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Clocking mechanism from a minimal spinning particle model

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Abstract

The clock hypothesis plays an important role in the theory of relativity. To test this hypothesis, a model of an ideal clock is needed. Such a model should have the phase of its intrinsic periodic motion increasing linearly with the affine parameter of the clock's center of mass worldline. A class of relativistic rotators introduced by Staruszkiewicz in the context of an ideal clock is studied. A singularity in the inverse Legendre transform leading from the Hamiltonian to the Lagrangian leads to new possible Lagrangians characterised by fixed values of mass and spin.

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1 Introduction

One of the most important concepts in the relativity theory is the proper time. Up to a unit of time, the proper time of a point-like object is identified with the length of its time-like worldline. The clock hypothesis states that an ideal clock always measures its proper time irrespective of the state of motion of that clock. This hypothesis has been verified to a high degree of precision, however it is not known whether it holds true. Violation of the hypothesis could occur for extreme accelerations of $10^{29}\frac{m}{s^2}$ corresponding to the electron's Zitterbewegung frequency. However, such high values of accelerations are not yet experimentally attainable. Nevertheless, an attempt can be made to theoretically test the clock hypothesis within the same framework one uses to describe real mechanical systems.

To test the clock hypothesis which refers to classical concepts, a classical model of the ideal clock must be devised.[1] Such a model has been proposed by Staruszkiewicz [2]. The model is based on the concept of a relativistic rotator – a dynamical system described by position, a single null direction and two dimensional parameters: mass m and length l. It seems the model provides the simplest mechanical system with intrdescribe whose clocking frequency could be fixed by the parameters of spin and mass. In this context, to realise on the classical level Wigner's irreducibility idea of quantum systems [3], Staruszkiewicz defined a classical *fundamental dynamical system* as one for which the Casimir invariants of the Poincaré group are given parameters, respectively m^2 and $-\frac{1}{4}m^4l^2$, rather than integrals of motion. Among the entirety of Lagrangians possible for the family of relativistic rotators considered in [2], there are only two which are fundamental in the above sense. However, later it was shown on the Lagrangian level [4] that the fundamental rotator in this class is defective as a dynamical system with 5 degrees of freedom, which explained why the rotation frequency remained indefinite, contrary to the original motivation.

A fundamental relativistic rotator (with definite frequency), as a purely mathematical construct unrelated to any material mechanism, would be a simple non-quantum device perfectly tailored to be used as an ideal clock. The mechanism of such a clock can be visualised in the following way. In the momentum rest frame with suitably chosen coordinate axes, the image of the spatial direction of the Pauli-Lubański four-vector W^μ could be identified with the equator on the Riemann sphere of null directions and used as the clock's face. On the other hand, the image of the null direction k^μ on the same sphere would be a point moving about the equator, counting the number of times the phase has been increased by 2π, and so it could be used as the clock's hand.[2]

To find a way to stabilise the clocking frequency a new idea was considered [5] based on a singularity of an inverse Legendre transformation (to be discussed later in this text) which distinguishes the motion with the speed of light. In this context a few words of clarification seem necessary here. The space-time geometry of Einstein's theory of relativity is characterised by the structure of invariant null cones. As an absolute element of this theory this structure distinguishes a unique universal velocity for the propagation of signals. In particular, the velocity of light in Maxwell's theory must coincide in the instantaneous spaces of arbitrary observers with this universal value and so be the same regardless of the motion of the light source or the observer. Furthermore, it is important to stress that the reparametrization invariant condition $\dot{x}\dot{x} = 0$ for a fourvelocity $\dot{x}$ sets to unity the value of the velocity $\tanh(\chi_e)$ irrespective of any time-like vector e representing an arbitrary observer, where $\tanh^2(\chi_e) \equiv 1 - \frac{(ee)(\dot{x}\dot{x})}{(e\dot{x})^2}$. There

[1] A spatially extended quantum field-theoretical model of a clock devised in the clock hypothesis context [1] goes beyond this conceptual limitation. The authors concluhasthat *no device built according to the rules of quantum field theory can measure proper time along its path*.

[2] Throughout this paper x^μ denotes the position vector, k^μ is the single null direction carrying the spinning degrees of freedom. The scalar product is denoted by $xy \equiv \eta_{\alpha\beta}x^\alpha y^\beta = x^\alpha y_\alpha$ (Einstein's summation convention is used), where $(\eta_{\alpha\beta}) = \mathrm{diag}(1,-1,-1,-1)$. Greek indices run over $0,1,2,3$ and 0 stands for the time component.

is no similar reparametrization-invariant condition that would set the value of velocity for a subluminal motion (with $\dot{x}\dot{x} \neq 0$) – the Lobachevsky space of unit fourvelocities e is homogeneous, therefore no particular e used to define the (finite) value of the hyperbolic angle χ_e can be singled out. This distinguished qualitative feature of the condition $\dot{x}\dot{x} = 0$ should also have its consequence for classical relativistic dynamical systems (then $\dot{x}$ and the time-like momentum of a massive system cannot be colinear). It is also worth pointing out a fact observed by Dirac in his book [6], which puts the distinguished role of the $\dot{x}\dot{x} = 0$ condition in another context. From Dirac's equation, it follows that a measurement of the instantaneous speed of the electron's motion gives a value equal to the speed of light. According to Dirac, this result is generally true in relativistic theory. This observation tempts one to conjecture that worldlines of classical analogues of quantum elementary particles should be null.

2 Staruszkiewicz class of relativistic rotators.

A class of relativistic rotators is defined by the following Hamiltonian action introduced by Staruszkiewicz [2]

$$S = -m \int \mathrm{d}\lambda \sqrt{\dot{x}\dot{x}}\, f(\xi), \qquad \xi \equiv -l^2 \frac{\dot{k}\dot{k}}{(k\dot{x})^2}, \quad f'(\xi) \not\equiv 0. \tag{1}$$

Here, the dot denotes differentiation with respect to λ – an arbitrary parameter along the worldline, and f can be arbitrary non-constant and positive function of a reparametrization invariant argument ξ depending on the spinning degrees of freedom through a null direction k (this means that ξ must be a Poincaré scalar invariant with respect to the rescaling of the null vector k by any function of λ).

Representations of the Poincaré group are enumerated by the eigenvalues of two Casimir operators (for the case of massive representations). These operators are the square of the momentum four-vector $C_1 = p^\mu p_\mu$ and the square of the Pauli-Lubański four-vector $C_2 = W^\mu W_\mu$, where:

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} p_\nu M_{\alpha\beta}, \qquad M_{\alpha\beta} = x_\alpha p_\beta - x_\beta p_\alpha + \Sigma_{\alpha\beta}.$$

The expression $\Sigma_{\alpha\beta}$ represents the internal angular momentum (spin). To find suitable Lagrangians in the considered class of rotators one can proceed as follows. The conserved quantities p_α and $M_{\alpha\beta}$ are determined from the action (1) (with $\Sigma_{\alpha\beta} = k_\alpha \pi_\beta - k_\beta \pi_\alpha$), where the momenta canonically conjugated to x^μ and k^μ read, respectively,

$$p_\mu = m\left[f(\xi)\frac{\dot{x}_\mu}{\sqrt{\dot{x}\dot{x}}} - 2\xi f'(\xi)\frac{\sqrt{\dot{x}\dot{x}}}{k\dot{x}} k_\mu \right] \qquad \text{and} \qquad \pi_\mu = 2m\frac{\sqrt{\dot{x}\dot{x}}}{\dot{k}\dot{k}}\xi f'(\xi)\dot{k}_\mu.$$

The corresponding Casimir invariants can now be calculated

$$C_1 = m^2[f^2(\xi) - 4\xi f(\xi) f'(\xi)], \qquad C_2 = -4m^4 l^2 \xi f^2(\xi)[f'(\xi)]^2.$$

By requiring that $C_1 = m^2$ and $C_2 = -\frac{1}{4}m^4 l^2$, we get two first-order differential equations that, remarkably, have a common solution of the form $f(\xi) = \sqrt{1 \pm \sqrt{\xi}}$. From the result obtained above, it follows that there are only two relativistic rotators that are fundamental. The Hamiltonian action describing these rotators takes on the form

$$S = -m \int \mathrm{d}\lambda \sqrt{\dot{x}\dot{x}} \sqrt{1 \pm \sqrt{-l^2 \frac{\dot{k}\dot{k}}{(k\dot{x})^2}}}. \tag{2}$$

As will be explained below, the dynamical system defined by the action (2) is not suitable as a clock. However, it is equivalent to a geometric model of a spinning particle introduced earlier in a different context by Lyakhovich, Segal, and Sharapov [7] and as such can be used with success.

3 Lagrangian singularity for fundamental rotators with subluminal intrinsic motion

In the Lagrangian form of dynamics, there are s Lagrangian equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \qquad i = 1, 2, \ldots, s$$

for a dynamical system with s degrees of freedom. In this form the Lagrangian L is assumed to be a function of s generalised coordinates $q^i = Q^i(\lambda)$ and the corresponding velocities $v^i = \dot{Q}^i(\lambda)$ that altogether characterise the physical state of the system. Differentiating the Lagrange equations with respect to the independent parameter λ, one gets a system of second-order equations

$$H_{ij}a^j = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial v^i \partial q^j}v^j - \frac{\partial^2 L}{\partial \lambda \partial v^i}, \qquad H_{ij} \equiv \frac{\partial^2 L}{\partial v^i \partial v^j}.$$

Provided that $\det[H_{ij}]$ for this system, one can express accelerations $a^i = \ddot{Q}^i(\lambda)$ as independent functions of positions and velocities. When the Hessian determinant $\det(H_{ij})$ is non-vanishing the Lagrangian is called regular, otherwise it is called singular. For a singular Lagrangian, there is an infinite number of accelerations available from which a dynamical system can choose at any stage of its motion. The regularity (or singularity) is a qualitative feature, independent of the particular coordinates in which the Lagrangian has been expressed.

One can verify the condition $\det[H_{ij}]$ for all members of the considered family of relativistic rotators regarded as dynamical systems with 5 degrees of freedom. Following the calculation presented in [4], one can start with Cartesian coordinates (x, y, z) and spherical angles (φ, θ) describing the position and the null direction in a reference system of some inertial observer. The arbitrary parameter λ can be set to be proportional to the time of that observer, $\lambda = l^{-1}t$. Then, in terms of the vector matrices $V = [\dot{x}, \dot{y}, \dot{z}]^T$, $N = [\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta]^T$ and $W = \left[\dot{\theta}, \dot{\varphi}\sin\theta\right]^T$, the Lagrangian form as defined in (1) gets reduced to

$$L = -m\sqrt{1 - V^T V}f(\xi), \qquad \text{with} \quad \xi = \frac{W^T W}{(1 - N^T V)^2} \quad \text{and} \quad f'(\xi) \not\equiv 0.$$

The Hessian determinant can be found by taking components of vectors V and W as independent velocity variables (linearly related to the original set of velocities) and use some identities for determinants of block matrices. As shown in [4], the resulting determinant reads

$$\det[H_{ij}] \propto f^3(\xi)\left[f'(\xi)\right]^2\left(1 + 2\xi\left(\frac{f'(\xi)}{f(\xi)} + \frac{f''(\xi)}{f'(\xi)}\right)\right),$$

where the proportionality factor (not shown) is independent of f. Hence, only with f satisfying the differential equation $\left(f(\xi) + 2\xi f'(\xi)\right)f'(\xi) + 2\xi f(\xi)f''(\xi) = 0$ the Lagrangian will be singular. This equation has only one solution such that $f'(\xi) \not\equiv 0$, namely

$$f(\xi) = a\sqrt{1 \pm b\sqrt{\xi}},$$

with a and b being positive integration constants to be set by the Casimir parameters.

Now it becomes clear that the only Lagrangian which is singular in the investigated family of relativistic rotators is that defined by the action (2). This singularity means that the phase of the clocking mechanism would remain indeterminate, which is the reason why this dynamical system cannot be interpreted as a clock.

4 Singularities in the inverse Legendre transformation. Zitterbewegung with the speed of light.

According to Dirac's method [8], the Hamiltonian for a (reparametrization invariant) relativistic system is a linear combination of first class constraints (that is, whose Poisson bracket with all other constraints is vanishing). The coefficients of this combination are arbitrary functions of the independent parameter. There are four such constraints present in the case of fundamental relativistic rotators: the first two follow from the requirement imposed on both Casimir invariants: $C_1 \simeq m^2$ and $C_2 \simeq -\frac{1}{4}m^4 l^2$; the other two concern the particular realisation of the spinning degrees of freedom described by a null direction k: $kk \simeq 0$ and $k\pi \simeq 0$. All of these constraints are first class. It immediately follows that the total Hamiltonian of the fundamental relativistic rotator must read

$$\mathcal{H} = \frac{u_1}{2m}\left[pp - m^2\right] + \frac{u_2}{2m}\left[pp + \frac{4}{l^2 m^2}(kp)^2 \pi\pi\right] + u_3 k\pi + u_4 kk, \tag{3}$$

with u_i's being arbitrary functions.[3] Now the Hamiltonian constraints follow from the equations $\partial_{u_i}\mathcal{H} = 0$ while the velocities are defined through the Hamiltonian equations:

$$\begin{aligned}
\dot{x}^\mu &= \frac{\partial \mathcal{H}}{\partial p_\mu} = \frac{u_1 + u_2}{m} p^\mu + u_2 \frac{4(kp)(\pi\pi)}{l^2 m^3} k^\mu, \\
\dot{k}^\mu &= \frac{\partial \mathcal{H}}{\partial \pi_\mu} = u_2 \frac{4(kp)^2}{l^2 m^3} \pi^\mu + u_3 k^\mu.
\end{aligned} \tag{4}$$

The Lagrangian corresponding to the Hamiltonian (3) can be obtained by applying the inverse Legendre transformation. The form of the resulting Lagrangian $L \equiv p\dot{x} + \pi\dot{k} - \mathcal{H}$, when expressed in terms of the velocities, is subject to the invertibility of the map (4) restricted to the submanifold defined by the Hamiltonian constraints. On this submanifold induced is a corresponding map between two sets of scalar variables $\{u_1, u_2, u_3, kp, p\pi\}$ and $\{\dot{k}\dot{k}, \dot{k}\dot{x}, \dot{x}\dot{x}, k\dot{x}, k\dot{k}\}$ which is easier to investigate:

$$\begin{aligned}
&\dot{x}\dot{x} = u_1^2 - u_2^2, \qquad k\dot{x} = (u_1 + u_2)\frac{kp}{m}, \qquad \dot{k}\dot{k} = -\frac{4(kp)^2}{l^2 m^2} u_2^2, \\
&\dot{k}\dot{x} = (u_1 + u_2)\Big[\frac{4(kp)(p\pi)}{m^3 l^2} u_2 + u_3\Big]\frac{kp}{m}, \qquad k\dot{k} = 0.
\end{aligned} \tag{5}$$

The number of new constraints for velocities depends on the rank of the Jacobi matrix of the above mapping. It can be shown that this rank depends only on the variables u_1, u_2, and equals 4 for $u_1^2 \neq u_2^2 \neq 0$, 3 for $u_1 = u_2 \neq 0$, and 2 for $u_1 = -u_2 \neq 0$.

[3] The Hamiltonian formulation of the whole class of relativistic rotators was presented in [9]. This formulation uses minimal phase space in terms of four-vectors. There is also a possible description of dynamical systems in extended phase spaces that upon reduction should recover the minimal Hamiltonians. In the case of the particular Hamiltonian (3) such an approach was presented in [10].

In passing from the Hamiltonian to the Lagrangian, one may first assume that $u_1 + u_2 \neq 0$ and $u_2 \neq 0$. Then the momenta expressed as functions of velocities and u_i's read

$$p^\mu = \frac{m}{u_1+u_2}\dot{x}^\mu - \frac{l^2 m (u_1+u_2)^2(\dot{k}\dot{k} - 2u_3 k\dot{k})}{4(k\dot{x})^2 u_2}\frac{k^\mu}{k\dot{x}},$$
$$\pi^\mu = \frac{l^2 m(u_1+u_2)^2}{4(k\dot{x})^2 u_2}(\dot{k}^\mu - u_3 k^\mu).$$

From the constraint equations $pp - m^2 = 0$ and $pp + \frac{4}{l^2m^2}(kp)^2(\pi\pi) = 0$ two conditions for u_1 and u_2 follow:

$$\frac{\dot{x}\dot{x}}{(u_1+u_2)^2} + \frac{u_1+u_2}{2u_2}\xi = 1 \qquad \text{and} \qquad \frac{(u_1+u_2)^2}{4u_2^2}\xi = 1. \tag{6}$$

The resulting u_1, u_2 can be expressed as independent functions of the velocities provided that the Jacobian determinant of the transformation (6) regarded as one leading from variables $(\dot{x}\dot{x}, \xi)$ to variables (u_1, u_2) – which, up to a constant factor, is equal to $\frac{\xi \dot{x}\dot{x}}{u_2^3(u_1+u_2)}$ – is non-zero. In this case the resulting Lagrangian overlaps with that in the action integral (2). However, assuming that the condition $\dot{x}\dot{x} \neq 0$ is not satisfied, two other Lagrangians are possible.

In the first case $u_1 = u_2$, and the corresponding new velocity constraints follow:

$$\frac{\dot{x}\dot{x}}{k\dot{x}} = 0, \qquad l^2\frac{\dot{k}\dot{k}}{k\dot{x}} + k\dot{x} = 0.$$

Then, from (5), $u_1 = \chi$, $u_2 = \chi$, $u_3 = \nu$, $kp = \frac{m}{2\chi}k\dot{x}$ and $p\pi = \frac{l^2m^2}{2k\dot{x}}\left[\frac{k\dot{x}}{k\dot{x}} - \nu\right]$ with χ and ν being arbitrary functions. After discarding a total derivative involving $k\dot{k}$ and the higher order terms in the velocity constraints, the resulting Lagrangian can be cast in the following form linear in these constraints

$$L = \frac{m\kappa}{2}\frac{\dot{x}\dot{x}}{k\dot{x}} + \frac{m}{4\kappa}\left[l^2\frac{\dot{k}\dot{k}}{k\dot{x}} + k\dot{x}\right] + \Lambda kk. \tag{7}$$

Here, $\kappa(\chi) \equiv \frac{kp}{m}$ is a new variable independent of velocities while Λ is a Lagrange multiplier.

In the second case, for $u_1 = -u_2$, a restricted Legendre transformation should be considered with p^μ left (for a while) unaltered. Using equations (4) and (5), one can find that $\boldsymbol{\pi} = \mp\frac{lm^2}{2}\frac{\dot{\boldsymbol{k}} - u_3\boldsymbol{k}}{kp\sqrt{-\dot{k}\dot{k}}}$ and $u_2 = \mp\frac{lm}{2kp}\sqrt{-\dot{k}\dot{k}}$. Now, integrating off the term linear in $k\dot{k}$ another Lagrangian is obtained in the form

$$L = p\dot{x} \pm \frac{lm^2}{2}\frac{\sqrt{-\dot{k}\dot{k}}}{kp} + \lambda kk. \tag{8}$$

Inferred from equations (4) and (5) the result $\dot{x}^\mu = \pm\frac{lm^2}{2}\frac{\sqrt{-\dot{k}\dot{k}}}{(kp)^2}k^\mu$ can be re-obtained by performing arbitrary variations of the Lagrangian with respect to p^μ, hence $e\dot{x} = \pm\frac{lm^2}{2}\frac{\sqrt{-\dot{k}\dot{k}}}{2(kp)^2}ek$ for any vector e^μ, and this fact can be used to eliminate p^μ from (8). Accordingly, the alternative form of the above Lagrangian can be taken to be

$$L = m\left[\frac{-4l^2\dot{k}\dot{k}}{(ek)^2(e\dot{x})^2}\right]^{1/4} e\dot{x} + \lambda kk,$$

which involves arbitrary (timelike) e^μ satisfying the condition $ek \neq 0$ and playing the role of the initial momentum p.

5 Conclusion

In this paper, the present status of Staruszkiewicz's relativistic rotators in free motion was discussed. The original motivation behind introducing the rotators was the idea of devising a model of an ideal clock that could be used to test the clock hypothesis [2]. However, the constraints imposed on the Casimir invariants for the purpose of realising the quantum irreducibility idea on the classical level, lead to singular Lagrangians when subluminal intrinsic motion is assumed from the start. Then the Hessian determinant for the particular rotator regarded as a system with 5 degrees of freedom is vanishing, in consequence of which the clocking rate remains an arbitrary function of the proper time of the momentum rest frame (while rotators with the usual less stringent constraint of reparametrization invariance remain well behaved).

However, at the level of constrained Hamiltonians one makes no a priori assumptions about the velocities. Possible constraints on velocities appear only when passing from the Hamiltonian to the Lagrangian. With this method[4] one recovers the original Lagrangian with subluminal motion when the rank of this inverse Legendre transformation is maximal. For a lower rank this transformation becomes singular and one obtains two new Lagrangians (7) and (8) with intrinsic motion with the speed of light (the motion of the momentum rest frame is still subluminal).

The dynamical systems described by the new Lagrangians exhibit behaviour that can be looked at as a counterpart of Zitterbewegung known for two states of Dirac's free electron (see the interesting and original discussion by Breit [11]). The existence of the two systems also conforms with the distinguished role of the constraint $\dot{x}\dot{x} = 0$ as discussed in the introduction. It remains to investigate how these systems would behave when coupled with the electromagnetic or gravitational field.

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