Spin degrees of freedom incorporated in conformal group: Introduction of an intrinsic momentum operator

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Abstract

Considering spin degrees of freedom incorporated in the conformal generators, we introduce an intrinsic momentum operator π_μ, which is feasible for the Bhabha wave equation. If a physical state $\psi_{\rm ph}$ for spin s is annihilated by the π_μ, the degree of $\psi_{\rm ph}$, $\deg\psi_{\rm ph}$, should equal twice the spin degrees of freedom, $2(2s+1)$ for a massive particle, where the muptiplicity 2 indicates the chirality. The relation $\deg\psi_{\rm ph} = 2(2s+1)$ holds in the representation $\mathrm{R}_5(s,s)$, irreducible representation of the Lorentz group in five dimensions.

1 Introduction

Conformal symmetry [1] has many applications in string theory and critical phenomena in condensed matter and statistical physics. For a scalar field, the conformal generators are composed of dilatation D, momentum P_μ, special conformal K_μ, and angular momentum $L_{\mu\nu}$. For a multicomponent field Φ, where spin degrees of freedom is incorporated as $L_{\mu\nu} \to L_{\mu\nu} + s_{\mu\nu}$, the D and K_μ are generalized as $D \to D + \Delta$ and $K_\mu \to K_\mu + \kappa_\mu$, while the P_μ, in an ordinary context [1], remains unchanged as $P_\mu \to P_\mu$. The unchangeability of P_μ may be because Φ transforms as a scalar under spacetime translation. If we assume that $\Phi(x) \to \Phi'(x') = \Phi(x)$ under $x \to x' = x + a$, that is, $\Phi'(x) = \Phi(x - a) = \mathrm{e}^{-a\cdot P}\Phi(x)$, we find it unnecessary to introduce an intrinsic momentum operator π_μ as $P_\mu \to P_\mu + \pi_\mu$. Even if we admit the scalar property of $\Phi(x)$ under $x \to x + a$, we can introduce π_μ in such a way that the π_μ may annihilate physical states.

This paper aims to introduce such an intrinsic momentum operator π_μ, to find that π_μ can realize for a matrix structure in parafermion-based Dirac-like equations, such as spin-1 Kemmer equation [2], and more generally, Bhabha equation [3]. In Sec. 2, we give some preliminaries concerning the conformal algebra, together with its Casimir operator. In Secs. 3-5, we deal with the π_μ in the case of spin $\frac{1}{2}, 1, \frac{3}{2}$, respectively. We devote Sec. 6 to the summary.

2 Preliminaries

We begin with the commutation relations between the intrinsic conformal generators Δ, π_μ, κ_μ, and $s_{\mu\nu}$, corresponding to D, P_μ, K_μ, and $L_{\mu\nu}$, respectively. If the intrinsic conformal generators satisfy the same commutation relations as ordinary conformal generators, we can write the non-vanishing commutation relations as

$$[\Delta,\,\pi_\mu]=\mathrm{i}\pi_\mu,\quad [\Delta,\,\kappa_\mu]=-\mathrm{i}\kappa_\mu,\quad [\kappa_\mu,\,\pi_\nu]=2\mathrm{i}(g_{\mu\nu}\Delta-s_{\mu\nu}), \tag{1}$$
$$[\pi_\rho,\,s_{\mu\nu}]=\mathrm{i}(g_{\rho\mu}\pi_\nu-g_{\rho\nu}\pi_\mu),\quad [\kappa_\rho,\,s_{\mu\nu}]=\mathrm{i}(g_{\rho\mu}\kappa_\nu-g_{\rho\nu}\kappa_\mu), \tag{2}$$
$$[s_{\mu\nu},\,s_{\rho\sigma}]=\mathrm{i}(g_{\nu\rho}s_{\mu\sigma}+g_{\mu\sigma}s_{\nu\rho}-g_{\mu\rho}s_{\nu\sigma}-g_{\nu\sigma}s_{\mu\rho}), \tag{3}$$

while the vanishing commutation relations are given by

$$[\Delta,\,s_{\mu\nu}]=[\pi_\mu,\,\pi_\nu]=[\kappa_\mu,\,\kappa_\nu]=0. \tag{4}$$

It should be remarked that (1)-(4) are invariant under the scaling of π_μ and κ_μ, and also under the substitution between π_μ and κ_μ as

$$(\Delta,\,\pi_\mu,\,\kappa_\mu,\,s_{\mu\nu})\to(\Delta,\,\lambda\pi_\mu,\,\lambda^{-1}\kappa_\mu,\,s_{\mu\nu}), \tag{5}$$
$$(\Delta,\,\pi_\mu,\,\kappa_\mu,\,s_{\mu\nu})\to(-\Delta,\,\kappa_\mu,\,\pi_\mu,\,s_{\mu\nu}), \tag{6}$$

where $\lambda\in\mathbb{C}\setminus\{0\}$, and use has been made of $s_{\nu\mu}=-s_{\mu\nu}$ in (6). Note that (5) represents the "chiral" transformation $g\to g'=\mathrm{e}^{\theta\Delta}g\mathrm{e}^{-\theta\Delta}$ ($g\in\{\Delta,\pi_\mu,\kappa_\mu,s_{\mu\nu}\}$), where $\lambda=\mathrm{e}^{\mathrm{i}\theta}$.

To check the irreducibility of the representation for the conformal group, it may be available to obtain the Casimir operator C. Note that although the C is invariant under (5) due to the chiral transformation, the invariance of C under (6) is somewhat naive. For simplicity, we consider $(3+1)$ spacetime dimensions, where the conformal algebra is isomorphic to $\mathfrak{so}(4,2)$ [1]. In this case, the order of C is given by $2,3,4$, as in the case of $\mathfrak{so}(6)$ [4]. Explicitly, we have $C=C_2,C_3,C_4$ (the index i in C_i represents the order) as [5]

$$C_2=\frac{1}{2}s_{\mu\nu}s^{\mu\nu}+\frac{1}{2}\{\kappa_\mu,\,\pi^\mu\}-\Delta^2,\qquad C_3=\epsilon^{\mu\nu\rho\sigma}\big(\Delta s_{\mu\nu}+\{\kappa_\mu,\,\pi_\nu\}\big)s_{\rho\sigma},$$
$$C_4=\frac{1}{2}\mathcal{J}_{\mu\nu}\mathcal{J}^{\mu\nu}-\frac{1}{2}\{\mathcal{J}_{K,\mu},\,\mathcal{J}_P^\mu\}-\frac{1}{16}\mathcal{J}^2, \tag{7}$$

where $\mathcal{J}^{\mu\nu}$, $\mathcal{J}_K^\mu$, $\mathcal{J}_P^\mu$, and $\mathcal{J}$ are given by $\mathcal{J}^{\mu\nu}=\epsilon^{\mu\nu\rho\sigma}\big(\Delta s_{\rho\sigma}+\frac{1}{2}\{\kappa_\rho,\,\pi_\sigma\}\big)$, $\mathcal{J}_K^\mu=\epsilon^{\mu\nu\rho\sigma}\kappa_\nu s_{\rho\sigma}$, $\mathcal{J}_P^\mu=\epsilon^{\mu\nu\rho\sigma}\pi_\nu s_{\rho\sigma}$, and $\mathcal{J}=\epsilon^{\mu\nu\rho\sigma}s_{\mu\nu}s_{\rho\sigma}$, with $\epsilon^{\mu\nu\rho\sigma}$ the totally anti-symmetric Levi-Civita tensor ($\epsilon^{0123}=1$), and $\{A,\,B\}=AB+BA$. It confirms that all the C's are invariant under (5). If the $\epsilon^{\mu\nu\rho\sigma}$ remains invariant under (6), the C_i's transform as $(C_2,\,C_3\,C_4)\to(C_2,\,-C_3,\,C_4)$. However, the invariance of $\epsilon^{\mu\nu\rho\sigma}$ under (6) is not so trivial, which will be discussed at the end of the next section and afterward.

3 Spin $\frac{1}{2}$

This section deals with the Dirac equation, which describes a spin-$\frac{1}{2}$ particle. In this case, the spin operator $s_{\mu\nu}$, which satisfies (3), can be written using the gamma matrix γ_μ as $s_{\mu\nu}=\mathrm{i}\frac{1}{4}[\gamma_\mu,\gamma_\nu]$, where $\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}\mathbb{1}$. The next thing is to obtain π_μ from the first equality in (2) and $[\pi_\mu,\,\pi_\nu]=0$. Considering that $[\gamma_\rho,\,s_{\mu\nu}]=\mathrm{i}(g_{\rho\mu}\gamma_\nu-g_{\rho\nu}\gamma_\mu)$, one may suspect that π_μ may be given by $\pi_\mu=\lambda\gamma_\mu$ ($\lambda\in\mathbb{C}$), which, however, would not be appropriate due to $[\pi_\mu,\,\pi_\nu]\neq0$. This conclusion is not the end of the story. For an even spacetime dimension, there is a matrix γ_5 such that $\gamma_5^2=\mathbb{1}$ and $\{\gamma_5,\,\gamma_\mu\}=0$. Under the existence of γ_5, the choice of $\pi_\mu=\lambda(\gamma_\mu\pm\gamma_5\gamma_\mu)$ satisfies the first equality in (2) and $[\pi_\mu,\,\pi_\nu]=0$. In a similar way, we obtain $\kappa_\mu=\lambda'(\gamma_\mu\pm\gamma_5\gamma_\mu)$ from the second equality in (2) and $[\kappa_\mu,\,\kappa_\nu]=0$.

The relation between λ and λ', along with the remaining generator Δ, can be derived from (1). To summarize, we have

$$\Delta = \pm\frac{1}{2}\mathrm{i}\gamma_5, \quad \pi_\mu = M\left(\frac{\mathbb{1}\pm\gamma_5}{2}\right)\gamma_\mu, \quad \kappa_\mu = \frac{1}{M}\left(\frac{\mathbb{1}\mp\gamma_5}{2}\right)\gamma_\mu, \quad s_{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\nu], \tag{8}$$

where the multiplier $M \in \mathbb{C}\setminus\{0\}$ corresponds to λ in (5). Note that the substitution (6) can be interpreted as $\gamma_5 \to -\gamma_5$. Note also that $[\Delta, s_{\mu\nu}] = 0$.

The fundamental property of π_μ (or κ_μ) is the nilpotence of order two. Let $a_\mu^\pm := (\mathbb{1}\pm\gamma_5)\gamma_\mu$. Then it follows that

$$a_\nu^+ a_\mu^+ = 0 = a_\nu^- a_\mu^-. \tag{9}$$

To be more exact, we can show that

$$\begin{cases} a_\mu^+ \mathrm{P}_1 = 0, \\ a_\mu^- \mathrm{P}_2 = 0, \end{cases} \qquad \begin{cases} a_\mu^+ \mathrm{P}_2 = 2\mathrm{P}_1\gamma_\mu, \\ a_\mu^- \mathrm{P}_1 = 2\mathrm{P}_2\gamma_\mu, \end{cases} \tag{10}$$

where $\mathrm{P}_1 = \frac{1}{2}(\mathbb{1}+\gamma_5)$ and $\mathrm{P}_2 = \frac{1}{2}(\mathbb{1}-\gamma_5)$ represent the projection operators such that $\mathrm{P}_1 + \mathrm{P}_2 = \mathbb{1}$ and $\mathrm{P}_i\mathrm{P}_j = \delta_{ij}\mathrm{P}_i$. In the Dirac theory, it is well known that P_1 and P_2 are employed in the chiral decomposition. In this sense, (10) can be derived without recognizing the concept of the intrinsic momentum operator π_μ; the existence of π_μ will play a substantial role in higher spin states.

Now we give some properties concerning the Casimir operators C_i's in (7). First, we discuss the invariance of C_3 under (6). Recalling that the substitution (6) corresponds to $\gamma_5 \to -\gamma_5$, and that $\gamma_5 = -\frac{1}{4!}\mathrm{i}\,\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$, we find that $\gamma_5 \to -\gamma_5$ implies that $\epsilon^{\mu\nu\rho\sigma} \to -\epsilon^{\mu\nu\rho\sigma}$. In this sense, C_3 remains invariant under (6). Next, we obtain the relation between C_2 and C_4. Note that $\mathcal{J}^{\mu\nu}$ can be rewritten as $3\Delta\epsilon^{\mu\nu\rho\sigma}s_{\rho\sigma}$, which leads to $\mathcal{J}_{\mu\nu}\mathcal{J}^{\mu\nu} = 9s_{\mu\nu}s^{\mu\nu}$. In a similar way, we have $\{\mathcal{J}_{K,\mu}, \mathcal{J}_P^\mu\} = -9\{\kappa_\mu, \pi^\mu\}$ and $\frac{1}{16}\mathcal{J}^2 = 9\Delta^2$. Thus we obtain $C_4 = 9C_2$. Anyway, there is no such operator (except a scalar multiple of identity $\mathbb{1}$) that is commutative with all the γ_μ's, so that the C_i's are given by a multiple of identity $\mathbb{1}$ as $(C_2, C_3, C_4) = \frac{15}{4}(1, 2^2, 3^2)\mathbb{1}$.

4 Spin 1

This section deals with relativistically invariant wave equations for spin $s = 1$. For the sake of simplicity, spacetime dimension d is restricted to $(3+1)$. We summarize the wave functions for a free massive particle in Table 1, to find that the π_μ is allowed for the KDP equation but not for the Proca and the WSG equations. This is because the $n\times n$ matrix π_μ such that $[\pi_\rho, s_{\mu\nu}] = \mathrm{i}(g_{\rho\mu}\pi_\nu - g_{\rho\nu}\pi_\mu)$ is allowed for $n = 10$, but not for $n = 4, 6$. In what follows, we concentrate on the KDP equation, where the β_μ's satisfy the trilinear relations

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu = g_{\mu\nu}\beta_\rho + g_{\rho\nu}\beta_\mu \qquad (\mu, \nu, \rho \in \{0,1,2,3\}). \tag{11}$$

Note that β_i $(i = 1,2,3)$ can be identified with the non-relativistic spin-1 operator s_i in the sense that the s_i's satisfy $s_is_js_k + s_ks_js_i = \delta_{ij}s_k + \delta_{kj}s_i$.

For $n = 10$, it is known that [2] there is a matrix ω $(= \beta_5)$ which is given by extending (11) to those for $\mu, \nu, \rho \in \{0,1,2,3,5\}$ with $g_{5\mu} = g_{\mu 5} = \delta_{5\mu}$. Explicitly, we have

$$\omega^3 = \omega, \qquad \begin{cases} \{\omega^2, \beta_\mu\} = \beta_\mu, \\ \omega\beta_\mu\omega = 0, \end{cases} \qquad \begin{cases} \beta_\mu\omega\beta_\nu + \beta_\nu\omega\beta_\mu = 0, \\ \omega\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\omega = g_{\mu\nu}\omega. \end{cases} \tag{12}$$

Table 1: Lorentz invariant wave equations for $s=1$ and $d=3+1$. For the Proca equation, the upperscript in $\psi=(A^0,A^1,A^2,A^3)$ represents the Lorentz vector component, and $\Lambda_{\mu\nu}$ represents the generator of the Lorentz transformation. For the WSG equation, s_i $(i=1,2,3)$ is given by the (3×3) representation matrix for the non-relativistic spin-1 operator.

Name	Equation	Degree of ψ	$s_{\mu\nu}$	π_μ
Proca	$(\Box+m^2)A^\mu=\partial^\mu(\partial\cdot A)$	4	$\Lambda_{\mu\nu}$	NA
WSG [6,7]	$(\Box+\gamma_{\mu\nu}\partial^\mu\partial^\nu)\psi=2m_0^2\psi$	6	$\begin{cases} s_{0i}=\frac{1}{\mathrm{i}}\sigma_3\otimes s_i \\ s_{ij}=\mathbb{1}\otimes\epsilon_{ijk}s_k\end{cases}$	NA
KDP [2,8,9]	$(\mathrm{i}\beta_\mu\partial^\mu+m)\psi=0$	10	$\mathrm{i}[\beta_\mu,\beta_\nu]$	✓

Then the intrinsic conformal generators are given by

$$\Delta=\pm\mathrm{i}\omega,\quad \pi_\mu=M\left(\beta_\mu\pm[\omega,\beta_\mu]\right),\quad \kappa_\mu=\frac{1}{M}\left(\beta_\mu\mp[\omega,\beta_\mu]\right),\quad s_{\mu\nu}=\mathrm{i}[\beta_\mu,\beta_\nu]. \tag{13}$$

Note that (13) reduces to (8) under $(\beta_\mu,\omega)\to\frac{1}{2}(\gamma_\mu,\gamma_5)$. It is not so difficult to obtain from (11) and (12) the nilpotence of π_μ as

$$\alpha^+_\mu\alpha^+_\nu\alpha^+_\rho=0=\alpha^-_\mu\alpha^-_\nu\alpha^-_\rho, \tag{14}$$

where $\alpha^\pm_\mu:=\beta_\mu\pm[\omega,\beta_\mu]$. To be more exact, we have the following relations:

$$\begin{cases}\alpha^+_\mu\mathrm{P}_1=0,\\ \alpha^-_\mu\mathrm{P}_3=0,\end{cases}\quad \begin{cases}\alpha^+_\mu\mathrm{P}_2=2\mathrm{P}_1\beta_\mu,\\ \alpha^-_\mu\mathrm{P}_2=2\mathrm{P}_3\beta_\mu,\end{cases}\quad \begin{cases}\alpha^+_\nu\alpha^+_\mu\mathrm{P}_3=2\mathrm{P}_1A_{\mu\nu},\\ \alpha^-_\nu\alpha^-_\mu\mathrm{P}_1=2\mathrm{P}_3A_{\mu\nu},\end{cases} \tag{15}$$

where $A_{\mu\nu}=\{\beta_\mu,\beta_\nu\}-g_{\mu\nu}\mathbb{1}$, and P_i represents a projection operators as $\mathrm{P}_1=\frac{1}{2}\omega(\omega+\mathbb{1})$, $\mathrm{P}_2=\mathbb{1}-\omega^2$, and $\mathrm{P}_3=\frac{1}{2}\omega(\omega-\mathbb{1})$, so that $\sum_{i=1}^{3}\mathrm{P}_i=\mathbb{1}$ and $\mathrm{P}_i\mathrm{P}_j=\delta_{ij}\mathrm{P}_i$. Notice that in (15), the lower relations can derive from the corresponding upper ones through the substitution $\omega\to-\omega$. Notice further that $A_{\mu\nu}$ anticommutes with ω, that is

$$\{A_{\mu\nu},\,\omega\}=0. \tag{16}$$

The relation (16) leads to $[A^\mu_{\ \mu},\,\omega^2]=0$. Note that $A^\mu_{\ \mu}$ and ω are Lorentz invariant in the sense that $[s_{\alpha\beta},A^\mu_{\ \mu}]=0=[s_{\alpha\beta},\,\omega]$. This relation implies that $A^\mu_{\ \mu}$ can be written as $A^\mu_{\ \mu}=\sum_{i=0}^{2}c_i\omega^i$ $(c_i\in\mathbb{C})$, where c_i $(i\geq 3)$ is not necessary due to $\omega^3=\omega$. Here we have assumed that there is no Lorentz invariant other than $\mathbb{1}$, ω, and ω^2. In this case, we find that $c_0+c_2=0=c_1$ from $\{A^\mu_{\ \mu},\,\omega\}=0$ by (16), and that $c_0=2$ from $\{\beta_\nu,\,\beta_\mu\beta^\mu\}=5\beta_\nu$ by (11) and $\{\beta_\nu,\,\omega^2\}=\beta_\nu$ by (12). Eventually, we have

$$\beta_\mu\beta^\mu=\mathrm{P}_2+2\mathbb{1}. \tag{17}$$

Actually, the relation (17) holds in the ten-dimensional representation [2] for (11) and (12), which corresponds to the adjoint representation of the Lorentz group in five dimensions (for the adjoint representation, we have $\binom{5}{2}=10$ Lorentz group generators). For later convenience, we rewrite $\frac{1}{2}s_{\mu\nu}s^{\mu\nu}$ using P_2 as

$$\frac{1}{2}s_{\mu\nu}s^{\mu\nu}=4\mathbb{1}-\mathrm{P}_2, \tag{18}$$

where we have used (17), together with $\mathsf{P}_2^2 = \mathsf{P}_2$.

As was mentioned in Sec. 1, the π_μ should annihilate the physical state. To check the validity, we show that the rank of P_k (or equivalently, the trace of P_k) for $k = 1, 3$ equals the spin degrees of freedom. In the ten-dimensional representation, the eigenvalues of ω are given by $1, 0, -1$ appearing $3, 4, 3$ times, respectively. Thus, we obtain

$$\mathrm{Rank}(\mathsf{P}_1) = \mathrm{Rank}(\mathsf{P}_3) = 3, \qquad \mathrm{Rank}(\mathsf{P}_2) = 4.$$

This result is quite reasonable because the number "3" equals the spin degree of freedom for a massive particle for $s = 1$. To confirm the validity, we calculate the 3-dimensional spin magnitude $\langle s\rangle^2 := s_{12}{}^2 + s_{23}{}^2 + s_{31}{}^2$. Let $|\psi^+_{\rm ph}\rangle = \mathsf{P}_1|\psi\rangle$, $|\psi^-_{\rm ph}\rangle = \mathsf{P}_3|\psi\rangle$, and $|\psi_{\rm un}\rangle = \mathsf{P}_2|\psi\rangle$, in which we have $\alpha^\pm_\mu|\psi^\pm_{\rm ph}\rangle = 0$. Recalling that $\langle s\rangle^2$ $(= \frac{1}{4}s_{\mu\nu}s^{\mu\nu}) = 2\mathbb{1} - \frac{1}{2}\mathsf{P}_2$ by (18), and that $\mathsf{P}_i\mathsf{P}_j = \delta_{ij}\mathsf{P}_i$, we obtain $\langle s\rangle^2|\psi^\pm_{\rm ph}\rangle = s(s+1)|\psi^\pm_{\rm ph}\rangle$ $(s = 1)$ and $\langle s\rangle^2|\psi_{\rm un}\rangle = \frac{3}{2}|\psi_{\rm un}\rangle$. These relations indicate that $|\psi^\pm_{\rm ph}\rangle$ represents the spin-1 state, while $|\psi_{\rm un}\rangle$ does not. Bearing these findings in mind, we can regard $|\psi^\pm_{\rm ph}\rangle$ and $|\psi_{\rm un}\rangle$ as physical and unphysical states, respectively.

Finally, we give some properties of the Casimir operator C. As in the case of $s = \frac{1}{2}$, the invariance of C_3 under (6) is guaranteed by the statement that $(\omega \to -\omega) \Longrightarrow (\epsilon^{\mu\nu\rho\sigma} \to -\epsilon^{\mu\nu\rho\sigma})$ by $\omega = -\frac{\mathrm{i}}{4}\epsilon^{\mu\nu\rho\sigma}\beta_\mu\beta_\nu\beta_\rho\beta_\sigma$ [10, 11]. After a somewhat tedious calculation, we can write the C_i's in (7) as $(C_2, C_3, C_4) = (9, 48, 144)\mathbb{1}$, which confirms the irreducibility of the ten-dimensional representation.

5 Spin $\frac{3}{2}$

In this section, we consider the $(3+1)$-dimensional Minkowski space, as in the case of $s = 1$. Although the Rarita-Schwinger equation is well known as a relativistic invariant wave equation for $s = \frac{3}{2}$, the intrinsic momentum operator is not allowed, as in the case of the Proca equation. Instead, we adopt a Dirac-like wave equation for parafermion of order 3, namely (massive) Bhabha wave equation [3] (see Table 2).

Table 2: Lorentz invariant wave equations for $s = \frac{3}{2}$. For the Rarita equation, ψ is composed of four Dirac spinors as $\psi := (\psi_0, \psi_1, \psi_2, \psi_3)$, where the subscript represents the Lorentz vector component, so that Λ $(= \{\Lambda_{\mu\nu}\}) : \psi \mapsto \psi'$ acts as $(\psi')_\mu = \Lambda_\mu^{\ \nu}\psi_\nu$.

Name	Equation	Degree of ψ	$s_{\mu\nu}$	π_μ
Rarita-Schwinger	$(\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\partial_\rho + mg^{\mu\sigma})\psi_\sigma = 0$	4×4	$\Lambda_{\mu\nu} + \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\nu]$	NA
Bhabha	$(\mathrm{i}s_\mu\partial^\mu + m)\psi = 0$	20	$\mathrm{i}[s_\mu, s_\nu]$	✓

Extending the polynomial relations among the non-relativistic spin operators s_i's $(i = 1, 2, 3)$ to those among s_μ's $(\mu = 0, 1, 2, 3)$ in a relativistically covariant way, we obtain

$$\begin{cases} s_\mu s_\nu s_\alpha + s_\alpha s_\nu s_\mu + g_{\mu\alpha}s_\nu = s_\mu s_\alpha s_\nu + s_\nu s_\alpha s_\mu + g_{\mu\nu}s_\alpha, \\ 0 = \left(s_\mu s_\nu s_\alpha s_\beta - \frac{5}{4}\{s_\mu, s_\nu\}g_{\alpha\beta} + \frac{9}{16}g_{\mu\nu}g_{\alpha\beta}\right) + (\text{perm. of } \mu, \nu, \alpha, \beta). \end{cases} \tag{19}$$

It may be convenient to rewrite the first relation of (19) as $[s_\mu, [s_\nu, s_\alpha]] = g_{\mu\nu}s_\alpha - g_{\mu\alpha}s_\nu$. Note that $\frac{1}{2}\gamma_\mu$ satisfies both relations in (19). This implies that there should exist a polynomial relation such that $p(s_0, s_1, s_2, s_3) = 0$ with $p(s_0, s_1, s_2, s_3)|_{s_\mu \to \frac{1}{2}\gamma_\mu} \neq 0$. However, we neglect, for the time being, such a polynomial relation because it is not irrelevant to the following discussion.

Suppose that there exists an operator s_5 which satisfies (19) for $\mu,\nu,\alpha,\beta\in\{0,1,2,3,5\}$, with $g_{5\mu}=g_{\mu 5}=\delta_{5\mu}$. Then the intrinsic conformal generators are given, as is analogous to the case of $s=\frac{1}{2},1$, by

$$\Delta=\pm\mathrm{i}s_5,\quad \pi_\mu=M\left(s_\mu\pm[s_5,s_\mu]\right),\quad \kappa_\mu=\frac{1}{M}\left(s_\mu\mp[s_5,s_\mu]\right),\quad s_{\mu\nu}=\mathrm{i}[s_\mu,s_\nu]. \tag{20}$$

Note that the first equality in (19), together with the existence of s_5, is sufficient for (20); the second equality in (19) is not necessary for (20). Recalling that the first relation in (19) is satisfied for $s_\mu\to\frac{1}{2}\gamma_\mu$ ($s=\frac{1}{2}$) and for $s_\mu\to\beta_\mu$ ($s=1$), we find it natural that the relation (20) is the same form as (8) and (13). For later convenience, we obtain some operators which anti-commute with s_5. Such operators are exemplified as

$$\{s_5,A_\mu\}=0=\{s_5,A_{\rho\nu\mu}+(\text{perm. of }\rho,\nu,\mu)\}, \tag{21}$$

where $A_\mu=s_5s_\mu s_5-\frac{3}{4}s_\mu$, and $A_{\rho\nu\mu}=s_\rho s_\nu s_\mu-\frac{7}{4}g_{\rho\nu}s_\mu$.

The projection operators P_i's ($i=1,2,3,4$) can be written using the minimum polynomial $f(x)$ with respect to s_5 as $\mathrm{P}_i=\frac{1}{f'(\lambda_i)}\frac{f(s_5)\mathbb{1}}{s_5-\lambda_i\mathbb{1}}$, where $f(x)=\prod_{i=1}^{4}(x-\lambda_i)$, with $\lambda_1=\frac{3}{2}$, $\lambda_2=\frac{1}{2}$, $\lambda_3=-\frac{1}{2}$, $\lambda_4=-\frac{3}{2}$. Let $s_\mu^\pm:=s_\mu\pm[s_5,s_\mu]$. Then it follows that (see Appendix A)

$$\begin{cases}s_\mu^+\mathrm{P}_1=0,\\ s_\mu^-\mathrm{P}_4=0,\end{cases}\quad \begin{cases}s_\mu^+\mathrm{P}_2=2\mathrm{P}_1X_\mu,\\ s_\mu^-\mathrm{P}_3=2\mathrm{P}_4X_\mu,\end{cases}\quad \begin{cases}s_\nu^+s_\mu^+\mathrm{P}_3=2\mathrm{P}_1X_{\nu\mu},\\ s_\nu^-s_\mu^-\mathrm{P}_2=2\mathrm{P}_4X_{\nu\mu},\end{cases}\quad \begin{cases}s_\rho^+s_\nu^+s_\mu^+\mathrm{P}_4=\frac{4}{3}\mathrm{P}_1X_{\rho\nu\mu},\\ s_\rho^-s_\nu^-s_\mu^-\mathrm{P}_1=\frac{4}{3}\mathrm{P}_4X_{\rho\nu\mu},\end{cases} \tag{22}$$

where X_μ,$X_{\nu\mu}$ and $X_{\rho\nu\mu}$ are given by

$$X_\mu=s_\mu,\quad X_{\nu\mu}=\{s_\nu,s_\mu\}-sg_{\nu\mu}\mathbb{1},\quad X_{\rho\nu\mu}=\left[Y_{\rho\nu\mu}+(\text{perm. of }\rho,\nu,\mu)\right],$$

with $s=\frac{3}{2}$ and $Y_{\rho\nu\mu}:=s_\rho s_\nu s_\mu-g_{\rho\nu}(ss_\mu+\frac{1}{2s}s_5s_\mu s_5)\ \to A_{\rho\nu\mu}-\frac{1}{3}g_{\rho\nu}A_\mu\quad(s=\frac{3}{2})$. The relations (22) lead to $s_\mu^+s_\nu^+s_\rho^+s_\sigma^+\mathrm{P}_i=0=s_\mu^-s_\nu^-s_\rho^-s_\sigma^-\mathrm{P}_i$ ($i=1,2,3,4$), from which, together wirh $\sum_{i=1}^4\mathrm{P}_i=\mathbb{1}$, we obtain the nilpotence of $s_\mu^\pm$ (of order 4) as

$$s_\mu^+s_\nu^+s_\rho^+s_\sigma^+=0=s_\mu^-s_\nu^-s_\rho^-s_\sigma^-. \tag{23}$$

Note that by (21), not only have we the anti-commutativity

$$\{X_{\rho\nu\mu},s_5\}=0,$$

but also the anti-commutativities $\{\gamma_\mu,\gamma_5\}=0$ and (16) can be rewritten using X_μ and $X_{\nu\mu}$ as

$$\{X_\mu^{(\frac{1}{2})},\gamma_5\}=0=\{X_{\nu\mu}^{(1)},\omega\}, \tag{24}$$

where $X_\mu^{(\frac{1}{2})}$ and $X_{\nu\mu}^{(1)}$, more generally, $X_{\nu\mu\ldots}^{(s)}$ represents the corresponding $X_{\nu\mu\ldots}$ for a given spin s. For example, we have $Y_{\rho\nu\mu}^{(\frac{1}{2})}=\frac{1}{8}\gamma_\rho\gamma_\nu\gamma_\mu-\frac{1}{8}g_{\rho\nu}\gamma_\mu$, and $Y_{\rho\nu\mu}^{(1)}=\beta_\rho\beta_\nu\beta_\mu-g_{\rho\nu}\beta_\mu$ by replacing $(s_\rho,s_\nu,s_\mu;s)$ in $Y_{\rho\nu\mu}$ with $\frac{1}{2}(\gamma_\rho,\gamma_\nu,\gamma_\mu;1)$ and $(\beta_\rho,\beta_\nu,\beta_\mu;1)$, respectively. Note further that we have the following vanishing relations:

$$X_{\nu\mu}^{(\frac{1}{2})}=X_{\rho\nu\mu}^{(\frac{1}{2})}=0,\qquad X_{\rho\nu\mu}^{(1)}=0,$$

which, in vew of (22), are due to the relations (9) and (14), respectively.

Now we discuss whether or not physical states can be given by $\mathrm{P}_k|\psi\rangle$ ($k=1,4$) by calculating the rank of P_k. In the Bhabha theory [3] for $s=\frac{3}{2}$, we have two irreducible representations $\mathrm{R}_5(\frac{3}{2},\frac{3}{2})$ and $\mathrm{R}_5(\frac{3}{2},\frac{1}{2})$, where $\mathrm{R}_5(s,\tilde{s})$ represents the spin-s Lorentz group representation in five dimensions. Let $S:=\{s_1,s_2,s_3,\mathrm{i}s_0\}$. For $\mathrm{R}_5(\frac{3}{2},\frac{3}{2})$, the eigenvalues of $x\in S$

are $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ appearing $4, 6, 6, 4$ times, respectively; while for $\mathrm{R}_5(\frac{3}{2}, \frac{1}{2})$, the eigenvalues of $x \in S$ are $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ appearing $2, 6, 6, 2$ times, respectively. If s_5 realizes, the eigenvalues of s_5 are identical with those of $x \in S$, so that

$$\mathrm{Rank}(\mathsf{P}_1) = \mathrm{Rank}(\mathsf{P}_4) = \begin{cases} 4 & \left(\mathrm{R}_5(\frac{3}{2}, \frac{3}{2})\right), \\ 2 & \left(\mathrm{R}_5(\frac{3}{2}, \frac{1}{2})\right), \end{cases} \qquad \mathrm{Rank}(\mathsf{P}_2) = \mathrm{Rank}(\mathsf{P}_3) = \begin{cases} 6 & \left(\mathrm{R}_5(\frac{3}{2}, \frac{3}{2})\right), \\ 6 & \left(\mathrm{R}_5(\frac{3}{2}, \frac{1}{2})\right). \end{cases}$$

Thus we obtain in the representation $\mathrm{R}_5(\frac{3}{2}, \frac{3}{2})$, the relation $\mathrm{Rank}(\mathsf{P}_1) = \mathrm{Rank}(\mathsf{P}_4) = 4$, the spin degrees of freedom for a spin-$\frac{3}{2}$ massive particle.

The analogous relation holds for a general spin s. Note that by a fundamental property of the projector, we have $\mathrm{Rank}(\mathsf{P}_i) = N_i$, where N_i represents the number of the eigenvalue $(s+1-i)$ of s_5. Note also that in the representaion $\mathrm{R}_5(s, \tilde{s})$ $(\tilde{s} = s, s-1, \ldots)$, the maximum and minimum eigenvalues of s_5 [that is, s and $(-s)$, respectively] occur $(2\tilde{s}+1)$ times [3]. Considering these two remarks, we obtain in the representation $\mathrm{R}_5(s, s)$, the relation $\mathrm{Rank}(\mathsf{P}_1) = \mathrm{Rank}(\mathsf{P}_{2s+1}) = 2s+1$, the spin degrees of freedom. To confirm that $|\psi^+_{\mathrm{ph}}\rangle = \mathsf{P}_1|\psi\rangle$ and $|\psi^-_{\mathrm{ph}}\rangle = \mathsf{P}_{2s+1}|\psi\rangle$, in which we have $s^\pm_\mu|\psi^\pm_{\mathrm{ph}}\rangle = 0$, can be regarded as physical states, we should further show $\langle s\rangle^2|\psi^\pm_{\mathrm{ph}}\rangle = s(s+1)|\psi^\pm_{\mathrm{ph}}\rangle$, which, however, will be discussed elsewhere.

6 Conclusion

We have found that the intrinsic momentum operator $\pi_\mu = s^+_\mu, s^-_\mu$, which we do not introduce in the ordinary conformal group, is feasible for the Bhabha wave equation, provided that s_5, corresponding to $\frac{1}{2}\gamma_5$ $(s = \frac{1}{2})$ and ω $(s = 1)$, exists. For a general spin s, we can write the intrinsic conformal generators as the same relations as (20) and those where $s_5 \to (-s_5)$, satisfying the invariance under (5) and (6). The fundamental property of π_μ is the nilpotence of order $(2s+1)$. To be more exact, let P_i's $(i = 1, 2, \ldots, 2s+1)$ be the projection operators concerning the s_5 as $\mathsf{P}_i = \frac{1}{f'(\lambda_i)} \frac{f(s_5)\mathbb{1}}{s_5 - \lambda_i\mathbb{1}}$, where $f(x) = \prod_{i=1}^{2s+1}(x - \lambda_i)$, $\lambda_i = s+1-i$. Then we have the same hierarchical relation as (22), where $X^{(\frac{1}{2})}_\mu$, $X^{(1)}_{\mu\nu}, \ldots$ anti-commute with γ_5, $\omega, \ldots$, respectively. As long as the wave function transforms as a scalar under the spacetime translation, either s^+_μ or s^-_μ should annihilate a physical state, so that the relation $\mathrm{Rank}(\mathsf{P}_k) = 2s+1$ $(k = 1, 2s+1)$ is required for a massive particle. Fortunately, this relation holds in the representation $\mathrm{R}_5(s, s)$, irreducible representation of the Lorentz group in five dimensions.

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A Derivation of (22)

It is not so difficult to obtain X_μ and $X_{\nu\mu}$ by rewriting $s^+_\mu\mathsf{P}_2$ and $s^+_\nu s^+_\mu\mathsf{P}_3$ in such a way that s_5 is located as leftward as possible. However, this procedure is not practical for the calculation of $X_{\rho\nu\mu}$ because $X_{\rho\nu\mu}$ hinges on s_5 so that we may not represent $X_{\rho\nu\mu}$ uniquely due to some relations between s_5 and s_μ's. In this sense, it would be better to adopt another approach. We start with the following relation:

$$s_\mu^+ \mathsf{P}_4 = 2X_\mu \mathsf{P}_4 \qquad (X_\mu = s_\mu). \tag{25}$$

Keeping the form of (25) without rearranging s_5 leftward, and applying s_ν^+ to both sides of (25) from the left, then we find it rather simple to obtain

$$s_\nu^+ s_\mu^+ \mathsf{P}_4 = 2X_{\nu\mu} \mathsf{P}_4 \quad \left(X_{\nu\mu} = \{s_\nu, s_\mu\} - s\mathbb{1}, \quad s = \frac{3}{2} \right),$$

where we have used $[s_\nu^+, s_\mu] = [s_\nu, s_\mu] + g_{\nu\mu} s_5$, together with the relation $s_5 \mathsf{P}_4 = -s\mathsf{P}_4$. Further application of s_ρ^+ leads to the relation

$$s_\rho^+ s_\nu^+ s_\mu^+ \mathsf{P}_4 = \tfrac{4}{3} X_{\rho\nu\mu} \mathsf{P}_4 \quad \left(X_{\rho\nu\mu} = Y_{\rho\nu\mu} + (\text{perm. of } \rho, \nu, \mu) \right),$$

where $Y_{\rho\nu\mu} = s_\rho s_\nu s_\mu - g_{\rho\nu}(s s_\mu + \frac{1}{2s} s_5 s_\mu s_5)$. A similar calculation yields $s_\rho^- s_\nu^- s_\mu^- \mathsf{P}_1 = \frac{4}{3} X_{\rho\nu\mu} \mathsf{P}_1$. Recalling that $\{s_5, X_{\rho\nu\mu}\} = 0$ by (21) and noticing that $\mathsf{P}_1 \leftrightarrow \mathsf{P}_4$ under the substitution $s_5 \to -s_5$, we finally get the last relation in (22).

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