Newton mechanics, Galilean relativity, and special relativity in α-deformed binary operation setting

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Abstract

We define new velocity and acceleration having dimension of $(Length)^{\alpha}/(Time)$ and $(Length)^{\alpha}/(Time)^2$, respectively, based on the fractional addition rule. We discuss the formulation of fractional Newton mechanics, Galilean relativity and special relativity in the same setting. We show the conservation of the fractional energy, characterize the Lorentz transformation and group, and derive the expressions of the energy and momentum. The two body decay is discussed as a concrete illustration.

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1 Introduction

This work is based on pseudo analysis (see [6] and [7], and references therein for a review). It is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a,b] \subset [-\infty,+\infty]$ endowed with pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$. It has different applications in mathematics and physics, e.g. in modeling nonlinearity, uncertainty in optimization problems, nonlinear partial differential equations, nonlinear difference equations, optimal control, fuzzy systems, decision making, game theory, etc. It also gives solutions in the forms, which are not achieved by other approaches, e.g., Bellman difference equation, Hamilton Jacobi equation with non-smooth Hamiltonians.

Definition 1.1 *The pseudo binary operations are defined by the help of a monotonous bijective map f, called their generator, as:*

$$x \oplus_f y = f^{-1}(f(x)+f(y)), \qquad x \ominus_f y = f^{-1}(f(x)-f(y)), \tag{1}$$

$$x \otimes_f y = f^{-1}(f(x)f(y)), \quad \textit{and} \quad x \oslash_f y = f^{-1}(f(x)/f(y)), \tag{2}$$

$$x^n \oplus_f y^m = x^n + y^m, \qquad x^n \ominus_f y^m = x^n - y^m, \tag{3}$$

$$x^n \otimes_f y^m = x^n y^m, \quad \textit{and} \quad x^n \oslash_f y^m = x^n / y^m, \tag{4}$$

$$\left(x \oplus_f y\right)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \qquad \left(x \ominus_f y\right)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k y^{n-k}, \tag{5}$$

$$\left(x \otimes_f y\right)^n = (xy)^n, \qquad \textit{and} \qquad \left(x \oslash_f y\right)^n = (x/y)^n. \tag{6}$$

It can be easily checked that the operation $\oplus_f$ and $\otimes_f$ satisfy the commutativity and associativity properties. Through the map f, we can perform many deformed binary operations [2,8]. The first use of this pseudo binary operations was made by Einstein (noticed by Chung *et al.* [4]) in the velocity addition. The second use was made in constructing the q-additive entropy theory [1].

Recently, in 2019, Chung and Hassanabadi [4] considered a special choice of f,

$$f(x) = |x|^{\alpha-1} x, \quad \alpha > 0, \tag{7}$$

so that the deformed multiplication and deformed division may be the same as the ordinary ones. Using this, these authors studied the anomalous diffusion process by using the α-deformed mechanics which possesses the α-translation in space $x \to x \oplus \delta x$. The α-deformed binary operations, i. e. α-addition, α-subtraction, α-multiplication and α-division take the form:

$$a \oplus_\alpha b = |a|a|^{\alpha-1} + b|b|^{\alpha-1}|^{1/\alpha-1}(a|a|^{\alpha-1} + b|b|^{\alpha-1}) \tag{8}$$
$$a \ominus_\alpha b = |a|a|^{\alpha-1} - b|b|^{\alpha-1}|^{1/\alpha-1}(a|a|^{\alpha-1} - b|b|^{\alpha-1}) \tag{9}$$
$$a \otimes_\alpha b = ab, \; a \oslash_\alpha b = \frac{a}{b}. \tag{10}$$

Interestingly, the multiplication and division are invariant under α-deformation.

In this same spirit, in 2022, Hounkonnou *et al* proved that a Minkowski phase space endowed with a bracket relatively to a conformable (α-deformed) differential realizes a conformable Poisson algebra, confering a bi-Hamiltonian structure to the resulting manifold. They deduced that the related α-Hamiltonian vector field for a free particle is an infinitesimal Noether symmetry and computed the corresponding $\alpha-$deformed recursion operator [5].

The present paper is organized as follows. In Section 2, we derive the Newton law of $\alpha-$deformed Newton mechanics. Section 3 is devoted to the characterization of α-deformed Galilean relativity. The α-deformed Galilei group is described, and energy conservation law is deduced. In Section 4, we study the special relativity with α-translation symmetry. Section 5 deals with an analysis of two body decay.

2 $\alpha-$deformed Newton mechanics

In an ordinary Newtonian mechanics in one dimension, the Newton velocity is defined as

$$v = \frac{dx}{dt}. \tag{11}$$

The infinitesimal displacement is invariant under spatial translation $x \to x + \delta x$ and the infinitesimal time interval is invariant under temporal translation $t \to t + \delta t$. If we impose new translation symmetry based on α-addition rule, we need to change the definition of velocity so that it may possess this new symmetry. Here we impose two translation symmetries: the α-translation in position, $x \to x \oplus_\alpha \delta x$, and α-translation in time, $t \to t \oplus_\alpha \delta t$.

Note in 2019, Chung and Hassanabadi [4] defined the deformed velocity, which is invariant under α-translation in position and ordinary translation in time. Their average velocity is given by

$$v_{ave} = \frac{f_\alpha(x' \ominus_\alpha x)}{t' - t} = \frac{\Delta_\alpha x}{\Delta t} = \frac{|x'|^{\alpha-1}x' - |x|^{\alpha-1}x}{t' - t}. \tag{12}$$

Taking $t' \to t$, we obtain the velocity:

$$v = \frac{d_\alpha x}{dt} = \alpha|x|^{\alpha-1}\frac{dx}{dt}. \tag{13}$$

If we impose the α-translation in both time and position, we have to change the definition of the velocity. In this case, the average α-velocity is furnished by the expression

$$v_{\alpha,ave} = \frac{f_\alpha(x' \ominus_\alpha x)}{f(t' \ominus_\alpha t)} = \frac{\Delta_\alpha x}{\Delta_\alpha t} = \frac{|x'|^{\alpha-1}x - |x|^{\alpha-1}x}{|t'|^{\alpha-1}t' - |t|^{\alpha-1}t}. \tag{14}$$

Taking $t' \to t$ leads to the α-velocity:

$$v_\alpha = \frac{d_\alpha x}{d_\alpha t} = t^{1-\alpha}|x|^{\alpha-1}\frac{dx}{dt} \tag{15}$$

Because v_α is α-translation invariant, the α-acceleration is defined as

$$a_\alpha = \frac{dv_\alpha}{d_\alpha t} = \frac{1}{\alpha}t^{1-\alpha}\frac{dv_\alpha}{dt} \tag{16}$$

Since the α-velocity and α-acceleration have dimension $[Length]^\alpha/[Time]^\alpha$ and dimension $[Length]^\alpha/[Time]^{2\alpha}$, respectively, the Newton equation is obtained by the relation

$$|F|^{\alpha-1}F = m^\alpha a_\alpha \quad \text{or, equivalently,} \quad F = m|a_\alpha|^{\frac{1}{\alpha}-1}a_\alpha \tag{17}$$

In mechanics with α-translation symmetry, the α-velocity and α-acceleration have the fractional dimensions which are different from the ordinary case $\alpha = 1$. But, for the force, we assumed that it has the same dimension as in the $\alpha = 1-$mechanics.

3 α-deformed Galilean Relativity

Based on the new definition of α-velocity and α-acceleration, we define the α-inertial frames of reference possessing the property that a body with zero net force acting upon these frames does not α-accelerate; that is, such a body is at rest or moving at a constant α-velocity. Here we assume *the physical laws must be the same in all α-inertial frames of reference.* Now let us consider two inertial frames $S(t,x)$ and $S'(t',x')$ moving at a relative constant α-velocity u_α with x-axes. The Newton equation is invariant under the transformations

$$v'_\alpha = v_\alpha - u_\alpha, \quad v'_\alpha = \frac{d_\alpha x'}{d_\alpha t}, \quad x' = x \ominus_\alpha |u_\alpha|^{\frac{1}{\alpha}-1} u_\alpha t, \quad t' = t. \tag{18}$$

3.1 α-deformed Galilei group

Based on the α-operations for matrices, we can rewrite the coordinate transformations of the Newton equation as:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = T_\alpha(u_\alpha) \otimes_\alpha \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 1 & -|u_\alpha|^{\frac{1}{\alpha}-1} u_\alpha \\ 0 & 1 \end{pmatrix} \otimes_\alpha \begin{pmatrix} x \\ t \end{pmatrix}. \tag{19}$$

The transformation matrix $T_\alpha(u_\alpha)$ forms a Lie group with the α-multiplication. The following properties are indeed satisfied:

- $T_\alpha(u_\alpha) \otimes_\alpha T_\alpha(v_\alpha) = T_\alpha(u_\alpha + v_\alpha)$.
- The α-multiplication is associative.
- The identity is $T_\alpha(0)$.
- The inverse is $T_\alpha(-u_\alpha)$.

3.2 Energy conservation

Because dx is not invariant under the α-translation, we use α-translational invariant infinitesimal displacement to get $d_\alpha x = \alpha |x|^{\alpha-1} dx$ and define the work,

$$|W|^{\alpha-1} W = -\int d_\alpha x |F|^{\alpha-1} F, \tag{20}$$

having the same dimension as in the $\alpha = 1$—mechanics. We define the potential energy through the conservative force,

$$|F|^{\alpha-1} F = -\frac{d_\alpha U}{d_\alpha x} = -|x|^{1-\alpha} |U|^{\alpha-1} \frac{dU}{dx}. \tag{21}$$

Thus, for the conservative force, we have

$$|W_{1\to 2}|^{\alpha-1} W_{1\to 2} = -(|U_2|^{\alpha-1} U_2 - |U_1|^{\alpha-1} U_1). \tag{22}$$

Inserting the Newton equation obtained previously (see (17)) into (20), we get

$$|W_{1\to 2}|^{\alpha-1} W_{1\to 2} = K_2 - K_1 \tag{23}$$

where the kinetic energy is given by

$$K = \frac{1}{2} m^\alpha v_\alpha^2. \tag{24}$$

Considering the dimension, the conservation of energy is provided by

$$|E|^{\alpha-1}E = K + |U|^{\alpha-1}U = \frac{1}{2}m^{\alpha}v_{\alpha}^{2} + |U|^{\alpha-1}U = \frac{p_{\alpha}^{2}}{2m^{\alpha}} + |U|^{\alpha-1}U, \tag{25}$$

where the linear momentum is expressed as $p_{\alpha} = m^{\alpha}v_{\alpha}$. The energy has the same dimension as in the $\alpha = 1$—mechanics, while the linear momentum has fractional dimension.

4 Special relativity with α-translation symmetry

The 3-position in non-relativistic mechanics is changed into 4-position (or event) in the relativistic mechanics. Let us consider the event $P(ct, x, y, z)$, where c is the Newton speed of light, (i. e. speed with $\alpha = 1$). Based on the definition of α-translation invariant infinitesimal displacement and α-translation invariant infinitesimal time interval, the α-translation invariant distance (α-distance) of infinitesimally close space-time events denoted by ds_{α} is given by

$$d_{\alpha}s^{2} = c^{2\alpha}d_{\alpha}t^{2} - d_{\alpha}x^{2} - d_{\alpha}y^{2} - d_{\alpha}z^{2}. \tag{26}$$

The α-deformed proper time τ_{α} is

$$d_{\alpha}\tau^{2} = \frac{d_{\alpha}s^{2}}{c^{2\alpha}}. \tag{27}$$

4.1 α-Lorentz transformations

The α-Lorentz transformations making invariant the space-time interval

$$(\Delta_{\alpha}s)^{2} = (c^{\alpha}(|t|^{\alpha-1}t))^{2} - ((|x|^{\alpha-1}x))^{2} \tag{28}$$

are given by

$$|x|^{\alpha-1}x = c^{\alpha}|t'|^{\alpha-1}t'sh_{\alpha}(\psi) + |x'|^{\alpha-1}x'ch_{\alpha}(\psi) \tag{29}$$

$$c^{\alpha}|t|^{\alpha-1}t = c^{\alpha}|t'|^{\alpha-1}t'ch_{\alpha}(\psi) + |x'|^{\alpha-1}x'sh_{\alpha}(\psi), \tag{30}$$

where the little α-deformed hyperbolic functions are defined by

$$sh_{\alpha}(\psi) := \frac{1}{2}(e_{\alpha}(\psi) - e_{\alpha}(-\psi)) = \sinh(|\psi|^{\alpha-1}\psi) \tag{31}$$

$$ch_{\alpha}(\psi) := \frac{1}{2}(e_{\alpha}(\psi) + e_{\alpha}(-\psi)) = \cosh(|\psi|^{\alpha-1}\psi) \tag{32}$$

$$th_{\alpha}(\psi) := \frac{sh_{\alpha}(\psi)}{ch_{\alpha}(\psi)} = \tanh(|\psi|^{\alpha-1}\psi), \quad e_{\alpha}(x) := e^{|x|^{\alpha-1}x}. \tag{33}$$

The little α-deformed hyperbolic functions obey the relations

$$ch_{\alpha}^{2}(\psi) - sh_{\alpha}^{2}(\psi) = 1. \tag{34}$$

In terms of the α-deformed binary operations, we get

$$x = ct'Sh_{\alpha}(\psi) \oplus x'Ch_{\alpha}(\psi), \quad ct = ct'Ch_{\alpha}(\psi) \oplus x'Sh_{\alpha}(\psi), \tag{35}$$

where the big α-deformed hyperbolic functions are

$$Ch_{\alpha}(\psi) := |ch_{\alpha}(\psi)|^{\frac{1}{\alpha}-1}ch_{\alpha}(\psi), \quad Sh_{\alpha}(\psi) := |sh_{\alpha}(\psi)|^{\frac{1}{\alpha}-1}sh_{\alpha}(\psi) \tag{36}$$

$$Th_\alpha(\psi) := \frac{Sh_\alpha(\psi)}{Ch_\alpha(\psi)}. \tag{37}$$

obeying $|Ch_\alpha(\psi)|^2 \ominus |Sh_\alpha(\psi)|^2 = 1$. Consider in the coordinate system (ct, x) the origin of the coordinate system (ct', x'). Then, $x' = 0$, and

$$x = ct' Sh_\alpha(\psi), \quad ct = ct' Ch_\alpha(\psi). \tag{38}$$

Dividing the two equations gives

$$\frac{x}{ct} = Th_\alpha(\psi), \quad \text{or,} \quad \frac{|x|^{\alpha-1}x}{c^\alpha |t|^{\alpha-1}t} = th_\alpha(\psi). \tag{39}$$

Since $\frac{|x|^{\alpha-1}x}{|t|^{\alpha-1}t} = v_\alpha$ is the relative uniform α-velocity of the two systems, we identify the physical meaning of the imaginary "rotation angle ψ" as

$$th_\alpha(\psi) = \frac{v_\alpha}{c^\alpha} = \beta_\alpha. \tag{40}$$

Using the following identities

$$ch_\alpha(\psi) = \gamma_\alpha, \;\; sh_\alpha(\psi) = \gamma_\alpha \beta_\alpha, \text{where} \quad \gamma_\alpha = \frac{1}{\sqrt{1-\beta_\alpha^2}}, \tag{41}$$

we obtain the α-deformed Lorentz transformation of the form

$$|x|^{\alpha-1}x = \gamma_\alpha(|x'|^{\alpha-1}x' + v_\alpha |t'|^{\alpha-1}t'), \;\; |t|^{\alpha-1}t = \gamma_\alpha(|t'|^{\alpha-1}t' + \frac{v_\alpha}{c^{2\alpha}}|x'|^{\alpha-1}x'). \tag{42}$$

Expressing the spatial and temporal coordinates in terms of the α-deformed binary operations, we get

$$x = \Gamma_\alpha(x' \oplus v_\alpha^{1/\alpha} t'), \;\; t = \Gamma_\alpha(t' \oplus \frac{v_\alpha^{1/\alpha}}{c^2} x'), \quad \text{where} \;\; \Gamma_\alpha = \gamma_\alpha^{1/\alpha} = (1-\beta_\alpha)^{-\frac{1}{2\alpha}}. \tag{43}$$

If we set

$$u_\alpha = \left(\frac{|x|^{\alpha-1}}{|t|^{\alpha-1}}\right)\frac{dx}{dt}, \quad u'_\alpha = \left(\frac{|x'|^{\alpha-1}}{|t'|^{\alpha-1}}\right)\frac{dx'}{dt'}, \tag{44}$$

the addition of α-velocity becomes

$$u_\alpha = \frac{u'_\alpha + v_\alpha}{1 + \frac{v_\alpha u_\alpha}{c^{2\alpha}}}. \tag{45}$$

If we regard the α-speed of light as c^α, the eq.(44) shows that the α-speed of light remains invariant, and, hence, the speed of light also remains invariant under the α-deformed Lorentz transformation.

4.2 α-Lorentz group

Now, let us introduce the four α-velocity. For that, we change the notation as:

$$ct = x^0, \;\; x = x^1, \;\; y = x^2, \;\; z = x^3. \tag{46}$$

Then, the four α-velocity is given by

$$u_\alpha^a = \frac{|x^a|^{\alpha-1}dx^a}{(\tilde{d}\tau)_\alpha} \quad \text{or, explicitly,} \;\; u_\alpha^0 = c^\alpha \gamma_\alpha, \;\; u_\alpha^i = v_\alpha^i \gamma_\alpha, \;\; i = 1,2,3. \tag{47}$$

Therefore, we have

$$\eta_{ab} u_\alpha^a u_\alpha^b = c^{2\alpha}. \tag{48}$$

4.3 Energy and α-momentum

The four α-momentum is defined as

$$p_\alpha^a = m^\alpha u_\alpha^a \tag{49}$$

explicitly giving

$$p_\alpha^0 = m^\alpha c^\alpha \gamma_\alpha, \quad p_\alpha^i = m^\alpha v_\alpha^i \gamma_\alpha, \quad \text{and thus} \;\; \eta_{ab} p_\alpha^a p_\alpha^b = m^{2\alpha} c^{2\alpha}. \tag{50}$$

Here, we have $p_\alpha^a \neq (E/c, \vec{p}_\alpha)$ because the energy in α-deformed mechanics has the same unit as in the undeformed case. Therefore, we set

$$p_\alpha^a = \left((\frac{E}{c})^\alpha, \vec{p}_\alpha \right) \tag{51}$$

Thus, the eq.(50) gives

$$E^{2\alpha} = c^{2\alpha}|\vec{p}_\alpha|^2 + m^{2\alpha}c^{4\alpha}, \quad \text{and when} \;\; |\vec{v}_\alpha| \ll c^\alpha, \;\; E^\alpha \approx \frac{|\vec{p}_\alpha|^2}{2m^\alpha}, \tag{52}$$

which is the same as the non-relativistic case.

5 Two body decay

The simplest particle reaction is the two-body decay of unstable particles. A well known example from nuclear physics is the alpha decay of heavy nuclei. In particle physics, one observes, for instance, decays of charged pions or kaons into muons and neutrinos, or decays of neutral kaons into pairs of pions, etc. Consider the decay of a particle of mass M which is initially at rest. Its four α-momentum is $P = (M^\alpha, \vec{0})$, where we set $c = 1$.

This reference frame is called the centre-of mass frame (CMS). Denote the four α-momenta of the two daughter particles by $p_1 = (E_1^\alpha, \vec{p}_{\alpha,1}), p_2 = (E_2^\alpha, \vec{p}_{\alpha,2})$. From the momentum conservation, we get

$$\vec{p}_{\alpha,1} + \vec{p}_{\alpha,2} = 0. \tag{53}$$

The energy conservation is

$$M^\alpha = \sqrt{|\vec{p}_{\alpha,1}|^2 + m_1^{2\alpha}} + \sqrt{|\vec{p}_{\alpha,2}|^2 + m_2^{2\alpha}} \tag{54}$$

If we set

$$p = |\vec{p}_{\alpha,1}| = |\vec{p}_{\alpha,2}|, \tag{55}$$

we have

$$p = \frac{1}{2M^\alpha}\sqrt{(M^{2\alpha} - (m_1^\alpha - m_2^\alpha)^2)(M^{2\alpha} - (m_1^\alpha + m_2^\alpha)^2)} \tag{56}$$

Thus, we have

$$M \geq m_1 \oplus_\alpha m_2. \tag{57}$$

6 Conclusion

This work has addressed the formulation of Newton mechanics, Galilean relativity, and special relativity in α-deformed binary operation setting.

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