# Classifying Invariants in $SU(3)$ Theories with Adjoints

Arvind Rajaraman

Department of Physics and Astronomy
University of California, Irvine, CA 92697-4575 USA
arajaram@uci.edu

June 15, 2023

## Abstract

**We develop a method for finding the independent invariant tensors of a gauge theory. Our method uses a theorem relating invariant tensors and constant configurations in field space. We apply our method to an $SU(3)$ gauge theory with matter in the adjoint, and find the independent invariant tensors of this theory.**

## Contents



## 1 Introduction

There has been much interest recently in finding the invariants for gauge field theories with multiple fields in various representations. These are important because in a confining theory, the physical spectrum is composed of gauge-invariant states. Similarly, tests of duality conjectures in supersymmetric gauge theories [?] require a knowledge of the gauge invariant operators [?, 3, 4]. Powerful calculational methods have been developed for these invariant tensors in gauge theories (for some of the mathematical literature relevant to invariant theory see: [5] – [8]). However, one of the important questions which is not fully addressed by these methods is: how many independent tensors exist in a given theory?

The method commonly used to address this latter question is the computation of the Hilbert series, which provides a generating function for the number of invariants of a given degree in the fields (see e.g. [9–22]). This can be used to try and find the independent invariants, at least for low degrees in the fields. However, except in a few cases, it is not possible to calculate to arbitrary degree (especially if there are large representations) and it is therefore often not possible to establish the complete set of invariants of the theory using this method.

Here we apply an alternative approach (see also [23, 24]) by using a theorem stating that there is a one-to-one correspondence between invariant operators and gauge-fixed configurations of the fields (a precise statement is in [25]). The logic of the theorem is simple. An arbitrary constant configuration of the fields can be gauge-fixed; that is, one can use gauge transformations to eliminate some parameters of the configuration until the gauge transformations are exhausted (this assumes that both the fields and the gauge symmetries are complexified). The remaining parameters are all, by construction, gauge-invariant operators. In this gauge, there is therefore an obvious one-to-one correspondence between parameters of the configuration and the gauge-invariant operators. This will then continue to hold in any gauge.

The approach to finding the basis set of invariants is then straightforward. We should consider all possible configurations of the fields and gauge-fix each configuration so that the gauge symmetry is broken. In this specific gauge, each remaining parameter of the configuration would be an invariant tensor. Finally we would then find a set of invariant tensors that would reduce to these tensors in the special gauge.

We here apply this approach to a theory with $SU(3)$ gauge group and fields in the adjoint representation. Partial results for the Hilbert basis have been obtained before for these theories [26], [27]. These theories are interesting, firstly because adjoint representations occur in many models that go beyond the Standard model, such as grand unified theories. Secondly, these appear to difficulties matching chiral operators in theories that include adjoint fields [28]. This suggests the importance of understanding the invariant operators. Finally these theories are simple enough that our methods can be applied fairly straightforwardly. Here we are able to explicitly find the basis set of invariants for this theory.

In the next section, we perform a preliminary evaluation of the number of invariants at low degree in the fields. Since we are interested in the symmetry structure of these invariants, and that is not supplied by the Hilbert series, we use a different approach using the LIE program, which allows us to find the symmetries for which independent invariants should exist. We then find explicit expressions for these invariants. These have up to six fields. To verify that these form a complete set, we apply the theorem described above, and gauge fix one of the fields. Generically, this gauge fixing leaves a residual $U(1) \times U(1)$ symmetry, and for a special set of choices, the gauge-fixing leaves a residual $SU(2) \times U(1)$ symmetry.

In each case, we find the gauge-fixed invariants; i.e. the combinations of fields invariant under the residual symmetry. We then verify that all gauge fixed invariants can be reproduced from the list of $SU(3)$ invariants, thereby establishing that we have found the full set of independent invariants. We conclude with a summary of results and ideas for future work.

## 2 Enumeration of Invariants

We start by counting invariants in a manner similar to the Hilbert series. However, since we are interested in finding invariants with specific symmetry structures, and this is not provided by the Hilbert series, we take a slightly different route.

To look for these invariants, we use the program LIE [29]. The program can calculate the decomposition into $SU(3)$ representations of any combinations of adjoints with any flavor symmetry. We use it to compute the the number of invariants of any combination of up to seven $SU(3)$ adjoints.

We must however note that many invariants found by LIE are not independent, but are rather products of smaller invariants. For example from LIE, we find one invariant with two fields with the symmetry $\ydiagram{2}$. With four fields, we find two invariants with symmetry $\ydiagram{2,2}$. However, we know that we can take a product of two of the two-field invariants which has the symmetry $\ydiagram{2,2}$. Subtracting this off, we find one new independent invariant with symmetry $\ydiagram{2,2}$.

We repeat this process for all possible combinations of up to six adjoints. For each possible symmetry of the fields (shown in the first column as a list of the number of boxes per row in the corresponding Young tableau) we use LIE to find the number of invariants (shown in the second column) with that symmetry. (Note that the Hilbert series would provide us with the total number of invariants with a given number of fields, but here we have a more detailed picture.) We then subtract off any invariants that can be written as a product of smaller invariants; these are shown in the following column of the table. The net number of independent invariants after subtraction is shown in the final column. We have found no invariants with seven adjoints; for conciseness, we have not shown the calculations for seven adjoints.

| Tableau | # Invts | Subtract | Net |
|---|---|---|---|
| (2) | 1 | - | 1 |
| (1,1) | 0 | - | 0 |
| (3) | 1 | - | 1 |
| (2,1) | 0 | - | 0 |
| (1,1,1) | 1 | - | 1 |
| (4) | 1 | (2)(2) | 0 |
| (3,1) | 0 | - | 0 |
| (2,2) | 2 | (2)(2) | 1 |
| (2,1,1) | 1 | - | 1 |
| (1,1,1,1) | 0 | - | 0 |
| (5) | 1 | (2)(3) | 0 |
| (4,1) | 1 | (2)(3) | 0 |
| (3,2) | 1 | (2)(3) | 0 |
| (3,1,1) | 2 | (1,1,1)(2) | 1 |
| (2,2,1) | 1 | - | 1 |
| (2,1,1,1) | 1 | (1,1,1)(2) | 0 |
| (1,1,1,1,1) | 1 | - | 1 |

| Tableau | # Invts | Subtract | Net |
|---|---|---|---|
| (6) | 2 | (2)(2)(2)<br>(3)(3) | 0 |
| (5,1) | 0 | - | 0 |
| (4,2) | 3 | (2)(2)(2)<br>(2,2)(2)<br>(3)(3) | 0 |
| (3,3) | 1 | - | 1 |
| (4,1,1) | 2 | (1,1,1)(3)<br>(2,1,1)(2) | 0 |
| (3,2,1) | 2 | (2,2)(2)<br>(2,1,1)(2) | 0 |
| (2,2,2) | 3 | (2)(2)(2)<br>(2,2)(2)<br>(1,1,1)(1,1,1) | 0 |
| (3,1,1,1) | 3 | (2,1,1)(2)<br>(1,1,1)(3) | 1 |
| (2,2,1,1) | 1 | (2,1,1)(2) | 0 |
| (2,1,1,1,1) | 1 | (1,1,1)(1,1,1) | 0 |
| (1,1,1,1,1,1) | 0 | - | 0 |

## 3 The Explicit Form of the Invariants

At this point, we have found that a number of invariants should exist. We here try to find explicit expressions for these invariants.

We are considering a theory with a gauge symmetry of $SU(3)$, and $N$ matter fields transforming in the adjoint of $SU(3)$. We denote the adjoints as $X^I_{i\dot{j}}$ where $i, \dot{j}$ (the gauge indices) are the fundamental and antifundamental indices of $SU(3)$, and $I = 1..N$ labels the different fields. We shall call the $I$ index a flavor index in analogy with the flavors in particle physics.

We shall use lower case indices for gauge indices and upper case for the flavor indices.

The general constant configuration of these adjoints is of the form

$$X^I = \begin{pmatrix} X^I_{1\dot{1}} & X^I_{1\dot{2}} & X^I_{1\dot{3}} \\ X^I_{2\dot{1}} & X^I_{2\dot{2}} & X^I_{2\dot{3}} \\ X^I_{3\dot{1}} & X^I_{3\dot{2}} & X^I_{3\dot{3}} \end{pmatrix} \tag{1}$$

Under a gauge transformation,

$$\delta X^I = [i\alpha_a t^a, X^I] \tag{2}$$

where $t^a$ are the Gell-Mann matrices.

A general invariant is of the form

$$Tr\left(X^I \cdot X^J .. \cdot X^L\right). \tag{3}$$

where we define the matrix multiplication

$$(X^I \cdot X^J)_i^{\ j} = \sum_k (X^I)_i^{\ k} (X^J)_k^{\ j} \tag{4}$$

The table above shows that there is an invariant with two symmetric indices (corresponding to the tableau (2)) . This must be

$$I^{(1)IJ} = Tr\left(X^I \cdot X^J\right) \tag{5}$$

which is automatically symmetric.

Similarly, there is an invariant with tableau (3) . This must be

$$I^{(2)IJK} = Tr\left(X^{\{I} \cdot X^J \cdot X^{K\}}\right) \tag{6}$$

where all the flavor indices are symmetrized; this symmetrization is indicated by the curly brackets.

The invariant with tableau (1,1,1) must be

$$I^{(3)IJK} = Tr\left(X^{[I} \cdot X^J \cdot X^{K]}\right) \tag{7}$$

where all the flavor indices are antisymmetrized; this antisymmetrization is indicated by the square brackets.

We continue this process for the other invariants obtained in the table above; we then have the further invariants

$$(2,2): I^{(4)IJKL} = Tr\left(X^{[I} \cdot X^{J]} \cdot X^{[K} \cdot X^{L]}\right) \tag{8}$$
$$(2,1,1): I^{(5)IJKL} = Tr\left(X^{[I} \cdot X^J \cdot X^{K]} \cdot X^L\right) \tag{9}$$
$$(3,1,1): I^{(6)IJKLM} = Tr\left(X^{[I} \cdot X^J \cdot X^{K]} \cdot X^{\{L} X^{M\}}\right) \tag{10}$$
$$(2,2,1): I^{(7)IJKLM} = Tr\left(X^{[I} \cdot X^J \cdot X^{K]} \cdot X^{[L} \cdot X^{M]}\right) \tag{11}$$
$$(1,1,1,1,1): I^{(8)IJKLM} = Tr\left(X^{[I} \cdot X^J \cdot X^K \cdot X^L \cdot X^{M]}\right) \tag{12}$$
$$(3,3): I^{(9)IJKLMN} = Tr\left(X^{[I} \cdot X^{J]} \cdot X^{[K} \cdot X^{L]} \cdot X^{[M} \cdot X^{N]}\right) \tag{13}$$
$$(3,1,1,1): I^{(10)IJKLMN} = Tr\left(X^{[I} \cdot X^{J]} \cdot X^{[K} \cdot X^{L]} \cdot X^{[M} \cdot X^{N]}\right) + (K \longleftrightarrow M) \tag{14}$$

This is a total of ten possible invariants, with up to six fields.

## 4 SU(3) adjoints: The gauge-fixed field space

The calculations in the previous section have yielded a set of independent invariants for the $SU(3)$ theory with fields in the adjoint. Furthermore, there were no new invariants with seven fields. However, this argument can never prove that this is truly a complete set. To achieve this proof, we turn to the theorem described in the introduction, stating that the invariants are in 1-1 correspondence with the gauge-fixed configurations.

We therefore try to gauge fix the symmetry. Using the complexified $SU(3)$ gauge transformations, we can bring one adjoint (which we label $X^0$) to a diagonal form, while the other fields remain general

$$X^0 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \qquad X^I = \begin{pmatrix} X^I_{1\dot{1}} & X^I_{1\dot{2}} & X^I_{1\dot{3}} \\ X^I_{2\dot{1}} & X^I_{2\dot{2}} & X^I_{2\dot{3}} \\ X^I_{3\dot{1}} & X^I_{3\dot{2}} & X^I_{3\dot{3}} \end{pmatrix} \tag{15}$$

with $a+b+c=0$.

This gauge fixing partially breaks the $SU(3)$ symmetry. We have typically a preserved continuous symmetry of $U(1)\times U(1)$ generated by

$$t^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t^8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{16}$$

$X^0$ (and hence $a, b, c$) are invariant under the action of $U(1)\times U(1)$.

We could then try to gauge fix these residual symmetries (using the other adjoints) to find the gauge invariant parameters. Instead, we can just as well directly look for the $U(1)\times U(1)$ gauge invariant operators describing the configuration (15). Since these are found after gauge fixing one adjoint, these operators will in general not be manifestly $SU(3)$ invariant. We will then need to find fully $SU(3)$ invariant operators that are in one-to-one correspondence with these operators with $U(1)\times U(1)$ gauge invariance.

Under the $U(1)\times U(1)$ transformations, we have

$$\delta_{t^3} X^I = \delta_{t^3} \begin{pmatrix} X^I_{1\dot{1}} & X^I_{1\dot{2}} & X^I_{1\dot{3}} \\ X^I_{2\dot{1}} & X^I_{2\dot{2}} & X^I_{2\dot{3}} \\ X^I_{3\dot{1}} & X^I_{3\dot{2}} & X^I_{3\dot{3}} \end{pmatrix} = i\alpha_3 \begin{pmatrix} 0 & 2X^I_{1\dot{2}} & X^I_{1\dot{3}} \\ -2X^I_{2\dot{1}} & 0 & -X^I_{2\dot{3}} \\ -X^I_{3\dot{1}} & X^I_{3\dot{2}} & 0 \end{pmatrix} \tag{17}$$

$$\delta_{t^8} X^I = \delta_{t^8} \begin{pmatrix} X^I_{1\dot{1}} & X^I_{1\dot{2}} & X^I_{1\dot{3}} \\ X^I_{2\dot{1}} & X^I_{2\dot{2}} & X^I_{2\dot{3}} \\ X^I_{3\dot{1}} & X^I_{3\dot{2}} & X^I_{3\dot{3}} \end{pmatrix} = i\alpha_8 \begin{pmatrix} 0 & 0 & 3X^I_{1\dot{3}} \\ 0 & 0 & 3X^I_{2\dot{3}} \\ -3X^I_{3\dot{1}} & -3X^I_{3\dot{2}} & 0 \end{pmatrix} \tag{18}$$

so that we can identify the charges of each component of the adjoint under the $U(1)\times U(1)$ as

| Field | $X^I_{1\dot{1}}$ | $X^I_{1\dot{2}}$ | $X^I_{1\dot{3}}$ | $X^I_{2\dot{1}}$ | $X^I_{2\dot{2}}$ | $X^I_{2\dot{3}}$ | $X^I_{3\dot{1}}$ | $X^I_{3\dot{2}}$ | $X^I_{3\dot{3}}$ |
|---|---|---|---|---|---|---|---|---|---|
| $U(1)_1$ | 0 | 2 | 1 | -2 | 0 | -1 | -1 | 1 | 0 |
| $U(1)_2$ | 0 | 0 | 3 | 0 | 0 | 3 | -3 | -3 | 0 |

Table 1: Charges of fields under $U(1)\times U(1)$

It is straightforward to find all the combinations of fields invariant under the $U(1)\times U(1)$. These are generated by a small set of operators. These operators can be categorized as

(a) invariants with one flavor index: $X^I_{1\dot{1}}, X^I_{2\dot{2}}, X^I_{3\dot{3}} = -X^I_{1\dot{1}} - X^I_{2\dot{2}}$.

(b) invariants with two flavor indices: $X^I_{2\dot{3}}X^J_{3\dot{2}}, X^I_{3\dot{1}}X^J_{1\dot{3}}, X^I_{1\dot{2}}X^J_{2\dot{1}}$
(c) invariants with three flavor indices: $X^I_{1\dot{2}}X^J_{2\dot{3}}X^K_{3\dot{1}}, X^I_{2\dot{1}}X^J_{3\dot{2}}X^K_{1\dot{3}}$
We shall refer to these as the *gauge-fixed invariants* at the generic parameter point.

We now ask if all the gauge-fixed invariants can be reproduced by the candidate set of invariants found in the previous section. That is, we replace one or more adjoints in the candidate invariants by (15), and see if the resulting expressions cover all the gauge fixed invariants above. We are then assured that there are no further independent invariants at the generic parameter point.

We try this for the gauge fixed invariants with one flavor index. We find

$$I^{(1)0I} = aX^I_{11} + bX^I_{22} + cX^I_{33} \tag{19}$$
$$I^{(2)00I} = a^2X^I_{11} + b^2X^I_{22} + c^2X^I_{33} \tag{20}$$

and these are indeed sufficient to reproduce the gauge fixed invariants with one flavor index for generic $a, b, c$.

For the invariants with two indices, we must consider both the symmetric and asymmetric flavor combinations. The three symmetric combinations are reproduced by

$$I^{(1)IJ} = X^I_{2\dot{3}}X^J_{3\dot{2}} + X^I_{3\dot{1}}X^J_{1\dot{3}} + X^I_{1\dot{2}}X^J_{2\dot{1}} \tag{21}$$
$$I^{(2)0IJ} = -2\left(aX^I_{2\dot{3}}X^J_{3\dot{2}} + bX^I_{3\dot{1}}X^J_{1\dot{3}} + cX^I_{1\dot{2}}X^J_{2\dot{1}}\right) + \ldots \tag{22}$$
$$I^{(4)0I0J} = 3\left(a^2X^I_{2\dot{3}}X^J_{3\dot{2}} + b^2X^I_{3\dot{1}}X^J_{1\dot{3}} + c^2X^I_{1\dot{2}}X^J_{2\dot{1}}\right) + \ldots \tag{23}$$

where the ellipses indicate expressions made of products of smaller invariants.

Similarly, the three $SU(3)$ invariants $I^{(3)0IJ}, I^{(5)0IJ0}, I^{(6)0IJ00}$ reproduce the three antisymmetric combinations.

In the invariants with three indices, the flavor indices can be completely symmetrized, completely antisymmetrized or in a mixed symmetry. The two completely symmetric combinations are reproduced by

$$I^{(2)IJK} = X^{\{I}_{1\dot{2}}X^J_{2\dot{3}}X^{K\}}_{3\dot{1}} + X^{\{I}_{2\dot{1}}X^J_{3\dot{2}}X^{K\}}_{1\dot{3}} + \ldots \tag{24}$$
$$I^{(9)I0J0K0} = (a-b)(b-c)(c-a)\left(X^{\{I}_{1\dot{2}}X^J_{2\dot{3}}X^{K\}}_{3\dot{1}} - X^{\{I}_{2\dot{1}}X^J_{3\dot{2}}X^{K\}}_{1\dot{3}}\right) + \ldots \tag{25}$$

Similarly, the two completely antisymmetric combinations are reproduced by $I^{(3)IJK}, I^{(10)IJK000}$, while the mixed structures are reproduced from $I^{(4)IJK0}, I^{(5)IJ0K}, I^{(6)IJ0K0}, I^{(7)IJ0K0}$.

The invariants in the previous section therefore reproduce all gauge fixed invariants at the generic parameter point.

## 5 The Point of Enhanced symmetry

While the analysis above yields the invariants at generic points in parameter space, there are special points that needs to be considered separately. These are where the symmetry is enhanced; this can happen when two eigenvalues of $X^0$ coincide e,g. when $b = c$.

At this point, the unbroken symmetry generators are

$$t^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad t^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad t^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad t^8 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{26}$$

and the continuous symmetry is enhanced to $SU(2)\times U(1)$. The invariants therefore need to be invariant under $SU(2)\times U(1)$.

The $SU(3)$ adjoint decomposes as an adjoint of $SU(2)$,

$$a^I = \begin{pmatrix} \frac{X^I_{2\dot{2}}-X^I_{3\dot{3}}}{2} & X^I_{2\dot{3}} \\ X^I_{3\dot{2}} & \frac{X^I_{3\dot{3}}-X^I_{2\dot{2}}}{2} \end{pmatrix} \tag{27}$$

two doublets,

$$q^I_a = (X^I_{1\dot{2}}, X_{1\dot{3}}) \qquad Q^J_{\dot{a}} = (X^J_{2\dot{1}}, X_{3\dot{1}}) \tag{28}$$

and a scalar

$$s^I = \frac{X^I_{2\dot{2}} + X^I_{3\dot{3}}}{2} \tag{29}$$

Under the unbroken $U(1)$, the $q, Q$ have charges $3, -3$ respectively, while the scalar and adjoint are neutral.

The $U(1)$ invariants are the scalar, the adjoint $a^i$, and $q^I\cdot Q^J \equiv q^I_a Q^J_{\dot{a}}$, and, $q^I\cdot\sigma^i\cdot Q^J \equiv q^I_a \sigma^i_{\dot{a}b} Q^J_{\dot{b}}$. These are scalars and fundamentals under $SO(3)$. The invariant tensors of $SO(3)$ with fundamentals are known to be $\delta_{ij}, \epsilon_{ijk}$, which allows us to form the complete set of invariants of this theory by contracting the invariant tensors with the fundamentals.

Using the Fierz identities

$$\sigma^i_{\dot{a}b}\sigma^i_{\dot{c}d} = -\delta_{\dot{a}b}\delta_{\dot{c}d} + 2\delta_{\dot{a}d}\delta_{\dot{c}b} \tag{30}$$

helps us to reduce some of these invariants to products of smaller invariants. Once this is done, the remaining invariants are

(a) invariants with one flavor index: $s^I$

(b) invariants with two flavor indices: $q^I \cdot Q^J$ , $a^K_i a^L_i$

(c) invariants with three flavor indices: $q^I \cdot \sigma^i \cdot Q^J a^K_i$ , $\epsilon^{ijk} a^K_i a^L_j a^M_k$

(d) invariants with four flavor indices: $\epsilon^{ijk} q^I \cdot \sigma^i \cdot Q^J a^K_j a^L_k$

We now see if these can be reproduced from the $SU(3)$ candidate invariants from the previous section, once we substitute 15.

We find that the invariants with one flavor index are reproduced as $I^{(1)0I} = 4s^I$.

When considering the invariants with two flavor indices, $q^I \cdot Q^J$ can have either symmetrized or antisymmetrized indices, while , $a^K_i a^L_i$ must have symmetrized indices. The symmetrized combinations are reproduced as

$$I^{(1)IJ} = 2a^K_i a^L_i + 2q^I \cdot Q^J + 2Q^I \cdot q^J + \ldots \tag{31}$$

$$I^{(2)0IJ} = 2a^I_i a^J_i - q^I \cdot Q^J - Q^I \cdot q^J + \ldots \tag{32}$$

while the antisymmetric combination is

$$I^{(3)0IJ} = -3q^I \cdot Q^J + 3Q^I \cdot q^J + \ldots \tag{33}$$

For the invariants with three flavor indices, $\epsilon^{ijk} a^K_i a^L_j a^M_k$ is completely antisymmetric in the flavor indices, while $q^I \cdot \sigma^i \cdot Q^J a^K_i$ can have any symmetry. The completely antisymmetric combinations are reproduced from

$$I^{(3)IJK} = q^J \cdot \sigma^i \cdot Q^K a^I_i + \frac{1}{2} a^I a^J a^K + [IJK] + \ldots \tag{34}$$

$$I^{(5)IJK0} = -\frac{4}{3} q^J \cdot \sigma^i \cdot Q^K a^I_i + \frac{1}{3}\epsilon^{ijk} a^I_i a^J_j a^K_k + [IJK] + \ldots \tag{35}$$

while the completely symmetric combination is reproduced from

$$I^{(2)IJK} = q^I \cdot \sigma^i \cdot Q^J a_i^K + \{IJK\} + \ldots \tag{36}$$

Similarly, the mixed symmetries are obtained from $I^{(4)IJK0}$.

In the invariant with four flavor indices, the flavor indices of the two $a_i$ are antisymmetrized and are combined with the remaining two indices.

If all indices are antisymmetric, this can be reproduced from

$$I^{(8)IJKL0} = 2\epsilon^{ijk} q^I \cdot \sigma^i \cdot Q^J a_j^K a_k^L + \text{antisymmetric in IJKL} \tag{37}$$

If three indices are antisymmetric (this corresponds to the Young tableau (2,1,1,1)), these is the invariant $\epsilon^{ijk} q^I \cdot \sigma^i \cdot Q^J a_j^K a_k^L$ with either IKL or JKL antisymmetrized. These are found to be reproduced from $I^{(5)IJKL}, I^{(6)IJKL0}$.

The other possibilities are that we have $\epsilon^{ijk} q^I \cdot \sigma^i \cdot Q^J a_j^K a_k^L$ with KL antisymmetrized, and IJ antisymmetrized, or with KL antisymmetrized, and IJ symmetrized. These are respectively reproduced from $I^{(4)IJKL}, I^{(6)IJ0KL}$.

At this point, we have reproduced all the gauge-fixed invariants, both at the generic parameter point as well as at the enhanced symmetry point, from the $SU(3)$ invariants listed in section (3). This proves that this list is a complete set of invariants for the theory.

## 6 Conclusions

We have discussed a method to determine a set of independent invariants of a theory with a general gauge group and matter field in a general representation. We have done this by using a theorem that relates gauge-fixed configurations to the independent invariants in a gauge theory. Specifically, this theorem asserts that the constant configurations of the fields, identified by complex gauge transformations, are in one-to-one correspondence with the invariants in the theory. We have shown that this method provides a straightforward approach to find the independent invariant tensors.

We have applied these methods to a theory with $SU(3)$ gauge symmetry and with matter in the adjoint, and found the independent invariant tensors, listed in section (3). There are 10 independent tensors with up to six fields. This generalizes and confirms results of [26,27].

Many other groups and representations remain to be studied; in particular little is known of the invariant tensors of the exceptional groups. We hope to return to this topic in future work.

**Acknowledgements**

This work was supported in part by NSF Grant No. PHY-1915005.

## References

[1] N. Seiberg, Nucl. Phys. B **435**, 129-146 (1995) doi:10.1016/0550-3213(94)00023-8 [arXiv:hep-th/9411149 [hep-th]].

[2] D. Kutasov, A. Schwimmer and N. Seiberg, Nucl. Phys. B **459**, 455-496 (1996) doi:10.1016/0550-3213(95)00599-4 [arXiv:hep-th/9510222 [hep-th]].

[3] P. Brax, C. Grojean and C. A. Savoy, Nucl. Phys. B **561**, 77-99 (1999) doi:10.1016/S0550-3213(99)00501-5 [arXiv:hep-ph/9808345 [hep-ph]].

[4] P. Pouliot, JHEP **01**, 021 (1999) doi:10.1088/1126-6708/1999/01/021 [arXiv:hep-th/9812015 [hep-th]].

[5] P. Cvitanovic, "Group theory: Birdtracks, Lie's and exceptional groups," Princeton University Press, Princeton, NJ, (2008), URL: www.birdtracks.eu.

[6] H. Weyl, "The Classical Groups, Their Invariants and Representations," Princeton University Press, 1946.

[7] W. Fulton and J. Harris, "Representation Theory, A First Course," Springer-Verlag, 1991.

[8] P. Olver, "Classical Invariant Theory," London Mathematical Society Student Texts #44, 1999.

[9] S. Benvenuti, B. Feng, A. Hanany and Y. H. He, JHEP **11**, 050 (2007) doi:10.1088/1126-6708/2007/11/050 [arXiv:hep-th/0608050 [hep-th]].

[10] B. Feng, A. Hanany and Y. H. He, JHEP **03**, 090 (2007) doi:10.1088/1126-6708/2007/03/090 [arXiv:hep-th/0701063 [hep-th]].

[11] E. E. Jenkins and A. V. Manohar, JHEP **10**, 094 (2009) doi:10.1088/1126-6708/2009/10/094 [arXiv:0907.4763 [hep-ph]].

[12] A. Hanany, E. E. Jenkins, A. V. Manohar and G. Torri, JHEP **03**, 096 (2011) doi:10.1007/JHEP03(2011)096 [arXiv:1010.3161 [hep-ph]].

[13] L. Lehman and A. Martin, Phys. Rev. D **91**, 105014 (2015) doi:10.1103/PhysRevD.91.105014 [arXiv:1503.07537 [hep-ph]].

[14] B. Henning, X. Lu, T. Melia and H. Murayama, Commun. Math. Phys. **347**, no.2, 363-388 (2016) doi:10.1007/s00220-015-2518-2 [arXiv:1507.07240 [hep-th]].

[15] B. Henning, X. Lu, T. Melia and H. Murayama, JHEP **08**, 016 (2017) [erratum: JHEP **09**, 019 (2019)] doi:10.1007/JHEP08(2017)016 [arXiv:1512.03433 [hep-ph]].

[16] S. Keppeler and M. Sjodahl, JHEP **09**, 124 (2012) doi:10.1007/JHEP09(2012)124 [arXiv:1207.0609 [hep-ph]].

[17] S. Keppeler and M. Sjödahl, J. Math. Phys. **55**, 021702 (2014) doi:10.1063/1.4865177 [arXiv:1307.6147 [math-ph]].

[18] J. Alcock-Zeilinger and H. Weigert, J. Math. Phys. **58**, no.5, 051703 (2017) doi:10.1063/1.4983479 [arXiv:1610.08802 [math-ph]].

[19] A. Trautner, JHEP **05**, 208 (2019) doi:10.1007/JHEP05(2019)208 [arXiv:1812.02614 [hep-ph]].

[20] B. Henning, X. Lu, T. Melia and H. Murayama, JHEP **10**, 199 (2017) doi:10.1007/JHEP10(2017)199 [arXiv:1706.08520 [hep-th]].

[21] Y. Wang, B. Yu and S. Zhou, JHEP **09**, 053 (2021) doi:10.1007/JHEP09(2021)053 [arXiv:2107.06274 [hep-ph]].

[22] P. Pouliot, J. Phys. A **34**, 8631 (2001) doi:10.1088/0305-4470/34/41/317 [hep-th/0107151].

[23] D. Berger, J. N. Howard and A. Rajaraman, LHEP **1**, no.2, 14-19 (2018) doi:10.31526/LHEP.2.2018.04 [arXiv:1806.04332 [hep-th]].

[24] Y. Almumin, J. Baretz, E. Bryan and A. Rajaraman, Acta Phys. Polon. B **54**, no.5, 2 (2023) doi:10.5506/APhysPolB.54.5-A2 [arXiv:2009.06545 [hep-th]].

[25] M. A. Luty and W. I. Taylor, "Varieties of vacua in classical supersymmetric gauge theories," Phys. Rev. D **53**, 3399 (1996) [hep-th/9506098].

[26] P. Dittner, Commun. Math. Phys. **22**, 238-252 (1971) doi:10.1007/BF01877709

[27] P. Dittner, Commun. Math. Phys. **27**, 44-52 (1972) doi:10.1007/BF01649658

[28] K. Intriligator and E. Nardoni, JHEP **09**, 043 (2016) doi:10.1007/JHEP09(2016)043 [arXiv:1604.04294 [hep-th]].

[29] van Leeuwen, M.A.A., Cohen, A.M., Lisser, B.: LiE, A Package for Lie Group Computations. Available from http://wwwmathlabo.univ-poitiers.fr/ maavl/LiE/