IFT-UAM/CSIC-21-066
December 23rd, 2021
Revised March 30th, 2023

Extremal stringy black holes in equilibrium at first order in α'

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Abstract

We compute the first-order α' corrections to well-known families of heterotic multi-center black-hole solutions in five and four dimensions. The solutions can be either supersymmetric or non-supersymmetric, depending on the relative sign between two of the black-hole charges. For both cases, we find that the equilibrium of forces persists after including the α' corrections, as the existence of multi-center solutions free of unphysical singularities shows. We analyze the possibility of black-hole fragmentation.

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1 Introduction

Over the last 30 years many extremal and multi-center black-hole solutions of supergravity theories have been obtained, supersymmetric and non-supersymmetric.[1] Many of those theories can also be viewed as low-energy effective field theories of different string theory compactifications to lowest order in α' and in the string coupling constant. This allowed Strominger and Vafa to establish a relation between the entropy of the corresponding solutions (*stringy black-hole solutions*) and the microscopic entropy of the corresponding compactification of string theory which, in general, includes branes and other extended objects [2].

It is natural to try to extend this relation between macroscopic and microscopic entropies to more general solutions and, more importantly perhaps, to higher orders in α' and in the string coupling constant. The calculations of both the macroscopic and microscopic entropies have to be improved to the next order, independently, to be later compared. If this comparison is to be understood as an unbiased test of this relation and of the viability of String Theory as a theory of Quantum Gravity, each of these separate calculations must be internally self-consistent: the correctness of one of them cannot solely rely on its agreement with the other one if we do not want to suffer confirmation bias.

In the macroscopic side, this suggests the following program:

1. Find the first corrections in α' and in the string coupling constant to the relevant superstring effective field theory.

2. Find single- and multi-center black-hole solutions to the corrected action. This can be done perturbatively so that the known zeroth-order solutions are recovered when α' is set to zero.

3. Compute the macroscopic entropy associated to the corrected black hole solution. This entropy will no longer be the Bekenstein-Hawking one, since the corrected action will contain higher-curvature terms and can no longer be understood as General Relativity coupled to a prescribed set of interacting matter. One has to use Wald's entropy.

4. The macroscopic entropy must be written in terms of charges that can be unambiguously related to a string background, *i.e.* charges that can be associated to numbers of branes.

Only after a satisfactory completion of this program can the macroscopic entropy be compared to the microscopic one, which must have been computed independently.

This program, though, presents many practical and conceptual difficulties:

[1] For a review on black-hole solutions with many references see, for instance, [1].

1. We have a very limited knowledge of the corrections to the zeroth order superstring effective action, specially of the corrections in the string coupling constant, which are mainly unknown.

   We know that the first corrections in α' start at third order for the type II superstring theories, but we do not have a complete knowledge of the effective action to that order. This prevents us from doing fully reliable calculations of α' corrections to the solutions of these theories.

   The situation with the heterotic superstring is much better: we know in full detail the α' corrections to the effective action up to third order and in a form consistent with local spacetime supersymmetry [3].[2] In contrast, there is very limited information about the string loop corrections [6].

2. Given the limitations in our knowledge of the string effective action we have just discussed, it is natural to focus on trying to obtain the first α' corrections to black-hole solutions of the heterotic superstring imposing the necessary conditions on the parameters to ensure that the string coupling constant is small enough that the string loop corrections can be safely neglected.[3]

   Furthermore, given the complexity of the corrected equations of motion, it is also natural to consider at first, single, static, spherically-symmetric, extremal black holes with the minimal number of charges that produce a regular event horizon and start by computing the first-order α' corrections only. These are the so-called 3-charge and 4-charge black holes in 5 and 4 dimensions, respectively and all of them can be embedded in the heterotic superstring effective action. The 3-charge ones include a black hole dual to the one originally considered by Strominger-Vafa [2], which is supersymmetric, together with similar, non-supersymmetric black holes. The 4-charge ones include the heterotic version of the Maldacena-Strominger-Johnson-Khuri-Myers black hole [7,8], which is supersymmetric, plus several similar non-supersymmetric black holes.[4]

   The first-order α' corrections to all these simplest but paradigmatic black-hole solutions have been found quite recently in a series of papers [10–13, 9] and now it is natural to take the next step and consider more complicated solutions: non-extremal, multi-center, stationary...

   There are good reasons to start with the non-extremal ones, as we are going to argue when we discuss the calculation of the macroscopic entropy. The first-order α' corrections to some non-extremal solutions have also been found [14, 15], but they are much harder to obtain since the general results proven in [11] cannot be

[2] See also earlier work in Refs. [4, 5].

[3] Actually, these and other similar conditions that guarantee that the α' corrections are also negligible must also be imposed on the zeroth-order solutions as well.

[4] An exhaustive study of the different non-supersymmetric 4-dimensional 4-charge back holes can be found in Ref. [9].

applied to the non-extremal solutions. Thus, it will be long before the corrections to the non-extremal 4-charge black holes will be found [16].

The general results proven in [11] do not seem to apply to stationary solutions either but, fortunately for us, they do apply with little modifications for the multi-center ones, and, in this paper we are going to profit from them to construct the first α'-corrected multi-center black-hole solutions both in 5- and in 4-dimensions and both supersymmetric and non-supersymmetric. They are the multi-center generalization of the single, static, extremal black-hole solutions whose α'-corrections were found in Refs. [10–13, 9] and include most of them.[5] Before we introduce this particular work, though, it is convenient to discuss the rest of the program outlined above since it affects the reliability of our results.

It is worth mentioning, though, that the solutions presented in Refs. [10–13, 9] satisfy a highly non-trivial consistency test: they are invariant (as families of solutions) under the α'-corrected Buscher T duality transformations derived in Ref. [17] (see also Ref. [18]). The multi-center solutions that we are going to present, being related to the general solutions of Ref. [11] pass this strict test as well.

3. At lowest order in α' and the string coupling constant, all string effective field theories can be regarded as General Relativity (GR) coupled to particular kinds of interacting matter and the entropy of any of their black-hole solutions is the Bekenstein-Hawking entropy: one quarter of the area of the horizon in Planck units. This relation originates in the proofs of the first and second laws of black hole mechanics [19, 20] (based on Einstein's equations) together with the discovery of the Hawking radiation [21] and the precise relation between its (Hawking) temperature and the surface gravity of the black hole.

   At higher orders in α', string effective actions contain terms of higher order in the curvatures and can no longer be regarded as GR. However, we know that their black-hole solutions still behave as thermal objects since the phenomenon of Hawking radiation does not depend on the dynamics of gravity. Furthermore, it can be shown that the zeroth law (the fact that surface gravity is constant over the horizon) [20] can be proven without the use of the Einstein equations [22] and, therefore, it will still hold in the α'-corrected string effective field theory actions.

   In GR, the relation between the area and the entropy was hinted at by the results obtained by Christodoulou and Hawking [23–25] but there are no analogous results for other theories of gravity suggesting a candidate to be identified with the entropy. For this reason, Wald's derivation of the first law of black hole mechanics for arbitrary diff-invariant theories of pure gravity [26], based on previous work with Lee [27], constituted a great breakthrough in this field. The quantity

[5] We have not found the multi-center generalization of all the possible non-supersymmetric black holes considered in Ref. [9].

whose variation with respect to the mass is the inverse of the Hawking temperature must be the black hole entropy and, in this framework, it is known as Wald entropy. In order to confirm this identification one has to prove that this quantity also satisfies the second law. This is a much harder problem, but there is real progress in this direction [28–30].

As we have mentioned, the string effective action is a theory of gravity coupled to very special interacting matter. Thus, an extension of Wald's results including matter is necessary to deal with it. In Ref. [31], Iyer and Wald considered diff-invariant theories of gravity coupled to matter fields transforming as tensors under diffeomorphisms, obtaining, as a result, the celebrated Iyer-Wald prescription to derive the entropy formula which has been widely used to compute the Wald entropy of α'-corrected near-horizon limits of stringy black holes (see, *e.g.* Refs. [32, 33] and references therein).

This formula was used in Refs. [10–13] to compute the Wald entropy of the extremal black hole solutions found in those same references, but it was soon realized that the Iyer-Wald prescription cannot be directly and reliably applied to the heterotic superstring effective action:

(a) The entropy formula produced is different if one uses the 10-dimensional or the toroidally-compactified actions [34].

(b) The entropy formula produced is Lorentz frame-dependent [35], which is utterly unacceptable.

Both problems arise from the inadequate treatment of the Lorentz-Chern-Simons terms present in the Kalb-Ramond field strength in the heterotic superstring effective action at first order in α'.[6] However, ultimately, this inadequacy follows from the over-restrictive assumptions made on the matter fields in Ref. [31], which leads to the total absence of non-gravitational work terms in the first law of black hole mechanics, such as those proportional to the variations of electric charges. As explained in [37], for instance, most matter fields are not simple tensor fields under diffeomorphisms, but sections of principal bundles or more complicated structures. The only true tensor field in the Standard Model is the metric.[7]

In Refs. [37, 40] we revised from this point of view Wald's derivation of the first law of black hole mechanics in simple theories, obtaining the expected work terms and, as a bonus, restricted forms of the generalized zeroth laws.[8] In Ref. [35] we applied these ideas to the (by far) more complicated case of the

[6] It should be stressed that these Chern-Simons terms are very different from those considered in Ref. [36], present in the action and introducing additional total derivatives in its variation under diffeomorphisms: they transform as 3-forms under diffeomorphisms, to start with.

[7] See Ref. [38] for a rigorous treatment of the principal bundle case and Ref. [39] for a different take on this problem.

[8] Again, in the principal bundle case, this result was obtained in Ref. [38].

heterotic superstring effective action to first order in α', obtaining the first law of black hole mechanics with work terms[9] and a Lorentz frame-independent and gauge-invariant expression for the entropy.

In the frame typically used to construct the solutions, the entropy formula found in Ref. [35] can be written in the same form as the one that follows from the Iyer-Wald prescription, except for one coefficient.

As we have argued before, it would be wrong to decide which formula is right by just comparing the Wald entropies of the α'-corrected stringy black holes obtained by applying them with the microscopic entropies. A internal consistency check of these entropy formulas is badly needed.[10]

The property that the correct Wald entropy must satisfy is, precisely, that it satisfies the first law: $\partial S/\partial M = 1/T$. It would be enough to check which entropy satisfies this property, but this can only be done with α'-corrected non-extremal stringy black holes. Finding an α'-corrected non-extremal 4-dimensional black hole and the non-extremal versions of the 3-charge black holes to carry out this test was one of the main motivations behind Refs. [14] and [15], respectively. In both cases it was the entropy computed using the Lorentz frame- and gauge-independent formula of Ref. [35] that was shown to satisfy the first law.

4. Having computed by internally-consistent methods the Wald entropy of stringy black-hole solutions whose α' corrections have been reliably computed as well it is tempting to proceed immediately to compare that macroscopic entropy with the microscopic one. However, only the entropies of corresponding systems must be compared; for instance, the entropy of the (heterotic version of the) Strominger-Vafa black hole, which can be considered the strong-coupling limit of a set of solitonic 5-branes (S5) and fundamental strings (F1) with momentum flowing along them can be compared withe the entropy of the heterotic superstring quantized on such a background.

   At zeroth order in α' the identification of the branes corresponding to a given black-hole solution is straightforward: it follows directly from the calculation of the conserved charges of the solution. The definitions of those conserved charges and their calculation are unambiguous at that order and it is based on

[9] Not all work terms were recovered in that article: at the time it was not known how to obtain the work terms proportional to the variations of the magnetic charges in this formalism (a problem solved in Ref. [41]), nor the terms proportional to the asymptotic values of the scalars found in Ref. [42] (a problem solved in Ref. [43]). Furthermore, there are charges associated to fields that only enter the theory after dimensional reduction, such as Kaluza-Klein vectors and it was not known how to recover the work terms proportional to the variations of those charges using the 10-dimensional action. This problem has been partially solved in Ref. [44], but more work is needed to understand the higher-dimensional origin of all the work terms that occur in the 4- or 5-dimensional first laws.

[10] As we have stressed, the Iyer-Wald entropy formula is Lorentz frame-dependent, and this is enough to discard it. However, since the entropy formula of Ref. [35] gives a numerically very close result, it does not hurt to momentarily keep it in the game for the sake of the argument.

integrals of terms that satisfy Gauss laws and give the same results whether they are integrated over the horizon or at spatial infinity. The identification of the charges with numbers of branes is also unambiguous.

At first order in α' the definitions of the conserved charges contain a large number of higher-order terms. All these terms are necessary to obtain integrands that satisfy Gauss laws but, since they are very hard to compute and they vanish asymptotically much faster than the zeroth-order terms, they are typically ignored. These integrands give conserved charges when integrated at spatial infinity but they give different results when integrated over the horizon because the terms that have been ignored do not vanish there. Thus, we have different notions of charge [45] with different properties of transformation under duality and the problem of figuring out which charges correspond to the numbers of charges[11] that occur in the microscopic entropy and study their conservation.[12] This is a difficult and not yet solved problem that will be studied in more detail elsewhere [46],[13] but the reader should be aware of its existence when comparing the macroscopic and microscopic entropies at first and higher orders in α' because their agreement depends on the choice of variables.

It should be clear form this discussion that this is a research program still under way and far from having been completed: the α' corrections of many black-hole solutions (multi-center, rotating...) and their entropies have not yet been computed, the identification of the branes that give rise to the corresponding string background has to be clarified and, at some point in the future, one would like to extend all these results to higher orders in α' and in the string coupling constant.

In this paper, as part of this program, we are going to focus on the α' corrections to the well-known multi-center generalizations of the 3- and 4-charge extremal, static, black-hole solutions in 5 and 4 dimensions. We are interested in the corrections to the geometry and to the entropy and their consequences. In particular, we want to know whether the α' corrections preserve the regularity of the horizons and of the rest of the spacetime. It has to be taken into account that solutions describing collinear Schwarzschild black holes in static equilibrium have long been known [47, 48]. These solutions, however, have conical singularities in the line joining the centers or extending from the centers to infinity, known as *struts*, associated to the external forces necessary to hold the system in equilibrium. The absence of these singularities and the regularity of the horizons can be interpreted as a proof of the cancellation of interaction energies and of the equilibrium of forces between the black holes [49,50] (see Figs. 1 and 2.[14]. We also want to know whether the fragmentation of large black holes into smaller black

[11]This problem is aggravated by the α'-dependent definitions that the fields theat descend from the 10-dimensional Kalb-Ramond 2-form suffer in the dimensional reduction [18].

[12]Incidentally, this discussion casts a shadow over the results obtained using near-horizon solutions since only the horizon charges can be computed with them.

[13]Nevertheless, see the discussion concerning the S5 brane at the end of Section 4.

[14]For an extended discussion on this we refer to the Introduction of [51] and references therein.

holes is entropically favored. We are going to consider supersymmetric and some non-supersymmetric multi-center black-hole solutions to see of and how the equilibrium of forces between the black holes depends on supersymmetry.[15]

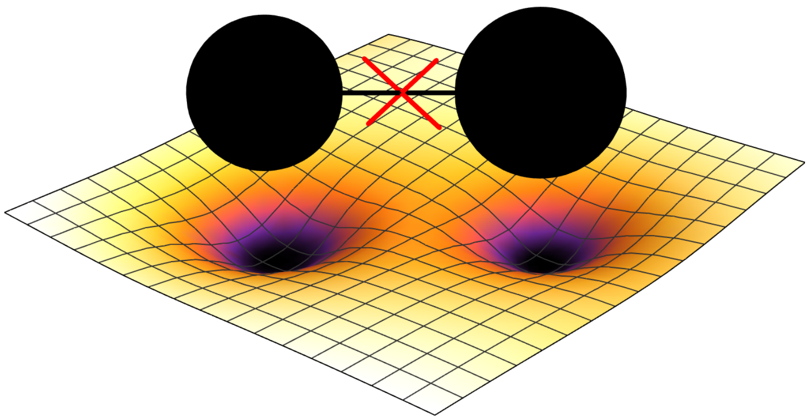

Figure 1: The absence of struts (conical singularities) joining the centers is a necessary condition to interpret the solution as black holes in equilibrium without the help of external forces..

Our calculation of the α' corrections in mainly based on Ref. [11] in which a very general class of α'-corrected solutions was found. This class includes multi-center black-hole solutions, as we are going to show.

This paper is organized as follows: in Section 2 we review the five- and four-dimensional multi-center black hole solutions we will work with at leading order in α', starting with their common ten-dimensional origin. In Section 3 we show the α'-corrected solutions and analyze their thermodynamic properties. Finally, in Section 4 we summarize our main results and discuss how the α' corrections would affect the fragmentation of these extremal black holes into smaller extremal black holes of the same kind. The appendices contain the low-energy effective action and corresponding equations of motion of the heterotic superstring to first order in α' and a collection of results which are used in the main text.

[15] In the solutions that we are going to consider, the existence of unbroken supersymmetry depends on the relative sign between two of the black-hole charges. This relative sign does not cover all the non-supersymmetric possibilities in four dimensions, but the corrections to those not included here are much more difficult to deal with [9].

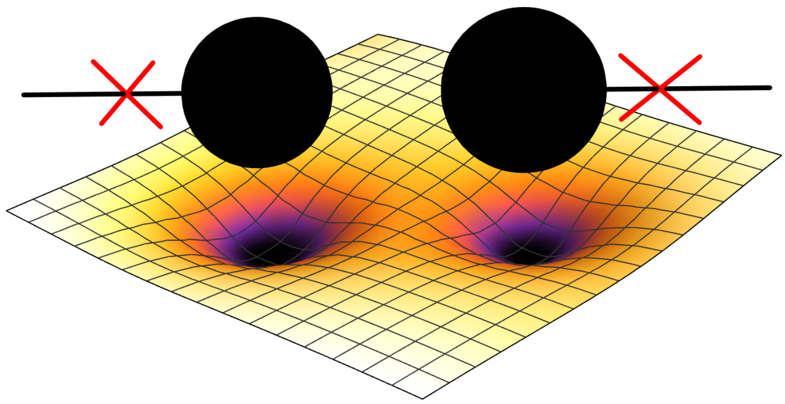

Figure 2: The absance of struts (conical singularities) extending from the centers to infinity is another necessary condition to interpret the solution as black holes in equilibrium without the help of external forces..

2 Heterotic multi-center black-hole solutions at leading order

Let us begin with a review of two well-known families of multi-center black-hole solutions in five and four dimensions which arise as solutions of the heterotic superstring effective field theory compactified on a torus.[16] Both families have as a common origin a very general class of ten-dimensional solutions which depend on a choice of four-dimensional hyper-Kähler metric ($m, n = 1, 2, 3, 4$),

$$d\sigma^2 = h_{\underline{mn}}\, dx^m dx^n\,, \tag{2.1}$$

and on a choice of three functions, $\mathcal{Z}_0, \mathcal{Z}_+, \mathcal{Z}_-$, which are harmonic in this hyper-Kähler space, *i.e.* they satisfy the linear equations

$$d \star_\sigma d\mathcal{Z}_{0,+,-} = 0\,. \tag{2.2}$$

This linearity allows us to construct multi-center solutions by "superposition" of several single-center solutions.

[16]This theory is described in Appendix A.

In terms of these four objects, the string-frame line element $d\hat{s}^2$, the Kalb-Ramond 3-form field strength $\hat{H}$ and the dilaton field $\hat{\phi}$ of the solutions take the form[17]

$$d\hat{s}^2 = \frac{1}{\mathcal{Z}_+\mathcal{Z}_-}dt^2 - \mathcal{Z}_0\, d\sigma^2 - \frac{k_\infty^2 \mathcal{Z}_+}{\mathcal{Z}_-}\left[dz + k_\infty^{-1}\left(\mathcal{Z}_+^{-1} - 1\right)dt\right]^2 - d\vec{y}^{\,2}_{(4)}\,, \tag{2.3a}$$

$$\hat{H} = d\left[(2\epsilon - 1)\, k_\infty \left(\mathcal{Z}_-^{-1} - 1\right) dt \wedge dz\right] + \star_\sigma d\mathcal{Z}_0\,, \tag{2.3b}$$

$$e^{-2\hat{\phi}} = e^{-2\hat{\phi}_\infty}\mathcal{Z}_-/\mathcal{Z}_0\,. \tag{2.3c}$$

The modulus $\hat{\phi}_\infty$ will correspond the asymptotic value of the dilaton if, as we will assume, the functions $\mathcal{Z}_-$ and $\mathcal{Z}_0$ asymptote to 1. Since the vacuum expectation value of the exponential of the dilaton is the string coupling constant, for asymptotically-flat solutions we can make the identification,

$$\hat{g}_s = e^{\hat{\phi}_\infty}\,. \tag{2.4}$$

The coordinates $\vec{y}_{(4)}$, periodically identified with equal periods $2\pi\ell_s$, where ℓ_s is the string length, parametrize a four-dimensional torus $\mathbb{T}^4$ which has trivial internal dynamics, meaning that all the Kaluza-Klein (KK) zero modes (scalars and vectors) are trivial.[18] The coordinate $z \sim z + 2\pi\ell_s$ further parametrizes a circle $\mathbb{S}^1_z$ whose asymptotic radius R_z is related to the asymptotic value of the KK scalar associated to this compact direction, k_∞, and to ℓ_s by

$$R_z = k_\infty \ell_s\,. \tag{2.5}$$

Finally, ϵ is the supersymmetry-breaking parameter. It takes two values, $\epsilon = \{0, 1\}$, which correspond to the relative sign between two of the black hole charges. It turns out that the configuration is supersymmetric only if $\epsilon = 1$ [9]. We have not considered the possibility of changing the sign of the second term in $\hat{H}$. This sign becomes the sign of a third black-hole charge upon dimensional reduction. At zeroth order, the effect of this change of sign is almost trivial, but the first-order in α' corrections are much more difficult to obtain than with the chosen sign [9].

Many different solutions can be obtained from Eq. (2.3) by making particular choices of hyper-Kähler space and harmonic functions. Many of them are singular or have no known physical interpretation. Let us see which choices lead to five- and four-dimensional multi-center black-hole solutions.

[17] We use hats to distinguish the ten-dimensional fields from the five- and four-dimensional ones.

[18] In particular, the KK scalars associated to the $\mathbb{T}^4$ are equal to equal to their vacuum expectation values that we set to 1 for convenience, as they do not play any rôle in our solutions.

2.1 Multi-center black-hole solutions in five dimensions

The simplest choice of hyper-Kähler space which allows for five-dimensional multi-center black-hole solutions is $\mathbb{E}^4$. The choice of harmonic functions in $\mathbb{E}^4$ that gives rise to multi-center three-charge black holes is the following,[19]

$$\mathcal{Z}_+ = 1 + \sum_{a=1}^{n_c} \frac{q_+^a}{\rho_a^2}\,, \qquad \mathcal{Z}_- = 1 + \sum_{a=1}^{n_c} \frac{q_-^a}{\rho_a^2}\,, \qquad \mathcal{Z}_0 = 1 + \sum_{a=1}^{n_c} \frac{q_0^a}{\rho_a^2}\,, \tag{2.6}$$

where n_c is the total number of centers and

$$\rho_a^2 = (x^m - x_a^m)\,(x^m - x_a^m)\,, \tag{2.7}$$

x_a^m being the position of the a^{th} center in $\mathbb{E}^4$. The parameters q_+^a, q_-^a and q_0^a must be taken to be strictly positive in order to avoid naked singularities and, then, they correspond to the absolute value of the charges associated to each black hole, up to normalization factors.

The five-dimensional form of the solution is obtained by compactification over the internal manifold, $\mathbb{T}^4 \times \mathbb{S}^1_z$. This operation can be conveniently carried out in two steps. Firstly, we compactify over the trivial $\mathbb{T}^4$. The corresponding six-dimensional solution has exactly the same form as the ten-dimensional one except for the term $-d\vec{y}^{\,2}_{(4)}$ in the metric, which is not present in the former. Secondly, we reduce the six-dimensional solution over $\mathbb{S}^1_z$. Using the relations between the higher- and lower-dimensional fields provided in Appendix B, we obtain

$$ds^2_{\text{E}} = (\mathcal{Z}_+\mathcal{Z}_-\mathcal{Z}_0)^{-2/3}\, dt^2 - (\mathcal{Z}_+\mathcal{Z}_-\mathcal{Z}_0)^{1/3}\, dx^m dx^m\,, \tag{2.8a}$$

$$H = \tfrac{1}{3!}\epsilon_{mnpq}\partial_q \mathcal{Z}_0\, dx^m \wedge dx^n \wedge dx^p\,, \tag{2.8b}$$

$$F = d\left[k_\infty^{-1}\left(\mathcal{Z}_+^{-1} - 1\right) dt\right]\,, \tag{2.8c}$$

$$G = d\left[(2\epsilon - 1)\, k_\infty \left(\mathcal{Z}_-^{-1} - 1\right) dt\right]\,, \tag{2.8d}$$

$$e^{-2\phi} = e^{-2\phi_\infty} \mathcal{Z}_+^{1/2} \mathcal{Z}_-^{1/2} \mathcal{Z}_0^{-1}\,, \tag{2.8e}$$

$$k = k_\infty \mathcal{Z}_+^{1/2} \mathcal{Z}_-^{-1/2}\,, \tag{2.8f}$$

[19] It has been recently proven in [52] that, in the Einstein-Maxwell theory, the standard multi-center black-hole solutions are the only ones within the Majumdar-Papapetrou class that give rise to asymptotically-flat black holes with regular event horizons. This extends a previous result by Chrusciel and Tod [53] to $d > 4$ dimensions.

where $ds^2_{\rm E}$ stands for the five-dimensional line element in the so-called *modified Einstein frame* [54],

$$F = dA\,, \qquad G = dC\,, \tag{2.9}$$

are, respectively, the field strengths of the KK and winding vector fields, A and C,

$$H = dB - \tfrac{1}{2}A\wedge G - \tfrac{1}{2}C\wedge F\,, \tag{2.10}$$

is the 3-form field strength of the Kalb-Ramond 2-form B, k is the KK scalar associated to the internal direction z and ϕ is the five-dimensional dilaton. In five dimensions, the Kalb-Ramond field can be dualized into a vector field whose field strength is

$$K = d\left[\left(\mathcal{Z}_0^{-1}-1\right)dt\right]\,. \tag{2.11}$$

2.1.1 Thermodynamics

The solution is asymptotically flat and its only potential singularities sit at the centers $x^m = x^m_a$. If the charge products $q^a_+q^a_-q^a_0$ are finite, though, these are just coordinate singularities: near the $a^{\rm th}$ center, the metric is that of $\mathrm{AdS}_2\times\mathbb{S}^3$, which is one of the possible near-horizon geometries of extremal black holes in five dimensions [55]. Hence, the $a^{\rm th}$ center is, actually a two-sphere of finite radius $(q^a_+q^a_-q^a_0)^{1/6}$ which corresponds to the event horizon of an extremal black hole. The Bekenstein-Hawking (BH) entropy of the $a^{\rm th}$ black hole is, then,

$$S^a_{\rm BH} = \frac{\pi^2}{2G^{(5)}_{\rm N}}\sqrt{q^a_+q^a_-q^a_0}\,. \tag{2.12}$$

The entropy of the solution is just the sum of the entropies of all the black holes:

$$S_{\rm BH} = \sum_a S^a_{\rm BH} = \frac{\pi^2}{2G^{(5)}_{\rm N}}\sum_a\sqrt{q^a_+q^a_-q^a_0}\,. \tag{2.13}$$

Strictly speaking, only the total mass of the solution can be rigorously defined. From the asymptotic expansion of the tt component of the metric we find that it is given by

$$M = \frac{\pi}{4G^{(5)}_{\rm N}}\sum_a\left(q^a_+ + q^a_- + q^a_0\right)\,. \tag{2.14}$$

However, since the vector fields of the solution are Abelian, we can compute the charges of each black hole independently and, as we have already mentioned, these are given given by the parameters q^a_+, q^a_-, q^a_0 up to signs and normalization. An isolated extremal black hole with those charges would have a mass given by

$$M^a = \frac{\pi}{4G_N^{(5)}}\left(q_+^a + q_-^a + q_0^a\right) \,. \tag{2.15}$$

Since the total mass is the sum of the would-be individual masses of all the black holes,

$$M = \sum_{a=1}^{n_c} M^a \,. \tag{2.16}$$

This fact implies that the black-holes mutual interaction energies (all the Einstein equations know about [51]) are zero because there is a cancellation between all the different contributions (gravitational, electromagnetic etc.) The cancellation of these interaction energies is associated to a cancellation of the forces between the centers and, then, since the solution is static, it is reasonable to conclude that the solution contains n_c extremal black holes with those charges and masses in equilibrium. The absence of singularities indicates that no external forces are necessary to keep those black holes in equilibrium.

2.1.2 Microscopic description

The string-theory description of these solutions is well known: they can be understood as a superposition of solitonic 5-branes (S5) —often referred to as NS5 branes— wrapped around the directions parametrized by the coordinates $y^1, \cdots, y^4, z$, fundamental strings (F1) wound around the circle parametrized by z and waves (W) carrying momentum propagating along the same circle, as we illustrate in Table 1.

	t	z	y^1	y^2	y^3	y^4	x^1	x^2	x^3	x^4
F1	$\times$	$\times$	$\sim$	$\sim$	$\sim$	$\sim$	$-$	$-$	$-$	$-$
W	$\times$	$\times$	$\sim$	$\sim$	$\sim$	$\sim$	$-$	$-$	$-$	$-$
S5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$-$	$-$	$-$	$-$

Table 1: Sources associated to five-dimensional black holes. The symbol $\times$ stands for the worldvolume directions and $-$ for the transverse directions. The symbol $\sim$, in turn, denotes a transverse direction over which the corresponding object has been smeared.

From the lower-dimensional point of view, these fundamental objects act as point-like sources for the three different types of charges carried by our solutions. The locations of the sources in the non-compact space coincides with the position of the centers, where the Laplace Eqs. (2.2) are not satisfied unless we take into account the coupling of the sources to the background fields. This give rise to a sum of Dirac delta functions, $\delta^{(4)}(x - x_a)$,[20] each of which is weighted by the fraction of the total tension/charge associated to all the sources located at $x^m = x_a^m$, which in turn is proportional to the

[20]Strictly speaking, this is just an effective description of the actual situation: the centers $x^m = x_a^m$ are not points, but finite-radii two-spheres from which the Abelian fields' fluxes "emerge" in radial direction.

amounts of momentum, winding and S5 branes in each of the centers, respectively denoted by n^a, w^a and N^a.[21] This is exactly the same kind of contribution that one gets from the Laplacian of $\mathcal{Z}_{+,-,0}$, with the only difference that now the delta functions are weighted by q^a_+, q^a_- and q^a_0. Imposing that both contributions cancel (*i.e.* that the equations of motion are also satisfied at the centers), one finds the following relation between both sets of parameters [10]:

$$q^a_+ = \frac{g_s^2 \alpha'^2}{R_z^2}\, n^a\,, \qquad q^a_- = g_s^2 \alpha'\, w^a\,, \qquad q^a_0 = \alpha'\, N^a\,. \tag{2.17}$$

Using these relations and the expression of the five-dimensional Newton constant in terms of stringy variables

$$G_{\rm N}^{(5)} = \frac{\pi g_s^2 \alpha'^2}{4R_z}\,, \tag{2.18}$$

one can rewrite the Bekenstein-Hawking entropy and the mass in terms of the source parameters as follows

$$S_{\rm BH} = 2\pi \sum_a \sqrt{n^a w^a N^a}\,, \tag{2.19a}$$

$$M = \frac{n}{R_z} + \frac{R_z}{\alpha'}\, w + \frac{R_z}{g_s^2 \alpha'}\, N\,, \tag{2.19b}$$

where $n = \sum_a n^a$, $w = \sum_a w^a$ and $N = \sum_a N^a$ are the total momentum, winding and number of S5-branes respectively. As we can see, the total energy of the configuration only depends on the absolute value of the total charges, no matter how they are distributed among the different centers. This will no longer be the case in presence of higher-derivative corrections, as we will see Section 3.

2.2 Multi-Center black-hole solutions in four dimensions

In order to describe multi-center black-hole solutions in four dimensions, one of the directions of the hyper-Kähler space must be compactified, which in practice means that it has to be isometric. Supersymmetry is preserved in the compactification along this isometric direction if the isometry is consistent with the hyper-Kähler structure, *i.e.* if the isometry is *triholomorphic*. In this case, the hyper-Kähler space must belong to the Gibbons-Hawking class [56, 57].

As we will see in the next section, there is another advantage for considering Gibbons-Hawking spaces: their contribution to the α'-corrected Bianchi identity takes

[21] These parameters are assumed to be positive.

a form (the Laplacian of a function) that allows us to solve the Bianchi identity in a fully analytic fashion [11].

If $w \sim w + 2\pi\ell_s$ is the compact coordinate adapted to the isometry, a Gibbons-Hawking metric can be written in the form

$$d\sigma^2 = h_{\underline{mn}}\, dx^m dx^n = \ell_\infty^2 \mathcal{H}^{-1} \left(dw + \ell_\infty^{-1}\chi\right)^2 + \mathcal{H}\, d\vec{x}^{\,2}_{(3)}\,, \tag{2.20}$$

where $\mathcal{H}$ is the Gibbons-Hawking function and χ is a 1-form which satisfying the equation

$$d\chi = \star_{(3)}\, d\mathcal{H}\,, \tag{2.21}$$

where $\star_{(3)}$ the Hodge star operator in $\mathbb{E}^3$. The above equation implies that $\mathcal{H}$ is harmonic in $\mathbb{E}^3$. This property is also satisfied by the functions $\mathcal{Z}_{+,-,0}$ if they are harmonic in $\mathbb{E}^4$ and we demand that they do not depend on the isometric coordinate w. The modulus ℓ_∞ corresponds to the asymptotic value of the KK scalar ℓ associated to the circle parametrized by w. The asymptotic radius of this circle is given in terms of ℓ_∞ an the string length ℓ_s by

$$R_w = \ell_\infty \ell_s\,. \tag{2.22}$$

The choice that allows us to describe multi-center black-hole solutions in four dimensions is

$$\mathcal{Z}_+ = 1 + \sum_{a=1}^{n_c} \frac{q_+^a}{r_a}\,, \quad \mathcal{Z}_- = 1 + \sum_{a=1}^{n_c} \frac{q_-^a}{r_a}\,, \quad \mathcal{Z}_0 = 1 + \sum_{a=1}^{n_c} \frac{q_0^a}{r_a}\,, \quad \mathcal{H} = 1 + \sum_{a=1}^{n_c} \frac{q_\mathcal{H}^a}{r_a}\,, \tag{2.23}$$

where $r_a = ||\vec{x} - \vec{x}_a||$. As before, $\vec{x}_a$ denotes the position of the a^{th} center in $\mathbb{E}^3$ and the parameters q_+^a, q_-^a, q_0^a and $q_\mathcal{H}^a$ are assumed to be strictly positive.

The 1-form χ is given, locally, by

$$\chi = \sum_a q_\mathcal{H}^a \cos\theta_a d\phi_a\,, \tag{2.24}$$

where r_a, θ_a, ϕ_a are spherical coordinates associated to the a^{th} center

$$\vec{x} - \vec{x}_a = (r_a \sin\theta_a \cos\phi_a, r_a \sin\theta_a \sin\phi_a, r_a \cos\theta_a)\,. \tag{2.25}$$

With these choices, the ten-dimensional solution can be rewritten locally in the form

$$
\begin{aligned}
d\hat{s}^2 = {} & \frac{1}{\mathcal{Z}_+\mathcal{Z}_-}dt^2 - \ell_\infty^2 \frac{\mathcal{Z}_0}{\mathcal{H}}\left(dw + \ell_\infty^{-1}\chi\right)^2 - \mathcal{Z}_0\mathcal{H}d\vec{x}^{\,2}_{(3)} \\
& - k_\infty^2 \frac{\mathcal{Z}_+}{\mathcal{Z}_-}\left[dz + k_\infty^{-1}\left(\mathcal{Z}_+^{-1} - 1\right)dt\right]^2 - d\vec{y}^{\,2}_{(4)}\,,
\end{aligned} \tag{2.26a}
$$

$$
\hat{H} = d\left[(2\epsilon - 1)\,k_\infty\left(\mathcal{Z}_-^{-1} - 1\right)dt\wedge dz\right] + \ell_\infty \star_{(3)} d\mathcal{Z}_0\wedge dw\,, \tag{2.26b}
$$

$$
e^{-2\hat{\phi}} = e^{-2\hat{\phi}_\infty}\mathcal{Z}_-/\mathcal{Z}_0\,. \tag{2.26c}
$$

In order to avoid Dirac-Misner string singularities we have to impose the following "quantization conditions" on the $q^a_{\mathcal{H}}$s':

$$
q^a_{\mathcal{H}} = \frac{R_w W^a}{2}\,, \qquad W^a \in \mathbb{Z}^+ \quad \forall a\,. \tag{2.27}
$$

The four-dimensional solution is obtained by compactifying the above solution on the six-torus $\mathbb{T}^6 = \mathbb{T}^4 \times \mathbb{S}^1_z \times \mathbb{S}^1_w$ by using (twice) the results in Appendix B. It has the following non-vanishing fields,

$$
ds^2_{\rm E} = (\mathcal{Z}_+\mathcal{Z}_-\mathcal{Z}_0\mathcal{H})^{-1/2}dt^2 - (\mathcal{Z}_+\mathcal{Z}_-\mathcal{Z}_0\mathcal{H})^{1/2}d\vec{x}^{\,2}_{(3)}\,, \tag{2.28a}
$$

$$
(A^\alpha) = \begin{pmatrix} \ell_\infty^{-1}\chi \\ k_\infty^{-1}\left(\mathcal{Z}_+^{-1} - 1\right)dt \end{pmatrix}, \tag{2.28b}
$$

$$
(C_\alpha) = \left((2\epsilon - 1)\,k_\infty\left(\mathcal{Z}_-^{-1} - 1\right)dt\,,\ \ \ell_\infty\chi_0\right)\,, \tag{2.28c}
$$

$$
\left(G_{\alpha\beta}\right) = \begin{pmatrix} \ell_\infty^2\mathcal{Z}_0/\mathcal{H} & 0 \\ 0 & k_\infty^2\mathcal{Z}_+/\mathcal{Z}_- \end{pmatrix}, \tag{2.28d}
$$

$$
e^{-2\phi} = e^{-2\phi_\infty}\sqrt{\frac{\mathcal{Z}_+\mathcal{Z}_-}{\mathcal{Z}_0\mathcal{H}}}\,, \tag{2.28e}
$$

where

$$e^{-2\phi_\infty} = e^{-2\hat{\phi}_\infty} k_\infty \ell_\infty \,, \tag{2.29}$$

and

$$\chi_0 = \sum_a q_0^a \cos\theta_a d\phi_a \,. \tag{2.30}$$

2.2.1 Thermodynamics

The reasoning leading to the identification of the locations of the poles of the harmonic functions with extremal black-hole event horizons is identical to that of the five-dimensional case. Now, the near-horizon geometries are $\mathrm{AdS}_2 \times \mathrm{S}^2$ and the radius of the a^{th} of these spaces is $(q_+^a q_-^a q_0^a q_{\mathcal{H}}^a)^{1/4}$. The Bekenstein-Hawking entropy to the a^{th} black hole is

$$\mathrm{S}_{\mathrm{BH}}^a = \frac{\pi}{G_N^{(4)}} \sqrt{q_+^a q_-^a q_0^a q_{\mathcal{H}}^a} \,, \tag{2.31}$$

and the entropy of the complete spacetime is its sum

$$S_{\mathrm{BH}} = \sum_a \mathrm{S}_{\mathrm{BH}}^a \,. \tag{2.32}$$

We can also assign a mass M^a to the a^{th} center

$$M^a = \frac{1}{4G_{\mathrm{N}}^{(4)}} \left(q_+^a + q_-^a + q_0^a + q_{\mathcal{H}}^a\right) \,, \tag{2.33}$$

and the cancellation of the mutual interaction energies follows from the relation between the total mass M and the putative masses of the individual black holes

$$M = \sum_a M^a = \frac{1}{4G_{\mathrm{N}}^{(4)}} \left(\sum_a q_+^a + \sum_a q_-^a + \sum_a q_0^a + \sum_a q_{\mathcal{H}}^a \right) \,. \tag{2.34}$$

2.2.2 Microscopic description

The main difference between the four- and five-dimensional cases is that in the four-dimensional case there is one more type of charge: the magnetic charge associated to the Kaluza-Klein vector. The sources for these charges are Kaluza-Klein monopoles. Their topological charges are the integers W^a, whose relation with the magnetic charges $q_{\mathcal{H}}^a$ was found in Eq. (2.27) by imposing the absence of Dirac-Misner string singularities. The remaining sources (fundamental strings, waves and S5-branes) are those of the five-dimensional black holes smeared over the coordinate w which is transverse to their worldvolume directions, see Table 2.

	t	z	y^1	y^2	y^3	y^4	w	x^1	x^2	x^3
F1	$\times$	$\times$	$\sim$	$\sim$	$\sim$	$\sim$	$\sim$	$-$	$-$	$-$
W	$\times$	$\times$	$\sim$	$\sim$	$\sim$	$\sim$	$\sim$	$-$	$-$	$-$
S5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\sim$	$-$	$-$	$-$
KK6	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\sim$	$-$	$-$	$-$

Table 2: Sources associated to the four-dimensional black holes.

Due to this smearing, the relations between the parameters q^a that occur in the harmonic functions and the source parameters are no longer given by Eqs. (2.17), but instead by [12][22]

$$q_+^a = \frac{g_s^2 \alpha'^2}{2R_z^2 R_w} n^a \,, \qquad q_-^a = \frac{g_s^2 \alpha'}{2R_w} w^a \,, \qquad q_0^a = \frac{\alpha'}{2R_w} N^a \,. \tag{2.35}$$

In terms of the source parameters, n^a, w^a, N^a and W^a, the entropies and putative masses of each of the centers read

$$S_{\text{BH}}^a = 2\pi \sqrt{n^a w^a N^a W^a} \,, \tag{2.36a}$$

$$M^a = \frac{n^a}{R_z} + \frac{R_z}{\alpha'} w^a + \frac{R_z}{g_s^2 \alpha'} N^a + \frac{R_w^2 R_z}{g_s^2 \alpha'^2} W^a \,. \tag{2.36b}$$

The total entropy and mass are just the sums.

3 Heterotic multi-center black-hole solutions at first order in α'

It was shown in Ref. [11] that the first-order α' corrections to the general ten-dimensional heterotic background considered in the previous section and given in Eqs. (2.3) can be analytically obtained in the supersymmetric case for arbitrary choices of harmonic functions as long as the hyper-Kähler metric is a Gibbons-Hawking space or $\mathbb{E}^4$. Here we extend the results of [11] to the $\epsilon = 0$ case, which does not preserve any supersymmetry. In this section we will just describe the corrected solutions.[23] They have exactly the same form as the leading-order solutions described in Section 2 with the

[22] Although we use the same symbols for the parameters that occur in the harmonic functions for the five- and four-dimensional cases, they are different. In particular, they have different dimensions.
[23] Additional details are provided in Appendix D.

only difference that the functions $\mathcal{Z}_+$ and $\mathcal{Z}_0$ get α' corrections.[24] For any solution to the zeroth-order equations of motion of the form Eq. (2.3), the corrections to these functions are given by[25]

$$\mathcal{Z}_+ = \mathcal{Z}_+^{(0)} - \alpha' \left\{ \frac{\epsilon}{2} \frac{\partial_m \mathcal{Z}_+^{(0)} \partial_m \mathcal{Z}_-^{(0)}}{\mathcal{Z}_0^{(0)} \mathcal{Z}_-^{(0)}} + \mathcal{H}_+ \right\} + \mathcal{O}(\alpha'^2) \,, \tag{3.1a}$$

$$\mathcal{Z}_0 = \mathcal{Z}_0^{(0)} + \alpha' \left\{ \frac{1}{4} \left[\frac{\partial_m \mathcal{Z}_0^{(0)} \partial_m \mathcal{Z}_0^{(0)}}{\left(\mathcal{Z}_0^{(0)}\right)^2} + \frac{\partial_m \mathcal{H}^{(0)} \partial_m \mathcal{H}^{(0)}}{\left(\mathcal{H}^{(0)}\right)^2} \right] + \mathcal{H}_0 \right\} + \mathcal{O}(\alpha'^2) \,, \tag{3.1b}$$

where now $\mathcal{Z}_{+,-,0}^{(0)}$ and $\mathcal{H}^{(0)}$ are the zeroth-order value of those functions (*i.e.* harmonic functions in the hyper-Kähler metric $h_{\underline{mn}}$) and $\mathcal{H}_0$ and $\mathcal{H}_+$ are arbitrary harmonic functions in the hyper-Kähler metric $h_{\underline{mn}}$ (which does not get any corrections) which will be determined later on by imposing appropriate boundary conditions.

The functions $\mathcal{H}$ and $\mathcal{Z}_-$ do not get any corrections, hence they are simply given by

$$\mathcal{Z}_- = \mathcal{Z}_-^{(0)} + \mathcal{O}(\alpha'^2) \,, \tag{3.2a}$$

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{O}(\alpha'^2) \,. \tag{3.2b}$$

Let us now discuss the corrections to the multi-center black-hole solutions studied in the previous section.

3.1 Multi-Center black-hole solutions in five dimensions

Plugging the functions $\mathcal{Z}_0^{(0)}$ and $\mathcal{Z}_+^{(0)}$ chosen in Eqs. (2.6) in Eqs. (3.1a) and (3.1b), we get

[24] This is only true for the 10-dimensional solutions, because the definitions of some the lower-dimensional fields (all those descending from the 10-dimensional Kalb-Ramond 2-form) have α' corrections [18]. These will not be needed for our purposes: we only need the lower-dimensional metrics, whose expression in terms of the $\mathcal{Z}$ functions do not change.

[25] Here we are using flat indices in the hyper-Kähler space $m, n, \ldots$, whose metric does not get any corrections. Thus, $\partial_m X \partial_m Y \equiv h^{\underline{mn}} \partial_{\underline{m}} X \partial_{\underline{n}} Y$.

$$\mathcal{Z}_+ = \mathcal{Z}_+^{(0)} + \alpha' \left[-2\epsilon \mathcal{Z}_-^{-1} \mathcal{Z}_0^{(0)\,-1} \sum_{a,b} \frac{q_+^a q_-^b n_a^m n_b^m}{\rho_a^3 \rho_b^3} + \mathcal{H}_+ \right] + \mathcal{O}(\alpha'^2)\,, \tag{3.3a}$$

$$\mathcal{Z}_0 = \mathcal{Z}_0^{(0)} + \alpha' \left[\mathcal{Z}_0^{(0)\,-2} \sum_{a,b} \frac{q_0^a q_0^b n_a^m n_b^m}{\rho_a^3 \rho_b^3} + \mathcal{H}_0 \right] + \mathcal{O}(\alpha'^2)\,, \tag{3.3b}$$

where we have defined the unit radial vectors

$$n_a^m \equiv (x^m - x_a^m)/\rho_a\,. \tag{3.4}$$

We just have to determine the harmonic functions $\mathcal{H}_+$ and $\mathcal{H}_0$. We shall impose the following two conditions:

1. The solutions will be asymptotically flat with the following normalization of the functions: $\lim_{||x||\to\infty} \mathcal{Z}_{+,0} = 1$. This implies

$$\lim_{||x||\to\infty} \mathcal{H}_{+,0} = 0\,. \tag{3.5}$$

2. The coefficients of the $1/\rho_a^2$ poles of $\mathcal{Z}_{0,+}$ which arise in the $\rho_a \to 0$ will not be renormalized. That is

$$\mathcal{Z}_+|_{x\to x_a} \sim \frac{q_+^a}{\rho_a^2}\,, \qquad \mathcal{Z}_0|_{x\to x_a} \sim \frac{q_0^a}{\rho_a^2}\,. \tag{3.6}$$

$\mathcal{H}_0$ and $\mathcal{H}_+$ must be harmonic functions of the same type as $\mathcal{Z}_0^{(0)}$ and $\mathcal{Z}_+^{(0)}$ to preserve asymptotic flatness:

$$\mathcal{H}_{0,+} = \alpha_{0,+} + \sum_a \frac{\beta_{0,+}^a}{\rho_a^2}\,. \tag{3.7}$$

Then, the above conditions determine the coefficients $\alpha_{0,+}, \beta_{0,+}^a$ as follows,

$$\alpha_0 = \alpha_+ = 0\,, \qquad \beta_0^a = -1\,, \qquad \beta_+^a = 2\epsilon \frac{q_+^a}{q_0^a}\,, \tag{3.8}$$

yielding

$$\mathcal{H}_+ = 2\epsilon \sum_a \frac{q_+^a}{q_0^a \rho_a^2}\,, \qquad \mathcal{H}_0 = -\sum_a \frac{1}{\rho_a^2}\,. \tag{3.9}$$

Hence, the final form of the functions that get α' corrections is

$$\mathcal{Z}_+ = \mathcal{Z}_+^{(0)} - 2\epsilon\alpha' \left[\mathcal{Z}_-^{-1} \mathcal{Z}_0^{(0)\,-1} \sum_{a,b} \frac{q_+^a q_-^b n_a^m n_b^m}{\rho_a^3 \rho_b^3} - \sum_a \frac{q_+^a}{q_0^a \rho_a^2} \right] + \mathcal{O}(\alpha'^2)\,, \tag{3.10a}$$

$$\mathcal{Z}_0 = \mathcal{Z}_0^{(0)} + \alpha' \left[\mathcal{Z}_0^{(0)\,-1} \sum_{a,b} \frac{q_0^a q_0^b n_a^m n_b^m}{\rho_a^3 \rho_b^3} - \sum_a \frac{1}{\rho_a^2} \right] + \mathcal{O}(\alpha'^2)\,. \tag{3.10b}$$

The two conditions imposed to determine $\mathcal{H}_{0,+}$ have a physical motivation. The first condition is equivalent to asking the non-renormalization of the asymptotic value of the string coupling, the radius of the internal direction and, therefore, of the five-dimensional Newton constant. These constants defined the vacuum and, therefore, we are dealing with corrected solutions in the original vacuum. This allows us to compare the masses and the charges of the black holes before and after the corrections.

The second condition is meant to keep unmodified the relation between the parameters of the solution $q^a_{0,+,-}$ and the physical parameters specifying the microscopic system, n^a, w^a and N^a Eq. (2.17). As we have already discussed, this is due to the fact that n^a, w^a and N^a are proportional to the coefficients of the poles of the functions [10, 12]. It is worth remarking that this condition implies that the $1/\rho^2$ coefficient in the asymptotic expansion of the functions can be corrected. Indeed, we find

$$\mathcal{Z}_0 \sim 1 + \frac{1}{\rho^2}\left(\sum_a q_0^a - \alpha' n_c\right) + \ldots\,, \qquad \mathcal{Z}_+ \sim 1 + \frac{1}{\rho^2}\sum_a \left(q_+^a + \frac{2\epsilon\alpha' q_+^a}{q_0^a}\right) + \ldots\,. \tag{3.11}$$

Another natural choice of harmonic functions $\mathcal{H}_{0,+}$ would have been to simply set both of them to zero. In such case, the $1/\rho^2$ coefficients in the asymptotic expansion would remain invariant, while the coefficients of the poles of the functions would be corrected, contrarily to what happens with the choice we have made. Nevertheless, it is important to emphasize that both choices are physically equivalent since they give rise to the same solution parametrized in very different ways. For instance, the identifications in Eq. (2.17) are no longer valid if we choose $\mathcal{H}_{0,+} = 0$.

In any case, these ambiguities disappear once the solution is expressed in terms of the physical quantities. The behavior of the functions near the a^{th} center is given by

$$\mathcal{Z}_+ \sim \frac{g_s^2 \alpha'^2 n^a}{R_z^2 \rho_a^2}\,, \qquad \mathcal{Z}_- \sim \frac{g_s^2 \alpha' w^a}{\rho_a^2}\,, \qquad \mathcal{Z}_0 \sim \frac{\alpha' N^a}{\rho_a^2}\,, \tag{3.12}$$

while, asymptotically, we find

$$\mathcal{Z}_+ \sim 1 + \frac{g_s^2 \alpha'^2 |\mathcal{Q}_+|}{R_z^2 \rho^2}\,, \qquad \mathcal{Z}_- \sim 1 + \frac{g_s^2 \alpha' |\mathcal{Q}_-|}{\rho^2}\,, \qquad \mathcal{Z}_0 \sim 1 + \frac{\alpha' |\mathcal{Q}_0|}{\rho^2}\,, \tag{3.13}$$

where we have defined, for later convenience,

$$\mathcal{Q}_{+} = n + 2\epsilon \sum_{a} \frac{n^{a}}{N^{a}}\,, \qquad \mathcal{Q}_{-} = (2\epsilon - 1)\, w\,, \qquad \mathcal{Q}_{0} = -\left(N - n_{c}\right)\,, \tag{3.14}$$

which coincide with the asymptotic charges of the solution, defined as

$$\ell_s^{-1}\mathcal{Q}_{+} = \frac{g_s^2}{16\pi G_{\rm N}^{(5)}} \int_{\mathrm{S}^3_\infty} e^{-2\phi} k^2 \star F\,, \tag{3.15a}$$

$$T_{F1}\mathcal{Q}_{-} = \frac{\hat{g}_s^2}{16\pi G_{\rm N}^{(10)}} \int_{\mathbb{T}^4\times \mathrm{S}^3_\infty} e^{-2\hat{\phi}}\, \hat{\star}\, \hat{H}\,, \tag{3.15b}$$

$$T_{S5}\mathcal{Q}_{0} = \frac{1}{16\pi G_{\rm N}^{(10)}} \int_{\mathrm{S}^3_\infty} \hat{H}\,, \tag{3.15c}$$

where

$$T_{F1} = \frac{1}{2\pi\alpha'}\,, \qquad T_{S5} = \frac{1}{(2\pi\ell_s)^5 g_s^2 \ell_s}\,, \tag{3.16}$$

are the string and S5-brane tensions, respectively.

3.1.1 Regularity of the corrected solutions

From the behavior of the functions near the centers Eqs. (3.12), we see that the corrected solutions describe n_c regular event horizons as long as the charge parameters are strictly positive, $q^a_{+,-,0} > 0$. When the corrections are taken into account, though, even if we take all these parameters to be positive, there exists the possibility that the functions $\mathcal{Z}_+$ and $\mathcal{Z}_0$ vanish at some points outside the event horizons, giving rise to naked (curvature) singularities. However, we can argue that in the solutions we are considering in this paper, all eventual curvature singularities that might appear must be spurious.[26] The reason is simply that they are not present in the zeroth-order solutions. Therefore, if they appear in the corrected solutions, it must be in a regime of charge parameters where the perturbative expansion is no longer justified, as the vanishing of the functions can only happen when the corrections cancel the leading-order contributions, which tells us that they are equally important. This can be seen very clearly in the single-center solutions. In this case, the corrections are given by [9]

[26]Note that this claim does not apply to the conical singularities (struts) we are also discussing in the paper.

$$\mathcal{Z}_+ = 1 + \frac{q_+}{\rho^2} + \frac{2\epsilon q_+ \alpha' \left(\rho^2 + q_0 + q_-\right)}{q_0 \left(\rho^2 + q_-\right)\left(\rho^2 + q_0\right)} + \mathcal{O}(\alpha'^2)\,, \tag{3.17a}$$

$$\mathcal{Z}_0 = 1 + \frac{q_0}{\rho^2} - \alpha' \frac{\rho^2 + 2q_0}{\left(\rho^2 + q_0\right)^2} + \mathcal{O}(\alpha'^2)\,. \tag{3.17b}$$

The function $\mathcal{Z}_+$ is always strictly positive if $q_{+,-,0} > 0$. However, the function $\mathcal{Z}_0$ can vanish and change sign, which is far worse. However, this only happens if $q_0 \ll \alpha'$ (see Fig. 3), in which case the perturbative expansion is no longer justified. Hence, naked singularities of the corrected solutions only arise for values of the parameters for which the perturbative expansion breaks down and we should not worry about them. A graphic depiction of a particular solution is givenin Figs. 6 and 7

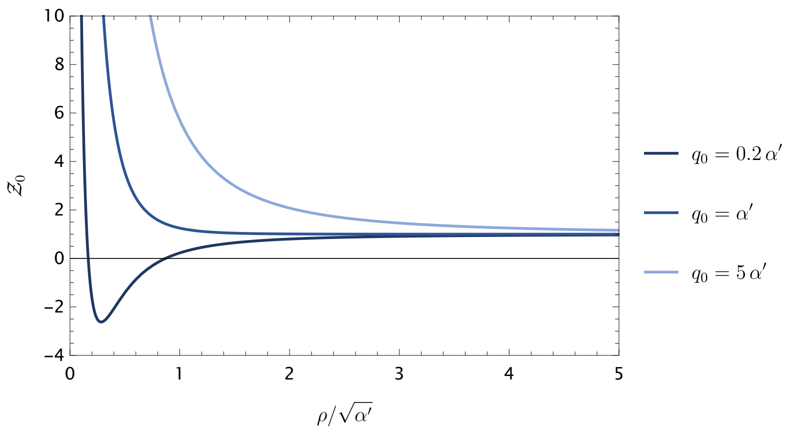

Figure 3: Plot of the function $\mathcal{Z}_0$ for the solution with a single center $n_c = 1$ and for several values of the charge $q_0/\alpha' = \{0.2, 1, 5\}$. As we can see, when the value of q_0 is sufficiently small, (spurious) naked singularities appear.

In the multi-center case, we have exactly the same situation. In any case, it is, of course, important to check that the we have found solutions are free of these pathologies when we take the charges to be large as compared to α'. We have done so numerically for several multi-center configurations and we have found no singularities whenever the charges are larger than ℓ_s, in agreement with our general argument: even if we consider a complicated configuration (see Fig. 4), approaching a center the corrections to $\mathcal{Z}_{+,0}$ reduce to those of a single BH (see Fig. 5).

3.1.2 Thermodynamic properties of the corrected black holes

The ADM mass of the α'-corrected five-dimensional black holes can be straightforwardly computed from the asymptotic expansion of tt component of the metric. The

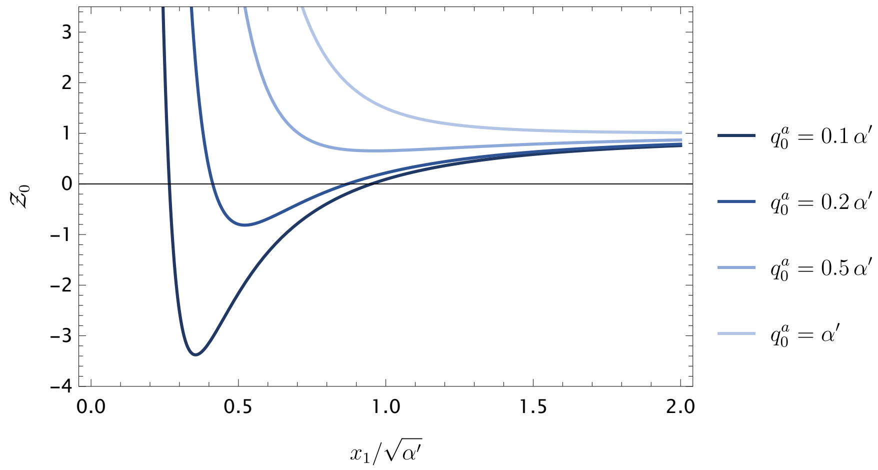

Figure 4: Plot of the function $\mathcal{Z}_0$ for the solution with multiple centers $n_c = 5$ along the plane $x^3 = x^4 = 0$. The plane spanned by x^1 and x^2 does not contain any of the centers. The centers are placed in the vertexes of a 5-cell.

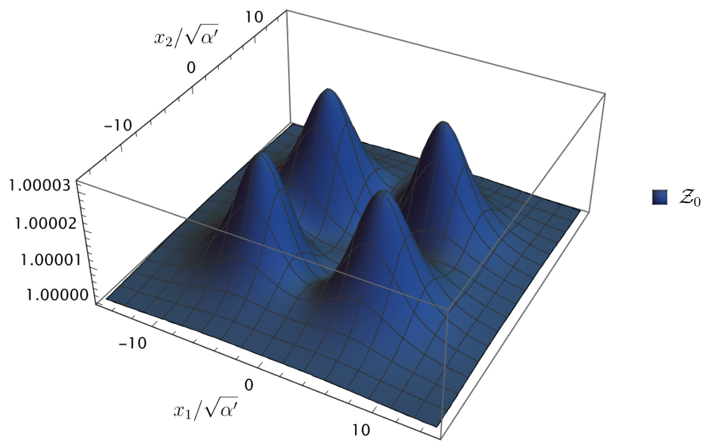

Figure 5: Plot of the function $\mathcal{Z}_0$ for the solution with multiple centers $n_c = 5$ along a direction passing through a center for several values of the charge of such center $q_0^a/\alpha' = \{0.1, 0.2, 0.5, 1\}$. As we can see, when the value of q_0 is much smaller than α', (spurious) naked singularities appear. The centers are placed in the vertexes of a 5-cell.

result can be expressed in several ways,

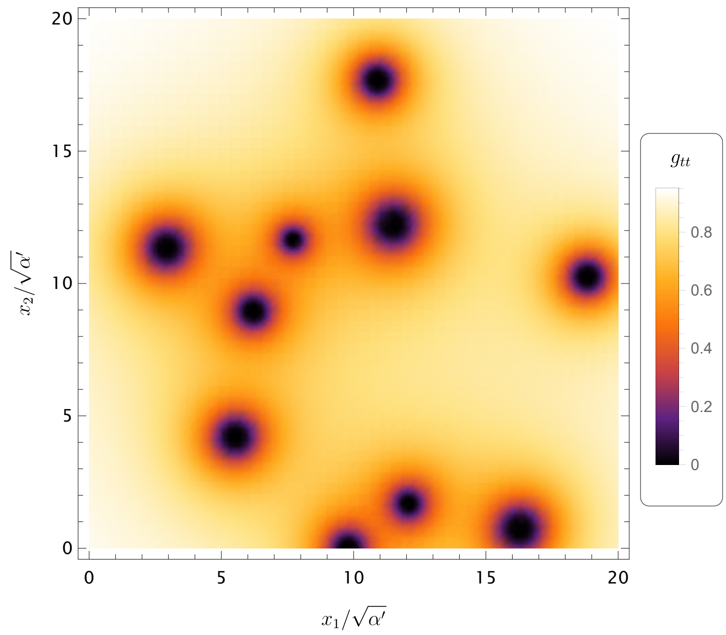

Figure 6: Plot of the g_{tt} component of the metric in the Einstein frame Eq. (2.8a) for a planar configuration of 3-charged extremal BHs in 5 dimensions. x_1 and x_2 are the same of Eq. (2.7). The charges and the positions are positive and randomly picked such that the average distance among the BHs is 5 times the average value of the charges in units of α'.

$$M = \frac{n}{R_z} + \frac{R_z}{\alpha'} w + \frac{R_z}{g_s^2 \alpha'} N + \frac{2\epsilon}{R_z} \sum_a \frac{n^a}{N^a} - \frac{n_c R_z}{g_s^2 \alpha'} \\ = \frac{|\mathcal{Q}_+|}{R_z} + \frac{R_z |\mathcal{Q}_-|}{\alpha'} + \frac{R_z |\mathcal{Q}_0|}{g_s^2 \alpha'} , \tag{3.18}$$

which highlight different aspects of the corrections to the mass, as we are going to discuss. First, observe that we can associate a mass to the a$^{\text{th}}$ black hole given by[27]

$$M^a = \frac{n^a}{R_z} \left(1 + \frac{2\epsilon}{N^a} \right) + \frac{R_z w^a}{\alpha'} + \alpha' \left(N^a - 1 \right) , \tag{3.19}$$

and that it is still true that the total mass is given by the sum of these putative individual black-hole masses, $M = \sum_a M^a$. Again, this is telling us that there is a cancellation of the interaction energies between the black holes, which is precisely what allows them to be in static equilibrium. However, contrarily to what happens at zeroth order, the total mass not only depends on the total amounts of winding, momentum and

[27]These individual masses can be found by setting $n_c = 1$ in Eq. (3.18).

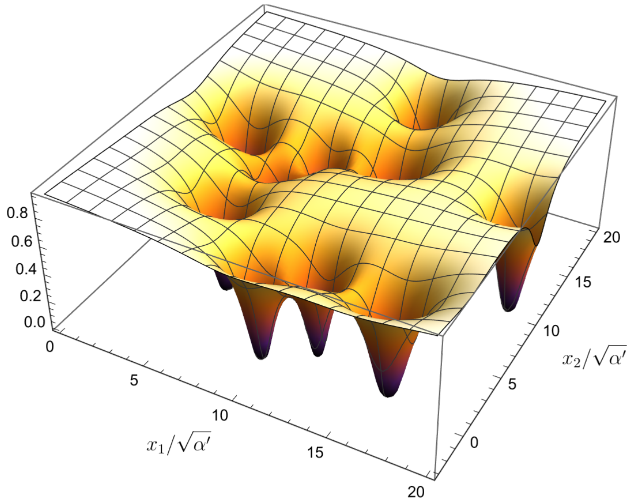

Figure 7: 3-dimensional version of the plot in Fig. 6. In this figure we do not have infinite throats because the coordinates we are using are defined outside of the BHs horizons where $g_{tt} = 0$.

S5-branes, but also on the particular distribution of these objects among the different centers.

In order to discuss the corrections to the mass, we have to specify the quantities that we keep fixed. There are two natural possibilities: either we fix the asymptotic (Maxwell) charges, $\mathcal{Q}_{+,-,0}$, or the total number of stringy sources, n, w and N. This distinction is necessary because these quantities do not coincide when taking into account α' corrections, as we have just seen in Eq. (3.14). In fact, the correction to the mass just vanishes when the asymptotic charges are kept fixed, as reported in [9]. In contrast, when fixing n, w and N, one finds that the correction to the mass is given by

$$\delta M|_{n,w,N} = \frac{2\epsilon}{R_z}\sum_a \frac{n^a}{N^a} - \frac{n_c R_z}{g_s^2 \alpha'} = \frac{1}{R_z}\left[2\epsilon \sum_a \frac{n^a}{N^a} - \frac{n_c k_\infty^2}{g_s^2}\right] . \tag{3.20}$$

As we can see, the correction in the non-supersymmetric case ($\epsilon = 0$) is always negative and, furthermore, it decreases linearly with the number of centers. This means that the fragmentation of these bound states is energetically favored in the non-supersymmetric case. We will come back to this issue in Section 4. In the supersymmetric case, in turn, the correction does not have a definite sign.

The entropy of these black holes can be obtained by means of the entropy formula derived in [35][28] based on the results of [37,40], as it has recently been done in [9] for

[28] As explained in that reference, in general, the Iyer-Wald formula derived in [31] using the formalism of [27,26] is not valid in presence of matter fields because the vast majority of them have gauge freedoms that were not correctly taken into account. In particular, in the case of the first-order in α' heterotic superstring effective action, the entropy formula derived using the Iyer-Wald prescription is not gauge

single black-hole solutions. Since this entropy is just an integral over the event horizon, it is just the sum of all the entropies of the individual black holes, and we can simply use the result in [9] to get

$$S_{\rm W} = \sum_a S^a_{\rm W}\,, \qquad S^a_{\rm W} = 2\pi\sqrt{n^a w^a N^a}\left(1+\frac{2\epsilon}{N^a}\right)\,. \tag{3.21}$$

As we can see, the correction to the entropy for fixed n^a, w^a and N^a vanishes in the non-supersymmetric case, in agreement with the results of [33, 58]. The result in the supersymmetric case coincides with previous results in the literature, see [59, 33, 58, 34, 18].

3.2 Multi-Center black-hole solutions in four dimensions

Plugging Eqs. (2.23) in Eqs. (3.1a) and (3.1b) and choosing the harmonic functions $\mathcal{H}_{0,+}$ with the same prescription as in the five-dimensional case, we obtain that the corrections to $\mathcal{Z}_+$ and $\mathcal{Z}_0$ are the following,[29]

$$\mathcal{Z}_+ = \mathcal{Z}_+^{(0)} - \frac{\epsilon\alpha'}{2}\left[\mathcal{Z}_0^{(0)\,-1}\mathcal{Z}_-^{-1}\mathcal{H}^{-1}\sum_{a,b}\frac{q^a_+ q^b_- n^m_a n^m_b}{r_a^2 r_b^2} - \sum_a \frac{q^a_+}{q^a_0 q^a_{\mathcal{H}} r_a}\right] + \mathcal{O}(\alpha'^2)\,, \tag{3.22a}$$

$$\mathcal{Z}_0 = \mathcal{Z}_0^{(0)} + \frac{\alpha'}{4}\left[\mathcal{Z}_0^{(0)\,-2}\mathcal{H}^{-1}\sum_{a,b}\frac{q^a_0 q^b_0 n^m_a n^m_b}{r_a^2 r_b^2} + \mathcal{H}^{-3}\sum_{a,b}\frac{q^a_{\mathcal{H}} q^b_{\mathcal{H}} n^m_a n^m_b}{r_a^2 r_b^2} - \sum_a \frac{2}{q^a_{\mathcal{H}} r_a}\right] + \mathcal{O}(\alpha'^2)\,, \tag{3.22b}$$

where, now, $n^m_a \equiv (x^m - x^m_a)/r_a$ and $m = 1, 2, 3$.

The behavior of all the functions that determine the four-dimensional solutions near the black-hole horizons ($r_a \to 0$) is,

$$\mathcal{Z}_+ \sim \frac{g_s^2\alpha'^2 n^a}{2R_z^2 R_w r_a}\,, \qquad \mathcal{Z}_- \sim \frac{g_s^2\alpha' w^a}{2R_w r_a}\,, \qquad \mathcal{Z}_0 \sim \frac{\alpha' N^a}{2R_w r_a}\,, \qquad \mathcal{H} \sim \frac{R_w W^a}{2r_a}\,, \tag{3.23}$$

where we have used Eqs. (2.35) and (2.27). Asymptotically, we find the following behavior

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invariant (it is frame-dependent). In certain frames, the entropy formula derived using the Iyer-Wald prescription and the entropy formula derived in [35] take the same form, except for a factor of 2 in one of the terms. The factor of 2 that occurs in the entropy formula derived in [35] leads to an entropy that satisfies the first law of black-hole thermodynamics in the α'-corrected non-extremal Reissner-Nordström black hole [14].

[29] The expressions for the harmonic functions in this case are $\mathcal{H}_+ = \sum_a \frac{\epsilon q^a_+}{2q^a_0 q^a_{\mathcal{H}} r_a}$ and $\mathcal{H}_0 = -\sum_a \frac{1}{2q^a_{\mathcal{H}} r_a}$.

$$\mathcal{Z}_+ \sim 1 + \frac{g_s^2\alpha'^2|\mathcal{Q}_+|}{2R_z^2 R_w r}\,, \quad \mathcal{Z}_- \sim 1 + \frac{g_s^2\alpha'|\mathcal{Q}_-|}{2R_w r}\,, \quad \mathcal{Z}_0 \sim 1 + \frac{\alpha'|\mathcal{Q}_0|}{2R_w r}\,, \quad \mathcal{H} \sim 1 + \frac{R_w|\mathcal{Q}_\mathcal{H}|}{2r}\,, \tag{3.24}$$

where we have defined

$$\mathcal{Q}_+ = n + 2\epsilon \sum_a \frac{n^a}{N^a W^a}\,, \quad \mathcal{Q}_- = (2\epsilon - 1)\, w\,, \quad \mathcal{Q}_0 = -\left(N - \sum_a \frac{2}{W^a}\right)\,, \quad \mathcal{Q}_\mathcal{H} = W\,, \tag{3.25}$$

which again coincide with the asymptotic charges of the solution.

3.2.1 Thermodynamic properties of the corrected solutions

The total mass of the four-dimensional black-hole solution is given by

$$\begin{aligned} M &= \frac{n}{R_z} + \frac{R_z w}{\alpha'} + \frac{R_z N}{g_s^2\alpha'} + \frac{R_w^2 R_z W}{g_s^2\alpha'^2} + \frac{2\epsilon}{R_z}\sum_a \frac{n^a}{N^a W^a} - \frac{2R_z}{g_s^2\alpha'}\sum_a \frac{1}{W^a} \\ &= \frac{|\mathcal{Q}_+|}{R_z} + \frac{R_z|\mathcal{Q}_-|}{\alpha'} + \frac{R_z|\mathcal{Q}_0|}{g_s^2\alpha'} + \frac{R_w^2 R_z|\mathcal{Q}_\mathcal{H}|}{g_s^2\alpha'^2}\,. \end{aligned} \tag{3.26}$$

It displays analogous features to the mass of the five-dimensional ones. First, the total mass is again the sum of the putative individual masses of each black hole, given by

$$M^a = \frac{n^a}{R_z}\left(1 + \frac{2\epsilon}{N^a W^a}\right) + \frac{R_z w^a}{\alpha'} + \frac{R_z\left(N^a - \frac{2}{W^a}\right)}{g_s^2\alpha'} + \frac{R_w^2 R_z W^a}{g_s^2\alpha'^2}\,. \tag{3.27}$$

Again, this fact can be interpreted as directly related to the no-force condition between the black holes. In addition to this, we observe that the correction to the mass also vanishes when we keep the asymptotic charges constant. Instead, when we keep the parameters n, w, N and W constant, we obtain a non-vanishing correction,

$$\delta M|_{n,w,N,W} = \frac{2}{R_z}\left[\sum_a \frac{\epsilon n^a}{N^a W^a} - \frac{k_\infty^2}{g_s^2}\sum_a \frac{1}{W^a}\right]\,, \tag{3.28}$$

which is, again, negative in the non-supersymmetric case and decreases with the number of centers.

The Wald entropy can be computed as in the five-dimensional case, making use of the results of [9], and it has the value

$$S_\text{W} = 2\pi \sum_a \sqrt{n^a w^a N^a W^a}\left(1 + \frac{2\epsilon}{N^a W^a}\right)\,. \tag{3.29}$$

4 Discussion

Our main results can be summarized as follows: we have computed the first-order α' corrections to multi-center black-hole solutions in five and four dimensions which provide effective descriptions of bound states of fundamental strings, momentum waves, solitonic 5-branes and KK monopoles. We have shown that the first-order α' corrections do not introduce any singularities and that, therefore, the equilibrium of forces between them is preserved.

We have also seen how the relation between the asymptotic charges and the numbers of fundamental objects is altered by the α' corrections, see Eqs. (3.14) and (3.25). As a consequence, we have seen that the mass can have corrections when n, w, N and W are kept fixed even if the corrections for the asymptotic charges vanish. Interestingly enough, we have observed that when the total number of fundamental objects is kept fixed, the masses of the non-supersymmetric solutions receive negative corrections that decrease with the number of centers, indicating us that black-hole fragmentation is energetically favored. The question now is whether this process is allowed or not.

Clearly, a necessary condition for the fragmentation process to be allowed is that the conserved charges of the initial and final configurations are identical. At the two-derivative level, the conserved charges are proportional to the total numbers of the different fundamental objects, which means that these numbers must not change. The number of centers n_c can, in principle, change, but it does not appear explicitly in the entropy formula at this order. Then, at this order, the fragmentation is, in principle, allowed, but entropically disfavored.

As already mentioned, in presence of α' corrections (which introduce Chern-Simons terms), one can define several notions of charge and not all of them are necessarily conserved [45]. Therefore, the first thing we have to do is to figure out which notions of charge are conserved and which are not. A thorough analysis of all the possible notions of charge, their physical interpretation and their properties in this context requires much more work and will be carried out elsewhere [46]. Thus, here we will just focus on one of them, the solitonic 5-brane charge. The presence of S5 branes modifies the Bianchi identity of the Kalb-Ramond 2-form $\hat{B}$ as follows:

$$\frac{\hat{g}_s^2}{16\pi G_{\mathrm{N}}^{(10)}}\left[d\hat{H}-\frac{\alpha'}{4}\hat{R}_{(-)}{}^{\hat{a}}{}_{\hat{b}}\wedge\hat{R}_{(-)}{}^{\hat{b}}{}_{\hat{a}}\right]=\hat{\star}\hat{J}_{S5}\,. \tag{4.1}$$

The current J_{S5} describes the coupling of external sources (S5 branes) to the magnetic dual of the KR 2-form. Following [45], we refer to it as the brane-source current. By definition, it is localized, which in this context means that it vanishes whenever the sourceless (supergravity) equations of motion are satisfied. For instance, in the five-dimensional configurations we have studied, the brane-source current associated to S5 branes is given by

$$\hat{\star}\hat{J}_{S5} = -\hat{g}_s^2 T_{S5} \sum_a N^a \star_4 \delta^{(4)}(x - x_a)\,, \quad \text{where} \quad \int_{\mathbb{E}^4} \star_4 \delta^{(4)}(x - x_a) = 1\,, \tag{4.2}$$

since it is precisely at the centers where Eqs. (2.2) are not satisfied. Therefore, the brane-source charge, defined as the integral of $\hat{\star}\hat{J}_{S5}$, is proportional to (minus) the total number of S5 branes,

$$\int_{\mathbb{E}^4} \hat{\star}\hat{J}_{S5} = -\hat{g}_s^2 T_{S5} \sum_a N^a = -\hat{g}_s^2 T_{S5} N\,. \tag{4.3}$$

As explained in [45], brane-source charges are not conserved quantities in presence of Chern-Simons terms. The heterotic case is, however, a bit peculiar because, when taking a exterior derivative in (4.1), we arrive to

$$d\hat{\star}\hat{J}_{S5} = \frac{\hat{g}_s^2 \alpha'}{32\pi G_{\mathrm{N}}^{(10)}} \hat{\mathcal{D}}_{(-)} \hat{R}_{(-)\,\hat{a}\hat{b}} \wedge \hat{R}_{(-)}{}^{\hat{a}\hat{b}} = 0\,, \tag{4.4}$$

if the Bianchi identity of the Riemann tensor is not modified by the presence of sources,

$$\hat{\mathcal{D}}_{(-)} \hat{R}_{(-)\hat{a}\hat{b}} = \hat{\mathcal{D}} \hat{R}_{\hat{a}\hat{b}} = 0\,. \tag{4.5}$$

This implies that the total number of 5-branes, N, must remain constant in the fragmentation process. The same happens with the Maxwell solitonic 5-brane charge, which in the five-dimensional case is given by $\mathcal{Q}_0 = -(N - n_c)$, see Eq. (3.14). Hence, it is evident that the fragmentation is forbidden if both Maxwell and brane-source charges are conserved. An analogous analysis in the four-dimensional case yields the same conclusion.

Acknowledgments

We are pleased to thank Roberto Emparan, Pedro F. Ramírez, Stefano Massai, Luca Martucci and Jose Juán Fernández-Melgarejo for very valuable discussions. This work has been supported in part by the MCIU, AEI, FEDER (UE) grant PGC2018-095205-B-I00 and by the Spanish Research Agency (Agencia Estatal de Investigación) through the grant IFT Centro de Excelencia Severo Ochoa SEV-2016-0597. AR is supported by a postdoctoral fellowship associated to the MIUR-PRIN contract 2017CC72MK003. The work of MZ was supported by the fellowship LCF/BQ/DI20/11780035 from "la Caixa" Foundation (ID 100010434). TO wishes to thank M.M. Fernández for her permanent support.

A The heterotic superstring effective action

The α' corrections that appear in the effective action of the heterotic superstring were first studied in [4, 5, 3], see also [6, 60–63] for more recent studies. Here we are following [3], adapting their results to the conventions of [1]. Working with a consistent truncation in which all the gauge fields are trivialized, we have that the effective action of the heterotic string that includes the first-order α' corrections[30] is given by

$$\hat{S} = \frac{\hat{g}_s^2}{16\pi G_{\mathrm{N}}^{(10)}} \int d^{10}x \sqrt{|\hat{g}|}\, e^{-2\hat{\phi}} \left[\hat{R} - 4(\partial\hat{\phi})^2 + \frac{1}{2\cdot 3!}\hat{H}^2 - \frac{\alpha'}{8}\hat{R}_{(-)\,\hat{\mu}\hat{\nu}}{}^{\hat{a}}{}_{\hat{b}}\hat{R}_{(-)}{}^{\hat{\mu}\hat{\nu}\,\hat{b}}{}_{\hat{a}} \right] , \tag{A.1}$$

where

$$\hat{R}_{(-)}{}^{\hat{a}}{}_{\hat{b}} = d\hat{\Omega}_{(-)}{}^{\hat{a}}{}_{\hat{b}} - \hat{\Omega}_{(-)}{}^{\hat{a}}{}_{\hat{c}} \wedge \hat{\Omega}_{(-)}{}^{\hat{c}}{}_{\hat{b}} , \tag{A.2}$$

is the curvature of the torsionful spin connection,

$$\hat{\Omega}_{(-)}{}^{\hat{a}}{}_{\hat{b}} \equiv \hat{\omega}^{\hat{a}}{}_{\hat{b}} - \frac{1}{2}\hat{H}^{(0)}_{\hat{c}}{}^{\hat{a}}{}_{\hat{b}}\,\hat{e}^{\hat{c}} , \tag{A.3}$$

and, in their turn, $\hat{\omega}^{\hat{a}}{}_{\hat{b}}$ is the Levi-Civita spin connection and

$$\hat{H}^{(0)} = d\hat{B} , \tag{A.4}$$

is the zeroth-order the 3-form field strength of the Kalb-Ramond 2-form $\hat{B}$. The first-order field strength is given by

$$\hat{H} = d\hat{B} + \frac{\alpha'}{4}\hat{\Omega}^{\mathrm{L}}_{(-)} , \tag{A.5}$$

where

$$\hat{\Omega}^{\mathrm{L}}_{(-)} = d\hat{\Omega}_{(-)}{}^{\hat{a}}{}_{\hat{b}} \wedge \hat{\Omega}_{(-)}{}^{\hat{b}}{}_{\hat{a}} - \frac{2}{3}\hat{\Omega}_{(-)}{}^{\hat{a}}{}_{\hat{b}} \wedge \hat{\Omega}_{(-)}{}^{\hat{b}}{}_{\hat{c}} \wedge \hat{\Omega}_{(-)}{}^{\hat{c}}{}_{\hat{a}} , \tag{A.6}$$

is the Chern-Simons 3-form of $\hat{\Omega}_{(-)}{}^{\hat{a}}{}_{\hat{b}}$. Thus, $\hat{H}$ satisfies the Bianchi identity

$$d\hat{H} = \frac{\alpha'}{4}\hat{R}_{(-)}{}^{\hat{a}}{}_{\hat{b}} \wedge \hat{R}_{(-)}{}^{\hat{b}}{}_{\hat{a}} . \tag{A.7}$$

[30] Some terms of second order in α' are implicitly included in this expression, in order to give it a more convenient form, but they should be consistently ignored.

A.1 Equations of motion

In order to derive the equations of motion, it is highly convenient to use the lemma proven in [3], which states that the variation of the action with respect to the torsionful spin connection, $\delta S/\delta\hat{\Omega}_{(-)}{}^{\hat{a}}{}_{\hat{b}}$, yields terms of order $\mathcal{O}(\alpha'^2)$ when evaluated on-shell. Since we are ignoring such higher-order terms, we can simply obtain the equations of motion by varying the action only with respect to explicit occurrences of the fields (*i.e.*, those which do not occur through the torsionful spin connection). Doing so, one obtains

$$\hat{R}_{\hat{\mu}\hat{\nu}} - 2\hat{\nabla}_{\hat{\mu}}\partial_{\hat{\nu}}\hat{\phi} + \frac{1}{4}\hat{H}_{\hat{\mu}\hat{\rho}\hat{\sigma}}\hat{H}_{\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}} - \frac{\alpha'}{4}\hat{R}_{(-)\,\hat{\mu}\hat{\rho}}{}^{\hat{a}}{}_{\hat{b}}\hat{R}_{(-)\,\hat{\nu}}{}^{\hat{\rho}\,\hat{b}}{}_{\hat{a}} = \mathcal{O}(\alpha'^2)\,, \tag{A.8a}$$

$$(\partial\hat{\phi})^2 - \frac{1}{2}\hat{\nabla}^2\hat{\phi} - \frac{1}{4\cdot 3!}\hat{H}^2 + \frac{\alpha'}{32}\hat{R}_{(-)\,\hat{\mu}\hat{\nu}}{}^{\hat{a}}{}_{\hat{b}}\hat{R}_{(-)}{}^{\hat{\mu}\hat{\nu}\,\hat{b}}{}_{\hat{a}} = \mathcal{O}(\alpha'^2)\,, \tag{A.8b}$$

$$d\left(e^{-2\hat{\phi}}\star\hat{H}\right) = \mathcal{O}(\alpha'^2)\,. \tag{A.8c}$$

B Torus compactification

The reduction of the heterotic superstring effective action over a torus to first order in α' has been carried out in Ref. [64]. Here, however, we only need to know the relation between the lower- and higher-dimensional fields at leading order in the α' expansion:[31]

[31]Here we are using the first letters of the Greek alphabet for the internal indices instead of the Latin letters $m, n, \cdots$, to avoid possible confusions with the indices of the hyper-Kähler space. We are also using an unconventional C for some of the fields descending from the Kalb-Ramond 2-form.

$$\hat{g}_{\mu\nu} = g_{\mu\nu} - G_{\alpha\beta}A^{\alpha}{}_{\mu}A^{\beta}{}_{\nu}\,, \tag{B.1a}$$

$$\hat{g}_{\mu\alpha} = -G_{\alpha\beta}A^{\beta}{}_{\mu}\,, \tag{B.1b}$$

$$\hat{g}_{\alpha\beta} = -G_{\alpha\beta}\,, \tag{B.1c}$$

$$\hat{B}_{\mu\nu} = B_{\mu\nu} - A^{\alpha}{}_{[\mu|}C_{\beta\,|\nu]} \tag{B.1d}$$

$$\hat{B}_{\mu\alpha} = C_{\alpha\,\mu} - B_{\alpha\beta}A^{\beta}{}_{\mu}\,, \tag{B.1e}$$

$$\hat{B}_{\alpha\beta} = C_{\alpha\beta}\,, \tag{B.1f}$$

$$\hat{\phi} = \phi + \tfrac{1}{2}\log\det(G_{\alpha\beta})\,. \tag{B.1g}$$

The inverse relations are

$$g_{\mu\nu} = \hat{g}_{\mu\nu} - \hat{g}^{\alpha\beta}\hat{g}_{\mu\alpha}\hat{g}_{\nu\beta}\,, \tag{B.2a}$$

$$A^{\alpha}{}_{\mu} = \hat{g}^{\alpha\beta}\hat{g}_{\mu\beta}\,, \tag{B.2b}$$

$$G_{\alpha\beta} = -\hat{g}_{\alpha\beta}\,, \tag{B.2c}$$

$$B_{\mu\nu} \equiv \hat{B}_{\mu\nu} + \hat{g}^{\alpha\beta}\hat{g}_{\alpha\,[\mu}\hat{B}_{\nu]\,\beta}\,, \tag{B.2d}$$

$$C_{\alpha\,\mu} \equiv \hat{B}_{\mu\alpha} + \hat{B}_{\alpha\beta}\hat{g}^{\beta\gamma}\hat{g}_{\mu\gamma}\,, \tag{B.2e}$$

$$C_{\alpha\beta} = \hat{B}_{\alpha\beta}\,, \tag{B.2f}$$

$$\phi = \hat{\phi} - \tfrac{1}{2}\log|\det(\hat{g}_{\alpha\beta})|\,. \tag{B.2g}$$

C Curvature components

For convenience, we change coordinates introducing the coordinate $u = t - k_\infty z$. The computations will be performed in the following vielbein basis

$$\hat{e}^+ = \frac{du}{\mathcal{Z}_-}\,, \quad \hat{e}^- = dt - \frac{\mathcal{Z}_+}{2}\,du\,, \quad \hat{e}^m = \mathcal{Z}_0^{1/2}\,v^m\,, \tag{C.1}$$

where v^m is a vierbein of the hyper-Kähler metric, namely $d\sigma^2 = v^m v^n \delta_{mn}$. In the following we will use ∂_m to indicate the partial derivative with respect to the flat indexes of the hyper-Kähler vielbeins v^m, i.e. we use

$$\partial_m = v_m{}^{\underline{m}}\partial_{\underline{m}}\,. \tag{C.2}$$

C.1 Torsionful spin connection

The components of the torsionful spin connection $\hat{\Omega}_{(-)\hat{a}\hat{b}} = \hat{\omega}_{\hat{a}\hat{b}} - \frac{1}{2}\hat{H}^{(0)}_{\hat{c}\hat{a}\hat{b}}\,\hat{e}^{\hat{c}}$ are given by

$$\hat{\Omega}_{(-)\,+-} = \frac{\epsilon}{\mathcal{Z}_0^{1/2}}\partial_m \log \mathcal{Z}_- \hat{e}^m\,, \tag{C.3a}$$

$$\hat{\Omega}_{(-)\,-m} = \frac{\epsilon}{\mathcal{Z}_0^{1/2}}\partial_m \log \mathcal{Z}_- \hat{e}^+\,, \tag{C.3b}$$

$$\hat{\Omega}_{(-)\,+m} = \frac{\mathcal{Z}_-}{2\mathcal{Z}_0^{1/2}}\partial_m \mathcal{Z}_+ \hat{e}^+ + \frac{1-\epsilon}{\mathcal{Z}_0^{1/2}}\partial_m \log \mathcal{Z}_- \hat{e}^-\,, \tag{C.3c}$$

$$\hat{\Omega}_{(-)\,mn} = \left[\tilde{\omega}_{pmn} + \mathbb{M}^-_{mnpq}\partial_q \log \mathcal{Z}_0\right]\frac{\hat{e}^p}{\mathcal{Z}_0^{1/2}}\,, \tag{C.3d}$$

where $\tilde{\omega}_{mn}$ is the hyper-Kähler spin connection[32] and $\mathbb{M}^-_{mnpq} = \delta_{m[p}\delta_{q]n} - \frac{1}{2}\epsilon_{mnpq}$ are the anti-self-dual $\mathfrak{so}(3)$ subalgebra of of $\mathfrak{so}(4)$.

C.2 Curvature 2-form

The non-vanishing components of the curvature 2-form are

[32] It is defined through $dv^m + \tilde{\omega}^m{}_n \wedge v^n = 0$.

$$\hat{R}_{(-)\,-m} = \frac{\hat{e}^n \wedge \hat{e}^+}{\mathcal{Z}_0}\epsilon \left[\nabla_n \partial_m \log \mathcal{Z}_- - \frac{1}{2}\partial_m \log \mathcal{Z}_- \partial_n \log \mathcal{Z}_0 - \mathbb{M}^-_{pmnq}\partial_p \log \mathcal{Z}_- \partial_q \log \mathcal{Z}_0\right], \tag{C.4a}$$

$$\hat{R}_{(-)+m} = \frac{\mathcal{Z}_- \hat{e}^n \wedge \hat{e}^+}{2\mathcal{Z}_0}\left[\nabla_n \partial_m \mathcal{Z}_+ - \frac{1}{2}\partial_n \log \mathcal{Z}_0 \partial_m \mathcal{Z}_+ + (\epsilon - 1)\partial_m \log \mathcal{Z}_- \partial_n \mathcal{Z}_0 \right.$$
$$-\epsilon \partial_m \mathcal{Z}_+ \partial_n \log \mathcal{Z}_-] + (1-\epsilon)\frac{e^n \wedge e^-}{\mathcal{Z}_0}[\nabla_n \partial_m \log \mathcal{Z}_-$$
$$\left. -\frac{1}{2}\partial_m \log \mathcal{Z}_- \partial_n \log \mathcal{Z}_0 - \mathbb{M}^-_{pmnq}\partial_p \log \mathcal{Z}_- \partial_q \log \mathcal{Z}_0\right], \tag{C.4b}$$

$$\hat{R}_{(-)mn} = \tilde{R}_{mn} + \tilde{F}_{mn}, \tag{C.4c}$$

where

$$\tilde{R}_{mn} = d\tilde{\omega}_{mn} + \tilde{\omega}_{mp} \wedge \tilde{\omega}_{pn}, \tag{C.5a}$$

$$\tilde{F}_{mn} = d\tilde{A}_{mn} + \tilde{A}_{mp} \wedge \tilde{A}_{pn}, \tag{C.5b}$$

and

$$\tilde{A}_{mn} = \mathbb{M}^-_{mnpq}\partial_q \log \mathcal{Z}_0\, v^p. \tag{C.6}$$

When computing the mn components of the curvature 2-form, Eq. (C.4) it is crucial that the spin connection $\tilde{\omega}_{mn}$ and the connection $\tilde{A}_{mn}$ satisfy opposite self-duality relations,

$$\tilde{\omega}_{mn} = +\frac{1}{2}\epsilon_{mnpq}\tilde{\omega}_{pq}, \qquad \tilde{A}_{mn} = -\frac{1}{2}\epsilon_{mnpq}\tilde{A}_{pq}, \tag{C.7}$$

i.e., each of these connections belongs to one of the two orthogonal subspaces, $\mathfrak{so}_\pm(3)$, in which $\mathfrak{so}(4) = \mathfrak{so}_+(3) \oplus \mathfrak{so}_-(3)$ splits.

D The general α'-corrected solution

The α' corrections to the equations of motion and Bianchi identity are encoded in the so-called T-tensors [3], which are given by

$$\hat{T}^{(2)}_{\hat{a}\hat{b}} = -\frac{1}{4}\hat{R}_{(-)\,\hat{a}\hat{c}\hat{d}\hat{e}}\hat{R}_{(-)\,\hat{b}}{}^{\hat{c}\hat{d}\hat{e}}\,, \tag{D.1}$$

$$\hat{T}^{(4)} = \hat{R}_{(-)\,\hat{a}\hat{b}} \wedge \hat{R}_{(-)}{}^{\hat{b}\hat{a}}\,, \tag{D.2}$$

$$\hat{T}^{(0)} = \hat{T}^{(2)}_{\hat{a}\hat{b}}\eta^{\hat{a}\hat{b}}\,. \tag{D.3}$$

Using the calculations displayed in Appendix C, one can check that all the T-tensors are equal to those of the supersymmetric case [11] except for $\hat{T}^{(2)}_{++}$, which simply vanishes in the non-supersymmetric case. This implies that the function $\mathcal{Z}_+$ does not receive α' corrections when $\epsilon = 0$. Hence, we find [11]

$$\mathcal{Z}_+ = \mathcal{Z}^{(0)}_+ - \frac{\epsilon\alpha'}{2}\frac{\partial_m \mathcal{Z}^{(0)}_+ \partial_m \mathcal{Z}^{(0)}_-}{\mathcal{Z}^{(0)}_0 \mathcal{Z}^{(0)}_-} + \mathcal{O}(\alpha'^2)\,, \tag{D.4}$$

where $\mathcal{Z}^{(0)}_{+,-,0}$ are the functions that determine the zeroth-order solution through Eqs. (2.3). As discussed in the main text, the solution for $\mathcal{Z}_+$ is only specified up to a harmonic function, which corresponds to the freedom we have to fix the boundary conditions. The correction to the function $\mathcal{Z}_0$ can be found by solving the Bianchi identity (A.7). Given that $\hat{T}^{(4)}$ does not depend on ϵ, the solution is the same as the one found in [11], namely

$$\mathcal{Z}_0 = \mathcal{Z}^{(0)}_0 + \frac{\alpha'}{4}\left[\frac{\partial_m \mathcal{Z}^{(0)}_0 \partial_m \mathcal{Z}^{(0)}_0}{(\mathcal{Z}^{(0)}_0)^2} + \frac{\partial_m \mathcal{H}^{(0)} \partial_m \mathcal{H}^{(0)}}{(\mathcal{H}^{(0)})^2}\right] + \mathcal{O}(\alpha'^2)\,. \tag{D.5}$$

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