T_{bc}^a \left[\vcenter{\hbox{\includegraphics{y3}}} \right] = \vcenter{\hbox{\includegraphics{y4}}} \tag{B5}$$

Explicit expressions for these T operators in terms of F-matrices are given in Fig. 22.

If there are no fusion multiplicities these T operators are just complex phases. More generally, in cases where there are fusion multiplicities, the vertices need indices as well, and the raising and lowering operators become unitary matrices, as shown here:

$$\sum_\mu [T_c^{ab}]_{\nu\mu} \left[\vcenter{\hbox{\includegraphics{y5}}} \right] = \vcenter{\hbox{\includegraphics{y6}}} \tag{B6}$$

For simplicity of notation, we will typically not write these matrix indices, but we should remember that they are implied. We will insert them explicitly when they become important.

Note that these turning-up and turning-down operators are gauge dependent. Under gauge transformation the vertices transform as

$$V_c^{ab} \longrightarrow u_c^{ab} V_c^{ab} \tag{B7}$$
$$V_{ab}^c \longrightarrow u_{ab}^c V_{ab}^c \tag{B8}$$

Where again the u factors become unitary matrices in cases where there are fusion multiplicities. Note that due to the Hermitian conjugation principle (Eq. B3) we have

$$u_c^{ab} = [u_{ab}^c]^\dagger$$

Under gauge transformation the T operators transform as

$$T_c^{ab} \longrightarrow [u_{c\bar{b}}^a] \, T_c^{ab} \, [u_c^{ab}]^\dagger \tag{B9}$$
$$T_{bc}^a \longrightarrow [u_c^{\bar{b}a}] \, T_{bc}^a \, [u_{bc}^a]^\dagger \tag{B10}$$

Again, if there are fusion multiplicities then the u's and T's are matrices in the μ vertex variables.

The key question here is whether we have enough gauge freedom to set all of the turning-up and turning-down factors T to unity. To answer this question, we refer to the hexagonal diagram in Fig. 27.

In this diagram, the equality at the bottom is assured by the pivotal property, Fig. 26. Thus, following the
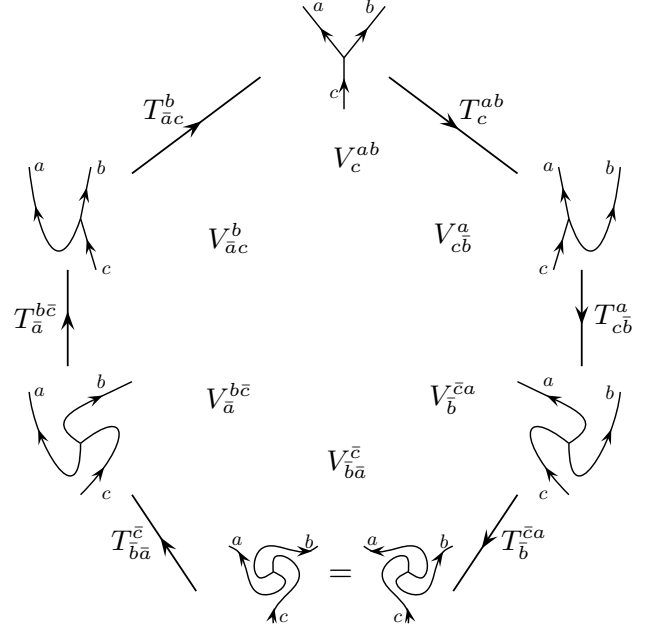
Figure 27. Rotating a vertex by 2π with turning-up and turning-down operators. The equality at the bottom is assured by the pivotal property, Fig. 26

arrows, all the way around the hexagon, we have the following identity.

$$T_{\bar{a}c}^b \, T_{\bar{a}}^{b\bar{c}} \, T_{\bar{b}\bar{a}}^{\bar{c}} \, T_{\bar{b}}^{\bar{c}a} \, T_{c\bar{b}}^a \, T_c^{ab} = 1 \tag{B11}$$

where again, with fusion multiplicities, this is a matrix equation in the μ variables and the right hand side is the identity matrix. Note that we can start the circle around the hexagon at any point on the circle and we will still get the identity. For example, we also have

$$T_{\bar{a}}^{b\bar{c}} \, T_{\bar{b}\bar{a}}^{\bar{c}} \, T_{\bar{b}}^{\bar{c}a} \, T_{c\bar{b}}^a \, T_c^{ab} \, T_{\bar{a}c}^b = 1 \tag{B12}$$

What we would like to know here is whether we have enough gauge freedom such that we can always choose a gauge such that all of the T operators are unity — i.e., so that one can turn up and turn down legs at vertices freely.

Let us first consider the case where all three quantum numbers a, b, c are distinct, meaning they are not the same and they are not related to each other by duality. In this case it is clear that it is always possible to set all of the T's to unity. To see this let us start at the top of the hexagon with V_c^{ab}. We pick an arbitrary gauge for this vertex (i.e., we fix u_c^{ab}). We then want to gauge transform $V_{c\bar{b}}^a$ such that T_c^{ab} becomes the identity. To do this, using Eq. B9 we may choose

$$u_{c\bar{b}}^a = ([T_c^{ab}]_{old-gauge})^\dagger [u_c^{ab}]$$

and we use the fact that the u's and T's are always unitary. In the new gauge T_c^{ab} is unity. Even in the case where there is fusion multiplicity, this strategy successfully trivializes T_c^{ab}.

Having set this T to unity, we continue around the hexagon clockwise and attempt to set $T^a_{c\bar{b}}$ to unity. To do this, we similarly gauge transform $V^{\bar{c}a}_{\bar{b}}$ (Using Eq. B10 in this case). We can continue all the way around the hexagon setting each T to unity. When we have set five of these T's to unity, the last one is guaranteed to be unity also by Eq. B11. Note that there is still one overall gauge freedom which is given by how we set the gauge of the first vertex at the start of the process.

1. Obstructions

The procedure described for gauge-fixing T's to unity can run into obstructions if not all of the three quantum numbers a, b, c are distinct. The problem one can run into is if two of the vertices V around the hexagon are the same (or are related by complex conjugation) then one cannot independently choose the gauge of each V.

With some thought it is clear that there are only two situations that may cause trouble. We will take these cases one at a time.

a. Case 1: $a = \bar{b}$, and c self-dual. Not an obstruction

To avoid notational confusion, let us set $a = x$, $b = \bar{x}$ and we assume $c = y$ is self-dual. We may also assume that c is not the vacuum field since we have already assumed that we can freely add and remove cups and caps, which can be thought of as a vertex with $c = I$.

At the top of the hexagon we have $V^{ab}_c = V^{x\bar{x}}_y$ whereas at the bottom we have $V^{\bar{c}}_{\bar{b}\bar{a}} = V^y_{x\bar{x}}$. These two are Hermitian conjugates of each other and cannot be independently gauge fixed. Similarly on the right hand side of the hexagon we have $V^a_{c\bar{b}} = V^x_{yx}$ and $V^{\bar{c}a}_{\bar{b}} = V^{yx}_x$ which are also Hermitian conjugates of each other; and on the left hand side of the hexagon we have $V^b_{\bar{a}c} = V^{\bar{x}}_{\bar{x}y}$ and $V^{b\bar{c}}_{\bar{a}} = V^{\bar{x}y}_{\bar{x}}$ which are again conjugate of each other. We thus have only three gauge freedoms in the hexagon instead of six. We might wonder if we still have enough freedom to set all of the T's to unity.

First, we claim that $T^{ab}_c = T^{x\bar{x}}_y$ is related to $T^{\bar{c}a}_{\bar{b}} = T^{yx}_x$ and $T^{\bar{c}}_{\bar{b}\bar{a}} = T^y_{x\bar{x}}$ is related to $T^b_{\bar{a}c} = T^{\bar{x}}_{\bar{x}y}$ via the equations

$$T^{x\bar{x}}_y = \left(\left[T^{yx}_x\right]^\dagger\right)^{-1} \tag{B13}$$

$$T^y_{x\bar{x}} = \left(\left[T^x_{\bar{x}y}\right]^\dagger\right)^{-1} \tag{B14}$$

Let us examine Eq. B13 in a bit of depth. The left hand side is the T^{ab}_c connecting two diagrams at the top right of the hexagon. However, if we flip the diagrams over a horizontal (i.e., Hermitian conjugate) we will find exactly the vertices at the bottom right of the hexagon which are connected by $T^{\bar{c}a}_{\bar{b}}$. This is on the right hand side of Eq. B13 and the fact that we flipped the diagrams accounts for the Hermitian conjugation. However, the

arrow on the operator $T^{\bar{c}a}_{\bar{b}}$ needs to be reversed, which accounts for the inverse sign. The argument is similar for Eq. B14.

Now let us try to set all of the T's to the identity. First, as above, by choosing a gauge for $V^a_{c\bar{b}}$ we can set the first link $T^{ab}_c = T^{x\bar{x}}_y$ to unity. However, given Eq. B13 this also sets $T^{\bar{c}a}_{\bar{b}} = T^{yx}_x$ to unity. Now let us think about the first three steps of the hexagon together, which are

$$T^{\bar{c}a}_{\bar{b}} T^a_{c\bar{b}} T^{ab}_c = T^{yx}_x T^x_{yx} T^{x\bar{x}}_y$$

If we consider gauge transforming the top of the hexagon $V^{ab}_c = V^{x\bar{x}}_y$ with $u^{x\bar{x}}_y$ this also transforms the bottom $V^{\bar{c}}_{\bar{b}\bar{a}} = V^y_{x\bar{x}}$ as the Hermitian conjugate. The result will be

$$[u^y_{x\bar{x}}] T^{yx}_x T^x_{yx} T^{x\bar{x}}_y [u^{x\bar{x}}_y]^\dagger = [u^y_{x\bar{x}}] T^{yx}_x T^x_{yx} T^y_x [u^y_{x\bar{x}}]$$

We can choose the gauge so that this is unity. Since T^{yx}_x and $T^{x\bar{x}}_y$ can both be set to unity by choosing the gauge of $V^a_{c\bar{b}}$, this means we must have also set the remaining T^x_{yx} to unity.

For the second half of the hexagon, we proceed similarly. By choosing the gauge of $V^{b\bar{c}}_{\bar{a}}$ we can set both $T^{\bar{c}}_{\bar{b}\bar{a}} = T^y_{x\bar{x}}$ and $T^b_{\bar{a}c} = T^{\bar{x}}_{\bar{x}y}$ to unity. However, now the product of T's all the way around the hexagon must be unity (Eq. B11) so the one remaining $T^{b\bar{c}}_{\bar{a}} = T^{\bar{x}y}_{\bar{x}}$ is also unity.

Thus we conclude that we still have enough gauge freedom to set all of the T's to unity and there is no obstruction.

b. Case 2: $a = b = \bar{c}$. Possible Obstruction

In the case where $a = b = \bar{c}$ we do have a possible obstruction. Again for clarity of notation we set $a = b = x$ and $c = \bar{x}$.

In the hexagon diagram, there are now only two different operators $T^{xx}_{\bar{x}}$ and $T^x_{\bar{x}x}$ which alternate around the hexagon. Taking any two consecutive steps along the hexagon, the vertex one ends up with is identical to the vertex one starts with. We can thus construct two gauge invariant quantities.

$$C_x = T^{xx}_{\bar{x}} T^x_{\bar{x}x} \tag{B15}$$

$$C'_x = T^x_{\bar{x}x} T^{xx}_{\bar{x}} \tag{B16}$$

Note that if $N^{xx}_{\bar{x}} > 1$ then these are matrix equations (with indices μ, ν not written).

From Eq. B11, i.e., going all the way around the hexagon, we have

$$[C_x]^3 = [C'_x]^3 = 1$$

so that the eigenvalues of C and C' must be cube roots of unity. We can then define the so-called *third Frobenius-Schur indicator*[10]

$$\nu_3(x) = \text{Tr}[C] = \text{Tr}[C']$$

We say the that this indicator is trivial if

$$\nu_3(x) = N_{xx}^{\bar{x}}$$

i.e., if C_x (or equivalently C_x') is the unit matrix. If this is not the case, and there is an eigenvalue which is not unity, then there is no way to choose a gauge such that turning up and down legs does not incur any phases. (Indeed, this eigenvalue tells us that we have a gauge invariant nontrivial phase associated with taking two steps around the hexagon, or twisting a vertex by 120 degrees). A crucial result here is that such a nontrivial third Frobenius-Schur indicator is in fact the *only* possible obstruction to obtaining isotopy of planar diagrams (given that we have a $\mathbb{Z}_2$ Frobenius-Schur grading and we have used our Convention 3 to account for signs associated with zig-zags). For planar diagram algebras there are simple cases of theories having nontrivial third Frobenius-Schur indicators — for example, the generating cocycle of the group $\mathbb{Z}_3$.

2. $\mathbb{Z}_3$ Frobenius-Schur in Ribbon Theories

It is rather difficult for ribbon theories to have nontrivial third Frobenius-Schur indicators. To see why this is, we will use the R matrices to explicitly calculate $\nu_3(x)$. We will consider taking two steps around the hexagon in Fig. 27 starting at the upper left and going to the upper right. I.e., $C_x = T_{\bar{x}}^{xx} T_{\bar{x}\bar{x}}^x$. Our plan will be to evaluate $T_{\bar{x}}^{xx}$ in terms of $T_{\bar{x}\bar{x}}^x$ by using the R matrix. Recall that we are using Convention 3 so that we may add and remove cups and caps freely. We start with

Note that with fusion multiplicity $N_{xx}^{\bar{x}} > 1$, here $R_{\bar{x}}^{xx}$ is a matrix in these internal indices which we suppress for simplicity of notation.

We then lower the left leg by using $[T_{\bar{x}\bar{x}}^x]^{-1}$ (i.e., walking from the top of the hexagon, one step to the left.) We then have

The diagram on the right can be deformed into the following diagram and untwisted as shown to yield θ_a^*. Note that

This diagram can then be untwisted with $[R_{\bar{x}}^{xx}]^{-1}$ to obtain the diagram on the upper right of the hexagon. Putting these pieces together we have

$$C_x = \theta_x^* [R_{\bar{x}}^{xx}]^{-1} [T_{\bar{x}\bar{x}}^x]^{-1} [R_{\bar{x}}^{xx}]^{-1} T_{\bar{x}\bar{x}}^x$$

Generally each term on the right (except θ_x) is a $N_{xx}^{\bar{x}}$ dimensional matrix. If $T_{\bar{x}\bar{x}}^x$ and $R_{\bar{x}}^{xx}$ commute then we can bring the two T terms together and they will cancel. We will then be left with $C_x = \theta_x^* [R_{\bar{x}}^{xx}]^{-2}$. We then use the ribbon identity $[R_c^{ab} R_c^{ba}]_{\mu\nu} = \delta_{\mu\nu} \theta_c / (\theta_a \theta_b)$ (see Ref. 19) to give C_x equal to the identity matrix.

We conclude that the third Frobenius-Schur indicator is trivial if $R_{\bar{x}}^{xx}$ commutes with $T_{\bar{x}\bar{x}}^x$. This is obviously satisfied if these quantities are scalars, i.e., if $N_{xx}^{\bar{x}} = 1$. (In the main text we use F instead of T, but these are equivalent up to constant factors).

If $N_{xx}^{\bar{x}} > 1$, the commutation of these two matrices may seem like a rather strong condition. However due to the so-called ribbon identity we must have the eigenvalues of R given by

$$\text{eigs}[R_{\bar{x}}^{xx}] = \pm \frac{1}{\sqrt{\theta_x}}$$

where θ_x is the twist factor for x. If all of the $\pm$ happen to be the same, then this matrix is proportional to the identity and it commutes with $T_{\bar{x}\bar{x}}^x$.

Appendix C: Unusual Examples

1. Theories without $\mathbb{Z}_2$ Frobenius-Schur Gradings

There are theories that do not admit $\mathbb{Z}_2$ Frobenius-Schur grading. For planar algebras (i.e., solutions of the pentagon without solution of the hexagon) it is fairly easy to find such exceptions.

A simple example is the generating cocycle of the group $\mathbb{Z}_4$. This theory has four objects $a = 0, 1, 2, 3$ with fusion rules $a \times b = (a + b) \bmod 4$. For the case of the generating cocycle, the Frobenius-Schur indicator for the second object is $\kappa_2 = -1$, but $1 \times 1 = 2$ so it is impossible to have a $\mathbb{Z}_2$ Frobenius-Schur grading. There are obvious generalizations to cocycles of the group $\mathbb{Z}_{4n}$.

However, for braided theories exceptions are much harder to find. By searching a database[55] of discrete (twisted and untwisted) gauge theories (i.e., Dijkgraaf-Witten theories[56] theories) we have found examples of modular theories which do not have $\mathbb{Z}_2$ Frobenius-Schur grading. Using this database we generate fusion relations using the Verlinde formula and Frobenius-Schur indicators using the Bantay formula[9] and then determine whether a grading is possible. As mentioned in item (v) of section A above, all gauge theories (twisted or untwisted) for groups of order 15 and less do have $\mathbb{Z}_2$ gradings. For groups of order 16, there are several exceptions, the simplest being the untwisted quantum double of the quasi-dihedral group of order 16 (group [16,8] in GAP

notation[57]). This modular theory has 46 simple objects in it. In addition certain twisted doubles of the groups [16,3],[16,4], [16,6], [16,10], and [16,11] also fail to have gradings (none of these have fewer than 46 objects). We note, however, that the group $Z_5 \rtimes Z_4$ (or [20,3]) which is a group of 20 elements, has some twisted quantum doubles with only 22 simple objects which also fails to have a $\mathbb{Z}_2$ gradings. This modular theory with 22 simple objects is the smallest modular (or braided) theory we have found which fails to admit a $\mathbb{Z}_2$ Frobenius-Schur grading.

2. Some modular theories with nontrivial third Frobenius-Schur Indicator

An example of a braided (and modular) theory with a nontrivial third Frobenius-Schur indicator is the (untwisted) quantum double of the group $Z_5 \rtimes Z_4$ (or [20,3] in GAP notation[57]). The group has 20 elements, and its quantum double has only 22 elements. In fact, this group (before taking the quantum double) is the smallest group where a representation has a nontrivial third Frobenius-Schur indicator $\frac{1}{|G|} \sum_g \chi(g^3)$ with χ the character of the rep. All of the Frobenius-Schur indicators of this quantum double are either 1 or 0 so the theory has a trivial $\mathbb{Z}_2$ Frobenius-Schur grading and yet we cannot put it into a form where turning-up and turning-down does not incur a phase. By searching the database (Ref. 55) we have found that among discrete gauge theories (twisted or untwisted) this is the smallest example with nontrivial third Frobenius-Schur indicator (There are twisted quantum doubles of the same group which also have nontrivial third Frobenius-Schur indicators). The method of calculation is similar to that of section C 1 above. Using the database we generate fusion relations using the Verlinde formula and third Frobenius-Schur indicators using a generalization of the Bantay formula[9,11].

There are also examples of Chern-Simons theories with nontrivial third Frobenius-Schur indicators. The simplest few are $SU(4)_5$, $SO(5)_4$, $SO(8)_4$, $(E_6)_4$, $(E_7)_4$, $(E_8)_5$, $(F_4)_4$, $(G_2)_9$. As mentioned in section A 2 all of these have $\mathbb{Z}_2$ Frobenius-Schur gradings. Among these examples $SO(5)_4$ and $(E_8)_5$ both have only 15 particle types, and $(F_4)_4$ has 16 (And further, the cases $(E_8)_5$ and $(F_4)_4$ have all particles with $\kappa_a = +1$). The method of finding these is similar to the previous paragraph: we use the program Kac[58] to generate S matrices and twist factors for a given Chern-Simons theory then generate fusion relations using the Verlinde formula and the third Frobenius-Schur indicators using a generalization of the Bantay formula[9,11].

While in section B 2 we showed that $N_{aa}^{\bar{a}}$ must be greater than 1 in order to have a nontrivial third Frobenius-Schur indicator. In fact, we have not found any case of a modular theory where $N_{aa}^{\bar{a}} = 2$ with nontrivial third Frobenius-Schur indicator. We conjecture that this can never happen.

3. Example of a modular theory with tetrahedral rotation, but not inversion

As an example of a modular theory which has full planar isotopy, and allows rotation of the tetrahedral diagram, but not inversion, we consider the example of $SU(3)_3$. The F-matrices for this are calculated explicitly in Ref. 59 section B.3 (Note that actually what is shown in this reference is the category $SU(3)_3/\mathbb{Z}_3$ but this is a subcategory of $SU(3)_3$). This theory has no negative $\mathbb{Z}_2$ Frobenius-Schur indicators, and no nontrivial third Frobenius-Schur indicators (The third Frobenius-Schur indicator is calculated using the formula in Ref. 11 using modular data for $SU(3)_3$ obtained from Ref. 58.) Because there are no nontrivial Frobenius-Schur indicators, we can put the theory in a form so that diagrams can be deformed in the plane as discussed above and tetrahedral diagrams can be rotated freely. However, we can give an example of a tetrahedron that is not invariant under inversion. We consider labeling every edge of the tetrahedron with the self-dual quantum number 8 except one edge which we label with 10 (these are names of the particle types in $SU(3)$ notation). There are two vertices where three 8's meet, and since $N_{88}^8 = 2$, we must label each of these vertices with additional quantum numbers, μ at one vertex and ν at the other. Up to positive constants (square-roots of d's) the value of this tetrahedron diagram is given by an F-matrix symbol $([F_{10}^{8,8,8}]_{8,8})_{\mu\nu}$. Inverting the tetrahedron exchanges μ and ν or equivalently flips 10 to its dual $\overline{10}$. If μ and ν are different then this inversion changes the sign of the result. One might wonder if one can choose a different gauge for the indices μ, ν so that this sign goes away. In fact, one cannot. A gauge transform would be a unitary matrix $U_{\alpha\nu}$ and would result in the F-matrix changed to $\sum_{\alpha,\beta} U_{\mu\alpha} U_{\nu\beta} ([F_{10}^{8,8,8}]_{8,8})_{\alpha\beta}$ which one can show cannot be made symmetric in μ, ν.

[1] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).

[2] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).

[3] S. H. Simon, "Lecture notes on topological quantum," (2016), http://www-thphys.physics.ox.ac.uk/people/SteveSimon/topological2021/TopoBook-Sep1-2021.pdf.

[4] E. Witten, Communications in Mathematical Physics **121**, 351 (1989).

[5] B. Bakalov and A. Kirillov Jr., *Lectures on tensor categories and modular functors*, University Lecture Series, Vol. 21 (American Mathematical Society, 2001).

[6] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, Vol. 18 (Walter de Gruyter, 1994).

[7] M. Atiyah, *The Geometry and Physics of Knots* (Cambridge University Press, 1990).

[8] Z. Wang, *Topological Quantum Computation*, CBMS Regional Conference Series in Mathematics, Vol. 112 (American Mathematical Society, New York, 2010).

[9] P. Bantay, Physics Letters B **394**, 87 (1997).

[10] S.-H. Ng and P. Schauenburg, in *Hopf Algebras and Generalizations*, Contemporary Mathematics, Vol. 441, edited by L. H. Kauffman, D. E. Radford, and F. J. O. Souza (2007).

[11] S.-H. Ng and P. Schauenburg, Advances in Mathematics **211**, 34 (2007).

[12] L. Huiszoon, A. Schellekens, and N. Sousa, Nuclear Physics B **575**, 401 (2000).

[13] G. Felder, J. Frohlich, J. Fuchs, and C. Schweigert, Compositio Mathematica **131**, 189–238 (2002).

[14] Y. Kashina, Y. Sommerhaeuser, and Y. Zhu, Mem. Amer. Math. Soc. **181** (2006).

[15] J. Farnsteiner and C. Schweigert, Letters in Mathematical Physics **112**, 39 (2022).

[16] P. Schauenburg, https://arxiv.org/abs/1604.02378 (2016).

[17] M. A. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005).

[18] C.-H. Lin, M. Levin, and F. J. Burnell, Phys. Rev. B **103**, 195155 (2021).

[19] A. Kitaev, Annals of Physics **321**, 2 (2006).

[20] P. Bonderson, *Non-Abelian anyons and interferometry*, Ph.D. thesis, California Institute of Technology (2007).

[21] It is assumed here for simplicity of presentation that there are no fusion multiplicities greater than 1, otherwise vertices would contain an additional index. See Refs. 3, 19, and 20.

[22] In the case where there is fusion multiplicity at the vertex the factors u_a^{bc} would instead be a unitary matrix. See Refs. 3, 19, and 20.

[23] In the language of Ref. 10 the Frobenius-Schur indicator that we are discussing for most of this paper would be called the *second* Frobenius-Schur indicator.

[24] P. Bantay, Physics Letters B **488**, 207 (2000).

[25] This result of $-1/2$ is precisely the $6j$ symbol

$$\left\{ \begin{array}{ccc} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{array} \right\} = -1/2$$

.

[26] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics, Vol. 155 (Springer Verlag, 1995).

[27] In category theory these flags correspond to choosing between ev and $\tilde{ev}$ or $coev$ and $co\tilde{ev}$. In fact in category theory it is more natural to keep such flags for all particles whether or not they are self dual, and for both $\kappa_a = \pm$. However, here we are concerned mainly with the case where we cannot gauge away complications.

[28] M. Freedman, C. Nayak, K. Shtengel, K. Walker, and Z. Wang, Annals of Physics **310**, 428 (2004).

[29] C.-H. Lin and M. Levin, Phys. Rev. B **89**, 195130 (2014).

[30] S. C. Morampudi, C. von Keyserlingk, and F. Pollmann, Phys. Rev. B **90**, 035117 (2014).

[31] J. W. Barrett and B. W. Westbury, Advances in Mathematics **143**, 357 (1999).

[32] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, Vol. 205 (American Mathematical Society, 2016).

[33] J. Vidal, Phys. Rev. B **97**, 125152 (2018).

[34] See for example, L. M. Isaacs.*Character Theory of Finite Groups*, Academic Press, 1976, p. 49 Lemma 4.4 for a discussion of how general n^{th} Frobenius-Schur indicators of elements of a finite group can be used to determine how many n^{th} roots an element of a group has.

[35] C. Galindo, Canadian Mathematical Bulletin **57**, 506–510 (2014).

[36] E. Lake and Y.-S. Wu, Phys. Rev. B **94**, 115139 (2016).

[37] N. Y. Reshetikhin and V. G. Turaev, Communications in Mathematical Physics **127**, 1 (1990).

[38] M. Bärenz, *Topolgical State Sum Models in Four Dimensions, Half-Twists and Their Applications*, Ph.D. thesis, University of Nottingham (2017).

[39] N. Snyder and P. Tingley, Algebra and Number Theory **3**, 809 (2009).

[40] P. Tingley, arxiv:1002.0555 (2010).

[41] J. Gu and H. Jockers, Communications in Mathematical Physics **338**, 393 (2015).

[42] L. Bombelli, J. Lee, D. Meyer, and R. D. Sorkin, Physical review letters **59**, 521 (1987).

[43] J. Ambjørn and R. Loll, Nuclear Physics B **536**, 407 (1998).

[44] G. Mason, arxiv:1703.08890v1 (2017).

[45] F. Ladisch, (2011), https://mathoverflow.net/ questions/ 53126/finite- groups- in- which- every- character- has- real- values- grading- the- representati.

[46] F. A. Bais and J. K. Slingerland, Phys. Rev. B **79**, 045316 (2009).

[47] I. S. Eliëns, J. C. Romers, and F. A. Bais, Phys. Rev. B **90**, 195130 (2014).

[48] J. Adams, Compositio Mathematica **150**, 2127–2142 (2014).

[49] Mathoverflow (2016), https://mathoverflow.net/questions/ 227869/is-there-a-formula-for-the-frobenius-schur- indicator-of-a-rep-of-a-lie-group.

[50] F. Lemire and J. Patera, Journal of Mathematical Physics **21**, 2026 (1980), https://doi.org/10.1063/1.524711.

[51] J. Fuchs, I. Runkel, and C. Schweigert, Journal of Physics A: Mathematical and General **35**, L255 (2002).

[52] C. Galindo and N. Jaramillo, Revista Colombiana de Matematicas **50**, 277–298 (2016).

[53] L. Wang and Z. Wang, Journal of Physics A: Mathematical and Theoretical **53**, 505203 (2020).

[54] J. Cano, M. Cheng, M. Mulligan, C. Nayak, E. Plamadeala, and J. Yard, Phys. Rev. B **89**, 115116 (2014).

[55] A. Gruen and S. Morrison, https://tqft.net/web/research/students/AngusGruen/ Computing Modular Data for Pointed Fusion Categories: arXiv:1808.05060 (2018).

[56] R. Dijkgraaf and E. Witten, Commun. Math. Phys. **129**, 393 (1990).

[57] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.11.1, 2021, https://www.gap-system.org.

[58] Computer Program Kac, by Bert Schellekens available at https://www.nikhef.nl/~t58/Site/Kac.html.

[59] E. Ardonne and J. Slingerland, Journal of Physics A: Mathematical and Theoretical **43**, 395205 (2010).