Towards higher super-σ-model categories

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Abstract

A simplicial framework for the gerbe-theoretic modelling of supercharged-loop dynamics in the presence of arbitrary worldsheet defects is discussed whose equivariantisation with respect to global supersymmetries of the bulk theory and subsequent orbit decomposition lead to a natural stratification of and cohomological superselection rules for target-space supergeometry. Physically relevant examples are provided.

1 Introduction

The hierarchical higher-geometric structures associated with background p-form gauge fields in simple (σ-)models of geometrodynamics of extended distributions of (super)charge, coupling to the worldvolume (super)charge current through distinguished Cheeger–Simons differential characters (as derived in all generality for $p = 3$ in [1]), have long been known not only to give rise to a canonical prequantisation of the models, as in [2, 3], and to give a natural cohomological classification of the models themselves, their boundary conditions and generic defects (as well as quantifying obstructions against their existence), but also – to lead to a categorification of their prequantisable group-theoretic symmetries, including the gauged ones, and more general dualities implemented by certain defects, *cp.* [1, 3–5]. This note gives a concise account of a proposal, advanced in [6] on the basis of the earlier studies [1, 3–5], for an effective structurisation of the stratified target spaces and the higher-geometric objects over them defining the $2d$ σ-model in the presence of defects compatible with configurational symmetries of the bulk field theory. The proposal is formulated in the most general (target-space) $\mathbb{Z}/2\mathbb{Z}$-graded setting of the Green–Schwarz-type super-σ-model and uses mixed group-theoretic, simplicial and cohomological tools. It is seen to pave the way to a systematic construction of maximally supersymmetric defects in the flat $\mathbb{Z}/2\mathbb{Z}$-graded target geometry, and to lead to interesting novel predictions for a higher-geometric target-space realisation of non-perturbative data of the bulk theory in the highly (super)symmetric setting of the Wess–Zumino–Witten(-type) models with Lie-supergroup targets.

2 A bicategory for the super-σ-model with defects

The $2d$ super-σ-model is a superfield theory with a spacetime given by a closed oriented (Graßmann-)even manifold $\Sigma \cong_d \Sigma_2 \sqcup \Sigma_1 \sqcup \Sigma_0$ partitioned into a disjoint union Σ_2 of open

$2d$ domains by an embedded defect quiver $\Gamma = \Sigma_1 \sqcup \Sigma_0$ which consists of oriented open defect lines composing a codimension-1 submanifold Σ_1 and intersecting at $0d$ defect junctions from the set $\Sigma_0 = \bigsqcup_{v\geq 3} \Sigma_0^{(v)}$ graded by the valence v of the junctions. The target space of the theory is a stratified supermanifold $M = M_0 \sqcup M_1 \sqcup \bigsqcup_{n\geq 2} M_n$ (with the M_k potentially further stratified) endowed with an even symmetric supertensor $\mathrm{g} \in \Gamma(\mathcal{T}^* M_0 \otimes \mathcal{T}^* M_0)_0$ and even 2π-integral forms: a de Rham 3-cocycle $\mathrm{H} \in Z^3_{\mathrm{dR}}(M_0)_0$ and a 2-form $\omega \in \Omega^2(M_1)_0$, subject to cohomological constraints indicated below. Superfields ξ of the theory, coming from the mapping supermanifold $[\Sigma, M] = \underline{\mathrm{Hom}}_{\mathbf{sMan}}(\Sigma, M)$, to be evaluated on the family $\{\mathbb{R}^{0|N}\}_{N\in\mathbb{N}}$ of odd hyperplanes, restrict as $\xi\restriction_{\Sigma_2} \in [\Sigma_2, M_0]$, $\xi\restriction_{\Sigma_1} \in [\Sigma_1, M_1]$ and $\xi\restriction_{\Sigma_0^{(v)}} \in [\Sigma_0^{(v)}, M_{v-1}]$, and are related at Γ by a collection of maps $\iota_A : M_1 \longrightarrow M_0$, $A \in \{1,2\}$ and $\pi^{(v)}_{k,k+1}, \pi^{(v)}_{v,1} : M_{v-1} \longrightarrow M_1$, $k \in \overline{1, v-1}$ in a natural manner: Given a decomposition $U = U_1 \cup U_\Gamma \cup U_2 \cong \mathbb{R}^{\times 2}$ of a (sufficiently small) neighbourhood U of $p \in \Sigma_1$ in Σ into the $1d$ defect component $U_\Gamma = U \cap \Sigma_1$ and a pair of disjoint path-connected $2d$ subdomains $U_1 \cup U_2 = U \cap \Sigma_2$ separated by the latter and mapped diffeomorphically to the upper (U_1) and lower (U_2) half-planes, respectively, in the model $\mathbb{R}^{\times 2}$ of U in which U_Γ is embedded, with its orientation preserved, as $\mathbb{R} \times \{0\}$, the respective extensions ξ_A to $[U_A \cup U_\Gamma, M_0]$ of the restrictions $\xi\restriction_{U_A}$ obey $\xi_A\restriction_{U_\Gamma} = \iota_A \circ \xi\restriction_{U_\Gamma}$. Similarly, given a neighbourhood V of $\upsilon \in \Sigma_0^{(v)}$ with v connected components $I_{k,k+1}$, $k \in \overline{1, v-1}$ and $I_{v,1}$ of $V \cap \Sigma_1$ intersecting at υ, the respective extensions $\xi_{k,k+1}$ to $[I_{k,k+1} \cup \{\upsilon\}, M_1]$ of the restrictions $\xi\restriction_{I_{k,k+1}}$ and the extension $\xi_{v,1}$ to $[I_{v,1} \cup \{\upsilon\}, M_1]$ of the restriction $\xi\restriction_{I_{v,1}}$ satisfy $\xi_{k,k+1}\restriction_{\{\upsilon\}} = \pi^{(v)}_{k,k+1} \circ \xi\restriction_{\{\upsilon\}}$ and $\xi_{v,1}\restriction_{\{\upsilon\}} = \pi^{(v)}_{v,1} \circ \xi\restriction_{\{\upsilon\}}$. Once the extensions of the v domain configurations composing $\xi\restriction_{V\cap\Sigma_2}$ are taken into account, consistency of the limiting relations is ensured by constraints imposed on the target structure maps. These depend on the orientation of the intersecting defect lines. Specifically, in the distinguished case of $v-1$ incoming (at υ) lines $I_{k,k+1}$ and a single outgoing one $I_{v,1} \equiv I_{1,v}$, which we restrict to in what follows for the sake of simplicity (denoting $\pi^{(v)}_{1,v} \equiv \pi^{(v)}_{v,1}$ in this case, in keeping with the standard conventions), the constraints read

$$\iota_2 \circ \pi^{(v)}_{k-1,k} = \iota_1 \circ \pi^{(v)}_{k,k+1}, \; k \in \overline{2, v-1}, \qquad \iota_1 \circ \pi^{(v)}_{1,v} = \iota_1 \circ \pi^{(v)}_{1,2}, \qquad \iota_2 \circ \pi^{(v)}_{1,v} = \iota_2 \circ \pi^{(v)}_{v-1,v}. \quad (1)$$

The structure maps give rise to a family of pullback operators: $\Delta_1 = \iota_2^* - \iota_1^*$ and $\Delta_{v-1} = \sum_{k=1}^{v-1} \pi^{(v)*}_{k,k+1} - \pi^{(v)*}_{v,1}$ which enter relative-cohomological constraints:

$$\Delta_1 \mathrm{H} = -\mathrm{d}\omega, \qquad \Delta_{v-1}\omega = 0. \quad (2)$$

The latter are to be understood as consistency conditions for the geometrisation of the data in the form of the superstring background $\mathfrak{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J})$ composed of

- the target $\mathcal{M} = (M_0, \mathrm{g}, \mathcal{G})$ in which (M_0, g) is a quasi-metric[1] supermanifold with a gerbe $\mathcal{G}$ of curvature H over it;
- the $\mathcal{G}$-bi-brane $\mathcal{B} = (M_1, \iota_\cdot, \omega, \Phi)$ with the $(\iota_1^*\mathcal{G}, \iota_2^*\mathcal{G})$-bimodule $\Phi : \iota_1^*\mathcal{G} \xrightarrow{\cong} \iota_2^*\mathcal{G} \otimes \mathcal{I}_\omega$, written in terms of the trivial gerbe $\mathcal{I}_\omega$;
- the $(\mathcal{G}, \mathcal{B})$-inter-bi-brane $\mathcal{J} = \bigsqcup_{n\geq 2} (M_n, \pi^{(n+1)}_{\cdot,\cdot}, \varphi_{n+1})$ with the fusion 2-isomorphisms

  $$\varphi_{n+1} : \left(\Phi^\vee_{n+1,1} \otimes \mathrm{id}_{\mathcal{I}_{\Delta_{T_{n,1}}\omega}}\right) \circ \cdots \circ \left(\Phi_{3,4} \otimes \mathrm{id}_{\mathcal{I}_{\omega_{1,2}+\omega_{2,3}}}\right) \circ \left(\Phi_{2,3} \otimes \mathrm{id}_{\mathcal{I}_{\omega_{1,2}}}\right) \circ \Phi_{1,2} \overset{\cong}{\Longrightarrow} \mathrm{id}_{(\iota_1 \circ \pi^{(n+1)}_{1,2})^*\mathcal{G}},$$

  written in terms of the dual $\Phi^\vee$ of the 1-isomorphism Φ, and of pullbacks $\mathcal{O}_{\cdot,\cdot} \equiv \pi^{(n+1)*}_{\cdot,\cdot}\mathcal{O}$ of differential objects $\mathcal{O}$ from M_1.

[1]The tensor g is typically degenerate in the odd coordinate directions.

The superfield theory is determined by the principle of least action for the Dirac–Feynman amplitude $\mathcal{A}_{\rm DF}[(\xi|\Gamma)] = \exp(i S_\sigma[(\xi|\Gamma)])$ in which the action 'functional' $S_\sigma[(\xi|\Gamma)] = S_{\rm metr}[(\xi)] + S_{\rm WZ}[(\xi|\Gamma)]$ splits into the 'metric' term $S_{\rm metr}[(\xi)] = \int_\Sigma \sqrt{|\det(\xi^* {\rm g})|}$ (written in the Nambu–Goto form here) and the 'topological' Wess–Zumino term $S_{\rm WZ}[(\xi|\Gamma)] = -i \log {\rm Hol}_{(\mathcal{G},\Phi,(\varphi.))}(\xi|\Gamma)$ given by the decorated-surface holonomy ${\rm Hol}_{(\mathcal{G},\Phi,(\varphi.))}(\xi|\Gamma)$ of [1].

The higher supergeometric objects $\mathcal{G}, \Phi$ and φ_{n+1} of $\mathfrak{B}$ are distinguished 0-, 1- and 2-cells, respectively, of the monoidal bicategory of (abelian) gerbes with connective structure over M, *cp.* [7–9]. The fundamental property of and the main rationale for physical interest in these objects, discussed at length in, *i.a.*, [2,3,10], is that they canonically determine – *via* cohomological transgression – a prequantisation of the above superfield theory and encode a lot of non-trivial information on its (nonperturbative) structure, *cp.* [11] for a recent review.

3 The Trinity: Simpliciality, Symmetry and Semisimplicity

Studies of topological defects implementing symmetries of the $2d$ σ-model in the setting without $\mathbb{Z}/2\mathbb{Z}$-gradation, and in particular evidence of the attendant induction of higher-geometric fusion data for defect junctions of valence $v > 3$ from the elementary 2-isomorphism φ_3, reported in [1,3,4], as well as elementary considerations of the identity defect indicate the possibility of further structurisation of the background through adjunction of intermediate maps between M_{n+1} and M_n. This leads to the definition of a simplicial superstring background whose supertarget M forms (a submanifold in) a simplicial (stratified) supermanifold $(M_\bullet, d_\cdot^{(\bullet)}, s_\cdot^{(\bullet)})$ with face maps $d_i^{(n+1)} : M_{n+1} \longrightarrow M_n$ and degeneracy maps $s_i^{(n)} : M_n \longrightarrow M_{n+1}$ defined for $i \in \overline{0, n+1}$ and for all $n \in \mathbb{N}$, and subject to the standard simplicial identities. The former maps reproduce the previously introduced structure maps *uniquely* as $(\iota_1, \iota_2) = (d_1^{(1)}, d_0^{(1)})$ and $(\pi_{1,2}^{(3)}, \pi_{2,3}^{(3)}, \pi_{1,3}^{(3)}) = (d_2^{(2)}, d_0^{(2)}, d_1^{(2)})$, and – for $v > 3$ – also $\pi_{1,v}^{(v)} = d_1^{(2)} \circ d_1^{(3)} \circ \cdots \circ d_1^{(v-1)}$ and $\pi_{k,k+1}^{(v)} = d_2^{(2)} \circ d_2^{(3)} \circ \cdots \circ d_2^{(v-k)} \circ d_0^{(v-k+1)} \circ d_0^{(v-k+2)} \circ d_0^{(v-1)}$ for $k \in \overline{1, v-1}$, consistently with identities (1). The latter ones neatly account for the existence of the flat identity (sub-)bi-brane $s_0^{(0)*}\Phi \equiv {\rm id}_\mathcal{G} : s_0^{(0)*} d_1^{(1)*}\mathcal{G} = \mathcal{G} \longrightarrow \mathcal{G} = s_0^{(0)*} d_0^{(1)*}\mathcal{G} \otimes \mathcal{I}_{s_0^{(0)*}\omega}$ and allow to write down fusion 2-isomorphisms for defect junctions with (at least) one identity defect line attached.

Amongst such backgrounds, we distinguish those, termed descent-complete in [6], for which every component inter-bi-brane worldvolume $M_{v-1\geq 3}$ contains, as a submanifold, the common intersection of the preimages of M_2 under all the maps $M_{v-1} \longrightarrow M_2$ of the simplicial descent pattern of every full binary tree which can be associated with the defect junction (of valence v) through the recursive binary resolution worked out in [6, Sec. 1.4] – the condition ensures that the elementary fusion 2-isomorphism φ_3 can be pulled back to M_{v-1} for every $v - 1 \geq 3$, whereupon it induces the corresponding composite fusion 2-isomorphism φ_v.

Simplicial (super)manifolds come with their simplicial deRham cohomology, *i.e.*, the total cohomology of the bicomplex $\Omega^{\bullet_1}(M_{\bullet_2}) \equiv \{\Omega^p(M_n)\}_{n\in\mathbb{N}}^{p\in\mathbb{N}}$ with coboundaries ${\rm d} \equiv {\rm d}_{\rm dR}$ and $\Delta_n^{(p)} \equiv \sum_{i=0}^{n+1} (-1)^i \, d_i^{(n+1)*}$, and the quadruple $({\rm H}, \omega, 0, 0)$ acquires the interpretation of a simplicial de Rham 3-cocycle, with, now, the second of the two constraints (2) for $v > 3$ *implied* by the same identity for $v = 3$. Hence, we may think of the higher-geometric component of a simplicial superstring background as a simplicial gerbe over the supertarget $M_\bullet$.

Simpliciality can be combined with symmetry to constrain rather effectively the geometry of the supertarget and the simplicial gerbe over it. Indeed, consider a Lie supergroup ${\rm G}_\sigma = (|{\rm G}_\sigma|, \mathfrak{g}_\sigma \equiv \mathfrak{g}_\sigma^{(0)} \oplus \mathfrak{g}_\sigma^{(1)})$ coming with an action $_{M_0}\lambda : {\rm G}_\sigma \times M_0 \longrightarrow M_0$ on the bulk target supermanifold $M_0 = (|M_0|, \mathcal{O}_{M_0})$, and so also with the induced action $|_{M_0}\lambda|_g \equiv {}_{M_0}\lambda \circ (\widehat{g} \times {\rm id}_{M_0}) : M_0 \circlearrowleft$, $g \in |{\rm G}_\sigma|$ of the body Lie group $|{\rm G}_\sigma|$, the latter being comprised of the topological points $\widehat{g} : \mathbb{R}^{0|0} \longrightarrow {\rm G}_\sigma$, and with the fundamental vector fields

$\mathcal{K}^0_X = -(X \otimes {\rm id}_{\mathcal{O}_{M_0}}) \circ M_0\lambda^*$ over M_0, labelled[2] by elements of the tangent Lie superalgebra $\mathfrak{g}_\sigma \ni X$ of G_σ. We say that G_σ is a Lie supergroup of prequantisable rigid (configurational) symmetries of the bulk super-σ-model if (i) the bulk tensors are G_σ-invariant, *i.e.*, they satisfy $|M_0\lambda|^*_g(\mathrm{g},\mathrm{H}) = (\mathrm{g},\mathrm{H})$ and $\mathscr{L}_{\mathcal{K}_X}(\mathrm{g},\mathrm{H}) = 0$ for arbitrary $(g,X) \in |\mathrm{G}_\sigma| \times \mathfrak{g}_\sigma$; (ii) the action admits generalised (co)momenta $\kappa_X \in \Omega^1(M_0)$ such that ${\rm d}\kappa_X = -\iota_{\mathcal{K}_X}\mathrm{H}$; and (iii) it lifts to the bulk gerbe as a $|\mathrm{G}_\sigma| \times \mathfrak{g}_\sigma$-indexed family of 1-isomorphisms $\Lambda_g \ :\ |M_0\lambda|^*_g\mathcal{G} \xrightarrow{\cong} \mathcal{G}$ and $K_X \ :\ \mathscr{L}_{\mathcal{K}_X}\mathcal{G} \xrightarrow{\cong} \mathcal{I}_0$. Here, $\mathscr{L}_{\mathcal{K}^0_X}\mathcal{G}$ is the gerbe represented by an even Beilinson–Deligne 2-cocycle (for some choice of an open cover of M_0) obtained from that of $\mathcal{G}$ by Lie derivation along $\mathcal{K}^0_X$. The assumption of simpliciality of the supertarget may now be used to define a *maximally* symmetric background by declaring the structure maps of $(M_\bullet, d^{(\bullet)}_\cdot, s^{(\bullet)}_\cdot)$ G_σ-equivariant and – based on this assumption – propagating $M_0\lambda$ all over $M_\bullet$ to engender a simplicial G_σ-space with an action $M_\bullet\lambda \ :\ \mathrm{G}_\sigma \times M_\bullet \longrightarrow M_\bullet$ and fundamental vector fields $\mathcal{K}^\bullet_X$. Once this is done, we demand G_σ-invariance of ω and consistency of the higher-geometric lifts of $M_\bullet\lambda$, as expressed by the existence of a $|\mathrm{G}_\sigma| \times \mathfrak{g}_\sigma$-indexed family of 2-isomorphisms $\lambda_g \ :\ (\iota_2^*\Lambda_g \otimes {\rm id}_{\mathcal{I}_\omega}) \circ |M_1\lambda|^*_g\Phi \overset{\cong}{\Longrightarrow} \Phi \circ \iota_1^*\Lambda_g$ and $k_X \ :\ \iota_2^*K_X \circ \mathscr{L}_{\mathcal{K}^1_X}\Phi \overset{\cong}{\Longrightarrow} \iota_1^*K_X$, and the identities ${\rm id} \circ |M_n\lambda|^*_g\varphi_{n+1} = (\varphi_{n+1} \circ {\rm id}) \bullet ({\rm id} \circ \lambda_{g\,1,2}) \bullet ({\rm id} \circ \lambda_{g\,2,3} \circ {\rm id}) \bullet \cdots \bullet ({\rm id} \circ \lambda_{g\,n,n+1} \circ {\rm id}) \bullet (\widetilde{\lambda}_{g\,1,n+1} \circ {\rm id})$ and ${\rm id} \circ \mathscr{L}_{\mathcal{K}^n_X}\varphi_{n+1} = ({\rm id} \circ k_{X\,1,2}) \bullet ({\rm id} \circ k_{X\,2,3} \circ {\rm id}) \bullet \cdots \bullet ({\rm id} \circ k_{X\,n,n+1} \circ {\rm id}) \bullet (\widetilde{k}_{X\,1,n+1} \circ {\rm id})$ (written for $n \geq 2$), in which $\widetilde{\lambda}_g$ and $\widetilde{k}_X$ are certain 'duals' of their un-tilded counterparts, given in [6, Sec. 2].

The final stage of the rather natural structurisation of a generic super-σ-model with maximally (super)symmetric defects consists in requiring semisimplicity of the supertarget, by which we mean a (disjoint-sum) decomposition of the simplicial G_σ-space $(M_\bullet, d^{(\bullet)}_\cdot, s^{(\bullet)}_\cdot, M_\bullet\lambda)$ into orbits of the simplicial G_σ-action. The demand that these support a simplicial gerbe described previously is then anticipated to give rise to cohomological superselection rules for the admissible orbits in the decomposition, which are the only ones that are kept. For G_σ compact, this is bound to result in a rationalisation of the superbackground.

4 The maximally supersymmetric simplicial Lie superbackgrounds

Prime examples of super-σ-models to which the above principles may be applied *constructively*, and in which their consequences may be explored, are those of the Wess–Zumino–Witten (WZW) type, with bulk supertargets given by Lie supergroups[3] $\mathrm{G} = (|\mathrm{G}|, \mathfrak{g} \equiv \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)})$, to be seen as orbits of an action $\mathrm{G}_0\lambda$ of a subgroup of the Lie supergroup $\mathrm{G} \times \mathrm{G}$ of left (ℓ) and right ($\wp$) regular translations on G. The corresponding maximally supersymmetric defects implement the *right* regular action $\wp$ of the target Lie supergroup on itself, *cp.* [1, 4, 14], and so the supertarget embeds in the nerve $\mathrm{N}(\mathrm{G} \rtimes_\wp \mathrm{G})$ of the action groupoid $\mathrm{G} \rtimes_\wp \mathrm{G}$, with the structure $(\mathrm{Ob}(\mathrm{G} \rtimes_\wp \mathrm{G}), \mathrm{Mor}(\mathrm{G} \rtimes_\wp \mathrm{G}), s, t, \mathrm{Id.}) \equiv (\mathrm{G}, \mathrm{G} \times \mathrm{G}, \mathrm{pr}_1, \wp, \mathrm{id}_\mathrm{G} \times \widehat{e})$ and morphism composition encoding supergroup multiplication $\mathrm{m}_\mathrm{G} \ :\ \mathrm{G} \times \mathrm{G} \longrightarrow \mathrm{G}$. The building blocks of the higher-geometric structure of these superfield theories are:

- a (super)gerbe $\mathcal{G}_\mathrm{C}$ over G geometrising the Cartan 3-form H_C;
- a supersymmetric trivialisation $\mathcal{T} \ :\ \iota_D^*\mathcal{G}_\mathrm{C} \xrightarrow{\cong} \mathcal{I}_{\omega_D}$ of $\mathcal{G}_\mathrm{C}$ over a (stratified) sub-super-

[2] For the sake of notational consistency, we should think of the $\mathcal{K}^0_X$ as global sections of $\mathcal{T}(\mathfrak{g}_\sigma \times M_0)$ valued in the distribution $\mathcal{T}M_0$ and linear in the global generators X^A of the structure sheaf $\mathcal{O}_{\mathfrak{g}_\sigma}$ of $\mathfrak{g}_\sigma$ (coordinates) present in the decomposition $X = X^A t_A$ of the vector in a homogeneous basis $\{t_A\}_{A \in \overline{1, \dim_\mathbb{R} \mathfrak{g}_\sigma}}$, and subsequently regard formulæ involving the $\mathcal{K}^0_X$ as constraints on the said distrubution $\mathcal{T}M_0$, *cp.* [6, Sec. 2].

[3] The more general homogeneous spaces G/H can then be reached through application of the universal gauge principle of [12, 13].

manifold $\iota_D : D \hookrightarrow \mathrm{G}$ with $\omega_D \in \Omega^2(D)_0$, giving rise to a boundary $(\mathcal{G}_{\mathrm{C}} \sqcup 0)$-bi-brane $(D, \iota_D, *, \omega_D, \mathcal{T})$ (or $\mathcal{G}_{\mathrm{C}}$-brane) for the target $\mathrm{G} \sqcup \mathbb{R}^{0|0}$ and $* : D \dashrightarrow \mathbb{R}^{0|0} \hookrightarrow \mathrm{G} \sqcup \mathbb{R}^{0|0}$;

- a supersymmetric *generalised* multiplicative structure on $\mathcal{G}_{\mathrm{C}}$, extending the latter to a simplicial gerbe over Segal's model of the classifying space $B\mathrm{G}$ of G (containing $\mathrm{N}(\mathrm{G} \rtimes_\wp \mathrm{G})$ as a simplicial sub-manifold), and thus categorifying m_{G} in a manner compatible with the simplicial lift $\mathrm{G}_\bullet\lambda$ of $\mathrm{G}_0\lambda$. The structure geometrises an extension of H_{C} to a simplicial 4-cocycle $\mathcal{Z}_{\mathrm{G}} \equiv (0, \mathrm{H}_{\mathrm{C}}, \varrho, \vartheta, \varphi)$ over $B\mathrm{G}$, and as such comprises a distinguished 1-isomorphism $\mathcal{M} : \mathrm{pr}_1^*\mathcal{G}_{\mathrm{C}} \otimes \mathrm{pr}_2^*\mathcal{G}_{\mathrm{C}} \xrightarrow{\cong} \mathrm{m}_{\mathrm{G}}^*\mathcal{G}_{\mathrm{C}} \otimes \mathcal{I}_\varrho$ over $\mathrm{G}^{\times 2}$ and a coherent (quasi-)associator 2-isomorphism α over $\mathrm{G}^{\times 3}$, described in detail in [6, 15]. Over the sub-supermanifold $\widetilde{\iota}_D : \mathrm{G} \times D \hookrightarrow \mathrm{G}^{\times 2}$, the 1-isomorphism induces the maximally supersymmetric $\mathcal{G}_{\mathrm{C}}$-bi-brane $(\mathrm{G} \times D, \mathrm{pr}_1, \wp, \widetilde{\iota}_D^*\varrho - \mathrm{pr}_2^*\omega_D, \Phi)$ with the bi-module $\Phi = (\widetilde{\iota}_D^*\mathcal{M} \otimes \mathrm{id}_{\mathcal{I}_{-\mathrm{pr}_2^*\omega_D}}) \circ (\mathrm{id}_{\mathrm{pr}_1^*\mathcal{G}_{\mathrm{C}}} \otimes \mathrm{pr}_2^*\mathcal{T}_D^{-1} \otimes \mathrm{id}_{\mathcal{I}_{-\mathrm{pr}_2^*\omega_D}})$. The existence of α turns the problem of constructing φ_{n+1} into a search – over (the second factor in) a disjoint union of $\mathrm{G}_n\lambda$-orbits within $\mathrm{G} \times \widetilde{D}_n \equiv \mathrm{G} \times (D^{\times n} \cap \mathrm{m}_{1\_n}^{-1}(D))$ with $\mathrm{m}_{1\_n} \equiv \mathrm{m}_{\mathrm{G}} \circ (\mathrm{m}_{\mathrm{G}} \times \mathrm{id}_{\mathrm{G}}) \circ \cdots \circ (\mathrm{m}_{\mathrm{G}} \times \mathrm{id}_{\mathrm{G}^{\times n-2}})$ – for 'boundary' fusion 2-isomorphisms $\varphi_n^{(\partial)} : \otimes_{i=1}^n \mathrm{pr}_i^*\mathcal{T}_D \xRightarrow{\cong} (\mathrm{m}_{1\_n}^*\mathcal{T}_D \otimes \mathrm{id}) \circ \mathcal{M}_{1\_n-1,n} \restriction_{\widetilde{D}_n}$, defined in terms of the 1-isomorphism $\mathcal{M}_{1\_n-1,n} = \mathcal{M}_{12\cdots n-1,n} \circ \cdots \circ (\mathcal{M}_{12,3} \otimes \mathrm{id}) \circ (\mathcal{M}_{1,2} \otimes \mathrm{id})$ in which we use the shorthand notation $\mathcal{M}_{12\cdots k-1,k} = ((\mathrm{m}_{1\_k-1} \times \mathrm{id}_{\mathrm{G}}) \circ \mathrm{pr}_{1,2,\ldots,k})^*\mathcal{M}$ with $\mathrm{pr}_{1,2,\ldots,k} = (\mathrm{pr}_1, \mathrm{pr}_2, \ldots, \mathrm{pr}_k)$.

The existence of all these structures is contingent upon the vanishing of topological obstructions[4] quantified by a suitable cohomology, the latter also capturing (in lower degree) a classification of inequivalent such structures. Basing on hitherto results of the geometric canonical analysis of [3, 5] and various approaches to quantisation of the WZW σ-models in the presence of defects, and in particular those of the functorial approach of [1, 18] (especially effective for compact bulk targets, *i.e.*, in the un-graded setting), one may expect to arrive, through coherent imposition of the organising principles listed above, at a purely supergeometric realisation of the ring of sectors of the (chiral bulk) super-σ-model in the form of the stratified supertarget (or, more accurately, its cohomology), in which fusion is represented by the inter-bi-brane sector with induction mediated by the face maps $d_\cdot^{(n)}$, $n \geq 3$. Below, we recapitulate the main findings in connection with and state key conjectures based upon this expectation, which follow from the extensive recent study [6].

The un-graded WZW defect & yet another link to the CS theory. The WZW σ-model for the compact 1-connected Lie group G is defined[5] in the bulk by its tensorial data: the Cartan–Killing metric $\mathrm{g}_{\mathrm{CK}} = -\frac{\mathsf{k}}{4\pi}\,\mathrm{tr}_{\mathfrak{g}}(\theta_{\mathrm{L}} \otimes \theta_{\mathrm{L}})$ and the (scaled) Cartan 3-form $\mathrm{H}_{\mathrm{C}} = \frac{\mathsf{k}}{24\pi}\,\mathrm{tr}_{\mathfrak{g}}(\theta_{\mathrm{L}} \wedge [\theta_{\mathrm{L}} \overset{\wedge}{,} \theta_{\mathrm{L}}])$, written in terms of the left-invariant $\mathfrak{g}$-valued Maurer–Cartan 1-form θ_{L} and co-normalised in a manner which ensures non-anomalous conformality of the (quantised) field theory, with the level $\mathsf{k} \in \mathbb{N}^\times$, *cp.* [19]. Integrality of the level implies the existence of a unique (isoclass of) gerbe geometrising H_{C} – the k-th tensor power of the Gawędzki–Hitchin–Meinrenken basic gerbe over G.

The rigid symmetries of the bulk theory compose the Lie group $\mathrm{G}_\sigma = \mathrm{G} \times \mathrm{G}$ and lift to $\mathsf{N}_\bullet(\mathrm{G} \rtimes \mathrm{G}) = \mathrm{G}^{\times \bullet+1}$ as $\mathrm{G}_n\lambda \equiv \ell\wp^{(n)} : (\mathrm{G} \times \mathrm{G}) \times \mathrm{G}^{\times n+1} \longrightarrow \mathrm{G}^{\times n+1} : ((x, y), g, h_1, h_2, \ldots, h_n) \longmapsto (x \cdot g \cdot y^{-1}, \mathrm{Ad}_y(h_1), \mathrm{Ad}_y(h_2), \ldots, \mathrm{Ad}_y(h_n))$. Consequently, one is led to look for $\mathcal{G}_{\mathrm{C}}$-modules over conjugacy classes $\mathcal{C}_\lambda = \mathrm{Ad}_{\mathrm{G}}(\mathrm{e}^{2\pi\mathrm{i}\frac{\lambda}{\mathsf{k}}})$ (for λ from the Cartan algebra $\mathfrak{t} \subset \mathfrak{g}$), and finds them over a *discretuum* thereof labelled by the fundamental affine Weyl alcove $\mathsf{P}_+^{\mathsf{k}}$ at level k. The brane curvature $\omega_D \restriction_{\mathcal{C}_\lambda} = \frac{\mathsf{k}}{8\pi}\,\mathrm{tr}_{\mathfrak{g}}(\theta_{\mathrm{L}} \wedge (\mathrm{id}_{\mathfrak{g}} + \mathsf{T}_e\mathrm{Ad}_\cdot)(\mathrm{id}_{\mathfrak{g}} - \mathsf{T}_e\mathrm{Ad}_\cdot)^{-1} \circ \theta_{\mathrm{L}})$ is fixed *uniquely* by the requirement that the bi-chiral loop-group extension of the equivariant lift of the rigid symmetry

[4] In the $\mathbb{Z}/2\mathbb{Z}$-graded setting, *cp.* the Rabin–Crane argument in [11, 16, 17].

[5] In the ungraded WZW setting, one customarily works with the Polyakov (energy) functional as S_{metr}.

G_σ to $\mathsf{N}(\mathrm{G}\rtimes\mathrm{G})$ preserve the Defect Gluing Condition[6] imposed at the maximally symmetric boundary defect. The existence of the symmetric multiplicative structure (a standard one as in [15], with $(\vartheta,\varphi)=(0,0)$) is unobstructed in the 1-connected un-graded setting, and so the boundary data from over $D=\bigsqcup_{\lambda\in\mathrm{P}^{\mathrm{k}}_+}\mathcal{C}_\lambda$ automatically induce those of the non-boundary maximally symmetric bi-brane with the worldvolume $\mathrm{G}_1=\mathrm{G}\times D$, its curvature $\omega=\varrho-\mathrm{pr}_2^*\omega_D$ also being fixed uniquely, in terms of the Polyakov–Wiegmann 2-form $\varrho=\frac{\mathrm{k}}{4\pi}\,\mathrm{tr}_{\mathfrak{g}}(\mathrm{pr}_1^*\theta_{\mathrm{L}}\wedge\mathrm{pr}_2^*\theta_{\mathrm{R}})$ (written with the right-invariant Maurer–Cartan 1-form θ_{R}), by an argument analogous to the one invoked for ω_D. Thus, the connected components $\mathrm{G}\times\mathcal{C}_\lambda$ of the bi-brane worldvolume are in a one-to-one correspondence with the highest weights of $\mathfrak{g}$ (integrable at level k) labelling the irreducible representations $\mathcal{V}_{\lambda,\mathrm{k}}$ of the Kac–Moody algebra $\widehat{\mathfrak{g}}_{\mathrm{k}}$ of the chiral currents of the centrally extended bi-chiral loop-group symmetry $\widehat{\mathrm{LG}}\times\widehat{\mathrm{LG}}$ of the bulk theory which enter the decomposition $\mathcal{H}_\sigma=\bigoplus_{\lambda\in\mathrm{P}^{\mathrm{k}}_+}\mathcal{V}_{\lambda,\mathrm{k}}\otimes\overline{\mathcal{V}}_{\lambda,\mathrm{k}}$ of the bulk Hilbert space.

The 'triple-S' argument of Sec. 3 localises the construction of the maximally symmetric inter-bi-brane over a disjoint union of $\ell\wp^{(n)}$-orbits within the $\widetilde{\mathrm{G}}_n$ for $n\geq 2$. This purely geometric/group-theoretic localisation receives a rather remarkable confirmation from an elementary analysis of the necessary conditions of existence of the component boundary fusion 2-isomorphism $\varphi_n^{(\partial)}$ captured by the second one of the contraints (2): Its left-hand side coincides with the partially symplectically reduced presymplectic form on the state space of the $3d$ Chern–Simons (CS) theory on the time cylinder $\mathbb{R}\times\Sigma$ over the Riemann surface Σ of the WZW σ-model, coupled to $n+1$ vertical Wilson lines $\mathbb{R}\times\{\sigma_i\}_{i\in\overline{1,n+1}}$ with holonomies along the respective non-contractible loops encircling simply the punctures σ_i constrained to lie in the conjugacy classes $\mathcal{C}_{\lambda_i}$ of the weights λ_i labelling the in-coming ($i\leq n$) and the out-going ($i=n+1$) defect lines of the WZW σ-model. The pre-symplectic form arises, in the first-order formalism of Tulczyjew *et al.* (*cp.*, *e.g.*, [20]), as a restriction of the Atiyah–Bott form on the moduli space of flat principal G-connections over $\Sigma\setminus\{\sigma_i\}_{i\in\overline{1,n+1}}$, augmented with a sum of Kirillov–Kostant–Souriau contributions from the coadjoint orbits attached to the Wilson lines, to the preimage of the zero vector under the moment map of the CS theory, the latter being given by the curvature of the connection with Dirac-δ-sources at the Wilson-line punctures. Its partial reduction, due to Alekseev and Malkin [21], is carried out with respect to the 'pointed' gauge group of the CS theory associated with a homological decomposition of $\Sigma\setminus\{\sigma_i\}_{i\in\overline{1,n+1}}$ relative to a point $\sigma_*\in\Sigma\setminus\{\sigma_i\}_{i\in\overline{1,n+1}}$, and so leaves us with the tangent of the residual gauge group $[\sigma_*,\mathrm{G}]\equiv\mathrm{G}$ as the characteristic distribution of the partially reduced presymplectic form – this is just the symmetry group G of the Ad-orbits in $\widetilde{\mathrm{G}}_n$ whose disjoint union composes the classical state space of the CS theory in the Alekseev–Malkin parametrisation.

Ultimately, the structure of (the inter-bi-brane component of) the simplicial target space is determined by the requirement of existence of the relevant gerbe 2-isomorphisms $\varphi_n^{(\partial)}$, which is expected to distinguish a subfamily within the disjoint union of Ad-orbits in $\widetilde{\mathrm{G}}_n$. Taking into account the identification of the Hilbert space of the (holomorphically) quantised CS theory as the space $\mathscr{C}(\otimes_{i=1}^n\mathcal{V}_{\lambda_i,\mathrm{k}},\mathcal{V}_{\lambda_{n+1},\mathrm{k}})$ of conformal blocks of the (chiral) bulk WZW theory, *cp.* [22], with their known relation to intertwiners $\mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{k}}}(\otimes_{i=1}^n\mathcal{V}_{\lambda_i,\mathrm{k}},\mathcal{V}_{\lambda_{n+1},\mathrm{k}})$ between sectors of the chiral bulk theory, in conjunction with the identification of the transgressed fusion 2-isomorphisms transmissive to rigid bulk symmetries as intertwiners of the symmetry representations on the state spaces of the phases converging at the defect junction (resp. those of the corresponding defect-twisted sectors), *cp.* [3,5], we are thus led to the following conjectures:

1. The boundary fusion 2-isomorphisms $\varphi_n^{(\partial)}$ exist only over manifolds $\times_{i=1}^n\mathcal{C}_{\lambda_i}\cap\mathrm{m}_{1\_n}^{-1}(\mathcal{C}_{\lambda_{n+1}})$

[6] Recall that the Condition constrains the admissible discontinuities of the kinetic momentum at a (spacelike) worldsheet defect line in terms of the curvature ω of the bi-brane attached to it, and determines a local symplectomorphism between the phases of the bulk field theory separated by the defect in the first-order canonical description, *cp.* [1,3].

with non-vanishing Verlinde numbers $\mathcal{N}_{\lambda_1,\lambda_2,\ldots,\lambda_n}^{\lambda_{n+1}} = \dim \mathscr{C}(\otimes_{i=1}^n \mathcal{V}_{\lambda_i,\mathsf{k}}, \mathcal{V}_{\lambda_{n+1},\mathsf{k}})$.

2. Given such a manifold, the number of Ad-orbits in its decomposition which support $\varphi_n^{(\partial)}$ is given by the corresponding Verlinde number.

One may also anticipate that the maximally symmetric simplicial WZW background is descent-complete, and the inter-bi-brane fusing matrices defined, as described in detail in [6], by the associator move relating inequivalent resolutions of 4-valent fusion 2-isomorphisms in terms of the elementary (3-valent) ones are intimately related to the standard fusing matrices of the bulk WZW σ-model. The first conjecture was corroborated for the special case of $\mathrm{G} = \mathrm{SU}(2)$ in [4], and the last expectation hinges on the highly nontrivial cohomological evidence from the simple-current sector gathered in [1] as well as on simple considerations of the topologicality of the maximally symmetric defect. Various geometric, algebraic and field-theoretic arguments in favour of the second conjecture were given in [6].

Candidate flat maximally supersymmetric Green–Schwarz bi-branes. The Green–Schwarz (GS) super-σ-model of the superstring in the Lie supergroup $\mathsf{T} \equiv \mathbb{R}^{d,1|D_{d,1}}$, associated with a $D_{d,1}$-dimensional Majorana-spinor representation of the Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{d,1}) = \langle \Gamma_a \rangle_{a\in\overline{0,d}}$ subject to the Fierz constraints $\eta^{ab}\,(\overline{\Gamma}_a)_{\alpha(\beta}\,(\overline{\Gamma}_b)_{\gamma\delta)} = 0$ (using the Minkowski metric η and a skew charge-conjugation matrix C which gives the symmetric $\overline{\Gamma}_a \equiv C\Gamma_a$), is defined by its tensorial data: the 'metric' $\mathrm{g} = \eta_{ab}\, e_{\mathrm{L}}^a \otimes e_{\mathrm{L}}^b$ and the GS Cartan 3-form $\mathrm{H}_{\mathrm{GS}} = \sigma_{\mathrm{L}}^\alpha \wedge (\overline{\Gamma}_a)_{\alpha\beta}\,\sigma_{\mathrm{L}}^\beta \wedge e_{\mathrm{L}}^a$, both constructed from the left-invariant 1-forms e_{L}^a and $\sigma_{\mathrm{L}}^\alpha$ dual to the generators P_a and Q_α, respectively, of the Lie superalgebra $\mathfrak{t} = \bigoplus_{a=0}^d \langle P_a \rangle \oplus \bigoplus_{\alpha=1}^{D_{d,1}} \langle Q_\alpha \rangle$ with the structure relations $[P_a, P_b] = 0 = [P_a, Q_\alpha]$ and $\{Q_\alpha, Q_\beta\} = \eta^{ab}\,(\overline{\Gamma}_a)_{\alpha\beta}\,P_b$, *cp.* [23]. Co-normalisation of the data ensures equibalance of fermionic and bosonic degrees of freedom in the superstring vacuum, mediated by the de Azcárraga–Lukierski–Siegel gauged right tangential supersymmetry known as κ-symmetry (*cp.* [17,24] for a geometric elucidation in the higher-geometric framework). The Cartan 3-form defines a nontrivial class in the Cartan–Eilenberg cohomology group $\mathrm{CaE}^3(\mathfrak{t})$ and geometrises as the GS super-1-gerbe $\mathcal{G}_{\mathrm{GS}}$ (*i.e.*, essentially a gerbe object in the category of Lie supergroups) in a manner proposed by the Author in [16] (*cp.* also the review [11] of the geometrisation programme).

The rigid supersymmetry of the bulk theory is modelled on the left regular action of T on itself, and so it extends to $\mathsf{T}^{\times\bullet+1}$ trivially as $\mathsf{T}_n\lambda \equiv \ell \times \mathrm{id}_{\mathsf{T}^{\times n}}$. This extension opens an essentially boundless field of constructions of (*rigidly*) maximally supersymmetric GS bi-branes according to the scheme delineated at the beginning of the present section. Its point of departure is the identification of a $\mathsf{T}_1\lambda$-invariant properly generalised (with $\vartheta \neq 0 = \varphi$) multiplicative structure on the GS super-1-gerbe, which was accomplished in [6]. The two distinct species of $\mathcal{G}_{\mathrm{GS}}$-brane found to date (for $d = 9$, with $D_{9,1} = 32$) are: (i) the instantonic 0-brane embedded as an odd hyperplane $\mathbb{R}^{0|N} \subset \mathbb{R}^{9,1|32}$ in the ambient target superspace (including as a sub-species the purely even 0-brane at an arbitrary topological point in $\mathbb{R}^{9,1|32}$); (ii) the $\frac{1}{2}$-BPS superstring-like 1-brane embedded as a hyperbolic superplane $\mathbb{R}^{1,1|16} \subset \mathbb{R}^{9,1|32}$. These induce the corresponding two species of $\mathcal{G}_{\mathrm{GS}}$-bi-brane, whose fusion was analysed at great length *ibidem*. Verification of their status as data of maximally supersymmetric defects in the GS super-σ-model calls for a coherent extension of the gerbe-theoretic realisation of κ-symmetry to the full-fledged bicategory of Sec. 2, which is currently under investigation.

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