

Three Dimensional Odd Viscosity in Ferrofluids with Vorticity-Magnetization Coupling

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Ferrofluids are a synthetic magnetic colloid consisting of magnetized nanoparticles surrounded by a repulsive surfactant layer. When subjected to an external magnetic field the ferrofluid acquires a macroscopic magnetization density which leads to magnetic behavior that is intricately coupled to the ambient fluid dynamics. Ferrofluids share several features with the chiral active fluids composed of unidirectionally spinning hematite cubes, which have been shown to possess a 2D non-dissipative odd viscosity term (Nature Physics, 15, 1188–1194 (2019)). In standard ferrofluid dynamics, 3D versions of parity breaking terms are not commonly observed, partly because of the small size of the magnetic particles. In this work, we investigate if there are unique mechanisms in ferrofluids that can lead to a 3D odd viscosity term. Our results show that coupling the fluid vorticity ($\vec{\omega}$) to the magnetization (\vec{M}) with a term proportional to $\vec{\omega} \cdot \vec{M}$ leads to parity breaking terms in ferrofluid hydrodynamics, and results in a three dimensional odd viscosity term when the magnetization is relaxed to the direction of a uniform and static applied field. Hele-Shaw cells are commonly used devices to investigate ferrofluids and we demonstrate that this coupling reproduces the parity odd generalization of Darcy’s Law discussed in a recent work (Phys. Rev. Fluids 7, 114201 (2022)). A potential experimental setup is discussed which may reveal the presence of this coupling in a ferrofluid confined to a Hele-Shaw cell.

I. INTRODUCTION

Parity violation is ubiquitous in physical systems, from astronomical and geological phenomena at the largest of scales [1–3], to the superfluid and quantum hall properties of electrons in condensed matter systems [4–20]. In describing the collective dynamics of many particle systems, parity breaking effects typically originate from the presence of intrinsic angular momentum at the level of the constituent particles. Recently much theoretical work has focused on describing so called “active matter”, systems in which energy is not conserved at the microscopic level. These systems naturally exhibit parity-breaking (parity odd) behavior [21–23]. Parity odd behavior in active matter systems can be described in a variety of ways. Odd elasticity is a framework that aims to capture these effects by studying the relation between stresses and strains for non-conservative microscopic interactions [24], while odd ideal gas descriptions study chiral collisions within a kinetic theory framework [25]. Additionally, many chiral active systems are best described using a fluid description [26, 27].

In contrast to ordinary incompressible fluids, these active fluids manifest additional transport coefficients beyond shear viscosity. Perhaps the most famous of them is the rotational viscosity, which acts as a relaxation mechanism for the fluid particle angular momenta. In 2D, there is an extra viscosity known as “odd” or “Hall” viscosity, which preserves the fluid isotropy. Different from the other viscosity coefficients, odd viscosity is neither dissipative nor invariant under parity symmetry. Furthermore, its effects are only present for particular boundary conditions, such as free surface problems [28–30]. On the other hand, parity breaking in 3D is incompatible

with isotropy, leading to a much larger class of transport coefficients [31]. However, we recently showed that within a Hele-Shaw cell, the 3D parity breaking effects in incompressible fluids are revealed in a strikingly simple fashion [31, 32].

A ferrofluid is a type of 3D active matter fluid system that appears to possess several characteristics associated with parity-odd behavior. It is made up of magnetic nanoparticles coated in a hydrophobic surfactant layer and suspended in a carrier fluid. Each nanoparticle has an intrinsic magnetic moment, which is the sum of the magnetic moments generated by the atoms within it. Without an external magnetic field, thermal fluctuations prevent alignment of these moments, and the fluid is non-magnetic. When a substantial magnetic field is applied, the nanoparticles show collective behavior and the fluid becomes magnetized. For a detailed description of ferrofluids see Refs. 33–39. Two common laboratory setups are that of a steady applied field, where the magnetization density \vec{M} aligns with the external field [40, 41], and a rotating applied field, in which \vec{M} rotates with the external field with the same angular frequency but with a phase lag [42–44].

The case of a rotating applied field, typically termed ‘spin-up’ flow, in principle should result in parity odd behavior as the rotating field causes the magnetic particles to rotate, which introduces intrinsic angular momentum into the system at the microscopic level. However, in the limit of vanishing moment of inertia per particle, due to its small size ($I \sim r^2 \rightarrow 0$), the microscopic angular momentum does not manifest itself at the hydrodynamic scale. Thus, there is no parity breaking behavior for ferrofluids in this setup. This should be contrasted to the configuration investigated in the recent chiral active col-

loids of Soni et al [27], where the magnetic colloids, which are orders of magnitude larger than ferrofluid nanoparticles, shows experimental signatures of parity odd effects in the form of 2D odd viscosity. Furthermore, the case of steady applied field trivially does not lead to any parity odd transport coefficients, since no net angular momentum is generated at the macroscopic level.

Thus, it seems that in order for ferrofluids to manifest a 3D analog of odd viscosity, one needs the magnetization to play the role of the fluid intrinsic angular momentum, since the latter usually gets washed out due to the small size of the ferrofluid particles. Following this intuition, in this paper we propose a coupling between magnetization density \vec{M} and fluid vorticity $\vec{\omega}$, which after relaxing the direction of magnetization leads to the 3D odd viscosity in the case of a steady applied field. This coupling is motivated by the recent work of Markovich and Lubensky [22], where they have considered the intrinsic angular momentum coupling to fluid vorticity using a Poisson bracket formalism. The key feature of the magnetization-vorticity coupling is that the appearance of parity odd behavior at the macroscopic scales does not necessarily result from the angular momentum of the ferrofluid nanoparticles. In fact, this behavior should be present even in the limit of vanishing particle angular momentum.

The equations of motion for ferrohydrodynamics are usually derived from a detailed analysis of the microscopic forces and torques acting on the fluid [33]. However, to understand the origins of the magnetization-vorticity coupling it's more insightful to use a Poisson bracket (PB) formalism [45, 46], where the dissipative terms are incorporated through a dissipation function. Within this setup the relevant three dimensional parity breaking arises when we add the following *intrinsic* term to the ferrohydrodynamic Hamiltonian

$$H_{M\omega} = \frac{\gamma}{2} \int d^3r \vec{\omega} \cdot \vec{M}, \quad (1)$$

where $\vec{\omega}$ is the fluid vorticity, \vec{M} is the magnetization per particle and γ is a coupling constant that depends on the microscopic properties of the ferrofluid nanoparticles. As pointed out before, this term is analogous in form to that seen in Markovich and Lubensky [22], with the intrinsic angular momentum $\vec{\ell}$ replaced by $\gamma\vec{M}$. In that analysis it was seen that such term leads to what the authors referred to as 3D odd viscosity. This 3D odd viscosity breaks the fluid isotropy, hence, the corresponding viscosity term contains a transverse part (with respect to \vec{M}), which has the same form as the 2D odd viscosity, and a longitudinal part, which is only present in three dimensional systems.

When this term is included in a ferrofluid system it can lead to a variety of parity-breaking behaviors. However, this paper will focus specifically on the case of a constant external field, and when the ferrofluid is confined to a Hele-Shaw cell. In this scenario the system is described by a modified version of Darcy's law, which was

previously derived for general parity-odd flows in Ref. 32. Using a Hele-Shaw cell provides a reliable experimental setup to measure the strength of this new coupling.

This paper is organized as follows. In section II we review the standard equations describing ferrofluids, and show how they can arise from a Hamiltonian and PB structure. In section III we introduce a vorticity magnetization term into the ferrofluid Hamiltonian and give the modified equations of motion. In section IV we confine the ferrofluid to a Hele-Shaw cell and examine the modification to Darcy's Law.

II. REVIEW OF STANDARD FERROFLUID DYNAMICS

In order to understand the significance of our modification, this section will provide an overview of the standard ferrofluid system and its governing equations. The ferrofluid nanoparticles are assumed to be evenly distributed throughout the carrier fluid, and the effects of the surfactant layer surrounding each particle are typically not considered. The ferrofluid is treated as a single entity with a constant density ρ , velocity v_i , pressure P , particle angular velocity Ω_i , and magnetization per particle M_i . The applied field is denoted by B_i , and throughout this paper we will employ Einstein summation notation on repeated indices.

The fluid is assumed to be incompressible with good approximation, $\partial_i v_i = 0$, and satisfy

$$D_t v_i + \partial_i P = \nu \nabla^2 v_i + M_j \partial_j B_i - \Gamma \epsilon_{ijk} \partial_j (\omega_k - 2\Omega_k), \quad (2)$$

where the material derivative is defined as $D_t = \partial_t + v_j \partial_j$, ϵ_{ijk} is the Levi-Civita symbol, ν is the kinematic shear viscosity, and Γ is the kinematic rotational viscosity, which creates a drag force whenever the local particle rotation differs from (half) the local vorticity ω_i . Note that in the above equation, and throughout this paper, the pressure P has been scaled by the density, which has subsequently been set to unity. Conservation of local angular momentum gives the equation of motion for Ω_i to be

$$I D_t \Omega_i = \epsilon_{ijk} M_j B_k + 2\Gamma (\omega_i - 2\Omega_i), \quad (3)$$

where I is the moment of inertia per particle, given by $I \sim r^2$, with r being the typical radius of the ferrofluid particles. For the sake of simplicity, in our analysis we assume the nanoparticles are spherical, since the general case does not add much complexity. The governing equation for M_i assumes that the magnetization is 'frozen' in to the particle, and so the particles rotate as rigid objects,

$$D_t M_i = \epsilon_{ijk} \Omega_j M_k. \quad (4)$$

A key assumption in ferrofluids is that the constituent particles are vanishingly small, and so we take the limit

$I \rightarrow 0$. From Eq. (3) we see that this assumption implies that magnetic torque balances rotational friction

$$2\Gamma(\omega_i - 2\Omega_i) = -\epsilon_{ijk}M_jB_k. \quad (5)$$

We now solve Eq. (5) for Ω_i and substitute into (2) and (4), which gives

$$D_tv_i + \partial_iP = \mu\nabla^2v_i + M_j\partial_jB_i + \frac{1}{2}\epsilon_{ijk}\partial_j(\epsilon_{klm}M_lB_m), \quad (6)$$

$$D_tM_i = \frac{1}{2}\epsilon_{ijk}\omega_jM_k - \frac{1}{4\Gamma}\epsilon_{ijk}M_j(\epsilon_{klm}M_lB_m). \quad (7)$$

Due to the smallness of the magnetic nanoparticles, thermal fluctuations can destroy the fluid magnetization for sufficiently low magnetic fields. A simplified model, appropriate for low magnetic field strengths, adds to Eq. (7) a dissipation term of the form $-(M_i - M_i^0)/\tau$, where τ is a characteristic relaxation time. The magnitude of the equilibrium magnetization M_i^0 is given in terms of the Langevin function

$$M_i^0 = \frac{\mu B_i}{B} \left[\coth\left(\frac{\mu B}{k_B T}\right) - \frac{k_B T}{\mu B} \right], \quad (8)$$

where μ is the magnetic moment of each particle. In the absence of an external applied field, or for high temperatures, the magnetic moments of each particle is ‘randomized’ and the magnetization M_i vanishes [43, 46]. A more detailed analysis takes the particle orientations to be random variables governed by Eq. (3), and averages over all angles. Taking the limit $I \rightarrow 0$ and substituting $\langle\Omega_i\rangle_T$ into the equation for M_i yields a similar dissipation term, and modifies the rotational viscosity term [33, 37]. For the purposes of this paper, either analysis will do, since the parity odd effects that we discuss are general and do not rely on specific microscopic details of the model.

The above equations of motion can also be derived starting from a Hamiltonian framework in combination with a dissipation function [45, 46]. Since the ferrofluid is assumed to be incompressible, we will scale the Hamiltonian by $1/\rho$ [47] and write the system in terms of the velocity v_i and the angular momentum per particle $\ell_i = I\Omega_i$. Starting from a Hamiltonian of the form

$$H = \int d^3r \left[\frac{1}{2}v_iv_i + \frac{1}{2I}\ell_i\ell_i - M_iB_i \right] \quad (9)$$

and Poisson brackets give by

$$\{v_i(\vec{r}), v_j(\vec{r}')\} = \epsilon_{ijk}\omega_k(\vec{r})\delta(\vec{r} - \vec{r}'), \quad (10)$$

$$\{v_i(\vec{r}), \ell_j(\vec{r}')\} = \partial_i\ell_j(\vec{r}')\delta(\vec{r} - \vec{r}'), \quad (11)$$

$$\{v_i(\vec{r}), M_j(\vec{r}')\} = \partial_iM_j(\vec{r}')\delta(\vec{r} - \vec{r}'), \quad (12)$$

$$\{\ell_i(\vec{r}), \ell_j(\vec{r}')\} = \epsilon_{ijk}\ell_k(\vec{r}')\delta(\vec{r} - \vec{r}'), \quad (13)$$

$$\{\ell_i(\vec{r}), M_j(\vec{r}')\} = \epsilon_{ijk}M_k(\vec{r}')\delta(\vec{r} - \vec{r}'), \quad (14)$$

one can derive the equations of motion (2-4) using the following form of Hamilton’s equations

$$\partial_tv_i = \{v_i, H\} - \frac{\delta R}{\delta v_i} - \partial_i\lambda, \quad (15)$$

$$\partial_t\ell_i = \{\ell_i, H\} - \frac{\delta R}{\delta \ell_i}, \quad (16)$$

$$\partial_tM_i = \{M_i, H\} - \frac{\delta R}{\delta M_i}, \quad (17)$$

where the effects of viscosity and dissipation are included by use of a dissipation function of the form

$$R = \int d^3r \left[\frac{1}{2}\nu(\partial_iv_j + \partial_jv_i)^2 + \frac{1}{2}\Gamma(2\Omega_i - \epsilon_{ijk}\partial_jv_k)^2 + \frac{1}{2\tau}(M_i - M_i^0)^2 \right]. \quad (18)$$

The equation of motion (15) is shifted by the gradient of a scalar function λ , which up to some redefinitions will be related to the fluid pressure. For incompressible fluids, the pressure does not follow from an equation of state, but works as a Lagrange multiplier to ensure that the velocity field dynamics is compatible with the incompressibility condition. Hence, the pressure function is completely determined by the flow and not an independent variable. Equations of motion (15-17) with the choice of Hamiltonian (9) and dissipation function (18) lead to equations (2-4) with the aforementioned relaxation term $-(M_i - M_i^0)/\tau$ added to the magnetization dynamics (4).

In the standard ferrofluid analysis, the assumption of vanishing moment of inertia per particle I is imposed before setting $\tau \rightarrow 0$. In other words, the nanoparticle angular momenta relax much earlier than their magnetization. For uniform magnetic field, the resulting equation is simply the Navier-Stokes equation

$$D_tv_i = \nu\nabla^2v_i - \partial_iP, \quad (19)$$

together with the incompressibility condition

$$\partial_iv_i = 0. \quad (20)$$

III. PARITY BREAKING IN FERROFLUIDS VIA VORTICITY-MAGNETIZATION COUPLING

The Hamiltonian and PB framework of ferrofluids provides a straightforward way to add modifications to the system. While many additional terms are in principle possible, not all terms are relevant or lead to interesting behavior. For example, one may add the term seen in Markovich et al [22]

$$H_{\ell\omega} = \int d^3r \frac{1}{2}\ell_i\omega_i, \quad (21)$$

which couples fluid vorticity to intrinsic angular momentum density. While this does lead parity breaking terms

in the stress tensor, they are washed out under the ferrofluid assumption of vanishing moment of inertia. In our analysis we will keep this term to compare and make contact with our modification. As discussed earlier, this term is relevant in chiral active fluids of Soni et al, where the constituent particles are orders of magnitude larger than the conventional ferrofluids. One may also add quadratic terms of the form

$$H_{MM} = \int d^3r \frac{1}{2} M_i M_i, \quad (22)$$

however without modifying the brackets (10) - (14), this term does not contribute to the equations of motion. If we wish to describe magnetic solids, or any motion of the magnetization relative to the particle orientation, we may add a nonvanishing bracket between M_i and M_j in the following way

$$\{M_i(\vec{r}), M_j(\vec{r}')\} = \frac{1}{\gamma} \epsilon_{ijk} M_k(\vec{r}) \delta(\vec{r} - \vec{r}'). \quad (23)$$

With the inclusion of this term, the equation of motion for M_i becomes the Landau-Lifshitz-Gilbert equation, which would more precisely track the relaxation of the magnetization [48]. In this work, we ignore the bracket (23) and assume the magnetization is ‘frozen’ into the ferrofluid particles. Physically this represents the fact that when attempting to align with the external magnetic field, it is more favorable for the particle to rotate than for the magnetization to rotate independently.

The full Hamiltonian we will study, including both $H_{M\omega}$ given in (1) and $H_{\ell\omega}$ given in (21), is then

$$H = \int d^3r \left[\frac{1}{2} v_i v_i + \frac{1}{2I} \ell_i \ell_i + \frac{1}{2} \ell_i \omega_i + \frac{\gamma}{2} \omega_i M_i - M_i B_i \right]. \quad (24)$$

For our analysis we leave the coupling constant γ as a free parameter to be investigated experimentally or from some more microscopic description. However, a heuristic analysis of the structure of the ferrofluid nanoparticles, and the atoms within, shows γ to be inversely proportional to the molecule Landé g -factor.

We now compute the governing equations resulting from the Hamiltonian (24), using the brackets (10) - (14), along with the dissipation function (18). We obtain

$$D_t v_i + \partial_i P = \nu \nabla^2 v_i + M_j \partial_i B_j - \Gamma \epsilon_{ijk} \partial_j (\omega_k - 2\Omega_k) + \frac{1}{2} \omega_j \partial_j (I\Omega_i + \gamma M_i), \quad (25)$$

$$ID_t \Omega_i = \epsilon_{ijk} M_j B_k + 2\Gamma (\omega_i - 2\Omega_i) + \frac{1}{2} \epsilon_{ijk} \omega_j (I\Omega_k + \gamma M_k) - \frac{I}{2} \epsilon_{jkm} \partial_j \Omega_i \partial_k (I\Omega_m + \gamma M_m), \quad (26)$$

$$D_t M_i = \epsilon_{ijk} \Omega_j M_k + \frac{1}{2} \epsilon_{ijk} \omega_j M_k - \frac{1}{\tau} (M_i - M_i^0) - \frac{1}{2} \epsilon_{jkm} \partial_j M_i \partial_k (I\Omega_m + \gamma M_m). \quad (27)$$

where $P = \lambda - \frac{1}{2} v_i v_i - \frac{I}{2} \Omega_i \Omega_i + M_i B_i$ is the modified pressure. The terms involving $I\Omega_i + \gamma M_i$ stem from $H_{\ell\omega}$ and $H_{M\omega}$, while the second term on the right hand side of (27) comes from $H_{\ell\omega}$ alone. The set of equations (25) - (27) model a wide class of magnetic fluids, of any particle size and any orientation of the magnetic field and magnetization.

Upon taking the ferrofluid limit ($I \rightarrow 0$), Eq. (26) gives the modified torque balance equation

$$2\Gamma (\omega_i - 2\Omega_i) + \epsilon_{ijk} M_j B_k + \frac{\gamma}{2} \epsilon_{ijk} \omega_j M_k = 0, \quad (28)$$

which shows that in the limit of vanishing particle size rotational friction is balanced not only by the magnetic torque, but also the torque coming from the misalignment of fluid vorticity and magnetization. Upon solving (28) for Ω_i and substituting into the rest of the equations of motion, we get the final form of the modified ferrofluid equations

$$D_t v_i + \partial_i P = M_j \partial_i B_j + \nu \nabla^2 v_i + \frac{\gamma}{2} \omega_j \partial_j M_i + \epsilon_{ijk} \partial_j \left(\frac{1}{2} \epsilon_{klm} M_l B_m + \frac{\gamma}{4} \epsilon_{klm} \omega_l M_m \right), \quad (29)$$

$$D_t M_i = \epsilon_{ijk} \omega_j M_k - \frac{1}{4\Gamma} \epsilon_{ijk} M_j (\epsilon_{klm} M_l B_m) + \frac{\gamma}{8\Gamma} \epsilon_{ijk} M_j (\epsilon_{klm} M_l \omega_m) - \frac{1}{\tau} (M_i - M_i^0) - \frac{\gamma}{2} \epsilon_{jkm} \partial_j M_i \partial_k M_m \quad (30)$$

In the case of a uniform magnetic field applied in the positive z direction, the small magnetic relaxation time τ sets $M_i = |\vec{M}^0| \delta_{iz}$, with \vec{M}^0 given by (8). Equation (29) can then be written as

$$D_t v_i + \partial_i P = \nu \nabla^2 v_i + \frac{\gamma}{4} M^0 \partial_z \omega_i. \quad (31)$$

This last term breaks parity and is similar to the term obtained by Markovich and Lubensky after taking the incompressible limit, albeit with M_z replacing the intrinsic angular momentum term $\ell_z = I\Omega_z$.

IV. FERROFLUIDS IN HELE-SHAW CELL: EXPERIMENTAL PROPOSAL

A Hele-Shaw (HS) cell is a simplified experimental setup that could potentially test the presence of vorticity-magnetization coupling leading to parity breaking effects in ferrofluids. A HS cell consists of two parallel plates separated by an infinitesimally small gap h , with the hydrodynamic variables assumed to vary in the xy plane

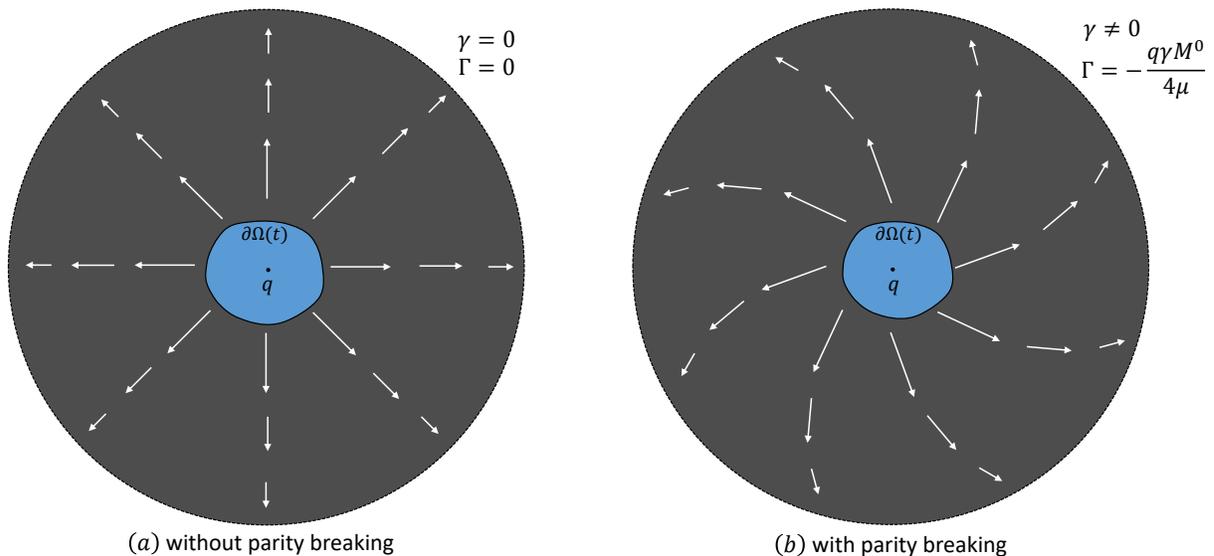


Figure 1. Schematic picture of injection of a test fluid (blue) into a ferrofluid (black), with a distinct interface $\partial\Omega$ formed between them. (a) In the absence of the parity breaking coupling γ , the ferrofluid flows radially outward and has no far field circulation ($\Gamma = 0$). (b) When the parity breaking coupling γ is nonzero the ferrofluid acquires a spiral behavior everywhere in the domain. In the far field the circulation can be computed even without knowing the details of the interface.

at much larger length scales than the distance between the two plates. Confining ferrofluids to a HS cell has much experimental and theoretical background [49–52]. A remarkable example is that of the labyrinthine instability [40]. For our purposes, we use the HS cell as a way to measure this new coupling constant γ .

Following the analysis seen in [32], we take the coordinates to scale as

$$t \sim \frac{1}{\epsilon}, \quad x \sim \frac{1}{\epsilon}, \quad y \sim \frac{1}{\epsilon}\bar{y}, \quad z \sim \epsilon^0, \quad (32)$$

where $\epsilon = h/L$, with L being some characteristic length in the xy plane. The hydrodynamic variables scale as

$$P \sim \frac{1}{\epsilon}, \quad v_x \sim \epsilon^0, \quad v_y \sim \epsilon^0, \quad v_z \sim \epsilon. \quad (33)$$

Using the relaxed form of our system, Eq. (31), the governing equations become

$$0 = \partial_x v_x + \partial_y v_y + \partial_z v_z, \quad (34)$$

$$\partial_x P = \mu \partial_z^2 v_x - \frac{\gamma}{4} M^0 \partial_z^2 v_y, \quad (35)$$

$$\partial_y P = \mu \partial_z^2 v_y + \frac{\gamma}{4} M^0 \partial_z^2 v_x, \quad (36)$$

$$\partial_z P = 0, \quad (37)$$

The solutions for velocity that satisfy the no-slip and no-penetration boundary conditions on the plates are

$$v_x = \frac{6z}{h^2}(h-z)V_x(x,y), \quad (38)$$

$$v_y = \frac{6z}{h^2}(h-z)V_y(x,y), \quad (39)$$

$$v_z = 0, \quad (40)$$

and pressure is now independent of z . We then substitute these solutions into (34) - (36) and average over the plate separation, which gives

$$0 = \partial_x V_x + \partial_y V_y, \quad (41)$$

$$-\frac{h^2}{12} \partial_x P = \mu V_x - \frac{\gamma}{4} M^0 V_y, \quad (42)$$

$$-\frac{h^2}{12} \partial_y P = \mu V_y + \frac{\gamma}{4} M^0 V_x, \quad (43)$$

The above set of equations is Darcy's law, the standard governing equation of flow within a HS cell, with an extra term coming from M^0 . This equation is of the same form as that seen in Ref. 32, with $\gamma M^0/4$ playing the role of the parity odd parameter seen there.

The experimental setups discussed in Ref. 32 provide methods to measure this coupling. Here we will only describe a simplified setup that highlights a key feature of parity odd flows. Consider HS cell with a central injection point, where a fluid (or air) is injected with a constant rate q (see Fig 1). Details of the inside fluid are not important for this discussion, since the following effect is independent of any intricate interface features or instabilities.

Equations (41-43) indicate that the ferrofluid flow is irrotational and the pressure is a harmonic function, that is,

$$\partial_x V_y - \partial_y V_x = 0, \quad (44)$$

$$\nabla^2 P = 0. \quad (45)$$

It is not hard to see that $V(\zeta) \equiv V_x(x,y) - iV_y(x,y)$, with $\zeta = x + iy$, is an analytic complex function, since

the Cauchy-Riemann conditions become the incompressibility and the irrotational equations (41,44). Assuming that the ferrofluid domain $\Omega(t)$ starts at the fluid interface and extends out to infinity, the analytic velocity function $V(\zeta)$ can be expanded in Laurent series as

$$V(\zeta) = \sum_{n=1}^{\infty} \frac{c_n}{\zeta^n}, \quad (46)$$

where the coefficients c_n are determined by the boundary conditions at the fluid interface. From the residue theorem, one can determine the coefficient c_1 by integrating $V(\zeta)$ over the fluid interface $\partial\Omega$, i.e.,

$$2\pi i c_1 = \oint_{\partial\Omega} V(\zeta) d\zeta = \Gamma - i \frac{dA}{dt}, \quad (47)$$

where Γ is the fluid circulation and $dA/dt = -q$ is the rate of change of the area occupied by the ferrofluid.

From the expression (46) it is possible to determine the ferrofluid pressure, which becomes

$$P = -\frac{3}{h^2} \text{Re} \left[(4\mu - i\gamma M^0) \left(\frac{q - i\Gamma}{2\pi} \ln \zeta - \sum_{n=1}^{\infty} \frac{c_{n+1}}{n\zeta^n} \right) \right]. \quad (48)$$

Since the pressure must be a single-valued function, this imposes that $(4\mu - i\gamma M^0)(q - i\Gamma)$ must be a real coefficient. This shows that the ferrofluid circulation must be proportional to

$$\Gamma = -\frac{q\gamma M^0}{4\mu}. \quad (49)$$

The existence of non-zero circulation in the outer region (far field flow of ferrofluids) can be used to directly measure the coupling constant γ , and serves as a straightforward way to test for the presence or absence of the vorticity-magnetization coupling in ferrofluids. We emphasize that the far field circulation, represented by the variable Γ , is independent of the two fluid interface. The physics at the boundary between two fluids can vary

greatly depending on the viscosity of the fluid being injected. For example, when a less viscous fluid such as an air bubble is injected into a ferrofluid, the Saffman-Taylor fingering instabilities can occur. These instabilities are not present when a more viscous fluid is injected. The advantage of this proposal is that the circulation in the far field, away from the interface, is not affected by the intricate and complex physics that occur near the interface.

V. CONCLUSION

In this work we introduced a new coupling in ferrofluids between fluid vorticity and magnetization. This modification is introduced in the context of a Hamiltonian and PB framework, and after following the standard ferrofluid assumptions we arrive at a modified set of governing equations containing extra parity-odd terms. When confined to a HS cell, these extra terms manifest as off diagonal components of Darcy's Law, and provide a robust experimental setting in which to measure this new coupling.

An important conclusion from our analysis is that simply including $H_{\ell\omega}$ is not enough to capture the parity-odd effects in ferrofluids due to the small size of the particles. Additionally, the coarse-graining procedure outlined in Ref. 22 can be used to determine the specifics of γ from a microscopic perspective. Future research in this direction will depend heavily on experimental confirmation of the existence of parity-breaking terms in ferrofluids.

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