The tower of Kontsevich deformations for Nambu–Poisson structures on $\mathbb{R}^d$: dimension-specific micro-graph calculus

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Abstract

In Kontsevich's graph calculus, internal vertices of directed graphs are inhabited by copies of a given Poisson structure; the Nambu-determinant Poisson brackets are differential polynomial in the Casimir(s) and density ϱ times Civita symbol. We resolve the old vertices into subgraphs such that every new internal vertex contains one Casimir or one Civita symbol $\times\,\varrho$. Using this micro-graph calculus, we show that Kontsevich's tetrahedral γ_3-flow on the space of Nambu-determinant Poisson brackets over $\mathbb{R}^3$ is a Poisson coboundary: we realize the trivializing vector field $\vec{X}$ over $\mathbb{R}^3$ using micro-graphs. This $\vec{X}$ projects to the known trivializing vector field for the γ_3-flow over $\mathbb{R}^2$.

Introduction

Kontsevich introduced [9] a universal – for any affine Poisson manifold of dimension d – construction of infinitesimal symmetries for the Jacobi identity: for suitable cocycles γ in the graph complex, one obtains bi-vector flows $\dot{P} = Q_\gamma([P])$ with differential-polynomial right-hand sides (with respect to components $P^{ij}(\boldsymbol{x})$ of Poisson structures $P \in \Gamma(\bigwedge^2 TM^d_{\text{aff}})$). We detect in [8] that for the tetrahedral graph cocycle γ_3 from [9] and for the pentagon-wheel graph cocycle γ_5 (see [7]), the corresponding flows (see [1,4,6]) have a well-defined restriction to the subclass of Nambu-determinant Poisson brackets[1] $P(\varrho,[\boldsymbol{a}])$ on $\mathbb{R}^d$ at least in the following three cases: (*i*) γ_3-cocycle flow $\dot{P} = Q_\gamma([P])$ for $P(\varrho,[a])$ over $\mathbb{R}^3$, (*ii*) the same γ_3-cocycle and the flow of $P(\varrho,[a_1],[a_2])$ over $\mathbb{R}^4$, and (*iii*) the next, γ_5-cocycle flow for $P(\varrho,[a])$ over $\mathbb{R}^3$.

[1]The Nambu-determinant Poisson brackets (with $\varrho \not\equiv 1$ and Casimir(s) a_ℓ) of $f,g \in C^1(\mathbb{R}^d)$ are, e.g.,

$$\{f,g\}_{P(\varrho,[a])} = \varrho(x,y,z)\cdot\begin{vmatrix} a_x & f_x & g_x \\ a_y & f_y & g_y \\ a_z & f_z & g_z \end{vmatrix} \qquad \text{on } \mathbb{R}^3 \ni \boldsymbol{x} = (x,y,z);$$

likewise $\{f,g\}_{P(\varrho,[a_1],[a_2])} = \varrho(x^1,x^2,x^3,x^4)\cdot\det\bigl(\partial(a_1,a_2,f,g)/\partial(x^1,x^2,x^3,x^4)\bigr)$ on $\mathbb{R}^4$, and so on; all such formulas are coordinate free (as $\varrho(\boldsymbol{x})\cdot\partial_{x^1}\wedge\ldots\wedge\partial_{x^d}$ is a top-degree multivector on $\mathbb{R}^d$).

To study universal – for all $d \geqslant 3$ – or incidental – at fixed d – (non)triviality of Kontsevich's graph flows in the second Poisson cohomology for Nambu-determinant brackets $\{\cdot,\cdot\}_{P(\varrho,[\boldsymbol{a}])}$, we consider the coboundary equation,

$$Q_\gamma([P])([\varrho],[\boldsymbol{a}]) = [\![P(\varrho,[\boldsymbol{a}]),\vec{X}([\varrho],[\boldsymbol{a}])]\!], \tag{1}$$

upon the trivializing vector field solution(s) $\vec{X}([\varrho],[\boldsymbol{a}])$ with differential-polynomial coefficients over $\mathbb{R}^d$. We discovered in [8] that the γ_3-flow over $\mathbb{R}^3$ is trivial w.r.t. a unique solution $\vec{X}$; we found it by using *micro-graphs* that resolve $\varrho(\boldsymbol{x})\cdot$ Civita symbol against the Casimir(s) a_ℓ within copies of Nambu-determinant Poisson brackets $\{\cdot,\cdot\}_{P(\varrho,[\boldsymbol{a}])} = \varrho(\boldsymbol{x})\cdot\sum_{i_1,\ldots,i_d=1}^{d} \varepsilon^{\vec{i}}\cdot\partial_{i_1}(a_1)\cdot \ldots\cdot\partial_{i_{d-2}}(a_{d-2})\cdot\partial_{i_{d-1}}\otimes\partial_{i_d}$ in the vertices of Kontsevich's directed graphs for $Q_\gamma([P])$.

Within the differential calculus of (multi-)vectors $\vec{X}$, P, Q_γ, and $[\![P,\vec{X}]\!]$ with differential-polynomial coefficients over $\mathbb{R}^d$, the Poisson (non)triviality problem for the γ_3-flow in dimension $d=4$ is computationally hard. Yet the dimension-specific micro-graphs offer us a much more economical way to encode and process all such differential-polynomial structures over $\mathbb{R}^d$. At the same time, at the level of micro-graphs before they are projected to polynomials, the cocycle equation acquires a previously invisible right-hand side:

$$Q_\gamma([P]) - [\![P,X^\gamma]\!] = \Diamond([\varrho],[\boldsymbol{a}],\tfrac{1}{2}[\![P,P]\!]) + \langle\text{zero micro-graphs}\rangle + \cdots. \tag{2}$$

Now its r.-h.s. contains differential consequences of Jacobi identity for Nambu bi-vector $P(\varrho,[\boldsymbol{a}])$, as well as can it contain non-empty micro-graphs that encode identically zero bi-vectors.

This text is a sequel to [8], which we refer for motivation, notation, and details. The present paper is structured as follows: in §1, we explain how, for $\gamma=\gamma_3$ and $d=3$, a solution of (1) was constructed using micro-graphs, and we discuss its properties. In §2 we explore the composition of both sides in (2) for larger problems: $\gamma=\gamma_5$ or $d\geqslant 4$.

Preliminaries

Kontsevich's directed graphs are built of $n\geqslant 0$ wedges $\xleftarrow{L}\bullet\xrightarrow{R}$, usually drawn in the upper half-plane $\mathbb{H}^2$, over $m\geqslant 0$ ordered sinks along $\mathbb{R}=\partial\mathbb{H}^2$; tadpoles are allowed. Leibniz graphs are akin: the out-degrees of all but one (or more) vertices equal 2 yet there is (at least) one aerial vertex of out-degree 3 and its outgoing edges are ordered Left $\prec$ Middle $\prec$ Right.[2]

We shall study only those flows $\dot{P}=Q([P])$ on spaces of bi-vectors $P\in\Gamma(\bigwedge^2 TM^{d<\infty}_{\text{aff}})$ which are encoded by Kontsevich's graphs. From [9] (cf. [4]) we know that from suitable *cocycles* γ in the Kontsevich graph complex, one obtains the flows[3] $\dot{P}=Q_\gamma([P])$ which preserve the (sub)set of *Poisson* bi-vectors on M^d_{aff}. (The tetrahedron γ_3 and pentagon-wheel cocycle γ_5, see [4,6], are examples of graph cocycles giving such flows.)

Definition 1. The *Nambu-determinant Poisson bracket* on $\mathbb{R}^{d\geqslant 3}$ is the derived bi-vector $P(\varrho,[a])$ $\stackrel{\text{def}}{=}[\![\ldots[\![\varrho\cdot\partial_{x^1}\wedge\ldots\wedge\partial_{x^d},a_1]\!],\ldots]\!],a_{d-2}]\!]$, where $\varrho(\boldsymbol{x})\cdot\partial_{\boldsymbol{x}}$ is a d-vector field and scalar functions a_ℓ are Casimirs ($q\leqslant\ell\leqslant d-2$). In global (e.g., Cartesian) coordinates $x^1,\ldots,x^d$ on $\mathbb{R}^d$,

[2] For example, the tripod is a Leibniz graph; like every Leibniz graph, it expands to a linear combination of Kontsevich's graphs, namely to the Jacobiator $\frac{1}{2}[\![P,P]\!]$ for a bi-vector P whose copies are realized by wedges.

[3] The formula of Kontsevich's graph flow $\dot{P}=Q_\gamma([P])$ can depend on a choice of representative γ for the graph cohomology class $[\gamma]$. Fortunately, the vertex-edge bi-gradings $(4,6)$ for γ_3 and $(6,10)$ for two graphs in γ_5 are not yet big enough to provide room for any nonzero coboundaries (from nonzero graphs on 3 vertices and 5 edges or on 5 vertices and 9 edges, respectively). In other words, the known markers for $[\gamma_3]$ and $[\gamma_5]$ are in fact uniquely defined modulo constant; we prove this by listing all the admissible (non)zero "potentials" and by taking their vertex-expanding differentials in the graph complex. This is why, in our present study of the γ_3- and γ_5-flows on the spaces of Nambu–Poisson brackets, we do not care about a would-be response of trivializing vector fields X^γ in (2) to shifts of the marker cocycle γ within its graph cohomology class $[\gamma]$.

the Nambu bracket of $f, g \in C^1(\mathbb{R}^d)$ is expressed by the formula

$$\{f,g\}_{P(\varrho,[\boldsymbol{a}])} = \varrho(\boldsymbol{x}) \cdot \sum\nolimits_{i_1\ldots,i_d=1}^{d} \varepsilon^{\vec{\imath}} \cdot \partial_{i_1}(a_1)\cdots\partial_{i_{d-2}}(a_{d-2})\cdot\partial_{i_{d-1}}(f)\cdot\partial_{i_d}(g), \tag{3}$$

where $\varepsilon^{i_1,\ldots,i_d}$ is the Civita symbol on $\mathbb{R}^d$: $\varepsilon^{\sigma(1,\ldots,d)} = (-)^\sigma$ for $\sigma \in S_d$, otherwise zero.

Remark 1. Nambu–Poisson brackets on $\mathbb{R}^{d\geqslant 3}$ can be obtained from Nambu–Poisson brackets on $\mathbb{R}^{d+1}$ by taking $a_{d-1} = \pm x^{d+1}$ on $\mathbb{R}^{d+1}$ and by excluding the last Cartesian coordinate x^{d+1} from the list of arguments for $\varrho(\boldsymbol{x})$ and $a_1, \ldots, a_d(\boldsymbol{x})$. • By doing the above for $d+1=3$, one obtains a generic bi-vector $P = \varrho(x^1, x^2)\partial_{x^1} \wedge \partial_{x^2}$, which is Poisson on $\mathbb{R}^2$, and the (Nambu) Poisson bracket $\{f,g\}(x,y) = \varrho(x,y)\cdot(f_x \cdot g_y - f_y \cdot g_x)$.

Definition 2. Fix the dimension $d \geqslant 2$. A *micro-graph* is a directed graph built over $m \geqslant 0$ sinks, over $n \geqslant 0$ aerial vertices with out-degree d and ordering of outgoing edges, and over n d-tuples of aerial vertices with in-degree 1 and no outgoing edges. • The correspondence between micro-graphs and differential-polynomial expressions in $\varrho, a_1, \ldots, a_{d-2}$ and the content of sink(s) is defined in the same way as the mapping of Kontsevich's graphs to multi-differential operators on $C^\infty(M^d_{\text{aff}})$, see [8, §2.2] or [9].

Example 1. Nambu–Poisson brackets $P(\varrho,[a_1],\ldots,[a_{d-2}])$ on $\mathbb{R}^d$ are realized using micro-graphs, namely by resolving $\varrho(\boldsymbol{x})\cdot\varepsilon^{i_1,\ldots,i_d}$ in one vertex against $d-2$ vertices with the Casimirs $a_1, \ldots, a_{d-2}$. The out-degree of vertex with $\varrho(\boldsymbol{x}) \cdot \varepsilon^{\vec{\imath}}$ equals d; the in-degree of each vertex with a Casimir equals 1 and its out-degree is zero: the Casimir vertices are terminal (not to be confused with the two sinks, of in-degree 1, for the Poisson bracket arguments). The ordered d-tuple of edges is decorated with summation indices: for the Civita symbol $\varepsilon^{i_1,\ldots,i_d}$ in their common arrowtail vertex, the range is $1 \leqslant i_\ell \leqslant d$ for $1 \leqslant \ell \leqslant d$.

Remark 2. If the wedge tops contain Nambu–Poisson bi-vectors $P(\varrho,[a])$ on $\mathbb{R}^d$, every Kontsevich graph expands to a linear combination of micro-graphs: the arrow(s) originally in-coming to an aerial vertex with a copy of P, now work(s) by the Leibniz rule over the $d-1$ vertices, with $\varrho \cdot \varepsilon^{\vec{\imath}}$ and with $a_1, \ldots, a_{d-2}$, in the subgraphs $P(\varrho,[a_1],\ldots,[a_{d-2}])$ of the micro-graph.[4]

Example 2. If $d=2$ and (Nambu–)Poisson brackets on $\mathbb{R}^2$ are $\{f,g\}(x,y) = \varrho\cdot(f_x\cdot g_y - f_y\cdot g_x)$ as in Remark 1, the only possible Kontsevich graph, 'sunflower' built with tadpoles from $n=3$ wedges over one sink (see [1, Appendix F, Remark 13]) tautologically expands to a nontrivial linear combination of nonzero micro-graphs on three aerial vertices with $\varrho(x,y)\cdot\varepsilon^{i^\alpha j^\alpha}$, $1 \leqslant \alpha \leqslant 3$. Independently, the linear combination X^γ of micro-graphs that encode the trivializing vector field $\vec{X}([\varrho],[a])$ for Kontsevich's γ_3-flow for Nambu bi-vectors $P(\varrho,[a])$ on $\mathbb{R}^3 \ni (x, y, z)$, see [8] and §1 below, under the reduction $a := z$ and $\varrho = \varrho(x,y)$ becomes a well-defined vector field on the plane $\mathbb{R}^2 \subset \mathbb{R}^3$: the z-component of $\vec{X}([\varrho(x,y)],[z])$ vanishes. Let us compare the two vector fields on $\mathbb{R}^2$.

Proposition 1. The old 'sunflower' vector field which trivializes the tetrahedral γ_3-flow for all Poisson brackets on $\mathbb{R}^2$ *coincides* with the new vector field $\vec{X}([\varrho(x,y)],[a=z])$ from the trivialization of γ_3-flow for the Nambu brackets $P(\varrho,[a])$ on $\mathbb{R}^3$ (both viewed as 1-vector fields on $\mathbb{R}^2$ with differential coefficients in $[\varrho]$).

Lemma 2. If $d=3$ and $P(\varrho,[a])$ is Nambu, the Jacobiator tri-vector graph $\text{Jac}(P)$ remains a nontrivial linear combination, $\text{Jac}(P)([\varrho],[a])$, of 3 or 6 nonzero micro-graphs, each on $m=3$ sinks of in-degree 1, on two trident vertices with $\varrho\cdot\varepsilon^{i_1^\alpha i_2^\alpha i_3^\alpha}$, and on two terminal vertices of in-degrees 1 and 2 (if $\varrho \equiv \text{const}$) or $(1,1)$ and $(1,2)$ if $\varrho \not\equiv \text{const}$.

This is why linear combinations of Leibniz micro-graphs $\Diamond([\varrho],[\boldsymbol{a}],\frac{1}{2}[\![P,P]\!])$ can contribute nontrivially to the right-hand side of (2).

[4]But not every micro-graph is obtained from a Kontsevich graph by resolving the old aerial vertices into subgraphs.

1 Trivializing vector field $\vec{X}$ for γ_3-flow over $\mathbb{R}^3$

We recall from [8, §4.1] that, given a suitable graph cocycle γ (e.g., γ_3 which we take here), Kontsevich's γ-flow $\dot{P} = \mathrm{O\vec{r}}(\gamma)(P^{\otimes\#\mathrm{Vert}(\gamma)})$ restricts to the set of Nambu–Poisson bi-vectors $P(\varrho,[a])$ such that the velocity of a Casimir a_ℓ is still encoded by Formality graphs [8, Proposition 2]: $\dot{a}_\ell = \mathrm{O\vec{r}}(\gamma)(P\otimes\cdots\otimes P\otimes a)$, whence the velocity $\dot{\varrho}([\varrho],[\boldsymbol{a}])$ is expressed from the known $\dot{\boldsymbol{a}}$ and $\dot{P}$ (see [8, Corollary 3]). The Leibniz rule, balancing $\dot{P}$ with $\dot{\varrho},\dot{\boldsymbol{a}}$ for P linear in ϱ and the first jets of all a_ℓ, is then a tautology.

Independently, if $\vec{Y}$ is any C^1-vector field on $\mathbb{R}^d$ with Nambu–Poisson bi-vectors $P(\varrho,[\boldsymbol{a}])$, then the evolution $L_{\vec{Y}}(a_\ell) = [\![\vec{Y},a_\ell]\!]$ of scalar functions and $L_{\vec{Y}}(\varrho\cdot\partial_{\boldsymbol{x}}) = [\![\vec{Y},\varrho\cdot\partial_{\boldsymbol{x}}]\!]$ of d-vectors correlates, by the Leibniz-rule shape of the Jacobi identity for the Schouten bracket $[\![\cdot,\cdot]\!]$, with evolution $L_{\vec{Y}}(P) = [\![\vec{Y},P]\!]$ of Nambu bi-vector $P = [\![\varrho\cdot\partial_{\boldsymbol{x}},\cdots\boldsymbol{a}\cdots]\!]$, see [8, §2.1].

Our finding in [8, Theorem 8] is that for the graph cocycle γ_3 and $d=3$, the action of vector field $\vec{X}$ (which trivialzes the γ_3-flow $\dot{P} = Q_{\gamma_3}([P]) = [\![P,\vec{X}]\!]$ of Nambu brackets P on $\mathbb{R}^3$) upon $P(\varrho,[a])$ factors through the initially known – from γ_3 – velocities of a and ϱ: having solved (1) for $\vec{X}$, we then verified that $\dot{a} = [\![a,\vec{X}]\!]$ and $\dot{\varrho}\,\partial_{\boldsymbol{x}} = [\![\varrho\cdot\partial_{\boldsymbol{x}},\vec{X}]\!]$.

By using this factorization – i.e. the lifting of the sought vector field's action on the elements of $P(\varrho,[a])$ – the other way round, we create an economical scheme to inspect the *existence* of trivialzing vector field $\vec{X}$ for larger problems (i.e. for bigger graph cocycles or higher dimension $d \geqslant 3$). When this shortcut works, so that $\vec{X}$ is found, it saves much effort. Otherwise, to establish the (non)existence of $\vec{X}$ one deals with a larger PDE, namely Eq. (1).

Open problem 1. Are Nambu–Poisson bi-vectors $P(\varrho,[a])$ such that the action of trivialzing vector field(s) $\vec{X}$ (when such exist(s) for a Kontsevich graph flow) always lifts to the Casimir(s) a_ℓ and d-vector field $\varrho\cdot\partial_{\boldsymbol{x}}$, so that $\dot{a}_\ell = [\![a_\ell,\vec{X}]\!]$ and $\dot{\varrho}\cdot\partial_{\boldsymbol{x}} = [\![\varrho\cdot\partial_{\boldsymbol{x}},\vec{X}]\!]$?

Let us illustrate how the shortcut scheme works. We now tune a 1-vector field $\vec{X}(\varrho,[a])$ for the flow $\dot{P} = Q_{\gamma_3}([P])$ of $P(\varrho,[a])$ over $\mathbb{R}^3$ such that $\dot{a} = [\![a,\vec{X}]\!]$ and $\dot{\varrho}\cdot\partial_{\boldsymbol{x}} = [\![\varrho\cdot\partial_{\boldsymbol{x}},\vec{X}]\!]$, whence we verify that $\dot{P} = [\![P,\vec{X}([\varrho],[a])]\!] \in \operatorname{im}\partial_P$ for the Nambu-determinant class of Poisson brackets on $\mathbb{R}^3$.

The micro-graph expansion of $Q_{\gamma_3}(P)$ for $P(\varrho,[a])$ over $\mathbb{R}^3$ consists of directed graphs on 2 sinks for f and g, on four terminal vertices for Casimirs a without outgoing arrows, and on four vertices for $\varrho\cdot\varepsilon^{ijk}$ with three ordered outgoing edges. In every micro-graph in bi-vector $Q_{\gamma_3}(P)$ there are 12 edges, with exactly two going towards f and g in the sinks. To have a solution $\vec{X}$ of the equation $Q_{\gamma_3}(P(\varrho,[a])) = [\![P,\vec{X}]\!]$ using micro-graphs that encode $\vec{X}([\varrho],[a])$ we thus need micro-graphs on one sink, three terminal vertices with a, and three trident vertices for $\varrho\cdot\varepsilon^{ijk}$. Of the nine edges in each micro-graph, exactly one goes to the sink, so that $\vec{X}$ is a 1-vector.

We first generate all suitable unlabeled micro-graphs (i.e. without distinction which sinks are for Casimirs) without tadpoles and with exactly one tadpole. Next, by deciding on the run which of the four sinks is the argument of 1-vector, we produce 366 1-vector fields with differential polynomial coefficients in ϱ and a, encoded by micro-graphs. Some of the coefficients are identically zero when the sums over three triples of indices in Civita symbols are fully expanded; there remain 244 nonzero marker micro-graphs in the ansatz for the trivializing vector field $\vec{X}$. Now, we do not attempt solving the big problem $Q_{\gamma_3}(P) = [\![P,\vec{X}]\!]$ directly with respect to the 244 coefficients of nonzero marker micro-graphs. Instead, let us find a vector field $\vec{X}$, realized by 1-vector micro-graphs X^γ, which reproduces the known velocities [8, Eq. (11)] of ϱ and Casimir a, that is, we solve the equations $\dot{a} = -[\![\vec{X},a]\!]$ and $\dot{\varrho}\,\partial_x\wedge\partial_y\wedge\partial_z = [\![\varrho\,\partial_x\wedge\partial_y\wedge\partial_z,\vec{X}]\!]$ with respect to the coefficients in the micro-graph ansatz for X^γ. To determine exactly the number of equations in either linear algebraic system we keep track of the number of differential monomials appearing when $\vec{X}$ acts on either a or ϱ

as above, and we recall also the differential monomials which already appeared in $\dot{a}$ and $\dot{\varrho}$ in the left-hand sides, that is in [8, Eq. (11)]. In this way, we detect that the linear algebraic system for $\dot{a}$ contains 2961 equations and the system for $\dot{\varrho}$ contains 6679 equations. Each equation is a balance of the coefficient of one differential monomial. We now merge these two systems of linear algebraic equations upon the coefficients of micro-graphs in the ansatz for the trivializing vector field $\vec{X}$, and we find a solution. Only 11 coefficients are nonzero. The analytic formula of this vector field was reported in [8, Theorem 8]. The three equalities, namely $\dot{a} = -[\![\vec{X}, a]\!]$ and $\dot{\varrho}\, \partial_x \wedge \partial_y \wedge \partial_z = [\![\varrho \partial_x \wedge \partial_y \wedge \partial_z, \vec{X}]\!]$ implying $Q_{\gamma_3}(P) = [\![P, \vec{X}]\!]$, are verified immediately. Here is the encoding of the weighted sum X^γ of these 11 micro-graphs which realize the trivializing vector field $\vec{X}$ for the tetrahedral flow $Q_{\gamma_3}(P) = [\![P, \vec{X}]\!]$ on the space of Nambu–Poisson structures over $\mathbb{R}^3$:

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 16 * [(0,4), (0,5), (0,6), (5,0), (5,1), (5,6), (6,0), (6,2), (6,3)]
 24 * [(0,4), (0,5), (0,6), (5,0), (5,1), (5,6), (6,1), (6,2), (6,3)]
 16 * [(0,4), (0,5), (0,6), (5,0), (5,1), (5,2), (6,1), (6,3), (6,5)]
-16 * [(0,4), (0,5), (0,6), (5,0), (5,1), (5,2), (6,1), (6,3), (6,5)]
 12 * [(0,4), (0,5), (0,6), (5,1), (5,2), (5,6), (6,1), (6,2), (6,3)]
-12 * [(0,4), (0,5), (0,6), (5,1), (5,2), (5,6), (6,1), (6,2), (6,3)]
 24 * [(4,0), (4,1), (4,6), (5,0), (5,1), (5,2), (6,0), (6,2), (6,3)]
-24 * [(4,0), (4,1), (4,6), (5,0), (5,2), (5,4), (6,0), (6,1), (6,3)]
  8 * [(4,0), (4,1), (4,5), (5,0), (5,2), (5,6), (6,0), (6,3), (6,4)]
 -8 * [(4,0), (4,1), (4,5), (5,2), (5,3), (5,6), (6,0), (6,1), (6,4)]
  8 * [(0,4), (0,5), (0,6), (5,0), (5,1), (5,6), (6,2), (6,3), (6,6)]
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Remark 3. At the level of micro-graphs, the solution X^{γ_3} contains a tadpole, i.e. a 1-cycle, in the last graph. In terms of differential polynomials this means the presence of a deriative ∂_{x^i} acting on the coefficient $\varrho(\boldsymbol{x})$ near the Civita symbol ε^{ijk} containing the index i of the base coordinate x^i in that derivative; that is, the last term in the vector field $\vec{X}$ contains $\partial_{x^i}(\varrho(\boldsymbol{x}))\cdot\varepsilon^{ijk}$.

Proposition 3. Without tadpoles in the micro-graph ansatz X^{γ_3}, there is no solution $\vec{X}$ to the trivialization problem $Q_{\gamma_3}(P(\varrho, [a])) = [\![P, \vec{X}]\!]$ at the level of differential polynomials.

Remark 4. The tadpole from X^γ survives into the l.-h.s. of Eq. (2) because there is no solution without tadpoles, so the 11th graph cannot be in the kernel of $[\![P, \cdot]\!]$. There must be Leibniz graph(s) with tadpole in the r.-h.s. to balance the equality.

Remark 5. Over $\mathbb{R}^3$, to generate an ansatz for the part of $\Diamond$ *without* tadpoles in the r.-h.s. of (2), it suffices to take Leibniz micro-graphs on two sinks (receiving exactly one edge into either sink), with one trident Jacobiator vertex, with two trident vertices for $\varrho(x, y, z) \times$ Civita symbol, and with two terminal vertices for copies of the Casimir a. There are 96 isomorphism classes of directed graphs with the above out-degree profiles; by filtering them w.r.t. the in-degree 1 of tri-vector's sinks, we keep 45.

Likewise, to generate the other part – with one tadpole – of $\Diamond$ in the r.-h.s. of (2) for the γ_3-flow $\dot{P}([\varrho], [a])$ over $\mathbb{R}^3$, we artificially decrease by one the number of edges in the graphs to generate – specifically, one of the "Civita" tridents becomes a wedge; the Jacobiator remains a trident that does not produce a tadpole – and, as soon as these new directed graphs are produced, we by hand add a tadpole at the lonely Civita vertex.

To conclude this section, we note that existence of a micro-graph solution X^γ for (2) in a fixed dimension $d \geqslant 2$ implies the existence of a solution $\vec{X}$ with differential-polynomial coefficients for problem (1) of triviality of Kontsevich's tetrahedral flow $\dot{P} = \mathrm{O}\vec{\mathrm{r}}(\gamma)(P^{\otimes n})$ on the space of Nambu-determinant Poisson structures $P(\varrho, [a])$ over $\mathbb{R}^d$. The vector field $\vec{X}([\varrho], [a])$ trivializing the tetrahedral flow $\dot{P} = \mathrm{O}\vec{\mathrm{r}}(\gamma_3)(P^{\otimes 4})$ over $\mathbb{R}^3$ can be derived in precisely this way.

2 (Non)triviality of γ_3-flow for Nambu–Poisson brackets on $\mathbb{R}^4$

From [8] we know that Kontsevich's tetrahedral γ_3-flow restricts to the space of Nambu-determinant Poisson bi-vectors $P(\varrho, [a_1], [a_2])$ over $\mathbb{R}^4$: the differential-polynomial velocities $\dot{\varrho}$ and $\dot{a}_1, \dot{a}_2$ inducing the graph cocycle evolution $\dot{P}([\varrho],[a_1],[a_2])$ are stored externally.[5] Recall that the evolutions $\dot{a}_1, \dot{a}_2(\varrho,[a_1],[a_2])$ are realized by Kontsevich graphs $\vec{\mathrm{Or}}(\gamma_3)(P\otimes P\otimes P \otimes a)$, hence they are immediately expanded to micro-graph realizations. The evolution $\dot{\varrho}([\varrho], [a_1],[a_2])$ can then be expressed by using micro-graphs with minimal effort.

The problem of Poisson (non)triviality of the tetrahedral γ_3-flow for Nambu brackets in dimension $d = 4$ is *a priori* independent from the similar problem in dimension three (the solution of which has been discussed in the preceding section).

Open problem 2. Is the restriction of Kontsevich's tetrahedral γ_3-flow on the space of Nambu-determinant Poisson structures $P(\varrho,[a_1],[a_2])$ over $\mathbb{R}^4$ trivial or not in the second Poisson cohomology? That is, at the level of differential polynomials in ϱ and a_1, a_2 in (multi-)vector coefficients, is there a vector field $\vec{X}([\varrho],[a_1],[a_2])$ satisfying the equation

$$Q_{\gamma_3}([P])([\varrho],[a_1],[a_2]) = [\![P,\vec{X}]\!]\,? \tag{1'}$$

Remark 6. Solving equation (1′) can be attempted, first, strictly at the level of differential polynomials by generating all the homogeneous differential monomials in ϱ,a_1,a_2 for the $d = 4$ coefficients X^i of the sought vector field $\vec{X}([\varrho],[a_1],[a_2]) = \sum_{i=1}^{d} X^i\,\partial_i$. For the graph cocycle γ_3, every such monomial contains $\varrho^3 = \varrho\cdot\varrho\cdot\varrho$, a_1^3, a_2^3 and 11 derivatives falling on these nine factors; the 12th derivative makes the 1-vector $X^i\,\partial_i$. These twelve derivations $\partial_{x^{i_1}}$, …, $\partial_{x^{i_{12}}}$ are such that $\{x^{i_1},\ldots,x^{i_{12}}\} = \{x^1,\ldots,x^4\}\sqcup\{x^1,\ldots,x^4\}\sqcup\{x^1,\ldots,x^4\}$. Another constraint – upon the derivative order profiles in $X^i([\varrho],[\boldsymbol{a}])$, hence in $[\![P,\vec{X}]\!]$ – comes from the bi-vector $Q_{\gamma_3}(P(\varrho,[\boldsymbol{a}]))$ with known differential-polynomial coefficients. In effect, terms in X^i can contain only those orders of derivatives which, under $[\![P,\cdot]\!]$, reproduce the actually existing profiles of derivatives in $Q_{\gamma_3}([\varrho],[\boldsymbol{a}])$. (This technique was illustrated for Eq. (1) with γ_3 over $\mathbb{R}^3$ in [3, §7.1.5].)

At the level of micro-graphs and Nambu-determinant Poisson structures $P(\varrho,[a_1],[a_2])$ over $\mathbb{R}^4$, a solution X^γ of (2) for the graph cocycle γ_3 would be realized by micro-graphs possibly with tadpoles, on one sink of in-degree 1, three vertices of out-degree 4, and two triples of terminal vertices for Casimirs a_1, a_1, a_1 and a_2, a_2, a_2. Expanding just one such micro-graph into a 1-vector on $\mathbb{R}^4$ means taking $n = 3$ sums with $d = 4$ indices in each sum, every index running from 1 to $d = 4$; this gives us $4^4{\cdot}4^4{\cdot}4^4 = 2^{24} \approx 16{\cdot}10^6$ terms. From Proposition 4 below we read that there are $\approx 20,000$ micro-graphs in the ansatz for X^γ. This totals with $\approx 320\cdot 10^9$ differential monomials in the ansatz for trivializing vector field $\vec{X}$ on $\mathbb{R}^4$.

So, to solve equation (1), let us try solving equation (2) at the level of micro-graphs, i.e. without projecting down to differential polynomials.

Open problem 3. Is there a vector field X^γ (realized by micro-graphs as above) satisfying the micro-graph equation

$$Q_{\gamma_3}(P(\varrho,[a_1],[a_2])) - [\![P,X^\gamma]\!] = \Diamond([\varrho],[\boldsymbol{a}],\mathrm{Jac}(P)([\boldsymbol{a}],[\varrho])) + \langle\text{zero micro-graphs}\rangle\,? \tag{2'}$$

Remark 7. In the right-hand side of (2′) the term $\Diamond([\varrho],[\boldsymbol{a}],\mathrm{Jac}(P)([\boldsymbol{a}],[\varrho]))$ can, firstly, be a linear combination of *Leibniz graphs*, that is, $\Diamond_1(P(\varrho,[\boldsymbol{a}]),\mathrm{Jac}(P))$, built from the micro-graph realizations of the Nambu–Poisson structure $P(\varrho,[a_1],[a_2])$ over $\mathbb{R}^4$ and then expanded into the linear combination of micro-graphs: the Leibniz graphs expand to Kontsevich graphs (by expanding the Jacobiator and Leibniz rules in each graph) and each Kontsevich graph expands to micro-graphs. Secondly, for a solution X^γ to appear, the right-hand side may need a term $\Diamond_2([\varrho],[\boldsymbol{a}],\mathrm{Jac}(P(\varrho,[\boldsymbol{a}])))$ with a linear combination of *Leibniz micro-graphs*, which by definition contain a micro-graph expansion of the Jacobiator, same as above, but also other micro-graph vertices with ϱ and Casimirs a_ℓ that do not merge into subgraphs realizing copies

[5] `https://rburing.nl/gcaops/adot_rhodot_g3_4D.txt`

of the Nambu--Poisson bi-vector over $\mathbb{R}^4$. To make equation (2′) a strict equality of left- and right-hand sides, its right-hand side could require addition of zero micro-graphs with outgoing edge ordering (which equal minus themselves under an automorphism).

Proposition 4. • There are $1,079$ isomorphism classes of directed graphs on one sink, three vertices of out-degree four, six terminal vertices, and at most one tadpole (of them, 352 are without a tadpole and 727 have one tadpole).
• Taking those graphs containing a vertex of in-degree one (for the sink), and dynamically appointing the Casimirs from the multi-set $\{a_1, a_1, a_1, a_2, a_2, a_2\}$ to the six terminal vertices of the above graphs, we obtain $38,120$ micro-graphs.
• Excluding repetitions in the above set of micro-graphs (e.g., if those micro-graphs are isomorphic), still not excluding micro-graphs which equal minus themselves under a symmetry (automorphism of micro-graph with outgoing edge ordering and known location of a_1's and a_2's) we obtain $19,957$ micro-graphs in the ansatz for X^γ that would encode the trivializing vector field solutions, if any, of coboundary equation (1′).
• Of these $19,957$ micro-graphs, one tadpole is present in $13,653$ micro-graphs, and there are no tadpoles in $6,304$ micro-graphs.

Construction sketch. The representatives of isomorphism classes of graphs without tadpoles are generated by the `nauty` [11] command-line call `geng 10 9:12 | directg -e12 | pickg -d0 -m7 -D4 -M3`. Likewise, the graphs with one tadpole are generated by first producing graphs with one edge fewer, using `geng 10 0:12 | directg -e11 | pickg -d0 -m7 -D4`, and adding a tadpole to the lonely vertex of out-degree 3 in each graph. For the appointment of Casimirs to 6 vertices in all *different* ways, one uses an efficient algorithm to generate the 20 permutations of the multi-set $\{a_1, a_1, a_1, a_2, a_2, a_2\}$. □

Remark 8. The gcaops software [2] for calculus in Kontsevich's graph language (see [1, 3, 4]) so far does not support the generation and manipulation of directed graphs with tadpoles — which, we expect, are essential for a solution X^γ of (1′) to appear.

Finally, let us examine how an unknown solution of (2) in high dimension $d \geqslant 4$ can be constrained by using the known solution(s) for similar problem with the same graph flow but in lower dimension $d-1$.

Remark 9. There is no known natural procedure to extend a given micro-graph realization X^γ_d of trivializing vector field $\vec{X}_d$ on $\mathbb{R}^d$ (for $d \geqslant 2$) to a larger micro-graph realization X^γ_{d+1} of some solution(s) $\vec{X}_{d+1}$ (if any) for Kontsevich's graph flow (non)triviality problem over $\mathbb{R}^{d+1}$. Note that in the next dimension $d+1$, Civita symbols are realized by micro-graph vertices with $d+1$ outgoing edges, instead of the vertices with d outgoing edges in a given solution X^γ_d.

Conversely, the *reduction* $d+1 \mapsto d$ of dimension for the Nambu-determinant Poisson structures amounts to setting the last Casimir $a_{d-1} = \pm x^{d+1}$ and excluding the last coordinate x^{d+1} from the list of arguments in either $\varrho(x^1, \ldots, x^d)$ or any of the remaining Casimirs $a_1, \ldots, a_{d-2}$. The (un)known trivializing vector field $\vec{X}_{d+1}$ then provides a known solution $\vec{X}_d$ and its micro-graph realization X^γ_d. By tracking backwards the correspondence of micro-graphs under the dimensional reduction, when the $(d+1)$th edges are essentially removed from the Civita symbol vertices, one can conjecture the shape of many micro-graphs in a sought solution X^γ_{d+1} of the larger problem.

Conclusion

We conjecture that over all $\mathbb{R}^{d\geqslant 3}$, Kontsevich's γ_3-flows of Nambu–Poisson brackets are coboundaries; the trivializing vector fields then project down under $\mathbb{R}^d \to \mathbb{R}^{d-1}$.

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