On Old Relations of Lie Theory, Classical Geometry and Gauge Theory

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Abstract

Having been led by hadron interactions and low-energy photoproduction to SU(4) and non-compact SU∗(4) symmetry, the general background turned out to be projective geometry (PG) of P^3, or when considering line and Complex geometry to include gauge theory, aspects of P^5. Point calculus and its dual completion by planes introduced quaternary (quadratic) 'invariants' $x_\mu x^\mu = 0$ and $p_\mu p^\mu = 0$, and put focus on the intermediary form (xu) and its treatment. Here, the major result is the identification of the symmetric $\underline{20}$ of SU(4) comprising nucleon and Delta states as related to the quaternary cubic forms discussed by Hilbert in his work on full invariant systems. So PG determines ***geometrically*** **the scene by representations (reps) and invariant theory without having to force affine restrictions and additional (spinorial or gauge) rep theory.**

1 Introduction

When analyzing low-energy hadronic experiments and their degrees of freedom in the context of effective chiral theories, it turned out that with respect to the pion-nucleon-delta system SU(4) linear states were able to describe the fermionic N- and Δ-states and their properties linearly. Explicitly, by starting over from current algebra and spectral descriptions (i.e. Goldberger-Treiman, PCAC,... [2]), Sudarshan [23] proposed to saturate the Adler-Weisberger sum rule [1], [25] of the (axial) charge commutators by quasi-particle calculations based on usual spin-isospin states. Thus, by requiring that the quasi-particle ansatz

$$N'_{\text{dyn}} = \lambda N_{\text{stat}} + \sqrt{1-\lambda^2} \int \pi(x) N(x) d^4x \tag{1}$$

describes the axial coupling g_A^2, we've showed that the 'dynamic' states N'_{dyn} fit perfectly to the members of the (linear) threefold symmetric rep $\underline{\mathbf{20}}$ of SU(4) [5], [6], [7], built symmetrically out of three fundamental reps $\underline{\mathbf{4}}$. The light non-strange mesons π, ω, and ρ fit into the linear SU(4) rep $\underline{\mathbf{15}}$, and due to SU(4) $\cong A_3$ it is evident from the Dynkin diagram that we can 'embed' (or identify) two commuting (chiral) SU(2) groups, $A_1 \otimes A_1$. But because SU(4) still yields well-defined 3-projections of spin and isospin, we cannot only treat the various actions of SU(2)×SU(2) or the 'spontaneously broken symmetry' SU(2)×SU(2)/SU(2) of the (nonlinear) chiral approaches – this rep theory can also handle partially conserved axial currents ('PCAC') in terms of the SU(2) pion field(s). Moreover, in contrary to somewhat tedious and 'higher order' calculations in effective chiral approaches, the threshold production amplitude of π^0 on the nucleon (the interactions of $\underline{\mathbf{20}}$ with $\underline{\mathbf{15}}$ when reduced to the observed 'spin-isospin states') yields strong suppression in first order by superselection rules. Last not least, pion scattering on the nucleon when treated by SU(4) reps yields small charge dependence ('isospin breaking'). With respect to usual quark descriptions, it is noteworthy that $\underline{\mathbf{20}}$ in the spinorial rep yields three symmetric constituents while $\underline{\mathbf{15}}$ comprises the fundamental rep $\underline{\mathbf{4}}$ and its conjugate $\underline{\mathbf{4}}^*$.

So the manifest question 'Why this?' lead us to a series of papers (see refs in [12]) to discuss SU(4) and the non-compact group SU∗(4), their common maximal compact subgroup USp(4) and certain aspects of the associated Riemannian spaces AII∼SU∗(4)/USp(4) and CI∼USp(4)/SU(2)×U(1). Here, however, it is time to step back from the transformation groups and related mathematical constructions, and to recall some old relations of Lie theory with physical and geometrical aspects.

An essential 'in-between' has been achieved by considering line geometry which allowed to associate reps of gauge bosons and reps of line Complexe [10], and to identify Lorentz transformations in Special Relativity as a special transformations of the Plücker-Klein quadric M_2^4 onto itself [12]. This emphasized the importance of treating line Complexe in P^5 and their 'reduction' via the Plücker-Klein quadric to line sets in P^3 so that in terms of (line) geometry of P^3 the symplectic transformations reflect mappings of Complexe onto each other, and the Lorentz (point) transformations of Special Relativity ensure 'invariance of line geometry' by restricting the Complexe to transformations of the Plücker-Klein quadric ([12], IV.C).

This concept on the one hand paved the path to identify the photon rep with a special line Complex, and it pointed to a possible geometrical/physical background of the 5-dim coset space SU∗(4)/USp(4), a rank-1 irreducible globally symmetric Riemannian space AII [8], and the occurrence of symplectic symmetries. On the other hand, it pointed to the necessary treatment of line Complexe, line sets, Congruences or ray systems, and associated reps from scratch. The 10-dim rank-2 CI-space can be represented once more by SU(2)×U(1) symmetric cosets which yield a simple background when restricting PG to affine geometry.

Needless to say, that besides the abstraction of 'a point x' and the associated evangelism of (Lagrangean) point motion, we have to consider its dual – the plane u – in P^3 as well, or – as a substitute of both – quadratic line geometry (lines being dual to lines in P^3) and using Hamilton's approach. So in all cases, reps of 'non-local' or 'extended' objects like lines or planes enter rep theory, although in PG of P^3 we can still found on their linearity. More generally, all such identifications require a priori a stricter treatment of the reps by (Lie) transformation theory and of their geometry, and a common treatment of lines versus points and planes (which compels a thorough discussion of conjugation, or duality, too!). In other words, as long as we treat linear reps and symmetries, we should apply projective geometry (PG) from scratch in order to *derive* and treat two aspects consistently: the breakdown to affine geometry by fixing an 'absolute plane' in order to connect geometrically to Weyl's concepts and gauge symmetries used throughout field and quantum theories, and the metric aspects from the viewpoint of Cayley and Klein with respect to a given (invariant) polar system ('absolute quadric') and the respective transformation groups.

2 The Results

Now, instead of taking the long approach to answer 'why' SU(4) obviously works with respect to hadron reps and symmetries, we argue 'top down' using both our construction scheme of SU(4) reps, and especially of $\underline{\mathbf{20}}$ (see details in [7], app. F.6), as well as Hilbert's (almost forgotten) foundations on invariant systems [15].

Whereas due to the rank-3 group A_3 the root system can be 'rotated' to P^3 and serve as a coordinate system by identifying $\underline{\mathbf{4}}$ with the fundamental tetrahedron, the construction of $\underline{\mathbf{20}}$ yields a 'threefold' tetrahedron subdivided by '$\underline{\mathbf{4}}$'s, see fig. 1, left. The individual states of $\underline{\mathbf{20}}$

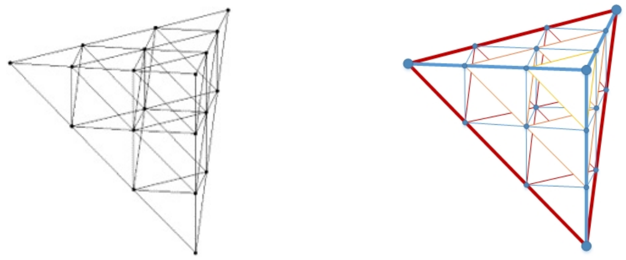

Figure 1: Symmetric $\underline{\mathbf{20}}$; left: construction by $\underline{\mathbf{4}}$ of SU(4) [7], F.6; right: subdivision by Hilbert [15], §19.

are given and discussed e.g. in [5], [6], and [7].

But using the symmetry group SU(4) requires a treatment of its invariant theory, especially of the full invariant system. As such, being concerned to construct the full invariant system with respect to (quaternary) linear transformations, we can use Hilbert's approach [15] and the related forms. The approach to rep theory via forms is suitable because the transformation determinant is 1, so all occurring determinant powers throughout invariant theory are 1, too, i.e. the respective forms are not altered by additional determinant factors.

Citing Hilbert's construction scheme ([15], §19), we find[1]: *'(...) For example, to construct the quaternary forms of the 3rd order, we construct a regular tetrahedron in 3-space with edge length 3, then divide each edge into three equal pieces and draw through the partial points two parallel planes to each of the four side faces; these planes cut the tetrahedron into regular tetrahedra with edge length 1. Each corner point (n_1, n_2, n_3, n_4) of these tetrahedra corresponds to a member of the quaternary cubic form. (...)'*.

So while we have constructed the rep $\underline{\mathbf{20}}$ (fig. 1, left) in a bottom-up approach by means of roots and the fundamental tetrahedron $\underline{\mathbf{4}}$ [7], Hilbert subdivides 'top-down' the 'large' tetrahedron (fig. 1, right) and identifies each of the 20 intersection/corner points (n_1,n_2,n_3,n_4) with a member of the quaternary cubic form, i.e. with one member of $\underline{\mathbf{20}}$, or what we denoted initially by a 'Chiron' [5], [6]. Based on our construction scheme, besides the bridge to well-established classical invariant theory, we thus have a symbolism at hand to treat the geometry of P^3 in terms of quaternary forms. From the physical point of view, when recalling the historic and ongoing quest for hadronic states and equations of motions (see e.g. [21] and references where one tries to separate spin content), we have identified the irrep $\underline{\mathbf{20}}$ yielding physical as well as geometrical background while additionally subordinating into the algebraic framework of invariant theory[2]. Whereas *synthetic* geometry proposes additional rich background and strategy (see e.g. [19] or [20] §2ff.), the *analytical* frameworks and tools beyond just linear algebra, affine geometry and gauge theory still have to be established consistently. Please note also with respect to physics and affine geometry, that by means of the structures above, it is straightforward to introduce quaternary barycentric coordinates. As such, masses and mass relations are linked to geometrical properties, especially with respect to the interior of the convex hull like in the case of $\underline{\mathbf{20}}$, and – appropriately normalized – the four 'coordinates' sum up to 1. Here, the major result is the *geometrical* identification of $\underline{\mathbf{20}}$ in SU(4) with reps of cubics, which is a dead giveaway with respect to PG and higher order representations [20].

[1] ibd. p. 366, translated from German...

[2] In other words, the symmetric threefold 'spinorial' structure of $\underline{\mathbf{20}}$ is based on nothing but the very origins of invariant theory of transformation groups without the need to introduce additional 'gauge glue'.

3 The Context

Carrying forward the results of the last section, we can test the symbolism with respect to P^3 if we identify the four members of $\underline{\mathbf{4}}$ with points, i.e. the rep $\underline{\mathbf{4}}$ with the quaternary point coordinates x_α. So with respect to quaternary forms when multiplying two (a priori different) point reps $\underline{\mathbf{4}} \sim \square$, we expect a bilinear (symmetric) form (or the polarized form of a quadric) with dim 10 for the symmetric part, and a line rep of dim 6 for the antisymmetric part. The symbolism yields $\square \otimes \square = \square\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix} = \underline{\mathbf{10}} \oplus \underline{\mathbf{6}}$, and from both approaches it is evident that the limit $x \to y$ 'destroys' the antisymmetric part, and the quadric 'survives'. Formally, we can introduce a bilinear form $B(x, y)$ so that *the set of points* $\{y|B(x, y) = 0\}$ defines 'the polar' of x, or an associated linear map $m_B : V \to V^*$ by $B(x, y) = \langle m_b x, y\rangle$, and by symmetry $\langle m_b x, y\rangle = \langle m_b y, x\rangle$. So even analytically, we have tools to treat linear mappings as well as quadrics in V and V^* (by the adjoint map m_b^* and the induced quadratic form). To grasp the physical notation, we can use the 'old geometrical' notion of points x and planes u, and their 'products' $(a_\alpha x_\alpha)^n \equiv (ax)^n$, $(bu)^n$, as well as $x \cdot u \equiv (xu)$, higher orders thereof and appropriate forms[3]. So from the symbolism above, $\underline{\mathbf{10}}$ relates to the (symmetric) quaternary quadric whereas $\underline{\mathbf{6}}$ relates to (antisymmetric) line reps (which by appropriate complexification of the Plücker coordinates or using Klein's linear Complexe [11] can serve as $\square$ of SO(6)).

If we look for the conjugate of $\underline{\mathbf{4}}$, SU(4) requires the conjugate rep $\square^*$ to transform according to the threefold antisymmetric rep. Written in terms of determinants, it is easy to see that $\square^*$ has to represent quaternary plane coordinates of 3-space. SU(4) yields $\square^* \otimes \square = \underline{\mathbf{15}} \oplus [0]$ where the [0] represents a vanishing 4×4-determinant (or a point-plane incidence $(ux) = 0$). On the same footing, using $\square^*$ as rep for three linear independent points ('a plane'), non-vanishing $\underline{\mathbf{15}}$ associates a 4^{th} point a_α to the plane $u_\alpha = \epsilon_{\alpha\beta\gamma\delta} x_\beta y_\gamma z_\delta$, and $a_\alpha u_\alpha = \epsilon_{\alpha\beta\gamma\delta} a_\alpha x_\beta y_\gamma z_\delta$ represents a determinant, or geometrically a tetrahedron ('volume'). This requires a thorough discussion of u_x (or $u_\alpha x_\beta$, or $a_x b_u$) in quaternary invariant theory[4].

Note, however, that this symbolism works by means of the initial analytic rep of linearly transforming point and plane coordinates in $\mathbb{R}^3$, or P^3, and their respective analytic reps by forms, not as a feature of space geometry itself.

4 The Background

Now please recall, that given an irreducible polynomial $f \in K[x_\alpha]$ and a related hypersurface V, in order to define a tangential plane (and the tangential space) of V at a regular point p, we can invoke the hyperplane definition $\left\{v \in K^n | \sum v_\alpha \frac{\partial f}{\partial x_\alpha}(p) = 0\right\}$ describing a plane normal to $\nabla f(p)$. The same mechanism in (finite) geometry can be achieved by considering null systems [12]. So we have to treat two 'competing' descriptions, 'moving' the tangential plane with respect to the quadric and the ('orthogonal') null plane with respect to motions along the normal and their so(4) Lie algebra [12]. Whereas for the tangential discussion and two point x, y of the quadric, we can use the incidence relations $(ux) = (u'y) = 0$, a (null) plane $u_\alpha = A_{\alpha\beta} x_\beta$ in general will be mapped to the (null) plane of a different linear Complex, $u'_\alpha = B_{\alpha\beta} y_\beta$. So relating A and B requires symplectic transformation groups.

[3] While we have discussed $(bu)^2$ in relation to Dirac's linearization of $p_\mu p^\mu$ (see e.g. [11]), $(bu)^n$ in general relates to moments of order n and the tetrahedral Complex; due to the 8-page limit here, we postpone this discussion.

[4] To pursue the combinatorial aspects in contemporary considerations, one can follow Rota (see e.g. [13], [17], [14], however, according to his foreword in the reprint of Study's marvellous work [22] it is worth considering the classical path, too, as well as Study's geometrical concepts [22].

Formally, if in the special incident/tangential case we represent the plane by $u = \frac{1}{2}(u^p + u^N)$ where u^p denotes the polar/tangential part and u^N the null plane, we can express the 'parts' of the plane according to $u_\mu \rightarrow \partial'_\mu \sim \frac{1}{2}(\partial_\mu + A_{\mu\nu}x_\nu)$, A describes the 6-dim antisymmetric rep of the null system, or the rep of a (general) linear Complex. Now expressing A via its dual/conjugate A^c, i.e. $A_{\mu\nu} \sim \epsilon_{\mu\nu\alpha\beta}A^c_{\alpha\beta}$, we find the plane rep $u^N_\mu \sim \epsilon_{\mu\nu\alpha\beta}A^c_{\alpha\beta}x_\nu$. Here, the ϵ-'tensor' formally ensures the threefold antisymmetry of the coordinate expression, and moreover, it ensures the point-plane incidence $(xu) = 0$. For electromagnetism and the electromagnetic field, an (affine) replacement $A_{\mu\nu} \rightarrow F_{\mu\nu}$ has been discussed in [24] by 3-vectors $\vec{E}$, $\vec{B}$, $\vec{M}$ and $\vec{H}$ to derive Maxwell's equations. So the difference in the tangent planes can be seen as a necessary rotation (or readjustment) of the null lines (i.e. of moments), and symbolically as $\partial'_\mu = \partial_\mu + \tilde{u}^N_\mu$, i.e. by correcting the plane appropriately. The Jacobian J benefits from the polar decomposition, i.e. for $x'_\alpha = f_\alpha(x_\beta) = A_{\alpha\beta}x_\beta$ and $S^2 = x_\alpha f_\alpha(x_\beta)$, we find $\frac{\partial f_\alpha}{\partial x_\beta} = \frac{\partial x'_\alpha}{\partial x_\beta} = A_{\alpha\beta} \cong J_{\alpha\beta}$.

So using a sphere to represent the quadric above (and to connect to what Weyl[5] and Wigner understood as features of quantum theories), we want to emphasize the underlying line geometric picture. By considering the sphere as a hull with center common to the center of a ray or line bundle (see fig. 2, right), this introduces immediately two well-known algebraic reps. In the first approach, we can define operators *on the sphere* S^2 to shift the point p quarterwise along the great circles while inherently respecting the quadric constraint of the sphere. It is easy to see that these quarterwise transformation operators fulfill the quaternion algebra (see fig. 2, left), where $-kP = N = jiP \longleftrightarrow ij = k, ijk = -\mathbb{1}$. The negative squares map the points to their 'antipodes' on the sphere, i.e. using this quadratic algebra, we have an operator system at hand which respects the (invariant) geometry by means of transformations of points. In general, we can use the quadratic algebras (or especially Clifford algebras or hypercomplex number systems) to represent the three possible signatures of the various real cases of quadrics when the base elements square to $q_N^2 = \pm 1, 0$, i.e. also in the hyperbolic and parabolic cases. For rays or oriented lines this approach yields a 2π-periodicity, i.e. $q^4 = \mathbb{1}$ or reps in terms

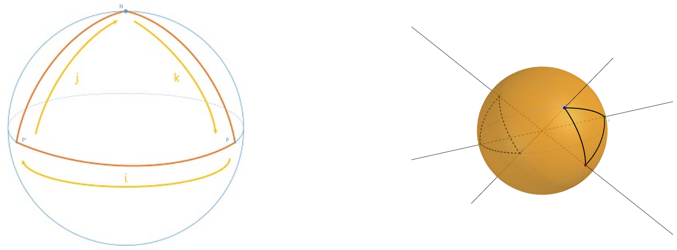

Figure 2: Left: Quaternionic action induced by rays and lines; right: line vs. ray intersections with quadric.

of sin() or cos(), whereas lines yield π-periodicity and tan(). So already this simple classical picture analytically introduces 'quantum notion', and if e.g. instead of three rays we use three lines the spherical triangle has a 'mirror image' on the opposite side of the sphere[6], so instead of a single quaternionic rep (or SU(2)), we can discuss a twofold quaternionic rep transformed by SL(1,$\mathbb{H}$)×SL(1,$\mathbb{H}$) (or SU(2)×SU(2)), SL(1,$\mathbb{H}$)×$\overline{\text{SL}(1,\mathbb{H})}$, or coverings like SL(2,$\mathbb{H}$) or S(SL(1,$\mathbb{H}$)×SL(1,$\mathbb{H}$)).

The second associated algebraic rep, Study's kinematical mapping[7], uses a similar reasoning to treat SU(2)×SU(2), where in addition special emphasis is given to projections onto the conic in the equatorial plane. Like in the first picture by using the equatorial great circle, we

[5]Recall e.g. [26], III § 16: the system space of quantum mechanics is a ray space, no vector space.

[6]In PG, we can also perform the shift of the center from 0 to ∞ easily (or an appropriate change of points in the anharmonic ratio), and discuss the associated orientation(s) of the second 'projected triangle'.

[7]See e.g. [16], Abb. 87, see also Study's transfer principle and dual numbers [4], §103.

can thus switch to an alternative, rational parametrization of the (planar) conic, $\Phi : \mathbb{R} \to \mathbb{R}^2$, by means of $t \to \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$ which recovers the 'spinor' introduction (see e.g. [9], III.C and III.F). By recalling the projective generation of a conic by two line pencils, we can introduce the respective pencil coordinates so that the theory of binary forms applies and Clebsch-Gordan decompositions enter naturally. So these examples reveal an obvious mismatch on the interpretations of algebraic reps vs. physical notion, and more important and induced by the focus on point calculus, between the number of different physical processes and the amount of allegedly independent algebraical descriptions. Here, based on the additional geometrical hint by Hilbert and invariant theory above, in the next section we follow Plücker, Klein and Lie and introduce another linear geometrical object, 'the plane', to keep in touch with PG and classical (quaternary) invariant theory of P^3. Evidently, this well-defined geometrical notion is able to treat certain tensorial notions common in contemporary ('quantum') rep theory.

5 Some Consequences

Thus, relying on point reps x_α, the geometrical rep theory of P^3 has to be completed by dual plane reps u_α, also in order to complete invariant theory[8]. We have discussed above few combinatorial aspects of a symbolism, so before entering algebraical details on invariant systems comprising x^2, u^2, $x \cdot u$, etc., it is worth to scrutinize plane reps $u \sim \square^*$ and their use.

Now, while from classical geometry we know various forms of plane reps in $\mathbb{R}^3$, in the usual (metric) interpretation when given e.g. in the Hesse form $n_i x_i - d = 0$ (and which relates to our tangential definition above), $\vec{n}$ describes the normal vector to the plane and d the distance of the plane with respect to the origin. If we *formally* introduce homogeneous point coordinates, $\vec{x}_i \to \frac{x_i}{x_0}$, and rewrite the form by $n_i x_i - d x_0 = 0 = u_\alpha x_\alpha = (ux)$, we thus have metric interpretations of the formal 'plane' coordinates u_α. Exhausting (or even overexciting) this formalism, we can think of the plane as a tangential plane to a sphere at distance d, and if we assume propagating spherical waves with velocity c and time t, then $d = ct$ corresponds formally to the 'energy component' u_0[9].

Here, as a further aspect with respect to the use of exponentials, their partial differentiation and 'plane waves' in physics, we want to use 'intermediary forms'[10] (ux). From above it is obvious, that this 'distance' measurement of points with respect to planes can be used to define the point coordinates, however, one has to work out the dependence from the usual metric terminology e.g. in the definition of the (Euclidean) coordinates or the distances. To approach this problem, we can go back to the Cayley-Klein approach, and define the distance d of two points q_1 and q_2 by $\mathrm{dist}(q_1, q_2) \sim -2i \log DV(q_1, q_2, R, S)$ with intersections R, S on the absolute quadric; DV denotes the anharmonic ratio. If we rewrite the distance d of a point x to a plane p by $d = \mathrm{dist}(q_1, q_2)$, then $id = ix \cdot p \sim \log DV(q_1, q_2, R, S)$. Exponentiation results in $\exp(id) = \exp(i\, x \cdot p) \sim DV(q_1, q_2, R, S)$ which yields some insight into the 'plane wave'-approaches, 'second quantization', 'exponential forms', etc. in the affine setup $x_0 = 1$. Whereas the rhs, $DV(q_1, q_2, R, S)$, is a priori well-defined in PG of P^3 for general projective generalization and accessible by von Staudt's 'Würfe' or derived concepts like binary forms or Hesse transfer, a naive 'relativistic' expansion $x_i \to x_\alpha$, $p_i \to p_\alpha$ has to be treated carefully. It is obvious that these well-defined, projective properties 'scatter' to individual analytical expansion of exponentials on the lhs, or parts of such series, a fortiori with respect to non-commuting

[8] Here, we exclude line and Complex reps and their use in force systems, kinematics and gauge theory.

[9] So based on this identification, care has to be taken when discussing linear roots of quadrics in order to not produce 'anti-particles' with opposite sign in the component u_0.

[10] german: 'Zwischenformen'; invariants using both variables and their duals when set to zero represent projective relations.

parameters, higher orders or necessary applications of Baker-Campbell-Hausdorff formulæ. As such, it is easier to start from an intermediary form (xp) and apply quaternary invariant theory from scratch; its exponentiation forces the use of expansions, power series, ordering and grouping schemes, and their respective comparison(s), i.e. it discharges into series of $(ux)^n$. The lhs thus complies with the operator and expansion methodology of usual invariant theory.

Now, whereas in the Cayley-Klein approach we can rely on the logarithm to produce metric and *additive* quantities from the DV and from PG, the rhs here 'lives' in pure and strict PG. So we can use the (partial) differential operators to restore the linearity and the 'vector' properties of the originally linear vectors x and p if we use 'new forms' $\exp(i(xp))$. Then, the action of ∂_μ reproduces 'contravariant' linear elements, $\partial_\mu \exp(i(xp)) \sim ip_\mu \exp(i(xp))$, so that effectively $p_\mu \sim -i\partial_\mu$ provides a 'quantization'. To a certain extent, differential operations in this rep theory thus replace linear operators or vector space behaviour from general PG. To get rid of the 'new forms', however, people have had to introduce additional rules and frameworks, e.g. the necessary $\mathbb{1}$-operator in terms of 'delta functions', 'integration' over homogeneous point variables x_μ, etc. (see e.g. [18], [3], ...). Because such reasoning leads us back to enrich the intermediary form and its exponential by additional invariants and possible exponentials, we are faced with the original problem (see above or [15]) of determining the full invariant system either analytically in terms of points *and* planes, or with respect to (quadratic) line geometry and its relations to Complex geometry in P^5.

6 Conclusion

By identifying the physical rep $\underline{\mathbf{20}}$ geometrically as cubic by means of Hilbert's construction [15], we have strengthened the foundations of our SU(4) Ansatz within PG. Although it is hard to recover physically relevant concepts from today's jungle of physical and algebraical phenomenology and empiricism, starting over from quaternary invariant theory provides a reliable basis and stable guidelines. By means of point and plane reps of P^3, we can treat invariants (especially covariants) where already the linear and quadratic orders have enormous physical relevance. Using PG and invariant theory to start over again seems to establish correct descriptions and an ordering scheme, the more as P^5 provides subtle and profound Complex background as well as important transformation theory and relevant mappings to P^3.

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References

[1] S. Adler, *Calculation of the Axial–Vector Coupling Constant Renormalization in β Decay*, Phys. Rev. Lett. **14**, 1051 (1965), https://doi.org/10.1103/PhysRevLett.14.1051

[2] V. de Alfaro, S. Fubini, G. Furlan, C. Rossetti, *Currents in Hadron Physics*, (Amsterdam, London: North Holland Publishing Company), 1973

[3] J. D. Bjorken, S. D. Drell, *Relativistic Quantum Mechanics*, (New York: McGraw-Hill) 1964; *Relativistic Quantum Fields*, (New York: McGraw-Hill) 1965

[4] W. Blaschke, *Vorlesungen über Differentialgeometrie I*, (Die Grundlehren der mathematischen Wissenschaft I) (Berlin: Springer) 1921

[5] R. Dahm, M. Kirchbach, *Linear Wave Equations and Effective Lagrangeans for Wigner Supermultiplets*, Int. J. Mod. Phys. **A10**, 4225 (1994), DOI:10.1142/S0217751X95001960

[6] R. Dahm, *Relativistic SU(4) and Quaternions*, Advances in Applied Clifford Algebra **7(S)** 337 (1995)

[7] R. Dahm, *Spin-Flavour-Symmetrien und das $\pi N\Delta$-System*, (Aachen: Shaker Verlag), 1996, ISBN 3-8265-1782-2

[8] R. Dahm, *A Symmetry Reduction Scheme of the Dirac Algebra without Dimensional Defects*, Phys. Atom. Nuclei **73(2)**, 276 (2010), DOI:10.1134/S1063778810020122

[9] R. Dahm, *On a Microscopic Description of Space-Time VII – On Spin*, J. Phys. Conf. Ser. **965(1)**, 012012 (2018), DOI:10.1088/1742-6596/965/1/012012

[10] R. Dahm, *On a Microscopic Description of Space-Time VIII – On Relativity*, Phys. Atom. Nuclei **81(6)**, 819 (2018), DOI:10.1134/S1063778818060108

[11] R. Dahm, *On A Microscopic Representation of Space-Time III*, Adv. Appl. Clifford Algebras **29(1)** 20 (2019); DOI:10.1007/s00006-018-0936-x; http://arxiv.org/abs/1508.06872

[12] R. Dahm, *On Some Geometrical Aspects of Space-Time Description and Relativity*, https://doi.org/10.48550/arXiv.2002.08889, accepted for publication in: *The Future of Relativity, Gravitation and Cosmology*, eds. V. V. Dvoeglazov et al., (Hauppauge, USA: Nova Science Publishers), ISBN: 979-8-88697-455-3, DOI: 10.52305/RLIT5885

[13] P. Doubilet, G.-C. Rota, *Skew-Symmetric Invariant Theory*, Adv. Math. **21**, 196 (1976), https://doi.org/10.1016/B978-0-12-428780-8.50015-5

[14] R. Ehrenborg, G.-C. Rota, *Apolarity and Canonical Forms for Homogeneous Polynomials*, Europ. J. Combinatorics **14**, 157 (1993), https://doi.org/10.1006/eujc.1993.1022

[15] D. Hilbert, *Ueber die vollen Invariantensysteme*, Math. Ann. **42**, 313 (1893)

[16] F. Klein, *Vorlesungen über höhere Geometrie*, (Die Grundlehren der mathematischen Wissenschaft XXII) (Berlin: Springer) 1926

[17] J. P. S. Kung, G.-C. Rota, *The Invariant Theory of Binary Forms*, Bull. Amer. Math. Soc. **10**, 27 (1984), https://doi.org/10.1090/S0273-0979-1984-15188-7

[18] D. Lurie, *Particles and Fields*, (New York-London-Sydney: Interscience Publishers) 1968

[19] H. Müller, *Zur Geometrie auf den Flächen zweiter Ordnung*, Math. Ann. **1**, 497 (1869)

[20] Th. Reye, *Collineare Grundgebilde und ihre Erzeugnisse*, Journal für die reine und angewandte Mathematik **74**, 1 (1872)

[21] V. M. Simulik, I. I. Vyikon, *On the choice of relativistic wave equation for the particle having spin s=3/2*, J. Phys. Commun. **6**, 075008 (2022), https://doi.org/10.1088/2399-6528/ac7eae

[22] E. Study, *Methoden zur Theorie der ternären Formen*, (Berlin: Springer), Reprint 1982

[23] E. C. G. Sudarshan, *Theory of Approximate Symmetries*. In: A. Salam (Dir.), *High Energy Physics and Elementary Particles*, (Vienna: IAEA), 1965

[24] M. von Laue, *Zur Minkowskischen Elektrodynamik der bewegten Körper*, Z. Physik **128**, 387 (1950), doi:10.1007/BF01339439

[25] W. I. Weisberger, *Renormalization of the Weak Axial–Vector Coupling Constant*, Phys. Rev. Lett. **14**, 1047 (1965), https://doi.org/10.1103/PhysRevLett.14.1047

[26] H. Weyl, *Gruppentheorie und Quantenmechanik*, (2^{nd} ed., Leipzig: Hirzel), 1931