Bethe vectors and recurrence relations for twisted Yangian based models

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Abstract

We study Olshanski twisted Yangian based models, known as one-dimensional "soliton non-preserving" open spin chains, by means of the algebraic Bethe ansatz. The even case, when the underlying bulk Lie algebra is $\mathfrak{gl}_{2n}$, was studied in [GMR19]. In the present work, we focus on the odd case, when the underlying bulk Lie algebra is $\mathfrak{gl}_{2n+1}$. We present a more symmetric form of the trace formula for Bethe vectors. We use the composite model approach and $Y(\mathfrak{gl}_n)$-type recurrence relations to obtain recurrence relations for twisted Yangian based Bethe vectors, for both even and odd cases.

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1 Introduction

Twisted Yangian based models, known as one-dimensional "soliton non-preserving" open spin chains, were first investigated by means of the analytic Bethe ansatz techniques in [Doi00, AA+05, AC+06a, AC+06b] and more recently in [ADK15]. The explicit form of Bethe vectors in the even case, when the underlying bulk Lie algebra is $\mathfrak{gl}_{2n}$, was obtained in [GMR19]. The latter paper uses the algebraic Bethe anstaz techniques put forward in [Rsh85, DVK87]. These techniques apply to the cases, when the R-matrix intertwining monodromy matrices of the model can be written in a six-vertex block-form. The monodromy matrix of the model is then also written in a block-form, in terms of matrix operators A, B, C, and D, that are matrix analogous of the conventional creation, annihilation and diagonal operators of the six-vertex model. The exchange relations between these matrix operators turn out to be reminiscent of those of the six-vertex model. Such techniques have been used to study one-dimensional $\mathfrak{so}_{2n}$- and $\mathfrak{sp}_{2n}$-symmetric spin chains in [Rsh91, GP16, GR20, Reg22].

In the present paper we extend the results of [GMR19] to the odd case, when the underlying bulk Lie algebra is $\mathfrak{gl}_{2n+1}$. This extension is based on the observation that defining relations of the odd twisted Yangian are unchanged by doubling the middle row and the middle column of its generating matrix. This doubling leads to "overlapping" matrix operators A, B, C, and D, satisfying the same exchange relations as their "standard" counterparts in the even case. The key ingredient of this approach is the action of the "middle" entry of the generating matrix on Bethe vectors, see Lemma 3.7. Computing this action requires knowledge of recurrence relations for Bethe vectors. We used the composite model techniques together with the known $Y(\mathfrak{gl}_n)$-type recurrence relations to obtain the wanted $Y^{\pm}(\mathfrak{gl}_{2n})$- and $Y^{+}(\mathfrak{gl}_{2n+1})$-type recurrence relations.

The main results of this paper are presented in Theorem 3.8 and Propositions 4.2 and 4.4. To the best of our knowledge, this is the first attempt to obtain recurrence relations for open spin chain models outside rank 1 case. The success is mostly down to the fact that in our approach $Y^{\pm}(\mathfrak{gl}_{2n})$- and $Y^{+}(\mathfrak{gl}_{2n+1})$-based models, after the first step of nesting, reduce to $Y(\mathfrak{gl}_n)$-based models, allowing us to exploit the already known properties of latter models. Lastly, in Proposition 3.11, we present a more symmetric form of the trace formula for Bethe vectors obtained in [GMR19].

2 Definitions and preliminaries

Throughout the manuscript the middle alphabet letters $i, j, k, \ldots$ will be used to denote integer numbers, letters $u, v, w, \ldots$ will denote either complex numbers or formal parameters, and letters a and b will be used to label vector spaces.

2.1 Lie algebras

Choose $N \geq 2$. Let $\mathfrak{gl}_N$ denote the general linear Lie algebra and let e_{ij} with $1 \leq i, j \leq N$ be the standard basis elements of $\mathfrak{gl}_N$ satisfying

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}. \tag{1}$$

The orthogonal Lie algebra $\mathfrak{so}_N$ and the symplectic Lie algebra $\mathfrak{sp}_N$ can be regarded as subalgebras of $\mathfrak{gl}_N$ as follows. For any $1 \leq i, j \leq N$ set $\theta_{ij} := \theta_i \theta_j$ with $\theta_i := 1$ in the orthogonal case and $\theta_i := \delta_{i>N/2} - \delta_{i\leq N/2}$ in the symplectic case. Introduce elements $f_{ij} := e_{ij} - \theta_{ij} e_{\bar{j}\bar{\imath}}$

with $\bar{\imath} := N - i + 1$ and $\bar{\jmath} := N - j + 1$. These elements satisfy the relations

$$[f_{ij}, f_{kl}] = \delta_{jk} f_{il} - \delta_{il} f_{kj} + \theta_{ij}(\delta_{j\bar{l}} f_{k\bar{\imath}} - \delta_{i\bar{k}} f_{\bar{\jmath}\, l}), \tag{2}$$

$$f_{ij} + \theta_{ij} f_{\bar{\jmath}\,\bar{\imath}} = 0, \tag{3}$$

which in fact are the defining relations of $\mathfrak{so}_N$ and $\mathfrak{sp}_N$. It will be convenient to denote both algebras by $\mathfrak{g}_N$. Write $N = 2n$ or $N = 2n + 1$. In this work we will focus on the following chain of Lie algebras

$$\mathfrak{gl}_N \supset \mathfrak{g}_N \supset \mathfrak{gl}_n \supset \mathfrak{gl}_{n-1} \supset \cdots \supset \mathfrak{gl}_2,$$

where $\mathfrak{gl}_n$, $\mathfrak{gl}_{n-1}$, ..., $\mathfrak{gl}_2$ are subalgebras of $\mathfrak{g}_N$ generated by f_{ij} with $1 \le i, j \le k$ and $k = n, n-1, \ldots, 2$, respectively.

2.2 Matrix operators

For any $k \in \mathbb{N}$ let $E^{(k)}_{ij} \in \mathrm{End}(\mathbb{C}^k)$ with $1 \le i, j \le k$ denote the standard matrix units with entries in $\mathbb{C}$ and let $E^{(k)}_i \in \mathbb{C}^k$ with $1 \le i \le k$ denote the standard basis vectors of $\mathbb{C}^k$ so that $E^{(k)}_{ij} E^{(k)}_l = \delta_{jl} E^{(k)}_i$. Introduce matrix operators

$$I^{(k,k)} := \sum_{i,j} E^{(k)}_{ii} \otimes E^{(k)}_{jj}, \qquad P^{(k,k)} := \sum_{i,j} E^{(k)}_{ij} \otimes E^{(k)}_{ji}, \qquad Q^{(k,k)} := \sum_{i,j} E^{(k)}_{ij} \otimes E^{(k)}_{\bar{\imath}\,\bar{\jmath}},$$

where $\bar{\imath} = k - i + 1$, $\bar{\jmath} = k - j + 1$ and the tensor product is defined over $\mathbb{C}$. We will always assume that the summation is over all admissible values, if not stated otherwise. Note that the operator $Q^{(k,k)}$ is an idempotent operator, $(Q^{(k,k)})^2 = k\, Q^{(k,k)}$, obtained by partially transforming the permutation operator $P^{(k,k)}$ with the transposition $w : E^{(k)}_{ij} \mapsto E^{(k)}_{\bar{\jmath}\,\bar{\imath}}$, that is, $Q^{(k,k)} = (id \otimes w)(P^{(k,k)}) = (w \otimes id)(P^{(k,k)})$.

Next, we introduce a matrix-valued rational function $R^{(k,k)}$ by

$$R^{(k,k)}(u) := I^{(k,k)} - u^{-1} P^{(k,k)}. \tag{4}$$

It is called the *Yang's R-matrix* and is a solution of the quantum Yang-Baxter equation on $\mathbb{C}^k \otimes \mathbb{C}^k \otimes \mathbb{C}^k$:

$$R^{(k,k)}_{12}(u-v) R^{(k,k)}_{13}(u-z) R^{(k,k)}_{23}(v-z) = R^{(k,k)}_{23}(v-z) R^{(k,k)}_{13}(u-z) R^{(k,k)}_{12}(u-v) \tag{5}$$

Here the subscript notation indicates the tensor spaces the matrix operators act on. We will use such a subscript notation throughout the manuscript. We will also make use the partially w-transposed R-matrix,

$$\widetilde{R}^{(k,k)}(u) := I^{(k,k)} - u^{-1} Q^{(k,k)}, \tag{6}$$

satisfying a transposed version of (5):

$$R^{(k,k)}_{12}(u-v) \widetilde{R}^{(k,k)}_{23}(v-z) \widetilde{R}^{(k,k)}_{13}(u-z) = \widetilde{R}^{(k,k)}_{13}(u-z) \widetilde{R}^{(k,k)}_{23}(v-z) R^{(k,k)}_{12}(u-v). \tag{7}$$

2.3 Twisted Yangian $Y^{\pm}(\mathfrak{gl}_N)$

We briefly recall the necessary details of the "ρ-shifted" twisted Yangian $Y^{\pm}(\mathfrak{gl}_N)$ adhering closely to [AC+06a, GMR19] (see also [Ols92]); here the upper (resp. lower) sign in $\pm$ corresponds to the orthogonal (resp. symplectic) case. For the purposes of the Bethe anstaz, we will give a non-standard presentation of $Y^{+}(\mathfrak{gl}_N)$ in the case when $N = 2n + 1$.

Set $\hat{n} := n$ when $N = 2n$ and $\hat{n} := n + 1$ when $N = 2n + 1$. Introduce symbols $s_{ij}[r]$ with $1 \le i, j \le N$ and $r \ge 1$, and combine them into generating series $s_{ij}(u) := \delta_{ij} + \sum_{r \ge 1} s_{ij}[r] u^{-r}$

where u is a formal variable. Then combine these series into a generating matrix by doubling the middle column and the middle row when $N=2n+1$ (i.e. when $\hat{n}=n+1$):

$$S^{(2\hat{n})}(u) := \sum_{i,j} E_{ij}^{(2\hat{n})} \otimes s_{\alpha(i),\alpha(j)}(u) \quad \text{where} \quad \alpha(i) = \begin{cases} i & i \le \hat{n}, \\ i+n-\hat{n} & i > \hat{n}. \end{cases} \tag{8}$$

The defining relations of $Y^{\pm}(\mathfrak{gl}_N)$ are then given by the reflection equation

$$\begin{aligned} &R_{12}^{(2\hat{n},2\hat{n})}(u-v)\, S_1^{(2\hat{n})}(u)\, \widehat{R}_{12}^{(2\hat{n},2\hat{n})}(\tilde{v}-u)\, S_2^{(2\hat{n})}(v) \\ &\quad = S_2^{(2\hat{n})}(v)\, \widehat{R}_{12}^{(2\hat{n},2\hat{n})}(\tilde{v}-u)\, S_1^{(2\hat{n})}(u)\, R_{12}^{(2\hat{n},2\hat{n})}(u-v) \end{aligned} \tag{9}$$

and the symmetry relation

$$\widehat{S}^{(2\hat{n})}(\tilde{u}) = S^{(2\hat{n})}(u) \pm \frac{S^{(2\hat{n})}(u) - S^{(2\hat{n})}(\tilde{u})}{u-\tilde{u}}. \tag{10}$$

Here $\tilde{u} := -u-\rho$, $\tilde{v} := -v-\rho$ with $\rho \in \mathbb{C}$ and $\widehat{S}^{(2\hat{n})} := \widehat{\omega}(R^{(2\hat{n})})$, $\widehat{R}^{(2\hat{n},2\hat{n})} := (id \otimes \widehat{\omega})(R^{(2\hat{n},2\hat{n})})$ with $\widehat{\omega} : E_{ij}^{(2\hat{n})} \mapsto \theta_{ij} E_{\bar{j}\bar{\imath}}^{(2\hat{n})}$. It is a direct computation to verify that the doubling in the $\hat{n}=n+1$ case has not introduced any additional relations.

2.4 Block decomposition

We write the matrix $S^{(2\hat{n})}(u)$ in the block form:

$$S^{(2\hat{n})}(u) = \begin{pmatrix} A^{(\hat{n})}(u) & B^{(\hat{n})}(u) \\ C^{(\hat{n})}(u) & D^{(\hat{n})}(u) \end{pmatrix}. \tag{11}$$

This allows us to rewrite the defining relations of $Y^{\pm}(\mathfrak{g}_N)$ in terms of these blocks. The relations that we will need are [GMR19]:

$$\begin{aligned} A_b^{(\hat{n})}(v) B_a^{(\hat{n})}(u) &= R_{ab}^{(\hat{n},\hat{n})}(u-v) B_a^{(\hat{n})}(u) \widetilde{R}_{ab}^{(\hat{n},\hat{n})}(\tilde{v}-u) A_b^{(\hat{n})}(v) \\ &\quad + \frac{P_{ab}^{(\hat{n},\hat{n})} B_a^{(\hat{n})}(v) \widetilde{R}_{ab}^{(\hat{n},\hat{n})}(\tilde{v}-u) A_b^{(\hat{n})}(u)}{u-v} \mp \frac{B_b^{(\hat{n})}(v) Q_{ab}^{(\hat{n},\hat{n})} D_a^{(\hat{n})}(u)}{u-\tilde{v}}, \end{aligned} \tag{12}$$

$$\begin{aligned} &R_{ab}^{(\hat{n},\hat{n})}(u-v) B_a^{(\hat{n})}(u) \widetilde{R}_{ab}^{(\hat{n},\hat{n})}(\tilde{v}-u) B_b^{(\hat{n})}(v) \\ &\quad = B_b^{(\hat{n})}(v) \widetilde{R}_{ab}^{(\hat{n},\hat{n})}(\tilde{v}-u) B_a^{(\hat{n})}(u) R_{ab}^{(\hat{n},\hat{n})}(u-v), \end{aligned} \tag{13}$$

$$\begin{aligned} &R_{ab}^{(\hat{n},\hat{n})}(u-v) A_a^{(\hat{n})}(u) A_b^{(\hat{n})}(v) - A_b^{(\hat{n})}(v) A_a^{(\hat{n})}(u) R_{ab}^{(\hat{n},\hat{n})}(u-v) \\ &\quad = \mp \frac{R_{ab}^{(\hat{n},\hat{n})}(u-v) B_a^{(\hat{n})}(u) Q_{ab}^{(\hat{n},\hat{n})} C_b^{(\hat{n})}(v) - B_b^{(\hat{n})}(v) Q_{ab}^{(\hat{n},\hat{n})} C_a^{(\hat{n})}(u) R_{ab}^{(\hat{n},\hat{n})}(u-v)}{u-\tilde{v}}, \end{aligned} \tag{14}$$

$$\begin{aligned} C_a^{(\hat{n})}(u) A_b^{(\hat{n})}(v) &= A_b^{(\hat{n})}(v) \widetilde{R}_{ab}^{(\hat{n},\hat{n})}(\tilde{v}-u) C_a^{(\hat{n})}(u) R_{ab}^{(\hat{n},\hat{n})}(u-v) \\ &\quad + \frac{P_{ab}^{(\hat{n},\hat{n})} A_a^{(\hat{n})}(u) \widetilde{R}_{ab}^{(\hat{n},\hat{n})}(\tilde{v}-u) C_b^{(\hat{n})}(v)}{u-v} \mp \frac{D_a^{(\hat{n})}(u) Q_{ab}^{(\hat{n},\hat{n})} C_b^{(\hat{n})}(v)}{u-\tilde{v}}. \end{aligned} \tag{15}$$

where a and b label two distinct copies of $\mathbb{C}^{\hat{n}}$. The symmetry relation implies that

$$\widetilde{D}^{(\hat{n})}(-u-\rho) = A^{(\hat{n})}(u) \pm \frac{A^{(\hat{n})}(u) - A^{(\hat{n})}(\tilde{u})}{u-\tilde{u}}, \tag{16}$$

$$\pm \widetilde{B}^{(\hat{n})}(-u-\rho) = B^{(\hat{n})}(u) \pm \frac{B^{(\hat{n})}(u) - B^{(\hat{n})}(\tilde{u})}{u-\tilde{u}}. \tag{17}$$

3 Bethe ansatz

3.1 Quantum space

We study spin chains with the full quantum space given by

$$L^{(n)} := L(\boldsymbol{\lambda}^{(1)}) \otimes \cdots \otimes L(\boldsymbol{\lambda}^{(\ell)}) \otimes M(\boldsymbol{\mu}) \tag{18}$$

where $\ell \in \mathbb{N}$ is the length of the chain, $L(\boldsymbol{\lambda}^{(1)})$, ..., $L(\boldsymbol{\lambda}^{(\ell)})$ and $M(\boldsymbol{\mu})$ are finite-dimensional irreducible highest-weight representations of $\mathfrak{gl}_N$ and $\mathfrak{g}_N$, respectively, and the N-tuples $\boldsymbol{\lambda}^{(1)}$, ..., $\boldsymbol{\lambda}^{(\ell)}$ and $\boldsymbol{\mu}$ are their highest weights. We will say that $L^{(n)}$ is a *level-n quantum space*.

The space $L^{(n)}$ can be equipped with a structure of a left $Y^{\pm}(\mathfrak{g}_N)$-module as follows. Introduce Lax operators

$$\mathcal{L}^{(2\hat{n})}(u) := \sum_{i,j} E_{ij}^{(2\hat{n})} \otimes (\delta_{ij} - u^{-1} e_{ji}), \tag{19}$$

$$\mathcal{M}^{(2\hat{n})}(u) := \sum_{i,j} E_{ij}^{(2\hat{n})} \otimes (\delta_{ij} - u^{-1} f_{\sigma(j)\sigma(i)}). \tag{20}$$

Choose an ℓ-tuple $\boldsymbol{c} = (c_1, \dots, c_\ell)$ of distinct complex parameters. Then for any $\xi \in L^{(n)}$ the action of $Y^{\pm}(\mathfrak{gl}_N)$ is given by

$$S_a^{(2\hat{n})}(u) \cdot \xi = \prod_i^{\rightarrow} \mathcal{L}_{ai}^{(2\hat{n})}(u - c_i)\, \mathcal{M}_{a,\ell+1}^{(2\hat{n})}(u + (\rho \pm 1)/2) \prod_i^{\leftarrow} \widehat{\mathcal{L}}_{ai}^{(2\hat{n})}(\tilde{u} - c_i) \cdot \xi \tag{21}$$

where the subscript a labels the matrix space of $S^{(2\hat{n})}$ and subscripts i and $\ell+1$ label the individual tensorands of $L^{(n)}$. This $Y^{\pm}(\mathfrak{gl}_N)$-module is called the *evaluation representation*. Moreover, since $L^{(n)}$ is finite-dimensional, the formal variable u can be evaluated to any complex number, not equal to any c_i, $\tilde{c}_i$, and $-(\rho \pm 1)/2$.

Let $1_{\boldsymbol{\lambda}^{(i)}}$ and $1_{\boldsymbol{\mu}}$ denote highest-weight vectors of $L(\boldsymbol{\lambda}^{(i)})$ and $M(\boldsymbol{\mu})$, respectively. Set

$$\eta := 1_{\boldsymbol{\lambda}^{(1)}} \otimes \cdots \otimes 1_{\boldsymbol{\lambda}^{(\ell)}} \otimes 1_{\boldsymbol{\mu}}. \tag{22}$$

Then $s_{ij}(u) \cdot \eta = 0$ if $i > j$ and $s_{ii}(u) \cdot \eta = \mu_i(u)\, \eta$ where

$$\mu_i(u) := \frac{u + (\rho \pm 1)/2 - \mu_i}{u + (\rho \pm 1)/2} \prod_{j \le \ell} \frac{u - c_j - \lambda_i^{(j)}}{u - c_i} \cdot \frac{\tilde{u} - c_j - \lambda_i^{(j)}}{\tilde{u} - c_i}. \tag{23}$$

Note that $\mu_{N-i+1} = -\mu_i$ and $\mu_{\hat{n}} = 0$ when $\hat{n} = n + 1$.

An important property of the evaluation representation is that the subspace $(L^{(n)})^0 \subset L^{(n)}$, annihilated by $s_{ij}(u)$ with $i > n$, $j \le \hat{n}$ and $i > j$, is isomorphic to an $(\ell+1)$-fold tensor product of irreducible $\mathfrak{gl}_n$ representations. Its subspace $(L^{(n)})^1 \subset (L^{(n)})^0$, annihilated by $s_{ni}(u)$ with $i < n$, is isomorphic to an $(\ell+1)$-fold tensor product of irreducible $\mathfrak{gl}_{n-1}$ representations. This can be continued to give the following chain of (sub)spaces

$$L^{(n)} \supset (L^{(n)})^0 \supset (L^{(n)})^1 \supset \cdots \supset (L^{(n)})^{n-1}$$

where $(L^{(n)})^0, (L^{(n)})^1, \dots, (L^{(n)})^{n-1}$ are isomorphic to $(\ell+1)$-fold tensor products of irreducible $\mathfrak{gl}_n$, $\mathfrak{gl}_{n-1}$, ..., $\mathfrak{gl}_2$ representations, respectively. We will say that $(L^{(n)})^k$ is a *level-k vacuum subspace*.

3.2 Nested quantum spaces

Choose an n-tuple $\boldsymbol{m} := (m_1, \ldots, m_n)$ of non-negative integers, the excitation (magnon) numbers. For each m_k assign an m_k-tuple $\boldsymbol{u}^{(k)} := (u_1^{(k)}, \ldots, u_{m_k}^{(k)})$ of complex parameters (off-shell Bethe roots) and an m_k-tuple $\boldsymbol{a}^k := (a_1^k, \ldots, a_{m_k}^k)$ of labels, except that for m_n we assign two m_n-tuples of labels, $\dot{\boldsymbol{a}} := (\dot{a}_1, \ldots, \dot{a}_{m_n})$ and $\ddot{\boldsymbol{a}} := (\ddot{a}_1, \ldots, \ddot{a}_{m_n})$. We will often use the following shorthand notation:

$$\boldsymbol{u}^{(k\ldots l)} := (\boldsymbol{u}^{(k)}, \ldots, \boldsymbol{u}^{(l)}). \tag{24}$$

We will assume that $\boldsymbol{u}^{(k\ldots l)}$ is an empty tuple if $k > l$ so that, for instance,

$$f(\boldsymbol{u}^{(1\ldots k)}; \boldsymbol{u}^{(k\ldots l)}) = f(\boldsymbol{u}^{(1\ldots k)})$$

for any function or operator f when $k > l$. Finally, for any tuples $\boldsymbol{u}$ and $\boldsymbol{v}$ of complex parameters we set

$$f^{\pm}(u_i, v_j) := \frac{u_i - v_j \pm 1}{u_i - v_j}, \qquad f^{\pm}(\boldsymbol{u}, \boldsymbol{v}) := \prod_{u_i \in \boldsymbol{u},\, v_j \in \boldsymbol{v}} f^{\pm}(u_i, v_j). \tag{25}$$

Let $V_{a_i^k}^{(k)}$ denote a copy of $\mathbb{C}^k$ labelled by "a_i^k" and let $W_{\boldsymbol{a}^k}^{(k)}$ be given by

$$W_{\boldsymbol{a}^k}^{(k)} := V_{a_1^k}^{(k)} \otimes \cdots \otimes V_{a_{m_k}^k}^{(k)}.$$

Let $V_{\dot{a}_i}^{(\hat{n})}$, $V_{\ddot{a}_i}^{(\hat{n})}$ and $W_{\dot{\boldsymbol{a}}}^{(\hat{n})}$, $W_{\ddot{\boldsymbol{a}}}^{(\hat{n})}$ be defined analogously. We define a *level-(n–1) quantum space* by

$$L^{(n-1)} := W_{\dot{\boldsymbol{a}}}^{(\hat{n})} \otimes W_{\ddot{\boldsymbol{a}}}^{(\hat{n})} \otimes (L^{(n)})^0. \tag{26}$$

When $\hat{n} = n+1$, we additionally introduce vector spaces

$$W_{\dot{\boldsymbol{a}}}^{(\hat{n})\prime} := V_{\dot{a}_1}^{(\hat{n})\prime} \otimes \cdots \otimes V_{\dot{a}_{m_n}}^{(\hat{n})\prime}, \qquad W_{\ddot{\boldsymbol{a}}}^{(\hat{n})\prime} := V_{\ddot{a}_1}^{(\hat{n})\prime} \otimes \cdots \otimes V_{\ddot{a}_{m_n}}^{(\hat{n})\prime}$$

where $V_{\dot{a}_i}^{(\hat{n})\prime} := \mathrm{span}_{\mathbb{C}}\{E_j^{(\hat{n})} : 2 \le j \le \hat{n}\} \subset V_{\dot{a}_i}^{(\hat{n})}$ and $V_{\ddot{a}_i}^{(\hat{n})\prime} := \mathrm{span}_{\mathbb{C}}\{E_1^{(\hat{n})}\} \subset V_{\ddot{a}_i}^{(\hat{n})}$. We then define a *reduced level-$(n-1)$ quantum space* by

$$L^{(n-1)\prime} := W_{\dot{\boldsymbol{a}}}^{(\hat{n})\prime} \otimes W_{\ddot{\boldsymbol{a}}}^{(\hat{n})\prime} \otimes (L^{(n)})^0 \subset L^{(n-1)}. \tag{27}$$

Next, we define a *level-(n–2) quantum space* by

$$L^{(n-2)} := W_{\boldsymbol{a}^{n-1}}^{(n-1)} \otimes (L^{(n-1)})^0 \tag{28}$$

where $(L^{(n-1)})^0$ is the *level-(n–1) vacuum subspace* given by

$$(W_{\dot{\boldsymbol{a}}}^{(\hat{n})})^0 \otimes (W_{\ddot{\boldsymbol{a}}}^{(\hat{n})})^0 \otimes (L^{(n)})^1 \subset L^{(n-1)}.$$

Here $(W_{\dot{\boldsymbol{a}}}^{(\hat{n})})^0 \subset W_{\dot{\boldsymbol{a}}}^{(\hat{n})}$ and $(W_{\ddot{\boldsymbol{a}}}^{(\hat{n})})^0 \subset W_{\ddot{\boldsymbol{a}}}^{(\hat{n})}$ are 1-dimensional subspaces spanned by vectors $E_{\hat{1}}^{(\hat{n})} \otimes \cdots \otimes E_{\hat{1}}^{(\hat{n})}$ and $E_1^{(\hat{n})} \otimes \cdots \otimes E_1^{(\hat{n})}$, respectively. When $\hat{n} = n+1$, note that $(L^{(n-1)})^0 \subset L^{(n-1)\prime}$.

Finally, for each $1 \le k \le n-3$ we define a *level-k quantum space* by

$$L^{(k)} := W_{\boldsymbol{a}^{k+1}}^{(k+1)} \otimes (L^{(k+1)})^0 \tag{29}$$

where $(L^{(k+1)})^0$ is a *level-(k+1) vacuum subspace* given by

$$(L^{(k+1)})^0 := (W_{\boldsymbol{a}^{k+2}}^{(k+2)})^0 \otimes \cdots \otimes (W_{\boldsymbol{a}^{n-1}}^{(n-1)})^0 \otimes (W_{\dot{\boldsymbol{a}}}^{(\hat{n})})^0 \otimes (W_{\ddot{\boldsymbol{a}}}^{(\hat{n})})^0 \otimes (L^{(n)})^{n-k} \subset L^{(k+1)}$$

and $(W_{\boldsymbol{a}^{k+2}}^{(k+2)})^0 \subset W_{\boldsymbol{a}^{k+2}}^{(k+2)}$ is the 1-dimensional subspace spanned by vector $E_1^{(k+2)} \otimes \cdots \otimes E_1^{(k+2)}$.

3.3 Monodromy matrices

We will say that the matrix $S^{(2\hat{n})}(u)$, acting on the space $L^{(n)}$ via (21), is a *level-n monodromy matrix*. In this setting, we will treat u as a non-zero complex number. We define a *level-$(n-1)$ nested monodromy matrix*, acting in the space $L^{(n-1)}$, by

$$T_a^{(\hat{n})}(v;\boldsymbol{u}^{(n)}) := \prod_{i\le m_n}^{\leftarrow} \widetilde{R}_{\dot{a}_i a}^{(\hat{n},\hat{n})}(u_i^{(n)}-v) \prod_{i\le m_n}^{\leftarrow} \widetilde{R}_{\ddot{a}_i a}^{(\hat{n},\hat{n})}(\tilde{u}_i^{(n)}-v) A_a^{(\hat{n})}(v). \tag{30}$$

When $\hat{n}=n+1$, we introduced a *reduced level-$(n-1)$ nested monodromy matrix*, acting in the space $L^{(n-1)\prime}$, by

$$T_a^{(n)\prime}(v;\boldsymbol{u}^{(n)}) := \prod_{i\le m_n}^{\leftarrow} \widetilde{R}_{\dot{a}_i a}^{(n,n)\prime}(u_i^{(n)}-v)\big[A_a^{(\hat{n})}(v)\big]^{(n)} \tag{31}$$

where $\widetilde{R}_{\dot{a}_i a}^{(n,n)\prime}$ is the restriction of $\widetilde{R}_{\dot{a}_i a}^{(n,n)}$ to $V_{\dot{a}_i}^{(\hat{n})\prime}\otimes V_a^{(n)} \subset V_{\dot{a}_i}^{(\hat{n})}\otimes V_a^{(\hat{n})}$, and the notation $[\ ]^{(n)}$ denotes the restriction to the upper-left $(n\times n)$-dimensional submatrix; this notation will be used throughout the manuscript. Then, for each $2\le k\le n-1$, we recursively define a *level-$(k-1)$ nested monodromy matrix*, acting in the space $L^{(k)}$, by

$$T_a^{(k)}(v;\boldsymbol{u}^{(k\dots n)}) := \prod_{i\le m_k}^{\leftarrow} \widetilde{R}_{a_i^k a}^{(k,k)}(u_i^{(k)}-v)\big[\,T_a^{(k+1)}(v;\boldsymbol{u}^{(k+1\dots n)})\big]^{(k)} \tag{32}$$

where $T_a^{(k+1)}$ should be $T_a^{(k+1)\prime}$ when $k+1=\hat{n}=n+1$.

Lemma 3.1. *For each $2\le k\le n$, the space $L^{(k)}$ is stable under the action of $T_a^{(k)}(v;\boldsymbol{u}^{(k\dots n)})$ and*

$$\begin{aligned} R_{ab}^{(k,k)}(v-w)\,T_a^{(k)}(v;\boldsymbol{u}^{(k\dots n)})\,T_b^{(k)}(w;\boldsymbol{u}^{(k\dots n)}) \\ = T_b^{(k)}(w;\boldsymbol{u}^{(k\dots n)})\,T_a^{(k)}(v;\boldsymbol{u}^{(k\dots n)})R_{ab}^{(k,k)}(v-w) \end{aligned} \tag{33}$$

in this space. Moreover, when $k+1=\hat{n}=n+1$, this is also true for the subspace $L^{(n-1)\prime}\subset L^{(n-1)}$ and $\big[T_a^{(\hat{n})}(v;\boldsymbol{u}^{(n)})\big]^{(n)}$. In particular, $\big[T_a^{(\hat{n})}(v;\boldsymbol{u}^{(n)})\big]^{(n)} = T_a^{(n)\prime}(v;\boldsymbol{u}^{(n)})$ in the space $L^{(n-1)\prime}$.

Proof. The first part is a standard result; it follows from (14), construction of quantum spaces, and application of the transposed quantum Yang-Baxter equation (7). We thus focus on proving that $L^{(n-1)\prime}$ is stable under the action of $\big[T_a^{(\hat{n})}(v;\boldsymbol{u}^{(n)})\big]^{(n)}$ when $\hat{n}=n+1$. Observe that

$$\big[\widetilde{R}_{ba}^{(\hat{n},\hat{n})}(v)\big]_{kl}E_j^{(\hat{n})} = \delta_{kl}\,E_j^{(\hat{n})} - v^{-1}\,\delta_{\hat{n}-l+1,j}E_{\hat{n}-k+1}^{(\hat{n})}$$

where $[\ ]_{kl}$ denotes restriction to the kl-th matrix element of $\widetilde{R}_{ba}^{(\hat{n},\hat{n})}$ in the a-space; this notation will be used throughout the manuscript. Therefore, for any $1\le k,l\le n$ and any $\eta\in W_{\dot{\boldsymbol{a}}}^{(\hat{n})\prime}$, $\zeta\in W_{\ddot{\boldsymbol{a}}}^{(\hat{n})\prime}$, $\xi\in(L^{(n)})^0$, cf. (27),

$$\begin{aligned} &\Big[T_a^{(\hat{n})}(v;\boldsymbol{u}^{(n)})\Big]_{kl}\cdot\eta\otimes\zeta\otimes\xi \\ &= \sum_{p,r}\Bigg[\prod_{i\le m_n}^{\leftarrow}\widetilde{R}_{\dot{a}_i a}^{(\hat{n},\hat{n})}(u_i^{(n)}-v)\Bigg]_{kp}\cdot\eta\otimes\Bigg[\prod_{i\le m_n}^{\leftarrow}\widetilde{R}_{\ddot{a}_i a}^{(\hat{n},\hat{n})}(\tilde{u}_i^{(n)}-v)\Bigg]_{pr}\cdot\zeta\otimes s_{rl}(v)\cdot\xi \\ &= \sum_{p\le n}\Bigg[\prod_{i\le m_n}^{\leftarrow}\widetilde{R}_{\dot{a}_i a}^{(\hat{n},\hat{n})}(u_i^{(n)}-v)\Bigg]_{kp}\cdot\eta\otimes\zeta\otimes s_{pl}(v)\cdot\xi \end{aligned}$$

since $s_{\hat{n}l}(v)\cdot\xi=0$. But

$$\Bigg[\prod_{i\le m_n}^{\leftarrow}\widetilde{R}_{\dot{a}_i a}^{(\hat{n},\hat{n})}(u_i^{(n)}-v)\Bigg]_{kp}\cdot\eta\notin W_{\dot{\boldsymbol{a}}}^{(\hat{n})\prime}$$

only if the product includes $\big[\widetilde{R}^{(\hat{n},\hat{n})}_{\dot{a}_i a}(u_i^{(n)}-v)\big]_{\hat{n}r}$ with $r \le n$ but then it must also include $\big[\widetilde{R}^{(\hat{n},\hat{n})}_{\dot{a}_i a}(u_i^{(n)}-v)\big]_{r\hat{n}}$ which acts by zero on η. Thus

$$\Bigg[\overleftarrow{\prod_{i\le m_n}}\widetilde{R}^{(\hat{n},\hat{n})}_{\dot{a}_i a}(u_i^{(n)}-v)\Bigg]_{kp}\cdot\eta=\Bigg[\overleftarrow{\prod_{i\le m_n}}\widetilde{R}^{(\hat{n},\hat{n})\prime}_{\dot{a}_i a}(u_i^{(n)}-v)\Bigg]_{kp}\cdot\eta\in W^{(\hat{n})\prime}_{\dot{\boldsymbol{a}}}$$

and so

$$\big[T_a^{(\hat{n})}(v;\boldsymbol{u}^{(n)})\big]^{(n)}\cdot\eta\otimes\zeta\otimes\xi=T_a^{(\hat{n})\prime}(v;\boldsymbol{u}^{(n)})\cdot\eta\otimes\zeta\otimes\xi.$$

It remains to prove (33) for $T_a^{(\hat{n})\prime}(v;\boldsymbol{u}^{(n)})$ in the space $L^{(n-1)\prime}$ which follows by the standard arguments. □

Remark 3.2. Lemma 3.1 together with (30), (31) say that $Y^{\pm}(\mathfrak{gl}_{2n})$- and $Y^{+}(\mathfrak{gl}_{2n+1})$-based models, after the first step of nesting, are equivalent to $Y(\mathfrak{gl}_n)$-based models with off-shell Bethe roots given by $\boldsymbol{v}^{(1\dots n-2)}:=\boldsymbol{u}^{(1\dots n-2)}$ and $\boldsymbol{v}^{(n)}:=(\boldsymbol{u}^{(n)},\tilde{\boldsymbol{u}}^{(n)})$ in the even case, and $\boldsymbol{v}^{(n)}:=\boldsymbol{u}^{(n)}$ in the odd case. This property will be explored in Section 4.

3.4 Creation operators

We define *level-n creation operator* by

$$\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}):=\overleftarrow{\prod_{i\le m_n}}\Bigg(\mathcal{b}^{(n)}_{\dot{a}_i\ddot{a}_i}(u_i^{(n)})\overrightarrow{\prod_{j>i}}\frac{R^{(\hat{n},\hat{n})}_{\dot{a}_i\ddot{a}_j}(\tilde{u}_i^{(n)}-u_j^{(n)})}{(f^{-}(\tilde{u}_i^{(n)},u_j^{(n)}))^{\delta_{\hat{n}n}}}\Bigg) \tag{34}$$

where

$$\mathcal{b}^{(n)}_{\dot{a}_i\ddot{a}_i}(u_i^{(n)}):=\sum_{k,l\le\hat{n}}(E_k^{(\hat{n})})^*\otimes(E_l^{(\hat{n})})^*\otimes\big[B_a^{(\hat{n})}(u_i^{(n)})\big]_{\hat{n}-k+1,l}\in(V^{(\hat{n})}_{\dot{a}_i})^*\otimes(V^{(\hat{n})}_{\ddot{a}_i})^*\otimes\mathrm{End}(L^{(n)}). \tag{35}$$

The R-matrices in (34) are necessary for the wanted order of the $\widetilde{R}$-matrices in (30), which in turn is necessary for Lemma 3.1 to hold. The denominator is an overall normalisation factor.

From (34) it is clear that $\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)})$ satisfies the recurrence relation

$$\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)})=\mathcal{b}^{(n)}_{\dot{a}_{m_n}\ddot{a}_{m_n}}(u^{(n)}_{m_n})\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u^{(n)}_{m_n})\mathscr{R}^{(\hat{n})}(u^{(n)}_{m_n};\boldsymbol{u}^{(n)}\backslash u^{(n)}_{m_n}) \tag{36}$$

where

$$\mathscr{R}^{(\hat{n})}(u^{(n)}_{m_n};\boldsymbol{u}^{(n)}\backslash u^{(n)}_{m_n}):=\overleftarrow{\prod_{i<m_n}}\frac{R^{(\hat{n},\hat{n})}_{\dot{a}_i\ddot{a}_{m_n}}(\tilde{u}_i^{(n)}-u^{(n)}_{m_n})}{(f^{-}(\tilde{u}_i^{(n)},u^{(n)}_{m_n}))^{\delta_{\hat{n}n}}}. \tag{37}$$

Next, for each $1\le k\le n-1$ we define *level-k creation operator* by

$$\mathscr{B}^{(k)}(\boldsymbol{u}^{(k)};\boldsymbol{u}^{(k+1\dots n)}):=\overleftarrow{\prod_{i\le m_k}}\mathcal{b}^{(k)}_{a_i^k}(u_i^{(k)};\boldsymbol{u}^{(k+1\dots n)}) \tag{38}$$

where

$$\mathcal{b}^{(k)}_{a_i^k}(u_i^{(k)};\boldsymbol{u}^{(k+1\dots n)}):=\sum_{j\le k}(E_j^{(k)})^*_{a_i^k}\otimes\big[T_a^{(k+1)}(u_i^{(k)};\boldsymbol{u}^{(k+1\dots n)})\big]_{k-j+1,k+1}\in(V^{(k)}_{a_i^k})^*\otimes\mathrm{End}(L^{(k)}). \tag{39}$$

Note that $T_a^{(n)}(u_i^{(n-1)};\boldsymbol{u}^{(n)})$ should be replaced with $T_a^{(n)\prime}(u_i^{(n-1)};\boldsymbol{u}^{(n)})$ when $\hat{n}=n+1$.

Parameters of creation operators may be permuted using the following standard result, which follows from (13); see Lemma 3.6 of [GMR19].

Lemma 3.3. *The level-n creation operator satisfies*

$$\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}) = \mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}_{i\leftrightarrow i+1})\check{R}^{(\hat{n},\hat{n})}_{\dot{a}_{i+1}\dot{a}_i}(u^{(n)}_i - u^{(n)}_{i+1})\check{R}^{(\hat{n},\hat{n})}_{\ddot{a}_{i+1}\ddot{a}_i}(u^{(n)}_{i+1} - u^{(n)}_i). \tag{40}$$

For each $1 \le k \le n-1$ the level-k creation operator satisfies

$$\mathscr{B}^{(k)}(\boldsymbol{u}^{(k)};\boldsymbol{u}^{(k+1\dots n)}) = \mathscr{B}^{(k)}(\boldsymbol{u}^{(k)}_{i\leftrightarrow i+1};\boldsymbol{u}^{(k+1\dots n)})\check{R}^{(k,k)}_{a^k_{i+1}a^k_i}(u^{(k)}_i - u^{(k)}_{i+1}). \tag{41}$$

Here the "check" $\check{R}$-matrices are defined by

$$\check{R}^{(k,k)}_{ab}(u) := \frac{u}{u-1}P^{(k,k)}_{ab}R^{(k,k)}_{ab}(u) \tag{42}$$

and $\boldsymbol{u}^{(k)}_{i\leftrightarrow i+1}$ denotes the tuple $\boldsymbol{u}^{(k)}$ with parameters $u^{(k)}_i$ and $u^{(k)}_{i+1}$ interchanged.

Introduce the following notation for a symmetrised combination of functions or operators

$$\{f(v)\}^v := f(v) + f(\tilde{v})$$

and a rational function

$$p(v) := 1 \pm \frac{1}{v-\tilde{v}}. \tag{43}$$

The Lemma below rephrases the results obtained in [GMR19].

Lemma 3.4. *The AB exchange relation for the level-n creation operator (34) is*

$$\begin{aligned}
&\{p(v)A^{(\hat{n})}_a(v)\}^v\,\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}) \\
&\quad= \mathscr{B}^{(n)}(\boldsymbol{u}^{(n)})\,\{p(v)\,T^{(\hat{n})}_a(v;\boldsymbol{u}^{(n)})\}^v \\
&\qquad+ \sum_i \frac{1}{p(u^{(n)}_i)}\left\{\frac{p(v)}{u^{(n)}_i - v}\,\mathscr{b}^{(n)}_{\dot{a}_{m_n}\ddot{a}_{m_n}}(v)\right\}^v \mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u^{(n)}_i)\,\mathscr{R}^{(\hat{n})}(u^{(n)}_i;\boldsymbol{u}^{(n)}\backslash u^{(n)}_i) \\
&\qquad\quad\times \operatorname*{Res}_{w\to u^{(n)}_i}\{p(w)\,T^{(\hat{n})}_a(w;\boldsymbol{u}^{(n)}_{\sigma_i})\}^w \prod_{j>i}^{\rightarrow}\check{R}^{(\hat{n},\hat{n})}_{\dot{a}_j\dot{a}_{j-1}}(u^{(n)}_i - u^{(n)}_j)\check{R}^{(\hat{n},\hat{n})}_{\ddot{a}_j\ddot{a}_{j-1}}(u^{(n)}_j - u^{(n)}_i)
\end{aligned} \tag{44}$$

where $\boldsymbol{u}^{(n)}\backslash u^{(n)}_i := (u_1,\dots,u_{i-1},u_{i+1},\dots,u_{m_n})$ and $\boldsymbol{u}^{(n)}_{\sigma_i} := (u_1,\dots,u_{i-1},u_{i+1},\dots,u_{m_n},u_i)$.

Proof. From [GMR19], the relation (12), as well as (10), lead to the following exchange relation with a single creation operator

$$\begin{aligned}
\{p(v)A^{(\hat{n})}_a(v)\}^v\,\mathscr{b}^{(n)}_{\ddot{a}_i\dot{a}_i}(u^{(n)}_i) &= \mathscr{b}^{(n)}_{\ddot{a}_i\dot{a}_i}(u^{(n)}_i)\,\{p(v)\,T^{(\hat{n})}_a(v;u^{(n)}_i)\}^v \\
&\quad+ \frac{1}{p(u^{(n)}_i)}\left\{\frac{p(v)}{u^{(n)}_i - v}\,\mathscr{b}^{(n)}_{\ddot{a}_i\dot{a}_i}(v)\right\}^v \operatorname*{Res}_{w\to u^{(n)}_i}\{p(w)\,T^{(\hat{n})}_a(w;u^{(n)}_i)\}^w
\end{aligned} \tag{45}$$

where $T^{(\hat{n})}_a(v;u^{(n)}_i) = \widetilde{R}_{\dot{a}_i a}(u^{(n)}_i - v)\widetilde{R}_{\ddot{a}_i a}(\tilde{u}^{(n)}_i - v)A^{(\hat{n})}_a(v)$. We extend this to the creation operator for m_n excitations by the standard argument. Indeed, the right hand side of the equation consists of terms with $A^{(\hat{n})}_a(u)$ as the rightmost operator, for u equal to each of $v, u^{(n)}_1,\dots,u^{(n)}_{m_n}$ and the corresponding tilded elements. Due to the $w \mapsto \tilde{w}$ symmetry of $\{p(w)A^{(\hat{n})}_a(w)\}^w$ in (45), it is sufficient to find those terms corresponding to $v, u^{(n)}_1,\dots,u^{(n)}_{m_n}$.

First, we find the term corresponding to v to be $\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)})\,\{p(v)\,T^{(\hat{n})}_a(v;\boldsymbol{u}^{(n)})\}^v$. The required order of $\widetilde{R}$-matrices inside $T^{(\hat{n})}_a(v;\boldsymbol{u}^{(n)})$ is a result of Yang-Baxter moves through the

R-matrices inside $\mathcal{B}^{(n)}(\boldsymbol{u}^{(n)})$. Using factorisation (36) we find the term corresponding to $u^{(n)}_{m_n}$ to be

$$\frac{1}{p(u^{(n)}_{m_n})}\left\{\frac{p(v)}{u^{(n)}_{m_n}-v}\,\mathcal{b}^{(n)}_{\dot a_{m_n}\ddot a_{m_n}}(v)\right\}^{v}\mathcal{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u^{(n)}_{m_n})$$
$$\times\,\mathcal{R}^{(\hat n)}(u^{(n)}_{m_n};\boldsymbol{u}^{(n)}\backslash u^{(n)}_{m_n})\operatorname*{Res}_{w\to u^{(n)}_{m_n}}\{p(w)\,T^{(\hat n)}_{a}(w;\boldsymbol{u}^{(n)})\}^{w}.$$

This is because, after applying (45) to the leftmost creation operator $\mathcal{b}^{(n)}_{\dot a_{m_n}\ddot a_{m_n}}(u^{(n)}_{m_n})$, there can be no further contributions from the parameter-swapped term in the subsequent applications of (45).

To find the remaining terms, we note that Lemma 3.3 allows us to apply any permutation to the spectral parameters of the level-n creation operator before applying the above argument. By applying the permutation $\sigma_i:(1,\ldots,i-1,i,i+1,\ldots,m_n)\mapsto(1\ldots,i-1,i+1,\ldots,m_n,i)$, we obtain the term corresponding to $u^{(n)}_i$. □

The Lemma below follows from Lemma 3.1 and is a standard result, see e.g. [BR08].

Lemma 3.5. *The AB exchange relation for the level-k creation operator* (38) *is*

$$\begin{aligned}&\big[T^{(k+1)}_{a}(v;\boldsymbol{u}^{(k+1\ldots n)})\big]^{(k)}\mathcal{B}^{(k)}(\boldsymbol{u}^{(k)};\boldsymbol{u}^{(k+1\ldots n)})\\&\quad=\mathcal{B}^{(k)}(\boldsymbol{u}^{(k)};\boldsymbol{u}^{(k+1\ldots n)})\,T^{(k)}_{a}(v;\boldsymbol{u}^{(k\ldots n)})\\&\qquad+\sum_i\frac{1}{u^{(k)}_i-v}\,\mathcal{b}^{(k)}_{a^k_{m_k}}(v;\boldsymbol{u}^{k+1\ldots n})\,\mathcal{B}^{(k)}(\boldsymbol{u}^{(k)}\backslash u^{(k)}_i;\boldsymbol{u}^{(k+1\ldots n)})\\&\qquad\quad\times\operatorname*{Res}_{w\to u^{(k)}_i}T^{(k)}_{a}(w;\boldsymbol{u}^{(k\ldots n)}_{\sigma^k_i})\prod^{\rightarrow}_{j>i}\check R^{(k,k)}_{a^k_j a^k_{j-1}}(u^{(k)}_i-u^{(k)}_j)\,.\end{aligned}\tag{46}$$

Moreover,

$$\begin{aligned}&\big[T^{(k+1)}_{a}(v;\boldsymbol{u}^{(k+1\ldots n)})\big]_{k+1,k+1}\mathcal{B}^{(k)}(\boldsymbol{u}^{(k)};\boldsymbol{u}^{(k+1\ldots n)})\\&\quad=\mathcal{B}^{(k)}(\boldsymbol{u}^{(k)};\boldsymbol{u}^{(k+1\ldots n)})\,f^-(v;\boldsymbol{u}^{(k)})\big[T^{(k+1)}_{a}(v;\boldsymbol{u}^{(k+1\ldots n)})\big]_{k+1,k+1}\\&\qquad+\sum_i\frac{1}{u^{(k)}_i-v}\,\mathcal{b}^{(k)}_{a^k_{m_k}}(v;\boldsymbol{u}^{k+1\ldots n})\,\mathcal{B}^{(k)}(\boldsymbol{u}^{(k)}\backslash u^{(k)}_i;\boldsymbol{u}^{(k+1\ldots n)})\\&\qquad\quad\times\operatorname*{Res}_{w\to u^{(k)}_i}f^-(w;\boldsymbol{u}^{(k)})\big[T^{(k+1)}_{a}(w;\boldsymbol{u}^{(k+1\ldots n)})\big]_{k+1,k+1}\prod^{\rightarrow}_{j>i}\check R^{(k,k)}_{a^k_j a^k_{j-1}}(u^{(k)}_i-u^{(k)}_j)\,.\end{aligned}\tag{47}$$

3.5 Bethe vectors

Recall (22) and define a *nested vacuum vector* by

$$\eta^{\boldsymbol{m}}:=(E^{(2)}_1)^{\otimes m_2}\otimes\cdots\otimes(E^{(n-1)}_1)^{\otimes m_{n-1}}\otimes(E^{(\hat n)}_{\hat 1})^{\otimes m_n}\otimes(E^{(\hat n)}_1)^{\otimes m_n}\otimes\eta\,.\tag{48}$$

For each $1\le k\le n$ we define a *level-k* (off-shell) Bethe vector with (off-shell) Bethe roots $\boldsymbol{u}^{(1\ldots k)}$ and free parameters $\boldsymbol{u}^{(k+1\ldots n)}$ by

$$\Psi(\boldsymbol{u}^{(1\ldots k)}\,|\,\boldsymbol{u}^{(k+1\ldots n)}):=\prod^{\leftarrow}_{i\le k}\mathcal{B}^{(i)}(\boldsymbol{u}^{(i)};\boldsymbol{u}^{(i+1\ldots n)})\cdot\eta^{\boldsymbol{m}}.\tag{49}$$

We will say that vector $\eta^{\boldsymbol{m}}$ is the *reference vector* of this Bethe vector.

The Lemma below follows by a repeated application of Lemma 3.3.

Lemma 3.6. *Bethe vector $\Psi(\boldsymbol{u}^{(1\ldots k)}\,|\,\boldsymbol{u}^{(k+1\ldots n)})$ is invariant under interchange of any two of its Bethe roots, $u_i^{(l)}$ and $u_j^{(l)}$, for all admissible i, j, and l.*

The last technical result that we will need is the action of $s_{\hat{n}\hat{n}}(v)$ on a Bethe vector, when $\hat{n}=n+1$. It is motivated by the following relation in $Y^+(\mathfrak{gl}_{2n+1})((u^{-1},v^{-1}))$ for $1\le k\le n$:

$$s_{\hat{n}\hat{n}}(v)\,s_{k\hat{n}}(u)=f^-(v,u)\,f^+(v,\tilde{u})\,s_{k\hat{n}}(u)\,s_{\hat{n}\hat{n}}(v)-\left\{\frac{p(v)}{u-v}\,s_{k\hat{n}}(v)\right\}^{v} s_{\hat{n}\hat{n}}(u)\,.$$

We postpone the proof of the Lemma below to Section 4.3.

Lemma 3.7. *When $\hat{n}=n+1$,*

$$\begin{aligned}
s_{\hat{n}\hat{n}}(v)\,\Psi(\boldsymbol{u}^{(1\ldots n)}) &= f^-(v,\boldsymbol{u}^{(n)})\,f^+(v,\tilde{\boldsymbol{u}}^{(n)})\,\mu_{\hat{n}}(v)\,\Psi(\boldsymbol{u}^{(1\ldots n)})\\
&\quad+\sum_i \frac{1}{p(u_i^{(n)})}\left\{\frac{p(v)}{u_i^{(n)}-v}\,\mathscr{b}^{(n)}_{\dot{a}_{m_n}\ddot{a}_{m_n}}(v)\right\}^{v}\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u_i^{(n)})\,\mathscr{R}^{(\hat{n})}(u_i^{(n)},\boldsymbol{u}^{(n)}\backslash u_i^{(n)})\\
&\quad\times\operatorname*{Res}_{w\to u_i^{(n)}} f^-(w,\boldsymbol{u}^{(n)})\,f^+(w,\tilde{\boldsymbol{u}}^{(n)})\,\mu_{\hat{n}}(w)\,\Psi^{(n-1)}(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{u}^{(n)}_{\sigma_i})\,.
\end{aligned}\tag{50}$$

3.6 Transfer matrix and Bethe equations

We define the *transfer matrix* by

$$\tau(v):=\operatorname{tr}_a M_a^{(2\hat{n})}S_a^{(2\hat{n})}(v)=\operatorname{tr}_a\left[M_a^{(2\hat{n})}\right]^{(\hat{n})}\left\{p(v)A_a^{(\hat{n})}(v)\right\}^{v}\tag{51}$$

where $M^{(2\hat{n})}=\sum_{i\le\hat{n}}\alpha_i\,\varepsilon_i\,(E_{ii}^{(2\hat{n})}+E_{\bar{\imath}\bar{\imath}}^{(2\hat{n})})$ is a twist matrix; here $\varepsilon_i\in\mathbb{C}^\times$ and $\alpha_i=1$ except $\alpha_{\hat{n}}=1/2$ when $\hat{n}=n+1$. The latter accounts the doubling of $s_{\hat{n}\hat{n}}(v)$ in $S_a^{(2\hat{n})}(v)$.

Theorem 3.8. *The Bethe vector $\Psi(\boldsymbol{u}^{(1\ldots n)})$ is an eigenvector of $\tau(v)$ with the eigenvalue*

$$\Lambda(v;\boldsymbol{u}^{(1\ldots n)}):=\sum_{k\le\hat{n}}\alpha_k\,\varepsilon_k\left\{p(v)\,\Gamma_k(v;\boldsymbol{u}^{(1\ldots n)})\right\}^{v}\tag{52}$$

where $p(v)$ is given by (43) and

$$\Gamma_k(v;\boldsymbol{u}^{(1\ldots n)}):=f^-(v,\boldsymbol{u}^{(k-1)})\,f^+(v,\boldsymbol{u}^{(k)})\,\mu_k(v)\quad\text{for } k<\hat{n}\tag{53}$$

and

$$\Gamma_{\hat{n}}(v;\boldsymbol{u}^{(1\ldots n)}):=\begin{cases} f^-(v,\boldsymbol{u}^{(n-1)})\,f^+(v,\boldsymbol{u}^{(n)})\,f^+(v,\tilde{\boldsymbol{u}}^{(n)})\,\mu_n(v) & \text{if } \hat{n}=n\\ f^-(v,\boldsymbol{u}^{(n)})\,f^+(v,\tilde{\boldsymbol{u}}^{(n)})\,\mu_{n+1}(v) & \text{if } \hat{n}=n+1\end{cases}\tag{54}$$

provided $\operatorname*{Res}_{v\to u_j^{(k)}}\Lambda(v;\boldsymbol{u}^{(1\ldots n)})=0$ for all admissible k and j; these equations are called Bethe equations.

Proof. When $\hat{n}=n$, this is a restatement of Theorems 4.3 and 4.4 in [GMR19]. When $\hat{n}=n+1$, using Lemmas 3.1–3.7 and the fact that $\Psi^{(n-1)}(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{u}^{(n)})\in L^{(n-1)}$, we find

$$\begin{aligned}
\tau(v)\,\Psi(\boldsymbol{u}^{(1\ldots n)}) &= \mathscr{B}^{(n)}(\boldsymbol{u}^{(n)})\,\tau'(v;\boldsymbol{u}^{(n)})\,\Psi(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{u}^{(n)})\\
&\quad+\sum_i\frac{1}{p(u_i^{(n)})}\left\{\frac{p(v)}{u_i^{(n)}-v}\,\mathscr{b}^{(n)}_{\dot{a}_{m_n}\ddot{a}_{m_n}}(v)\right\}^{v}\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u_i^{(n)})\\
&\quad\times\overleftarrow{\prod_{j<m_n}}R^{(\hat{n},\hat{n})}_{\dot{a}_j\ddot{a}_{m_n}}(\tilde{u}_i^{(n)}-u^{(n)}_{\sigma_i(j)})\operatorname*{Res}_{w\to u_i^{(n)}}\tau'(w;\boldsymbol{u}^{(n)}_{\sigma_i})\,\Psi(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{u}^{(n)}_{\sigma_i})
\end{aligned}\tag{55}$$

where

$$\tau'(v;\boldsymbol{u}^{(n)}) := \left\{p(v)\,\mathrm{tr}_a\,[M_a^{(\hat n)}]^{(n)}\,T_a^{(n)\prime}(v;\boldsymbol{u}^{(n)})\right\}^{\nu} + f^-(v,\boldsymbol{u}^{(n)})\,f^+(v,\tilde{\boldsymbol{u}}^{(n)})\,\varepsilon_{\hat n}\,s_{\hat n\hat n}(v)\,.$$

The operator $s_{\hat n\hat n}(v)$ acts by a scalar multiplication with $\mu_{\hat n}(v)$ in the space $L^{(n-1)}$. Requiring $\tau'(v;\boldsymbol{u}^{(n)})$ to act by a scalar multiplication on $\Psi(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{u}^{(n)})$ and repating the sames steps as in the $\hat n = n$ case, via Lemma 3.5, lead to the wanted result. □

Remark 3.9. Let $(a_{ij})_{i,j=1}^n$ denote Cartan matrix of type A_n. Let $(b_{ij})_{i,j=1}^n$ denote a zero matrix when $\hat n = n+1$ and let $b_{nn} = 2$, $b_{n-1,n} = b_{n,n-1} = -1$, and $b_{ij} = 0$ otherwise, when $\hat n = n$. Set $m_0 := 0$ and $z_j^{(k)} := u_j^{(k)} - \frac12(k-\rho)$. Then Bethe equations can be written as, for $k < n$,

$$\prod_{l=k-1}^{k+1}\prod_{i=1}^{m_l} \frac{z_j^{(k)} - z_i^{(l)} + \frac12 a_{kl}}{z_j^{(k)} - z_i^{(l)} - \frac12 a_{kl}} \cdot \frac{z_j^{(k)} + z_i^{(l)} + n + \frac12 b_{kl}}{z_j^{(k)} + z_i^{(l)} + n - \frac12 b_{kl}} = -\frac{\varepsilon_{k+1}}{\varepsilon_k}\cdot\frac{\mu_{k+1}(u_j^{(k)})}{\mu_k(u_j^{(k)})}, \tag{56}$$

$$\frac{z_j^{(n)} + \frac12(n+1)}{z_j^{(n)} + \frac12(\hat n-1)} \prod_{l=n-1}^{n}\prod_{i=1}^{m_l} \frac{z_j^{(n)} - z_i^{(l)} + \frac12 a_{nl}}{z_j^{(n)} - z_i^{(l)} - \frac12 a_{nl}} \prod_{l=\hat n-1}^{n}\prod_{i=1}^{m_l} \frac{z_j^{(n)} + z_i^{(l)} + n + \frac12 b_{nl}}{z_j^{(n)} + z_i^{(l)} + \hat n - \frac12 b_{nl}} = -\frac{\varepsilon_{\hat n}}{\varepsilon_n}\cdot\frac{\mu_{\hat n}(\tilde u_j^{(n)})}{\mu_n(u_j^{(n)})}. \tag{57}$$

3.7 Trace formula

Set

$$S^{(N)}(u) := \sum_{i,j} E_{ij}^{(N)} \otimes s_{ij}(u).$$

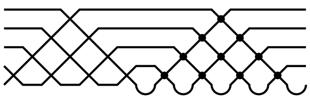
Define the "master" creation operator

$$\begin{aligned}
\mathscr{B}_N(\boldsymbol{u}^{(1\ldots n)}) := \prod_{k\le n}\prod_{j<i}\frac{1}{f^+(u_j^{(k)},u_i^{(k)})}\prod_{j<i}\frac{1}{(f^+(u_j^{(k)},\tilde u_i^{(k)}))^{\delta_{\hat n,n}}} \\
\times\,\mathrm{tr}\Bigg[\prod_{(k,i)\succ(l,j)} R^{(N,N)}_{a_i^k a_j^l}(u_i^{(k)} - u_j^{(l)}) \\
\times \prod_{(k,i)}\Bigg(S^{(N)}_{a_i^k}(u_i^{(k)}) \prod_{(k,i)\succ(l,j)} \widehat{R}^{(N,N)}_{a_i^k a_j^l}(\tilde u_i^{(k)} - u_j^{(l)})\Bigg) \\
\times (E^{(N)}_{n+1,n})^{\otimes m_n} \otimes\cdots\otimes (E^{(N)}_{21})^{\otimes m_1}\Bigg]
\end{aligned} \tag{58}$$

where $(k,i) \succ (l,j)$ means that $k > l$ or $k = l$ and $i > j$, and the products over tuples are defined in terms of the following rule

$$\prod_{(k,i)} = \overleftarrow{\prod_{k<n}}\ \overleftarrow{\prod_{i<m_k}}$$

In other words, these products are ordered in the reversed lexicographical order. The trace is taken over all a_i^k spaces, including a_i^n, which are associated with level-n excitations. Note that (k,i) is fixed in the third product inside the trace. Diagrammatically, the operator inside the trace is of the form



where $\times = R_{a_i^k a_j^l}(u_i^{(k)} - u_j^{(l)})$, $\times = \widehat{R}_{a_i^k a_j^l}(\tilde u_i^{(k)} - u_j^{(l)})$, and $\smile = S_{a_i^k}(u_i^{(k)})$.

Example 3.10.

$$\mathscr{B}_3(u_1^{(1)}) = s_{12}(u_1^{(1)}), \qquad \mathscr{B}_3(u_1^{(1)},u_2^{(1)}) = s_{12}(u_2^{(1)})s_{12}(u_1^{(1)}) + \frac{s_{13}(u_3^{(1)})s_{22}(u_1^{(1)})}{u_1^{(1)}-\tilde{u}_2^{(1)}},$$

$$\mathscr{B}_4(u_1^{(1)},u_1^{(2)}) = s_{23}(u_1^{(2)})s_{12}(u_1^{(1)}) + \frac{s_{24}(u_1^{(2)})s_{22}(u_1^{(1)})}{u_1^{(1)}-\tilde{u}_1^{(2)}} + \frac{(u_1^{(1)}-\tilde{u}_1^{(2)}+1)s_{13}(u_1^{(2)})s_{22}(u_1^{(1)})}{(u_1^{(1)}-u_1^{(2)})(u_1^{(1)}-\tilde{u}_1^{(2)})},$$

$$\mathscr{B}_5(u_1^{(1)},u_1^{(2)}) = s_{23}(u_1^{(2)})s_{12}(u_1^{(1)}) + \frac{s_{13}(u_1^{(2)})s_{22}(u_1^{(1)})}{u_1^{(1)}-u_1^{(2)}} + \frac{s_{25}(u_1^{(2)})s_{32}(u_1^{(1)})}{u_1^{(1)}-\tilde{u}_1^{(2)}} + \frac{s_{14}(u_1^{(2)})s_{32}(u_1^{(1)})}{(u_1^{(1)}-u_1^{(2)})(u_1^{(1)}-\tilde{u}_1^{(2)})}.$$

Proposition 3.11. *The level-n Bethe vector* (49) *can be written as*

$$\Psi(\boldsymbol{u}^{(1..n)}) = \mathscr{B}_N(\boldsymbol{u}^{(1\dots n)})\cdot\eta\,. \tag{59}$$

Proof. First, notice that R-matrices $R^{(N,N)}_{a_i^k a_j^k}(u_i^{(k)}-u_j^{(k)})$ in (58) evaluate to $f^+(u_j^{(k)}-u_i^{(k)})$ under the trace. This cancels the first overall factor in (58). The second overall factor is the choice of normalisation in (34). Next, let $V_a^{(N)}$ and $V_b^{(N)}$ denote copies of $\mathbb{C}^N$. Then, for any $\zeta\in(L^{(n)})^0$ and $E_i^{(N)}\otimes E_j^{(N)}\in V_a^{(N)}\otimes V_b^{(N)}$ with $1\le i,j\le n$, we have

$$Q_{ab}^{(N,N)}\,E_i^{(N)}\otimes E_j^{(N)} = 0$$

and

$$Q_{ab}^{(N,N)}S_a^{(N)}(v)\cdot E_i^{(N)}\otimes E_j^{(N)}\otimes\zeta = \sum_k Q_{ab}^{(N,N)}\cdot E_k^{(N)}\otimes E_j^{(N)}\otimes s_{ki}(v)\,\zeta = 0.$$

Thus $\widehat{R}^{(N,N)}_{a_i^k a_j^l}(\tilde{u}_i^{(k)}-u_j^{(l)})$ with $1\le k,l<n$ act as identity operators in (59). This gives an expression analogous (up to Yang-Baxter moves) to that in Proposition 4.7 of [GMR19]. The $N=2n$ case then follows from that proposition. The $N=2n+1$ case is proven analogously. □

4 Recurrence relations

4.1 Notation

Given any tuple $\boldsymbol{u}$ of complex parameters, let $(\boldsymbol{u}_{\rm I},\boldsymbol{u}_{\rm II})\vdash\boldsymbol{u}$ be a partition of this tuple and let $\boldsymbol{u}_{\rm I,II} := \boldsymbol{u}_{\rm I}\cup\boldsymbol{u}_{\rm II} = \boldsymbol{u}$. Assume that $1\le k<|\boldsymbol{u}|$ and set

$$\sum_{|\boldsymbol{u}_{\rm II}|=k} f(\boldsymbol{u}_{\rm I}) := \sum_{i_1<i_2<\dots<i_k} f(\boldsymbol{u}\backslash(u_{i_1},u_{i_2},\dots,u_{i_k}))$$

for any function or operator f. We will use a natural generalisation of this notation for any partition of $\boldsymbol{u}$. For instance, for $(\boldsymbol{u}_{\rm I},\boldsymbol{u}_{\rm II},\boldsymbol{u}_{\rm III})\vdash\boldsymbol{u}$, $\boldsymbol{u}_{\rm I,II}=\boldsymbol{u}_{\rm I}\cup\boldsymbol{u}_{\rm II}$, $\boldsymbol{u}_{\rm I,III}=\boldsymbol{u}_{\rm I}\cup\boldsymbol{u}_{\rm III}$, $\boldsymbol{u}_{\rm II,III}=\boldsymbol{u}_{\rm II}\cup\boldsymbol{u}_{\rm III}$, and $\boldsymbol{u}_{\rm I,II,III}=\boldsymbol{u}$. We will assume that the union of all components (of a partition) in a product of functions or operators always equals $\boldsymbol{u}$. For instance, $f(\boldsymbol{u}_{\rm III})\,g(\boldsymbol{u}_{\rm I})$ will mean that $\boldsymbol{u}_{\rm II}=\emptyset$ and $\boldsymbol{u}_{\rm I}\cup\boldsymbol{u}_{\rm III}=\boldsymbol{u}$.

We extend the notation above to partitions of tuples $\boldsymbol{u}^{(1\dots n)}$ in a natural way. For instance, $f(\boldsymbol{u}^{(n)}_{\rm III})\,g(\boldsymbol{u}^{(1\dots n)}_{\rm I})$ will mean that $\boldsymbol{u}^{(1)}_{\rm III}=\dots=\boldsymbol{u}^{(n-1)}_{\rm III}=\boldsymbol{u}^{(1)}_{\rm II}=\dots=\boldsymbol{u}^{(n)}_{\rm II}=\emptyset$ and

$\boldsymbol{u}_{\rm III}^{(n)} \cup \boldsymbol{u}_{\rm I}^{(1\dots n)} = \boldsymbol{u}^{(1\dots n)}$. We will write $\boldsymbol{u}_{\rm II,III}^{(1)} = \boldsymbol{u}_{\rm II}^{(1)} \cup \boldsymbol{u}_{\rm III}^{(1)}$ and $\boldsymbol{u}_{\rm II}^{(1,2)} = \boldsymbol{u}_{\rm II}^{(1)} \cup \boldsymbol{u}_{\rm II}^{(2)}$. The notation $|\boldsymbol{u}_{\rm II}^{(1,2)}| = (k,l)$ will mean that $|\boldsymbol{u}_{\rm II}^{(1)}| = k$ and $|\boldsymbol{u}_{\rm II}^{(2)}| = l$, so that

$$\sum_{|\boldsymbol{u}_{\rm II}^{(1,2)}|=(k,l)} = \sum_{|\boldsymbol{u}_{\rm II}^{(1)}|=k} \sum_{|\boldsymbol{u}_{\rm II}^{(2)}|=l} .$$

We will also use the following specific level-n notation:

$$\boldsymbol{u}_{\rm III}^{(n)} := u_j^{(n)}, \qquad \boldsymbol{u}_{\rm II}^{(n)} := \tilde{u}_j^{(n)}, \qquad \boldsymbol{u}_{\rm I}^{(n)} := \boldsymbol{u}^{(n)} \backslash u_j^{(n)}$$

for all $1 \le j \le m_n$.

4.2 Recurrence relations

We will combine the composite model method with the known $Y(\mathfrak{gl}_n)$-type recurrence relations to obtain recurrence relations for $Y^{\pm}(\mathfrak{g}_N)$-based Bethe vectors. The composite model method was introduced in [IK84]. For a pedagogical review, see [Sla20]. Recurrence relations for $Y(\mathfrak{gl}_n)$-based Bethe vectors were obtained in [HL+18a]. We will need the following statement which follows directly from those in [HL+18a].

Proposition 4.1. *Consider a $Y(\mathfrak{gl}_n)$-based Bethe vector $\Phi(\boldsymbol{v}^{(1\dots n-1)}\,|\,\boldsymbol{v}^{(n)})$ in the quantum space*

$$V_{a_{m_n}}^{(n)} \otimes \cdots \otimes V_{a_1}^{(n)} \otimes L(\boldsymbol{\lambda})$$

with $V_{a_i}^{(n)} \cong \mathbb{C}^n$, a finite-dimensional irreducible $Y(\mathfrak{gl}_n)$-module $L(\boldsymbol{\lambda})$, Bethe roots $\boldsymbol{v}^{(1\dots n-1)}$ and inhomogeneities $\boldsymbol{v}^{(n)}$ associated with spaces $V_{a_i}^{(n)}$. An expansion of $\Phi(\boldsymbol{v}^{(1\dots n-1)}\,|\,\boldsymbol{v}^{(n)})$ in the space $V_{a_{m_n}}^{(n)}$ is given by

$$\Phi(\boldsymbol{v}^{(1\dots n-1)}\,|\,\boldsymbol{v}^{(n)}) = \sum_{i\le n} \sum_{\substack{|\boldsymbol{v}_{\rm II}^{(r)}|=1 \\ i\le r<n}} \prod_{i<k\le n} \frac{\Lambda_k(\boldsymbol{v}_{\rm II}^{(k-1)};\boldsymbol{v}_{\rm I}^{(1\dots n)})}{\boldsymbol{v}_{\rm II}^{(k-1)} - \boldsymbol{v}_{\rm II}^{(k)}} E_{\bar{\imath}}^{(n)} \otimes \Phi(\boldsymbol{v}_{\rm I}^{(1\dots n-1)}\,|\,\boldsymbol{v}_{\rm I}^{(n)}) \tag{60}$$

where $\Lambda_k(z;\boldsymbol{v}^{(1\dots n)}) := f^{-}(z,\boldsymbol{v}^{(k-1)})\,f^{+}(z,\boldsymbol{v}^{(k)})\,\lambda_k(z)$, $\boldsymbol{v}^{(0)} = \emptyset$ and $\boldsymbol{v}_{\rm II}^{(n)} = v_{m_n}^{(n)}$.

Applying (60) twice gives an expansion of $\Phi(\boldsymbol{v}^{(1\dots n-1)}\,|\,\boldsymbol{v}^{(n)})$ in the space $V_{a_{m_n}}^{(n)} \otimes V_{a_{m_n-1}}^{(n)}$:

$$\begin{aligned}
&\sum_{1\le i\le n} \sum_{\substack{|\boldsymbol{v}_{\rm II,III}^{(r)}|=(2,0) \\ i\le r<n}} \prod_{i<k\le n} \Lambda_k(\boldsymbol{v}_{\rm II}^{(k-1)};\boldsymbol{v}_{\rm I}^{(1\dots n)})\, K(\boldsymbol{v}_{\rm II}^{(k-1)}\,|\,\boldsymbol{v}_{\rm II,III}^{(k)})\, E_{\bar{\imath}}^{(n)} \otimes E_{\bar{\imath}}^{(n)} \otimes \Phi^{(n-1)}(\boldsymbol{v}_{\rm I}^{(1\dots n-1)}\,|\,\boldsymbol{v}_{\rm I}^{(n)}) \\
&+ \sum_{1\le i<j\le n} \sum_{\substack{|\boldsymbol{v}_{\rm III}^{(r)}|=1 \\ i\le r<n}} \sum_{\substack{|\boldsymbol{u}_{\rm II}^{(s)}|=1 \\ j\le s<n}} \prod_{i<k<j} \frac{\Lambda_k(\boldsymbol{v}_{\rm III}^{(k-1)};\boldsymbol{v}_{\rm I}^{(1\dots n-1)})}{\boldsymbol{v}_{\rm III}^{(k-1)} - \boldsymbol{v}_{\rm III}^{(k)}} \cdot \Lambda_j(\boldsymbol{v}_{\rm III}^{(j-1)};\boldsymbol{v}_{\rm I}^{(1\dots n-1)}) \\
&\times \prod_{j<k\le n} \frac{\Lambda_k(\boldsymbol{v}_{\rm II}^{(k-1)};\boldsymbol{v}_{\rm I}^{(1\dots n-1)})\,\Lambda_k(\boldsymbol{v}_{\rm III}^{(k-1)};\boldsymbol{v}_{\rm I,II}^{(1\dots n-1)})}{(\boldsymbol{v}_{\rm II}^{(k-1)} - \boldsymbol{v}_{\rm II}^{(k)})(\boldsymbol{v}_{\rm III}^{(k-1)} - \boldsymbol{v}_{\rm III}^{(k)})} \\
&\times \left(\frac{f^{+}(\boldsymbol{u}_{\rm III}^{(j-1)},\boldsymbol{v}_{\rm II}^{(j)})}{\boldsymbol{v}_{\rm III}^{(j-1)} - \boldsymbol{v}_{\rm III}^{(j)}} E_{\bar{\imath}}^{(n)} \otimes E_{\bar{\jmath}}^{(n)} + \frac{1}{\boldsymbol{v}_{\rm III}^{(j-1)} - \boldsymbol{v}_{\rm II}^{(j)}} E_{\bar{\jmath}}^{(n)} \otimes E_{\bar{\imath}}^{(n)} \right) \otimes \Phi^{(n-1)}(\boldsymbol{v}_{\rm I}^{(1\dots n-1)}\,|\,\boldsymbol{v}_{\rm I}^{(n)})
\end{aligned} \tag{61}$$

where $\boldsymbol{v}_{\rm III}^{(n)} = v_{m_n}^{(n)}$, $\boldsymbol{v}_{\rm II}^{(n)} = v_{m_n-1}^{(n)}$ and

$$K(\boldsymbol{u}\,|\,\boldsymbol{v}) := \frac{\prod_{i,j}(u_i - v_j + 1)}{\prod_{i<j}(u_i - u_j)(v_j - v_i)} \det_{i,j}\left(\frac{1}{(u_i - v_j)(u_i - v_j + 1)} \right) \tag{62}$$

is the domain wall boundary partition function.

Proposition 4.2. *$Y^{\pm}(\mathfrak{gl}_{2n})$-based Bethe vectors satisfy the recurrence relation*

$$\Psi(\boldsymbol{u}^{(1\dots n)}) = \sum_{1\le i\le n}\sum_{\substack{|\boldsymbol{u}^{(r)}_{\mathrm{II,III}}|=(2,0)\\ i\le r<n}}\prod_{i<k\le n}\Gamma_k(\boldsymbol{u}^{(k-1)}_{\mathrm{II}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})\,K(\boldsymbol{u}^{(k-1)}_{\mathrm{II}}\,|\,\boldsymbol{u}^{(k)}_{\mathrm{II,III}})\,s_{i,2n-i+1}(\boldsymbol{u}^{(n)}_{\mathrm{III}})\,\Psi(\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})$$
$$+\sum_{1\le i<j\le n}\sum_{\substack{|\boldsymbol{u}^{(r)}_{\mathrm{III}}|=1\\ i\le r<n}}\sum_{\substack{|\boldsymbol{u}^{(s)}_{\mathrm{II}}|=1\\ j\le s<n}}\prod_{i<k<j}\frac{\Gamma_k(\boldsymbol{u}^{(k-1)}_{\mathrm{III}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})}{\boldsymbol{u}^{(k-1)}_{\mathrm{III}}-\boldsymbol{u}^{(k)}_{\mathrm{III}}}\cdot\Gamma_j(\boldsymbol{u}^{(j-1)}_{\mathrm{III}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})$$
$$\times\prod_{j<k\le n}\frac{\Gamma_k(\boldsymbol{u}^{(k-1)}_{\mathrm{II}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})\,\Gamma_k(\boldsymbol{u}^{(k-1)}_{\mathrm{III}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I,II}})}{(\boldsymbol{u}^{(k-1)}_{\mathrm{II}}-\boldsymbol{u}^{(k)}_{\mathrm{II}})(\boldsymbol{u}^{(k-1)}_{\mathrm{III}}-\boldsymbol{u}^{(k)}_{\mathrm{III}})}$$
$$\times\left(\frac{f^{+}(\boldsymbol{u}^{(j-1)}_{\mathrm{III}},\boldsymbol{u}^{(j)}_{\mathrm{II}})}{\boldsymbol{u}^{(j-1)}_{\mathrm{III}}-\boldsymbol{u}^{(j)}_{\mathrm{III}}}\,s_{i,2n-j+1}(\boldsymbol{u}^{(n)}_{\mathrm{III}})+\frac{1}{\boldsymbol{u}^{(j-1)}_{\mathrm{III}}-\boldsymbol{u}^{(j)}_{\mathrm{II}}}\,s_{j,2n-i+1}(\boldsymbol{u}^{(n)}_{\mathrm{III}})\right)\Psi(\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}}) \tag{63}$$

where $\boldsymbol{u}^{(n)}_{\mathrm{III}}=u^{(n)}_j$, $\boldsymbol{u}^{(n)}_{\mathrm{II}}=\tilde{u}^{(n)}_j$ and $\boldsymbol{u}^{(n)}_{\mathrm{I}}=\boldsymbol{u}^{(n)}\backslash u^{(n)}_j$ with $1\le j\le m_n$, and $\Gamma_n(\boldsymbol{u}^{(n-1)}_{\mathrm{III}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I,II}})$ denotes $f^{+}(\boldsymbol{u}^{(n-1)}_{\mathrm{III}},\boldsymbol{u}^{(n)}_{\mathrm{II}})\,\Gamma_n(\boldsymbol{u}^{(n-1)}_{\mathrm{III}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})$.

Example 4.3. When $n=2$, the recurrence relation (63) gives

$$\Psi(\boldsymbol{u}^{(1,2)})=s_{23}(\boldsymbol{u}^{(2)}_{\mathrm{III}})\,\Psi(\boldsymbol{u}^{(1,2)}_{\mathrm{I}})+\sum_{|\boldsymbol{u}^{(1)}_{\mathrm{II}}|=2}\Gamma_2(\boldsymbol{u}^{(1)}_{\mathrm{II}};\boldsymbol{u}^{(1,2)}_{i})\,K(\boldsymbol{u}^{(1)}_{\mathrm{II}}\,|\,\boldsymbol{u}^{(2)}_{\mathrm{II,III}})\,s_{14}(\boldsymbol{u}^{(2)}_{\mathrm{III}})\,\Psi(\boldsymbol{u}^{(1,2)}_{\mathrm{I}})$$
$$+\sum_{|\boldsymbol{u}^{(1)}_{\mathrm{II}}|=1}\Gamma_2(\boldsymbol{u}^{(1)}_{\mathrm{II}};\boldsymbol{u}^{(1,2)}_{\mathrm{I}})\left(\frac{f^{+}(\boldsymbol{u}^{(1)}_{\mathrm{II}},\boldsymbol{u}^{(2)}_{\mathrm{II}})}{\boldsymbol{u}^{(1)}_{\mathrm{II}}-\boldsymbol{u}^{(2)}_{\mathrm{III}}}\,s_{13}(\boldsymbol{u}^{(2)}_{\mathrm{III}})+\frac{1}{\boldsymbol{u}^{(1)}_{\mathrm{II}}-\boldsymbol{u}^{(2)}_{\mathrm{II}}}\,s_{24}(\boldsymbol{u}^{(2)}_{\mathrm{III}})\right)\Psi(\boldsymbol{u}^{(1,2)}_{\mathrm{I}}). \tag{64}$$

Proof of Proposition 4.2. By Lemma 3.6, it is sufficient to consider the $j=m_n$ case. Recall (36), (49) and consider vector

$$\mathcal{R}^{(\hat{n})}(u^{(n)}_{m_n};\boldsymbol{u}^{(n)}\backslash u^{(n)}_{m_n})\,\Psi^{(n-1)}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}). \tag{65}$$

With the help of Yang-Baxter equation we can move the operator $\mathcal{R}^{(\hat{n})}(u^{(n)}_{m_n};\boldsymbol{u}^{(n)}\backslash u^{(n)}_{m_n})$ all way to the nested vacuum vector $\eta^{\boldsymbol{m}}$. As a result of this, the level-n nested monodromy matrix (30) factorises as

$$\widetilde{R}^{(\hat{n},\hat{n})}_{\dot{a}_{m_n}a}(u^{(n)}_{m_n}-v)\,\widetilde{R}^{(\hat{n},\hat{n})}_{\ddot{a}_{m_n}a}(\tilde{u}^{(n)}_{m_n}-v)\,T^{(\hat{n})}_{a}(v;\boldsymbol{u}^{(n)}\backslash u^{(n)}_{m_n}). \tag{66}$$

Since $\mathcal{R}^{(\hat{n})}(u^{(n)}_{m_n};\boldsymbol{u}^{(n)}\backslash u^{(n)}_{m_n})\cdot\eta^{\boldsymbol{m}}=\eta^{\boldsymbol{m}}$ when $\hat{n}=n$, we may view vector (65) as a $Y(\mathfrak{gl}_n)$-based Bethe vector with monodromy matrix (66) and apply expansion (61) in the space $V^{(n)}_{\dot{a}_{m_n}}\otimes V^{(n)}_{\ddot{a}_{m_n}}$. Recall (53), (54) and act with $\mathscr{b}^{(n)}_{\dot{a}_{m_n}\ddot{a}_{m_n}}(u^{(n)}_{m_n})\,\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u^{(n)}_{m_n})$ on the resulting expression. This gives the wanted result. □

Proposition 4.4. *$Y^{+}(\mathfrak{gl}_{2n+1})$-based Bethe vectors satisfy the recurrence relation*

$$
\begin{aligned}
\Psi(\boldsymbol{u}^{(1\dots n)}) = & \sum_{1\le i\le n}\sum_{\substack{|\boldsymbol{u}^{(r)}_{\mathrm{III}}|=1\\ i\le r<n}}\prod_{i<k\le n}\frac{\Gamma_k(\boldsymbol{u}^{(k-1)}_{\mathrm{III}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})}{\boldsymbol{u}^{(k-1)}_{\mathrm{III}}-\boldsymbol{u}^{(k)}_{\mathrm{III}}}\, s_{i,\hat{n}}(\boldsymbol{u}^{(n)}_{\mathrm{III}})\,\Psi(\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}}) \\
& + \sum_{1\le i<n}\sum_{\substack{|\boldsymbol{u}^{(r)}_{\mathrm{II}}|=1\\ i\le r\le n}}\prod_{i<k\le n}\frac{\Gamma_k(\boldsymbol{u}^{(k-1)}_{\mathrm{II}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})}{\boldsymbol{u}^{(k-1)}_{\mathrm{II}}-\boldsymbol{u}^{(k)}_{\mathrm{II}}}\cdot\frac{\Gamma_{\hat{n}}(\boldsymbol{u}^{(n)}_{\mathrm{II}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})}{\boldsymbol{u}^{(n)}_{\mathrm{II}}-\tilde{\boldsymbol{u}}^{(n)}_{\mathrm{III}}} \\
& \quad\times\left(\frac{\boldsymbol{u}^{(n-1)}_{\mathrm{II}}-\boldsymbol{u}^{(n)}_{\mathrm{II}}+1}{\boldsymbol{u}^{(n-1)}_{\mathrm{II}}-\boldsymbol{u}^{(n)}_{\mathrm{III}}}\, s_{i,\hat{n}+1}(\boldsymbol{u}^{(n)}_{\mathrm{III}})+s_{n,\hat{n}+i+1}(\boldsymbol{u}^{(n)}_{\mathrm{III}})\right)\Psi(\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}}) \\
& + \sum_{1\le i<n}\sum_{\substack{|\boldsymbol{u}^{(r)}_{\mathrm{II,III}}|=(2,0)\\ i\le r<n}}\sum_{|\boldsymbol{u}^{(n)}_{\mathrm{II}}|=1}\prod_{i<k\le n}\Gamma_k(\boldsymbol{u}^{(k-1)}_{\mathrm{II}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})\, K(\boldsymbol{u}^{(k-1)}_{\mathrm{II}}\,|\,\boldsymbol{u}^{(k)}_{\mathrm{II,III}}) \\
& \quad\times\frac{\Gamma_{\hat{n}}(\boldsymbol{u}^{(n)}_{\mathrm{II}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})}{\boldsymbol{u}^{(n)}_{\mathrm{II}}-\tilde{\boldsymbol{u}}^{(n)}_{\mathrm{III}}}\, s_{i,2\hat{n}-i}(\boldsymbol{u}^{(n)}_{\mathrm{III}})\,\Psi(\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}}) \\
& + \sum_{1\le i<j<n}\sum_{\substack{|\boldsymbol{u}^{(r)}_{\mathrm{III}}|=1\\ i\le r<n}}\sum_{\substack{|\boldsymbol{u}^{(s)}_{\mathrm{II}}|=1\\ j\le s\le n}}\prod_{i<k<j}\frac{\Gamma_k(\boldsymbol{u}^{(k-1)}_{\mathrm{III}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})}{\boldsymbol{u}^{(k-1)}_{\mathrm{III}}-\boldsymbol{u}^{(k)}_{\mathrm{III}}}\cdot\Gamma_j(\boldsymbol{u}^{(j-1)}_{\mathrm{III}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}}) \\
& \quad\times\prod_{j<k<n}\frac{\Gamma_k(\boldsymbol{u}^{(k-1)}_{\mathrm{II}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})\,\Gamma_k(\boldsymbol{u}^{(k-1)}_{\mathrm{III}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I,II}})}{(\boldsymbol{u}^{(k-1)}_{\mathrm{II}}-\boldsymbol{u}^{(k)}_{\mathrm{II}})(\boldsymbol{u}^{(k-1)}_{\mathrm{III}}-\boldsymbol{u}^{(k)}_{\mathrm{III}})}\cdot\Gamma_n(\boldsymbol{u}^{(k-1)}_{\mathrm{II,III}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}})\,\Gamma_{\hat{n}}(\boldsymbol{u}^{(n)}_{\mathrm{II}};\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}}) \\
& \quad\times\left[\left(\left(\beta_0+\frac{\beta_2}{2\gamma}\right)\frac{f^{+}(\boldsymbol{u}^{(j-1)}_{\mathrm{III}},\boldsymbol{u}^{(j)}_{\mathrm{II}})}{\boldsymbol{u}^{(j-1)}_{\mathrm{III}}-\boldsymbol{u}^{(j)}_{\mathrm{III}}}+\frac{\beta_1}{2\gamma}\cdot\frac{1}{\boldsymbol{u}^{(j-1)}_{\mathrm{III}}-\boldsymbol{u}^{(j)}_{\mathrm{II}}}\right)s_{i,2\hat{n}-j}(\boldsymbol{u}^{(n)}_{\mathrm{III}})\right. \\
& \qquad\left.+\left(\frac{\beta_1}{2\gamma}\cdot\frac{f^{+}(\boldsymbol{u}^{(j-1)}_{\mathrm{III}},\boldsymbol{u}^{(j)}_{\mathrm{II}})}{\boldsymbol{u}^{(j-1)}_{\mathrm{III}}-\boldsymbol{u}^{(j)}_{\mathrm{III}}}+\left(\beta_0+\frac{\beta_2}{2\gamma}\right)\frac{1}{\boldsymbol{u}^{(j-1)}_{\mathrm{III}}-\boldsymbol{u}^{(j)}_{\mathrm{II}}}\right)s_{j,2\hat{n}-i}(\boldsymbol{u}^{(n)}_{\mathrm{III}})\right] \\
& \qquad\times\Psi(\boldsymbol{u}^{(1\dots n)}_{\mathrm{I}}) \qquad (67)
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_0 &= \frac{f^{-}(\boldsymbol{u}^{(n-1)}_{\mathrm{III}},\boldsymbol{u}^{(n-1)}_{\mathrm{II}})\,f^{+}(\boldsymbol{u}^{(n-1)}_{\mathrm{III}},\tilde{\boldsymbol{u}}^{(n)}_{\mathrm{III}})}{(\boldsymbol{u}^{(n-1)}_{\mathrm{II}}-\tilde{\boldsymbol{u}}^{(n)}_{\mathrm{III}})(\boldsymbol{u}^{(n-1)}_{\mathrm{III}}-\boldsymbol{u}^{(n)}_{\mathrm{III}})(\boldsymbol{u}^{(n)}_{\mathrm{II}}-\tilde{\boldsymbol{u}}^{(n)}_{\mathrm{III}})}, \\
\beta_1 &= \frac{\boldsymbol{u}^{(n-1)}_{\mathrm{III}}-\boldsymbol{u}^{(n)}_{\mathrm{III}}}{\boldsymbol{u}^{(n-1)}_{\mathrm{III}}-\boldsymbol{u}^{(n)}_{\mathrm{II}}}\left(\boldsymbol{u}^{(n-1)}_{\mathrm{II}}-\tilde{\boldsymbol{u}}^{(n)}_{\mathrm{III}}+1+\frac{\boldsymbol{u}^{(n-1)}_{\mathrm{III}}-\tilde{\boldsymbol{u}}^{(n)}_{\mathrm{III}}}{\boldsymbol{u}^{(n-1)}_{\mathrm{II}}-\boldsymbol{u}^{(n)}_{\mathrm{II}}}\right), \\
\beta_2 &= f^{+}(\boldsymbol{u}^{(n-1)}_{\mathrm{II}},\boldsymbol{u}^{(n)}_{\mathrm{II}})\,\frac{\boldsymbol{u}^{(n-1)}_{\mathrm{II}}-\tilde{\boldsymbol{u}}^{(n)}_{\mathrm{III}}}{\boldsymbol{u}^{(n-1)}_{\mathrm{III}}-\boldsymbol{u}^{(n)}_{\mathrm{II}}}+\frac{(\boldsymbol{u}^{(n-1)}_{\mathrm{II}}-\boldsymbol{u}^{(n)}_{\mathrm{III}})(\boldsymbol{u}^{(n-1)}_{\mathrm{III}}-\tilde{\boldsymbol{u}}^{(n)}_{\mathrm{III}})+1}{\boldsymbol{u}^{(n-1)}_{\mathrm{II}}-\boldsymbol{u}^{(n)}_{\mathrm{II}}}, \\
\gamma &= (\boldsymbol{u}^{(n-1)}_{\mathrm{II}}-\tilde{\boldsymbol{u}}^{(n)}_{\mathrm{III}})(\boldsymbol{u}^{(n-1)}_{\mathrm{III}}-\tilde{\boldsymbol{u}}^{(n)}_{\mathrm{III}})(\boldsymbol{u}^{(n-1)}_{\mathrm{II}}-\boldsymbol{u}^{(n)}_{\mathrm{III}})(\boldsymbol{u}^{(n-1)}_{\mathrm{III}}-\boldsymbol{u}^{(n)}_{\mathrm{III}}).
\end{aligned}
\qquad (68)
$$

and $\boldsymbol{u}^{(n)}_{\mathrm{III}}=u^{(n)}_j$, $\boldsymbol{u}^{(n)}_{\mathrm{II}}=\tilde{u}^{(n)}_j$, *and* $\boldsymbol{u}^{(n)}_{\mathrm{I}}=\boldsymbol{u}^{(n)}\backslash u^{(n)}_j$ *with* $1\le j\le m_n$.

Example 4.5. When $n=1$, the recurrence relation (67) gives

$$
\Psi(\boldsymbol{u}^{(1)})=s_{12}(\boldsymbol{u}^{(1)}_{\mathrm{III}})\,\Psi(\boldsymbol{u}^{(1)}_{\mathrm{I}})+\sum_{|\boldsymbol{u}^{(1)}_{\mathrm{II}}|=1}\frac{\Gamma_2(\boldsymbol{u}^{(1)}_{\mathrm{II}};\boldsymbol{u}^{(1)}_{\mathrm{I}})}{\boldsymbol{u}^{(1)}_{\mathrm{II}}-\tilde{\boldsymbol{u}}^{(1)}_{\mathrm{III}}}\, s_{13}(\boldsymbol{u}^{(1)}_{\mathrm{III}})\,\Psi(\boldsymbol{u}^{(1)}_{\mathrm{I}}). \qquad (69)
$$

When $n=2$, the recurrence relation (67) gives

$$\Psi(\boldsymbol{u}^{(1,2)}) = s_{23}(\boldsymbol{u}^{(2)}_{\rm III})\,\Psi(\boldsymbol{u}^{(1,2)}_{\rm I}) + \sum_{|\boldsymbol{u}^{(1)}_{\rm II}|=1} \frac{\Gamma_2(\boldsymbol{u}^{(1)}_{\rm II};\boldsymbol{u}^{(1,2)}_{\rm I})}{\boldsymbol{u}^{(1)}_{\rm II}-\boldsymbol{u}^{(2)}_{\rm III}}\, s_{13}(\boldsymbol{u}^{(2)}_{\rm III})\,\Psi(\boldsymbol{u}^{(1,2)}_{\rm I})$$
$$+\sum_{|\boldsymbol{u}^{(2)}_{\rm II}|=1} \frac{\Gamma_3(\boldsymbol{u}^{(2)}_{\rm II};\boldsymbol{u}^{(2)}_{\rm I})}{\boldsymbol{u}^{(2)}_{\rm II}-\tilde{\boldsymbol{u}}^{(2)}_{\rm III}}\, s_{24}(\boldsymbol{u}^{(2)}_{\rm III})\,\Psi(\boldsymbol{u}^{(1,2)}_{\rm I})$$
$$+\sum_{|\boldsymbol{u}^{(1,2)}_{\rm II}|=(1,1)} \frac{\Gamma_2(\boldsymbol{u}^{(1)}_{\rm II};\boldsymbol{u}^{(1,2)}_{\rm I})\,\Gamma_3(\boldsymbol{u}^{(2)}_{\rm II};\boldsymbol{u}^{(1,2)}_{\rm I})}{\boldsymbol{u}^{(2)}_{\rm II}-\tilde{\boldsymbol{u}}^{(2)}_{\rm III}}$$
$$\times\left(\frac{f^+(\boldsymbol{u}^{(1)}_{\rm II},\boldsymbol{u}^{(2)}_{\rm II})}{\boldsymbol{u}^{(1)}_{\rm II}-\boldsymbol{u}^{(2)}_{\rm III}}\, s_{14}(\boldsymbol{u}^{(2)}_{\rm III}) + \frac{1}{\boldsymbol{u}^{(1)}_{\rm II}-\boldsymbol{u}^{(2)}_{\rm II}}\, s_{25}(\boldsymbol{u}^{(2)}_{\rm III})\right)\Psi(\boldsymbol{u}^{(1,2)}_{\rm I})$$
$$+\sum_{|\boldsymbol{u}^{(1,2)}_{\rm II}|=(2,1)} \frac{\Gamma_2(\boldsymbol{u}^{(1)}_{\rm II};\boldsymbol{u}^{(1,2)}_{\rm I})\,\Gamma_3(\boldsymbol{u}^{(2)}_{\rm II};\boldsymbol{u}^{(1,2)}_{\rm I})}{\boldsymbol{u}^{(2)}_{\rm II}-\tilde{\boldsymbol{u}}^{(2)}_{\rm III}}\, K(\boldsymbol{u}^{(1)}_{\rm II}\,|\,\boldsymbol{u}^{(2)}_{\rm II,III})\, s_{15}(\boldsymbol{u}^{(2)}_{\rm III})\,\Psi(\boldsymbol{u}^{(1,2)}_{\rm I})\,. \quad (70)$$

The Lemma below will assist us in proving Proposition 4.4.

Lemma 4.6. *Let $\Psi_j(\boldsymbol{u}^{(1\dots n)})$ denote a $Y^+(\mathfrak{gl}_{2n+1})$-based Bethe vector with the reference vector $\eta^{\boldsymbol{m}}_j := (E^{(\hat n)}_{12})_{\dot a_j}\eta^{\boldsymbol{m}}$. Then*

$$\Psi_j(\boldsymbol{u}^{(1\dots n)}) = \sum_{1\le i\le j} \frac{1}{u^{(n)}_j - u^{(n)}_i + 1}\cdot\frac{\Gamma_{\hat n}(u^{(n)}_i,\boldsymbol{u}^{(1\dots n)}\backslash u^{(n)}_i)}{\prod_{k>j} f^+(u^{(n)}_k,u^{(n)}_i)}\,\Psi(\boldsymbol{u}^{(1\dots n)}\backslash u^{(n)}_i)\,. \quad (71)$$

Proof. Recall (34) and consider vector

$$\overrightarrow{\prod_{j>1}} R^{(\hat n,\hat n)}_{\dot a_1\ddot a_j}(\tilde u^{(n)}_1 - u^{(n)}_j)\,\Psi^{(n-1)}_1(\boldsymbol{u}^{1\dots n-1}\,|\,\boldsymbol{u}^{(n)})\,. \quad (72)$$

With the help of Yang-Baxter equation we can move the product of R-matrices all way to the reference vector $\eta^{\boldsymbol{m}}_1$. As a result of this, the level-n nested monodromy matrix (30) takes the form

$$\overleftarrow{\prod_{i>1}} \widetilde{R}^{(\hat n,\hat n)}_{\dot a_i a}(u^{(n)}_i - v)\overleftarrow{\prod_{i>1}} \widetilde{R}^{(\hat n,\hat n)}_{\ddot a_i a}(\tilde u^{(n)}_i - v)\,\widetilde{R}^{(\hat n,\hat n)}_{\dot a_1 a}(u^{(n)}_1 - v)\,\widetilde{R}^{(\hat n,\hat n)}_{\ddot a_1 a}(\tilde u^{(n)}_1 - v)\,A^{(\hat n)}_a(v). \quad (73)$$

In the space $L^{(n-1)\prime}$, it is equivalent to $T^{(n)\prime}_a(v;\boldsymbol{u}^{(n)}\backslash u^{(n)}_1)$. Next, recall (48) and note that

$$\overrightarrow{\prod_{j>1}} R^{(\hat n,\hat n)}_{\dot a_1\ddot a_j}(\tilde u^{(n)}_1 - u^{(n)}_j)\cdot\eta^{\boldsymbol{m}}_1 = f^+(u^{(n)}_1,\tilde{\boldsymbol{u}}^{(n)}\backslash\tilde u^{(n)}_1)\,\eta^{\boldsymbol{m}}_1. \quad (74)$$

Hence, vector (72) can be expanded in the space $V^{(\hat n)}_{\dot a_1}\otimes V^{(\hat n)}_{\ddot a_1}$ as

$$f^+(u^{(n)}_1,\tilde{\boldsymbol{u}}^{(n)}\backslash\tilde u^{(n)}_1)\cdot E^{(\hat n)}_1\otimes E^{(\hat n)}_1\otimes\Psi^{(n-1)}_1(\boldsymbol{u}^{1\dots n-1}\,|\,\boldsymbol{u}^{(n)}\backslash u^{(n)}_1)\,. \quad (75)$$

From (35) note that $\mathscr{b}^{(n)}_{\dot a_1\ddot a_1}(v)\cdot E^{(\hat n)}_1\otimes E^{(\hat n)}_1 = s_{\hat n\hat n}(v)$. Defining relations of $Y^+(\mathfrak{gl}_{2n+1})$ imply that

$$s_{\hat n\hat n}(u^{(n)}_1)\overleftarrow{\prod_{i<n}}\mathscr{B}^{(i)}(\boldsymbol{u}^{(i)};\boldsymbol{u}^{(i+1\dots n)}\backslash u^{(n)}_1) = \overleftarrow{\prod_{i<n}}\mathscr{B}^{(i)}(\boldsymbol{u}^{(i)};\boldsymbol{u}^{(i+1\dots n)}\backslash u^{(n)}_1)\,s_{\hat n\hat n}(u^{(n)}_1) + UWT$$

where UWT denotes "unwanted" terms, all of which act by 0 on $\eta_1^{\boldsymbol{m}}$. We have thus shown that

$$\Psi_1(\boldsymbol{u}^{(1\dots n)}) = \mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u_1^{(n)})\, \mathscr{b}^{(n)}_{\dot a_1 \ddot a_1}(u_1^{(n)}) \overrightarrow{\prod_{j>1}} R^{(\hat n,\hat n)}_{\dot a_1 \ddot a_j}(\tilde u_1^{(n)} - u_j^{(n)})\, \Psi_1^{(n-1)}(\boldsymbol{u}^{1\dots n-1}; \boldsymbol{u}^{(n)})$$
$$= \mu_{\hat n}(\nu) f^+(u_1^{(n)}, \tilde{\boldsymbol{u}}^{(n)}\backslash \tilde u_1^{(n)})\, \Psi(\boldsymbol{u}^{1\dots n}\backslash u_1^{(n)}). \qquad (76)$$

This gives the $j=1$ case of the claim. Then, using Yang-Baxter equation, Lemma 3.3, and the identity

$$\eta^{\boldsymbol{m}}_{j+1} = f^+(u_j^{(n)}, u_{j+1}^{(n)})\check R^{(\hat n,\hat n)}_{\dot a_{j+1}\dot a_j}(u_{j+1}^{(n)} - u_j^{(n)})\check R^{(\hat n,\hat n)}_{\ddot a_{j+1}\ddot a_j}(u_j^{(n)} - u_{j+1}^{(n)}) \cdot \eta_j^{\boldsymbol{m}} + \frac{1}{u_{j+1}^{(n)} - u_j^{(n)}}\eta_j^{\boldsymbol{m}}$$

we find

$$\Psi_{j+1}(\boldsymbol{u}^{(1\dots n)}) = f^+(u_j^{(n)}, u_{j+1}^{(n)})\, \Psi_j(\boldsymbol{u}^{(1\dots n)}_{u_j^{(n)} \leftrightarrow u_{j+1}^{(n)}}) + \frac{1}{u_{j+1}^{(n)} - u_j^{(n)}}\, \Psi_j(\boldsymbol{u}^{(1\dots n)}). \qquad (77)$$

A simple induction on j together with Lemma 3.6 gives the wanted result. □

Proof of Proposition 4.4. The main idea of the proof is similar to that of the proof of Proposition 4.2. However, there will be additional steps because in the $\hat n = n+1$ case (recall (65), (49) and (48))

$$\mathscr{R}^{(\hat n)}(u_{m_n}^{(n)}, \boldsymbol{u}^{(n)}\backslash u_{m_n}^{(n)}) \cdot \eta^{\boldsymbol{m}} = \eta^{\boldsymbol{m}} + \sum_{j<m_n} \frac{\prod_{j<k<m_n} f^+(u_k^{(n)}, \tilde u_{m_n}^{(n)})}{u_j^{(n)} - \tilde u_{m_n}^{(n)}} P^{(\hat n,\hat n)}_{\dot a_j \ddot a_{m_n}} \eta^{\boldsymbol{m}}. \qquad (78)$$

Thus, moving operator $\mathscr{R}^{(\hat n)}(u_{m_n}^{(n)}; \boldsymbol{u}^{(n)}\backslash u_{m_n}^{(n)})$ in (65) all way to the reference vector $\eta^{\boldsymbol{m}}$ results in the expression

$$\dot\Psi^{(n-1)}_{2,1}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}) + \sum_{j<m_n} \frac{\prod_{j<k<m_n} f^+(u_k^{(n)}, \tilde u_{m_n}^{(n)})}{u_j^{(n)} - \tilde u_{m_n}^{(n)}}\, \dot\Psi^{(n-1)}_{2,2;j}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}) \qquad (79)$$

where $\dot\Psi_{k,l}$ and $\dot\Psi_{k,l;j}$ denote Bethe vectors based on the transfer matrix (66) and the reference vectors $(E_{k2}^{(\hat n)})_{\dot a_{m_n}}(E_{l1}^{(\hat n)})_{\ddot a_{m_n}}\eta^{\boldsymbol{m}}$ and $(E_{k2}^{(\hat n)})_{\dot a_{m_n}}(E_{l1}^{(\hat n)})_{\ddot a_{m_n}}(E_{12}^{(\hat n)})_{\dot a_j}\eta^{\boldsymbol{m}}$, respectively. Consider the second term in (79). Acting with $\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u_{m_n}^{(n)})$ and applying Lemma 4.6 gives

$$\sum_{i\le j<m_n} \prod_{j<k<m_n} \frac{f^+(u_k^{(n)}, \tilde u_{m_n}^{(n)})}{f^+(u_k^{(n)}, u_i^{(n)})} \cdot \frac{\Gamma_{\hat n}(u_i^{(n)}, \boldsymbol{u}^{(1\dots n)}\backslash (u_i^{(n)}, u_{m_n}^{(n)}))}{(u_j^{(n)} - u_i^{(n)} + 1)(u_j^{(n)} - \tilde u_{m_n}^{(n)})}$$
$$\times \mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash (u_i^{(n)}, u_{m_n}^{(n)}))\, \dot\Psi^{(n-1)}_{2,2}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}\backslash u_i^{(n)}). \qquad (80)$$

Using the identity

$$\frac{1}{u_i^{(n)} - \tilde u_{m_n}^{(n)}} = \sum_{i\le j<m_n} \frac{1}{(u_j^{(n)} - u_i^{(n)} + 1)(u_j^{(n)} - \tilde u_{m_n}^{(n)})} \prod_{j<k<m_n} \frac{f^+(u_k^{(n)}, \tilde u_{m_n}^{(n)})}{f^+(u_k^{(n)}, u_i^{(n)})} \qquad (81)$$

which follows by a descending induction on i, expression (80) becomes

$$\sum_{i<m_n} \frac{\Gamma_{\hat n}(u_i^{(n)}; \boldsymbol{u}^{(n)}\backslash (u_i^{(n)}, u_{m_n}^{(n)}))}{u_i^{(n)} - \tilde u_{m_n}^{(n)}}\, \mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash (u_i^{(n)}, u_{m_n}^{(n)}))\, \dot\Psi^{(n-1)}_{2,2}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}\backslash u_i^{(n)}). \qquad (82)$$

Thus, acting with $\mathscr{b}^{(n)}_{\dot a_{m_n}\ddot a_{m_n}}(u^{(n)}_{m_n})\,\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u^{(n)}_{m_n})$ on (79) and recalling (65) gives

$$\begin{aligned}\Psi(\boldsymbol{u}^{(1\dots n)}) = \mathscr{b}^{(n)}_{\dot a_{m_n}\ddot a_{m_n}}(u^{(n)}_{m_n})\Bigg(&\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u^{(n)}_{m_n})\,\dot{\Psi}^{(n-1)}_{2,1}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}) \\ &+\sum_{i<m_n}\frac{\Gamma_{\hat n}(u^{(n)}_i;\boldsymbol{u}^{(1\dots n)}\backslash(u^{(n)}_i,u^{(n)}_{m_n}))}{u^{(n)}_i-\tilde u^{(n)}_{m_n}} \\ &\times\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash(u^{(n)}_i,u^{(n)}_{m_n}))\,\dot{\Psi}^{(n-1)}_{2,2}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}\backslash u^{(n)}_i)\Bigg).\end{aligned}\tag{83}$$

We will view vectors $\dot{\Psi}^{(n-1)}_{2,1}$ and $\dot{\Psi}^{(n-1)}_{2,2}$ as $Y(\mathfrak{gl}_n)$-based Bethe vectors and apply $Y(\mathfrak{gl}_n)$-based recurrence relations.

First, consider vector $\dot{\Psi}^{(n-1)}_{2,2}$. Its reference vector is annihilated by the (j,i)-th entries, with $1\le i<j\le n$, of the monodromy matrix (66), and we may use (61) to obtain an expansion in the space $V^{(\hat n)}_{\dot a_{m_n}}\otimes V^{(\hat n)}_{\ddot a_{m_n}}$. Taking $\boldsymbol{u}^{(n)}_{\rm III}=u^{(n)}_{m_n}$, the second term inside the brackets of (83) becomes

$$\begin{aligned}\sum_{1\le i\le n}\;\sum_{\substack{|\boldsymbol{u}^{(r)}_{\rm II,III}|=(2,0)\\ i\le r<n}}\;\sum_{|\boldsymbol{u}^{(n)}_{\rm II}|=1}\;\prod_{i<k\le n}&\Gamma_k(\boldsymbol{u}^{(k-1)}_{\rm II};\boldsymbol{u}^{(1\dots n)}_{\rm I})\,K(\boldsymbol{u}^{(k-1)}_{\rm II}\,|\,\boldsymbol{u}^{(k)}_{\rm II,III}) \\ &\times\frac{\Gamma_{\hat n}(\boldsymbol{u}^{(n)}_{\rm II};\boldsymbol{u}^{(1\dots n)}_{\rm I})}{\boldsymbol{u}^{(n)}_{\rm II}-\tilde{\boldsymbol{u}}^{(n)}_{\rm III}}\,E^{(\hat n)}_{\bar\imath}\otimes E^{(\hat n)}_{\bar\imath}\otimes\Psi(\boldsymbol{u}^{(1\dots n)}_{\rm I})\end{aligned}\tag{84}$$

$$\begin{aligned}&+\sum_{1\le i<n}\;\sum_{\substack{|\boldsymbol{u}^{(r)}_{\rm II}|=1\\ i\le r\le n}}\;\prod_{i<k<n}\frac{\Gamma_k(\boldsymbol{u}^{(k-1)}_{\rm II};\boldsymbol{u}^{(1\dots n)}_{\rm I})}{\boldsymbol{u}^{(k-1)}_{\rm II}-\boldsymbol{u}^{(k)}_{\rm II}}\cdot\frac{\Gamma_n(\boldsymbol{u}^{(n-1)}_{\rm II};\boldsymbol{u}^{(1\dots n)}_{\rm I})\,\Gamma_{\hat n}(\boldsymbol{u}^{(n)}_{\rm II};\boldsymbol{u}^{(1\dots n)}_{\rm I})}{\boldsymbol{u}^{(n)}_{\rm II}-\tilde{\boldsymbol{u}}^{(n)}_{\rm III}} \\ &\quad\times\left(\frac{f^+(\boldsymbol{u}^{(n-1)}_{\rm II},\tilde{\boldsymbol{u}}^{(n)}_{\rm III})}{\boldsymbol{u}^{(n-1)}_{\rm II}-\boldsymbol{u}^{(n)}_{\rm III}}\,E^{(\hat n)}_{\bar\imath}\otimes E^{(\hat n)}_{2}+\frac{1}{\boldsymbol{u}^{(n-1)}_{\rm II}-\tilde{\boldsymbol{u}}^{(n)}_{\rm III}}\,E^{(\hat n)}_{2}\otimes E^{(\hat n)}_{\bar\imath}\right)\otimes\Psi(\boldsymbol{u}^{(1\dots n)}_{\rm I})\end{aligned}\tag{85}$$

$$\begin{aligned}&+\sum_{1\le i<j<n}\;\sum_{\substack{|\boldsymbol{u}^{(r)}_{\rm III}|=1\\ i\le r<n}}\;\sum_{\substack{|\boldsymbol{u}^{(s)}_{\rm II}|=1\\ j\le s\le n}}\;\prod_{i<k<j}\frac{\Gamma_k(\boldsymbol{u}^{(k-1)}_{\rm III};\boldsymbol{u}^{(1\dots n)}_{\rm I})}{\boldsymbol{u}^{(k-1)}_{\rm III}-\boldsymbol{u}^{(k)}_{\rm III}}\cdot\Gamma_j(\boldsymbol{u}^{(j-1)}_{\rm III};\boldsymbol{u}^{(1\dots n)}_{\rm I}) \\ &\quad\times\prod_{j<k<n}\frac{\Gamma_k(\boldsymbol{u}^{(k-1)}_{\rm II};\boldsymbol{u}^{(1\dots n)}_{\rm I})\,\Gamma_k(\boldsymbol{u}^{(k-1)}_{\rm III};\boldsymbol{u}^{(1\dots n)}_{\rm I,II})}{(\boldsymbol{u}^{(k-1)}_{\rm II}-\boldsymbol{u}^{(k)}_{\rm II})(\boldsymbol{u}^{(k-1)}_{\rm III}-\boldsymbol{u}^{(k)}_{\rm III})}\cdot\frac{\Gamma_n(\boldsymbol{u}^{(n-1)}_{\rm II,III};\boldsymbol{u}^{(1\dots n)}_{\rm I})\,\Gamma_{\hat n}(\boldsymbol{u}^{(n)}_{\rm II};\boldsymbol{u}^{(1\dots n)}_{\rm I})}{\boldsymbol{u}^{(n)}_{\rm II}-\tilde{\boldsymbol{u}}^{(n)}_{\rm III}} \\ &\quad\times\frac{f^-(\boldsymbol{u}^{(n-1)}_{\rm III},\boldsymbol{u}^{(n-1)}_{\rm II})\,f^+(\boldsymbol{u}^{(n-1)}_{\rm III},\tilde{\boldsymbol{u}}^{(n)}_{\rm III})}{(\boldsymbol{u}^{(n-1)}_{\rm II}-\tilde{\boldsymbol{u}}^{(n)}_{\rm III})(\boldsymbol{u}^{(n-1)}_{\rm III}-\boldsymbol{u}^{(n)}_{\rm III})} \\ &\quad\times\left(\frac{f^+(\boldsymbol{u}^{(j-1)}_{\rm III},\boldsymbol{u}^{(j)}_{\rm II})}{\boldsymbol{u}^{(j-1)}_{\rm III}-\boldsymbol{u}^{(j)}_{\rm III}}\,E^{(\hat n)}_{\bar\imath}\otimes E^{(\hat n)}_{\bar\jmath}+\frac{1}{\boldsymbol{u}^{(j-1)}_{\rm III}-\boldsymbol{u}^{(j)}_{\rm II}}\,E^{(\hat n)}_{\bar\jmath}\otimes E^{(\hat n)}_{\bar\imath}\right)\otimes\Psi(\boldsymbol{u}^{(1\dots n)}_{\rm I}).\end{aligned}\tag{86}$$

Next, consider vector $\dot{\Psi}^{(n-1)}_{2,1}$. This time we can not apply (61). Instead, we will use the composite model approach to obtain the wanted expansion. Set $L^{\mathsf{II}} := V^{(\hat n)}_{\dot a_{m_n}}\otimes V^{(\hat n)}_{\ddot a_{m_n}}$ and $L^{\mathsf{I}} := W^{(\hat n)}_{\dot{\boldsymbol{a}}\backslash\dot a_{m_n}}\otimes W^{(\hat n)}_{\ddot{\boldsymbol{a}}\backslash\ddot a_{m_n}}\otimes(L^{(n)})^0$ so that $L^{(n-1)}\cong L^{\mathsf{II}}\otimes L^{\mathsf{I}}$. Recall (39) and set

$$\mathscr{a}^{\mathsf{II}}_{a_i^{n-1},k}(v) := \sum_{j<n}(E^{(n-1)}_j)^*_{a_i^{n-1}}\otimes\big[R^{(\hat n,\hat n)}_{\dot a_{m_n}a}(v-u^{(n)}_{m_n})\,R^{(\hat n,\hat n)}_{\ddot a_{m_n}a}(v-\tilde u^{(n)}_{m_n})\big]_{n-j,k},$$

$$\mathscr{b}^{\mathsf{I}}_k(v) := \big[T^{(k+1)}_a(u^{(k)}_i;\boldsymbol{u}^{(n)}\backslash u^{(n)}_{m_n})\big]_{k,n}.$$

The cases when $k = n, \hat{n}$ we denote by

$$\mathcal{b}^{\mathrm{II}}_{a_i^{n-1}}(v) := \mathcal{a}^{\mathrm{II}}_{a_i^{n-1},n}(v), \qquad \mathcal{p}^{\mathrm{II}}_{a_i^{n-1}}(v) := \mathcal{a}^{\mathrm{II}}_{a_i^{n-1},\hat{n}}(v), \qquad \mathcal{d}^{\mathrm{I}}(v) := \mathcal{b}^{\mathrm{I}}_{n}(v), \qquad \mathcal{c}^{\mathrm{I}}(v) := \mathcal{b}^{\mathrm{I}}_{\hat{n}}(v)$$

so that

$$\mathcal{b}^{(n-1)}_{a_i^{n-1}}(v;\boldsymbol{u}^{(n)}) = \sum_{k<n} \mathcal{a}^{\mathrm{II}}_{a_i^{n-1},k}(v)\,\mathcal{b}^{\mathrm{I}}_{k}(v) + \mathcal{b}^{\mathrm{II}}_{a_i^{n-1}}(v)\,\mathcal{d}^{\mathrm{I}}(v) + \mathcal{p}^{\mathrm{II}}_{a_i^{n-1}}(v)\,\mathcal{c}^{\mathrm{I}}(v)\,.$$

This notation is reminiscent of the Bethe ansatz notation commonly used in the composite model approach, and $\mathcal{p}^{\mathrm{II}}_{a_i^{n-1}}$ is an additional creation operator. Consider the II-labelled operators. Their action on the reference state $E_2^{(\hat{n})} \otimes E_1^{(\hat{n})}$ in the space L^{II} is given by

$$\begin{aligned}
\mathcal{a}^{\mathrm{II}}_{a_i^{n-1},j}(v)\cdot E_2^{(\hat{n})}\otimes E_1^{(\hat{n})} &= (E^{(n-1)}_{n-j})^*_{a_i^{n-1}}\cdot E_2^{(\hat{n})}\otimes E_1^{(\hat{n})},\\
\mathcal{b}^{\mathrm{II}}_{a_i^{n-1}}(v)\cdot E_2^{(\hat{n})}\otimes E_1^{(\hat{n})} &= \frac{1}{v-u^{(n)}_{m_n}}\sum_{j<n}(E^{(n-1)}_{j})^*_{a_i^{n-1}}\cdot E_{j+2}^{(\hat{n})}\otimes E_1^{(\hat{n})},\\
\mathcal{p}^{\mathrm{II}}_{a_i^{n-1}}(v)\cdot E_2^{(\hat{n})}\otimes E_1^{(\hat{n})} &= \frac{1}{v-\tilde{u}^{(n)}_{m_n}}\sum_{j<n}(E^{(n-1)}_{j})^*_{a_i^{n-1}}\left(\frac{1}{v-u^{(n)}_{m_n}}\,E_{j+2}^{(\hat{n})}\otimes E_2^{(\hat{n})} + E_2^{(\hat{n})}\otimes E_{j+2}^{(\hat{n})}\right),\\
\mathcal{p}^{\mathrm{II}}_{a_l^{n-1}}(w)\,\mathcal{b}^{\mathrm{II}}_{a_i^{n-1}}(v)\cdot E_2^{(\hat{n})}\otimes E_1^{(\hat{n})} &= \frac{1}{(w-\tilde{u}^{(n)}_{m_n})(v-u^{(n)}_{m_n})}\sum_{j,k<n}(E^{(n-1)}_{j})^*_{a_l^{n-1}}(E^{(n-1)}_{k})^*_{a_i^{n-1}}\\
&\qquad\times\left(\frac{1}{w-u^{(n)}_{m_n}}\,E_{j+2}^{(\hat{n})}\otimes E_{k+2}^{(\hat{n})} + E_{k+2}^{(\hat{n})}\otimes E_{j+2}^{(\hat{n})}\right).
\end{aligned}$$

Moreover, $\mathcal{b}^{\mathrm{II}}_{a_j^{n-1}}(v)\,\mathcal{b}^{\mathrm{II}}_{a_i^{n-1}}(u)$, $\mathcal{p}^{\mathrm{II}}_{a_j^{n-1}}(v)\,\mathcal{p}^{\mathrm{II}}_{a_i^{n-1}}(u)$, and $\mathcal{p}^{\mathrm{II}}_{a_k^{n-1}}(w)\,\mathcal{p}^{\mathrm{II}}_{a_j^{n-1}}(v)\,\mathcal{b}^{\mathrm{II}}_{a_i^{n-1}}(u)$ act by zero on $E_2^{(\hat{n})}\otimes E_1^{(\hat{n})}$. The homogeneous ($aa$ and bb, pp) exchange relations are analogous to (40) and (41), respectively. The mixed (ab, ap, bp) exchange relations have the form

$$\mathcal{a}^{\mathrm{II}}_{a_j^{n-1}}(v)\,\mathcal{b}^{\mathrm{II}}_{a_i^{n-1}}(u) = \mathcal{b}^{\mathrm{II}}_{a_i^{n-1}}(u)\,\mathcal{a}^{\mathrm{II}}_{a_j^{n-1}}(v)\,R^{(n-1,n-1)}_{a_i^{n-1},a_j^{n-1}}(u-v) + \frac{1}{u-v}\,\mathcal{b}^{\mathrm{II}}_{a_i^{n-1}}(v)\,\mathcal{a}^{\mathrm{II}}_{a_j^{n-1}}(u)\,P^{(n-1,n-1)}_{a_j^{n-1},a_i^{n-1}}.$$

Consider the I-labelled operators. The dc, cb, db exchange relations have the form

$$\mathcal{d}^{\mathrm{I}}(v)\,\mathcal{c}^{\mathrm{I}}(u) = f^-(v,u)\,\mathcal{c}^{\mathrm{I}}(u)\,\mathcal{d}^{\mathrm{I}}(v) + \frac{1}{v-u}\,\mathcal{c}^{\mathrm{I}}(v)\,\mathcal{d}^{\mathrm{I}}(u).$$

The standard Bethe ansatz arguments then imply

$$\overleftarrow{\prod_i}\mathcal{b}^{(n-1)}_{a_i^{n-1}}(u_i^{(n-1)};\boldsymbol{u}^{(n)})\cdot E_2^{(\hat{n})}\otimes E_1^{(\hat{n})}\otimes\Psi^{(n-2)}(\boldsymbol{u}^{(1\dots n-2)}\,|\,\boldsymbol{u}^{(n-1,n)}\backslash u^{(n)}_{m_n})$$

$$= \Bigg[E_2^{(\hat{n})}\otimes E_1^{(\hat{n})}\otimes\overleftarrow{\prod_i}\mathcal{b}^{\mathrm{I}}_{a_i^{n-1}}(u_i^{(n-1)}) \tag{87}$$

$$+\sum_j\frac{f^-(u_j^{(n-1)},\boldsymbol{u}^{(n-1)}\backslash u_j^{(n-1)})}{u_j^{(n-1)}-u^{(n)}_{m_n}}\sum_{k<n}E_{k+2}^{(\hat{n})}\otimes E_1^{(\hat{n})}\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}$$

$$\times\overleftarrow{\prod_{i\neq j}}\mathcal{b}^{\mathrm{I}}_{a_i^{n-1}}(u_i^{(n-1)})\,\mathcal{d}^{\mathrm{I}}(u_j^{(n-1)}) \tag{88}$$

$$+\sum_j\frac{f^-(u_j^{(n-1)},\boldsymbol{u}^{(n-1)}\backslash u_j^{(n-1)})}{u_j^{(n-1)}-\tilde{u}^{(n)}_{m_n}}\sum_{k<n}\left(\frac{1}{u_j^{(n-1)}-u^{(n)}_{m_n}}\,E_{k+2}^{(\hat{n})}\otimes E_2^{(\hat{n})}+E_2^{(\hat{n})}\otimes E_{k+2}^{(\hat{n})}\right)$$

$$\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}\overleftarrow{\prod_{i\neq j}}\mathcal{b}^{\mathrm{I}}_{a_i^{n-1}}(u_i^{(n-1)})\,\mathcal{c}^{\mathrm{I}}(u_j^{(n-1)}) \tag{89}$$

$$
\begin{aligned}
&+\sum_{j<j'} f^{-}((u_j^{(n-1)},u_{j'}^{(n-1)}),\boldsymbol{u}^{(n-1)}\backslash(u_j^{(n-1)},u_{j'}^{(n-1)}))\\
&\quad\times\sum_{k,l<n}\Bigg(\frac{1}{\gamma}\Big(\alpha_{11}E_{k+2}^{(\hat n)}\otimes E_{l+2}^{(\hat n)}+\alpha_{12}E_{l+2}^{(\hat n)}\otimes E_{k+2}^{(\hat n)}\Big)\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}(E_l^{(n-1)})^*_{a_{j'}^{n-1}}\\
&\qquad\qquad\qquad\qquad\times\overleftarrow{\prod_{i\neq j,j'}}\mathscr{b}^{\dagger}_{a_i^{n-1}}(u_i^{(n-1)})\,\mathscr{c}^{\dagger}(u_{j'}^{(n-1)})\,\mathscr{d}^{\dagger}(u_j^{(n-1)})\\
&\qquad+\frac{1}{\gamma}\Big(\alpha_{21}E_{k+2}^{(\hat n)}\otimes E_{l+2}^{(\hat n)}+\alpha_{22}E_{l+2}^{(\hat n)}\otimes E_{k+2}^{(\hat n)}\Big)\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}(E_l^{(n-1)})^*_{a_{j'}^{n-1}}\\
&\qquad\qquad\qquad\qquad\times\overleftarrow{\prod_{i\neq j,j'}}\mathscr{b}^{\dagger}_{a_i^{n-1}}(u_i^{(n-1)})\,\mathscr{c}^{\dagger}(u_j^{(n-1)})\,\mathscr{d}^{\dagger}(u_{j'}^{(n-1)})\Big)\Bigg] \qquad (90)\\
&\times\Psi^{(n-2)}(\boldsymbol{u}^{(1\ldots n-2)}\,|\,\boldsymbol{u}^{(n-1,n)}\backslash u_{m_n}^{(n)})
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_{11}&:=(u_{j'}^{(n-1)}-u_{m_n}^{(n)})(u_j^{(n-1)}-\tilde u_{m_n}^{(n)})-(u_{j'}^{(n-1)}-u_{m_n}^{(n)})/(u_j^{(n-1)}-u_{j'}^{(n-1)}),\\
\alpha_{12}&:=u_j^{(n-1)}-\tilde u_{m_n}^{(n)}-((u_j^{(n-1)}-u_{m_n}^{(n)})(u_{j'}^{(n-1)}-\tilde u_{m_n}^{(n)})+1)/(u_j^{(n-1)}-u_{j'}^{(n-1)}),\\
\alpha_{21}&:=f^{+}(u_j^{(n-1)},u_{j'}^{(n-1)})\,(u_{j'}^{(n-1)}-u_{m_n}^{(n)}),\\
\alpha_{22}&:=f^{+}(u_j^{(n-1)},u_{j'}^{(n-1)})((u_j^{(n-1)}-u_{m_n}^{(n)})(u_{j'}^{(n-1)}-\tilde u_{m_n}^{(n)})+1),\\
\gamma&:=(u_j^{(n-1)}-u_{m_n}^{(n)})(u_j^{(n-1)}-\tilde u_{m_n}^{(n)})(u_{j'}^{(n-1)}-u_{m_n}^{(n)})(u_{j'}^{(n-1)}-\tilde u_{m_n}^{(n)}).
\end{aligned}
\qquad (91)
$$

We will consider the terms (87–90) individually.

First, onsider the term (87). Acting with $\mathscr{b}^{(n)}_{\dot a_{m_n}\ddot a_{m_n}}(u_{m_n}^{(n)})\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u_{m_n}^{(n)})$ gives the $i=n$ case of the first term on the right hand side of (67).

Next, consider the term (88). The operator $\mathscr{d}^{\dagger}(u_j^{(n-1)})$ acts on $\Psi^{(n-2)}(\boldsymbol{u}^{(1\ldots n-2)}\,|\,\boldsymbol{u}^{(n-1,n)}\backslash u_{m_n}^{(n)})$ via multiplication by $f^{+}(u_j^{(n-1)},\boldsymbol{u}^{(n)}\backslash u_{m_n}^{(n)})\,\mu_n(u_j^{(n-1)})$ giving

$$
\begin{aligned}
\sum_j\frac{\Gamma_n(u_j^{(n-1)};\boldsymbol{u}^{(1\ldots n)}\backslash(u_j^{(n-1)},u_{m_n}^{(n)}))}{u_j^{(n-1)}-u_{m_n}^{(n)}}&\sum_{k<n}E_{k+2}^{(\hat n)}\otimes E_1^{(\hat n)}\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}\\
&\times\Psi^{(n-2)}(\boldsymbol{u}^{(1\ldots n-1)}\backslash u_j^{(n-1)}\,|\,u_j^{(n-1)},\boldsymbol{u}^{(n)}\backslash u_{m_n}^{(n)}).
\end{aligned}
\qquad (92)
$$

Using (60), we expand $\Psi^{(n-2)}(\boldsymbol{u}^{(1\ldots n-1)}\backslash u_j^{(n-1)}\,|\,u_j^{(n-1)},\boldsymbol{u}^{(n)}\backslash u_{m_n}^{(n)})$ in the space $V^{(n-1)}_{a_j^{n-1}}$:

$$
\sum_{i<n}\sum_{\substack{|\boldsymbol{u}_{\rm III}^{(r)}|=1\\ i\le r<n-1}}\prod_{i<k<n}\frac{\Gamma_k(\boldsymbol{u}_{\rm III}^{(k-1)};\boldsymbol{u}_{\rm I}^{(1\ldots n)})}{\boldsymbol{u}_{\rm III}^{(k-1)}-\boldsymbol{u}_{\rm III}^{(k)}}E_{n-i}^{(n-1)}\otimes\Psi^{(n-1)}(\boldsymbol{u}_{\rm I}^{(1\ldots n-1)}\,|\,\boldsymbol{u}_{\rm I}^{(n)}) \qquad (93)
$$

where $\boldsymbol{u}_{\rm III}^{(n-1)}:=u_j^{(n-1)}$ and $\boldsymbol{u}_{\rm I}^{(n)}:=\boldsymbol{u}^{(n)}\backslash u_{m_n}^{(n)}$. Substituting (93) into (92) yields

$$
\sum_{i<n}\sum_{\substack{|\boldsymbol{u}_{\rm II}^{(r)}|=1\\ i\le r<n}}\prod_{i<k\le n}\frac{\Gamma_k(\boldsymbol{u}_{\rm III}^{(k-1)};\boldsymbol{u}_{\rm I}^{(1\ldots n)})}{\boldsymbol{u}_{\rm III}^{(k-1)}-\boldsymbol{u}_{\rm III}^{(k)}}E_{\hat n-i+1}^{(\hat n)}\otimes E_1^{(\hat n)}\otimes\Psi^{(n-1)}(\boldsymbol{u}_{\rm I}^{(1\ldots n-1)}\,|\,\boldsymbol{u}_{\rm I}^{(n)}). \qquad (94)
$$

Acting with $\mathscr{b}^{(n)}_{\dot a_{m_n}\ddot a_{m_n}}(u_{m_n}^{(n)})\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u_{m_n}^{(n)})$ gives the $i<n$ cases of the first term on the right hand side of (67).

We are now ready to consider the term (89). Let η^{I} denote the restriction of $\eta^{\boldsymbol{m}}$ to the space L^{I}. Set $\eta^{\mathsf{I}}_l := (E^{(\hat{n})}_{12})_{\ddot{a}_l} \cdot \eta^{\mathsf{I}}$. Using the explicit form of $\mathcal{c}^{\mathsf{I}}(u^{(n-1)}_j)$ we find

$$\mathcal{c}^{\mathsf{I}}(u^{(n-1)}_j) \cdot \eta^{\mathsf{I}} = \sum_{l<m_n} \frac{\prod_{k<l} f^+(u^{(n-1)}_j, u^{(n)}_k)}{u^{(n-1)}_j - u^{(n)}_l} \, \mu_n(u^{(n-1)}_j) \, \eta^{\mathsf{I}}_l \tag{95}$$

giving

$$\begin{aligned}\sum_j \frac{f^-(u^{(n-1)}_j, \boldsymbol{u}^{(n-1)} \backslash u^{(n-1)}_j)}{u^{(n-1)}_j - \tilde{u}^{(n)}_{m_n}} \sum_{k<n} \left(\frac{1}{u^{(n-1)}_j - u^{(n)}_{m_n}} E^{(\hat{n})}_{k+2} \otimes E^{(\hat{n})}_2 + E^{(\hat{n})}_2 \otimes E^{(\hat{n})}_{k+2} \right) \otimes (E^{(n-1)}_k)^*_{a^{n-1}_j} \\ \times \sum_{l<m_n} \frac{\prod_{k<l} f^+(u^{(n-1)}_j, u^{(n)}_k)}{u^{(n-1)}_j - u^{(n)}_l} \, \mu_n(u^{(n-1)}_j) \, \Psi^{(n-2)}_l(\boldsymbol{u}^{(1\ldots n-1)} \backslash u^{(n-1)}_j \,|\, u^{(n-1)}_j, \boldsymbol{u}^{(n)} \backslash u^{(n)}_{m_n}).\end{aligned} \tag{96}$$

Acting with $\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)} \backslash u^{(n)}_{m_n})$ and applying Lemma 4.6 to the second line of (96) gives

$$\begin{aligned}\sum_{i \le l < m_n} \frac{\prod_{k<l} f^+(u^{(n-1)}_j, u^{(n)}_k)}{\prod_{l<k<m_n} f^+(u^{(n)}_k, u^{(n)}_i)} \cdot \frac{\Gamma_{\hat{n}}(u^{(n)}_i, \boldsymbol{u}^{(1\ldots n)} \backslash (u^{(n)}_i, u^{(n)}_{m_n}))}{(u^{(n)}_l - u^{(n)}_i + 1)(u^{(n-1)}_j - u^{(n)}_l)} \\ \times \mu_n(u^{(n-1)}_j) \, \Psi^{(n-2)}(\boldsymbol{u}^{(1\ldots n-1)} \backslash (u^{(n-1)}_j, u^{(n)}_i, u^{(n)}_{m_n}) \,|\, u^{(n-1)}_j).\end{aligned} \tag{97}$$

Using the identity

$$\frac{f^+(u^{(n-1)}_j, \boldsymbol{u}^{(n)} \backslash (u^{(n)}_i, u^{(n)}_{m_n}))}{u^{(n-1)}_j - u^{(n)}_i} = \sum_{i \le l < m_n} \frac{\prod_{k<l} f^+(u^{(n-1)}_j, u^{(n)}_k)}{\prod_{l<k<m_n} f^+(u^{(n)}_k, u^{(n)}_i)} \cdot \frac{1}{(u^{(n)}_l - u^{(n)}_i + 1)(u^{(n-1)}_j - u^{(n)}_l)}$$

which follows by a descending induction on i, expression (97) becomes

$$\begin{aligned}\sum_{i<m_n} \frac{f^+(u^{(n-1)}_j, \boldsymbol{u}^{(n)} \backslash (u^{(n)}_i, u^{(n)}_{m_n}))}{u^{(n-1)}_j - u^{(n)}_i} \, \Gamma_{\hat{n}}(u^{(n)}_i, \boldsymbol{u}^{(1\ldots n)} \backslash (u^{(n)}_i, u^{(n)}_{m_n})) \\ \times \mu_n(u^{(n-1)}_j) \, \Psi^{(n-2)}(\boldsymbol{u}^{(1\ldots n-1)} \backslash u^{(n-1)}_j \,|\, u^{(n-1)}_j, u^{(n)}_i, u^{(n)}_{m_n}).\end{aligned} \tag{98}$$

Therefore, acting with $\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)} \backslash u^{(n)}_{m_n})$ on (96) gives

$$\begin{aligned}\sum_j \sum_{i<m_n} \frac{\Gamma_n(u^{(n-1)}_j; \boldsymbol{u}^{(1\ldots n)} \backslash (u^{(n-1)}_j, u^{(n)}_i, u^{(n)}_{m_n})) \, \Gamma_{\hat{n}}(u^{(n)}_i; \boldsymbol{u}^{(1\ldots n)} \backslash (u^{(n)}_i, u^{(n)}_{m_n}))}{(u^{(n-1)}_j - u^{(n)}_i)(u^{(n-1)}_j - \tilde{u}^{(n)}_{m_n})} \\ \times \sum_{k<n} \left(\frac{1}{u^{(n-1)}_j - u^{(n)}_{m_n}} E^{(\hat{n})}_{k+2} \otimes E^{(\hat{n})}_2 + E^{(\hat{n})}_2 \otimes E^{(\hat{n})}_{k+2} \right) \otimes (E^{(n-1)}_k)^*_{a^{n-1}_j} \\ \times \Psi^{(n-2)}(\boldsymbol{u}^{(1\ldots n-1)} \backslash (u^{(n-1)}_j, u^{(n)}_i, u^{(n)}_{m_n}) \,|\, u^{(n-1)}_j).\end{aligned} \tag{99}$$

Finally, we expand $\Psi^{(n-2)}(\boldsymbol{u}^{(1\ldots n-1)} \backslash u^{(n-1)}_j \,|\, u^{(n-1)}_j, u^{(n)}_i, u^{(n)}_{m_n})$ in the space $V^{(n-1)}_{a^{n-1}_j}$ analogously to (93) yielding

$$\begin{aligned}\sum_{i<n} \sum_{\substack{|\boldsymbol{u}^{(r)}_{\mathrm{II}}|=1 \\ i \le r \le n}} \prod_{i<k<n} \frac{\Gamma_k(\boldsymbol{u}^{(k-1)}_{\mathrm{II}}; \boldsymbol{u}^{(1\ldots n)}_{\mathrm{I}})}{\boldsymbol{u}^{(k-1)}_{\mathrm{II}} - \boldsymbol{u}^{(k)}_{\mathrm{II}}} \cdot \frac{\Gamma_n(\boldsymbol{u}^{(n-1)}_{\mathrm{II}}; \boldsymbol{u}^{(1\ldots n)}_{\mathrm{I}}) \, \Gamma_{\hat{n}}(\boldsymbol{u}^{(n)}_{\mathrm{II}}; \boldsymbol{u}^{(1\ldots n)}_{\mathrm{I}})}{(\boldsymbol{u}^{(n-1)}_{\mathrm{II}} - \boldsymbol{u}^{(n)}_{\mathrm{II}})(\boldsymbol{u}^{(n-1)}_{\mathrm{II}} - \tilde{\boldsymbol{u}}^{(n)}_{\mathrm{III}})} \\ \times \left(\frac{1}{\boldsymbol{u}^{(n-1)}_{\mathrm{II}} - \boldsymbol{u}^{(n)}_{\mathrm{III}}} E^{(\hat{n})}_{\bar{\imath}} \otimes E^{(\hat{n})}_2 + E^{(\hat{n})}_2 \otimes E^{(\hat{n})}_{\bar{\imath}} \right) \otimes \Psi(\boldsymbol{u}^{(1\ldots n)}_{\mathrm{I}}).\end{aligned} \tag{100}$$

392 Combining (100) with (85) and acting with $\mathscr{b}^{(n)}_{\dot a_{m_n}\ddot a_{m_n}}(u^{(n)}_{m_n})$ gives the second term on the right
393 hand side of (67).

394 It remains to consider the term (90). Using the same arguments as above, and renaming
395 $j \to p$, $j' \to p'$, we obtain

$$\sum_{i<m_n}\sum_{p<p'}\Gamma_n((u^{(n-1)}_p,u^{(n-1)}_{p'});\boldsymbol{u}^{(1\dots n)}\backslash(u^{(n-1)}_p,u^{(n-1)}_{p'},u^{(n)}_i,u^{(n)}_{m_n}))\,\Gamma_{\hat n}(u^{(n)}_i;\boldsymbol{u}^{(1\dots n)}\backslash(u^{(n)}_i,u^{(n)}_{m_n}))$$
$$\times\sum_{k,l<n}\frac{1}{\gamma}\Big(\beta_1\,E^{(\hat n)}_{k+2}\otimes E^{(\hat n)}_{l+2}+\beta_2\,E^{(\hat n)}_{l+2}\otimes E^{(\hat n)}_{k+2}\Big)\otimes(E^{(n-1)}_k)^*_{a^{n-1}_p}(E^{(n-1)}_l)^*_{a^{n-1}_{p'}}$$
$$\times\,\Psi^{(n-2)}(\boldsymbol{u}^{(1\dots n)}\backslash(u^{(n-1)}_p,u^{(n-1)}_{p'},u^{(n)}_i,u^{(n)}_{m_n})\,|\,u^{(n-1)}_p,u^{(n-1)}_{p'})\quad(101)$$

396 where

$$\begin{aligned}\beta_1 &:= \frac{f^+(u^{(n-1)}_p,u^{(n)}_i)}{u^{(n-1)}_{p'}-u^{(n)}_i}\,\alpha_{11}+\frac{f^+(u^{(n-1)}_{p'},u^{(n)}_i)}{u^{(n-1)}_p-u^{(n)}_i}\,\alpha_{21}\\ &=\frac{u^{(n-1)}_{p'}-u^{(n)}_{m_n}}{u^{(n-1)}_{p'}-u^{(n)}_i}\left(u^{(n-1)}_p-\tilde u^{(n)}_{m_n}+1+\frac{u^{(n-1)}_{p'}-\tilde u^{(n)}_{m_n}}{u^{(n-1)}_p-u^{(n)}_i}\right),\\ \beta_2 &:= \frac{f^+(u^{(n-1)}_p,u^{(n)}_i)}{u^{(n-1)}_{p'}-u^{(n)}_i}\,\alpha_{12}+\frac{f^+(u^{(n-1)}_{p'},u^{(n)}_i)}{u^{(n-1)}_p-u^{(n)}_i}\,\alpha_{22}\\ &=f^+(u^{(n-1)}_p,u^{(n)}_i)\,\frac{u^{(n-1)}_p-\tilde u^{(n)}_{m_n}}{u^{(n-1)}_{p'}-u^{(n)}_i}+\frac{(u^{(n-1)}_p-u^{(n)}_{m_n})(u^{(n-1)}_{p'}-\tilde u^{(n)}_{m_n})+1}{u^{(n-1)}_p-u^{(n)}_i}.\end{aligned}\quad(102)$$

397 Note that

$$\beta_1+\beta_2=\frac{\gamma}{u^{(n)}_i-\tilde u^{(n)}_{m_n}}\Big(K(u^{(n-1)}_p,u^{(n-1)}_{p'}\,|\,u^{(n)}_i,u^{(n)}_{m_n})-K(u^{(n-1)}_p,u^{(n-1)}_{p'}\,|\,\tilde u^{(n)}_{m_n},u^{(n)}_{m_n})\Big).\quad(103)$$

398 We can now use (61) to expand vector

$$\Psi^{(n-2)}(\boldsymbol{u}^{(1\dots n-1)}\backslash(u^{(n-1)}_p,u^{(n-1)}_{p'},u^{(n)}_i,u^{(n)}_{m_n})\,|\,u^{(n-1)}_p,u^{(n-1)}_{p'})$$

399 in the space $V^{(n-1)}_{a^{n-1}_{p'}}\otimes V^{(n-1)}_{a^{n-1}_p}$:

$$\sum_{1\le i<n}\sum_{\substack{|\boldsymbol{u}^{(r)}_{\rm II,III}|=(2,0)\\ i\le r<n-1}}\prod_{i<k<n}\Gamma_k(\boldsymbol{u}^{(k-1)}_{\rm II};\boldsymbol{u}^{(1\dots n)}_{\rm I})\,K(\boldsymbol{u}^{(k-1)}_{\rm II}\,|\,\boldsymbol{u}^{(k)}_{\rm II,III})\,E^{(n-1)}_{n-i}\otimes E^{(n-1)}_{n-i}\otimes\Psi(\boldsymbol{u}^{(1\dots n)}_{\rm I})\quad(104)$$

$$+\sum_{1\le i<j<n}\sum_{\substack{|\boldsymbol{u}^{(r)}_{\rm III}|=1\\ i\le r<n-1}}\sum_{\substack{|\boldsymbol{u}^{(s)}_{\rm II}|=1\\ j\le s<n-1}}\prod_{i<k<j}\frac{\Gamma_k(\boldsymbol{u}^{(k-1)}_{\rm III};\boldsymbol{u}^{(1\dots n)}_{\rm I})}{\boldsymbol{u}^{(k-1)}_{\rm III}-\boldsymbol{u}^{(k)}_{\rm III}}\cdot\Gamma_j(\boldsymbol{u}^{(j-1)}_{\rm III};\boldsymbol{u}^{(1\dots n)}_{\rm I})$$
$$\times\prod_{j<k<n}\frac{\Gamma_k(\boldsymbol{u}^{(k-1)}_{\rm II};\boldsymbol{u}^{(1\dots n)}_{\rm I})\,\Gamma_k(\boldsymbol{u}^{(k-1)}_{\rm III};\boldsymbol{u}^{(1\dots n)}_{\rm I,II})}{(\boldsymbol{u}^{(k-1)}_{\rm II}-\boldsymbol{u}^{(k)}_{\rm II})(\boldsymbol{u}^{(k-1)}_{\rm III}-\boldsymbol{u}^{(k)}_{\rm III})}$$
$$\times\left(\frac{f^+(\boldsymbol{u}^{(j-1)}_{\rm III},\boldsymbol{u}^{(j)}_{\rm II})}{\boldsymbol{u}^{(j-1)}_{\rm III}-\boldsymbol{u}^{(j)}_{\rm III}}\,E^{(n-1)}_{n-i}\otimes E^{(n-1)}_{n-j}+\frac{1}{\boldsymbol{u}^{(j-1)}_{\rm III}-\boldsymbol{u}^{(j)}_{\rm II}}\,E^{(n-1)}_{n-j}\otimes E^{(n-1)}_{n-i}\right)\otimes\Psi(\boldsymbol{u}^{(1\dots n)}_{\rm I})\quad(105)$$

400 where $\boldsymbol{u}^{(n-1)}_{\rm II}:=u^{(n-1)}_p$, $\boldsymbol{u}^{(n-1)}_{\rm III}:=u^{(n-1)}_{p'}$ and $\boldsymbol{u}^{(n)}_{\rm I}:=\boldsymbol{u}^{(n)}\backslash(u^{(n)}_i,u^{(n)}_{m_n})$.

Substituting the term (104) into (101) and applying (103) gives

$$\sum_{1\le i<n}\sum_{\substack{|\boldsymbol{u}_{\mathrm{II}}^{(r)}|=2\\ i\le r<n}}\sum_{|\boldsymbol{u}_{\mathrm{II}}^{(n)}|=1}\prod_{i<k\le n}\Gamma_k(\boldsymbol{u}_{\mathrm{II}}^{(k-1)};\boldsymbol{u}_{\mathrm{I}}^{(1\ldots n)})\prod_{i<k<n}K(\boldsymbol{u}_{\mathrm{II}}^{(k-1)}\,|\,\boldsymbol{u}_{\mathrm{II}}^{(k)})$$
$$\times\frac{\Gamma_{\hat{n}}(\boldsymbol{u}_{\mathrm{II}}^{(n)};\boldsymbol{u}_{\mathrm{I}}^{(1\ldots n)})}{\boldsymbol{u}_{\mathrm{II}}^{(n)}-\tilde{\boldsymbol{u}}_{\mathrm{III}}^{(n)}}\Big(K(\boldsymbol{u}_{\mathrm{II}}^{(n-1)}\,|\,\boldsymbol{u}_{\mathrm{II,III}}^{(n)})-K(\boldsymbol{u}_{\mathrm{II}}^{(n-1)}\,|\,\tilde{\boldsymbol{u}}_{\mathrm{III}}^{(n)},\boldsymbol{u}_{\mathrm{III}}^{(n)})\Big)E_{\bar{\imath}}^{(\hat{n})}\otimes E_{\bar{\imath}}^{(\hat{n})}\otimes\Phi(\boldsymbol{u}_{\mathrm{I}}^{(1\ldots n)})\,. \tag{106}$$

Upon combining (106) with (84) and acting with $\mathscr{b}^{(n)}_{\dot{a}_{m_n}\ddot{a}_{m_n}}(u^{(n)}_{m_n})$ gives the third term on the right hand side of (67). Finally, substituting the term (105) into (101) and exploiting symmetry of Bethe vectors gives

$$\sum_{1\le i<j<n}\sum_{\substack{|\boldsymbol{u}_{\mathrm{III}}^{(r)}|=1\\ i\le r<n}}\sum_{\substack{|\boldsymbol{u}_{\mathrm{II}}^{(s)}|=1\\ j\le s\le n}}\prod_{i<k<j}\frac{\Gamma_k(\boldsymbol{u}_{\mathrm{III}}^{(k-1)};\boldsymbol{u}_{\mathrm{I}}^{(1\ldots n)})}{\boldsymbol{u}_{\mathrm{III}}^{(k-1)}-\boldsymbol{u}_{\mathrm{III}}^{(k)}}\cdot\Gamma_j(\boldsymbol{u}_{\mathrm{III}}^{(j-1)};\boldsymbol{u}_{\mathrm{I}}^{(1\ldots n)})$$
$$\times\prod_{j<k<n}\frac{\Gamma_k(\boldsymbol{u}_{\mathrm{II}}^{(k-1)};\boldsymbol{u}_{\mathrm{I}}^{(1\ldots n)})\,\Gamma_k(\boldsymbol{u}_{\mathrm{III}}^{(k-1)};\boldsymbol{u}_{\mathrm{I,II}}^{(1\ldots n)})}{(\boldsymbol{u}_{\mathrm{II}}^{(k-1)}-\boldsymbol{u}_{\mathrm{II}}^{(k)})(\boldsymbol{u}_{\mathrm{III}}^{(k-1)}-\boldsymbol{u}_{\mathrm{III}}^{(k)})}\cdot\Gamma_n(\boldsymbol{u}_{\mathrm{II,III}}^{(n-1)};\boldsymbol{u}_{\mathrm{I}}^{(1\ldots n)})\,\Gamma_{\hat{n}}(\boldsymbol{u}_{\mathrm{II}}^{(n)};\boldsymbol{u}_{\mathrm{I}}^{(1\ldots n)})$$
$$\times\frac{1}{2\gamma}\Bigg[\Bigg(\Bigg(\beta_2\frac{f^+(\boldsymbol{u}_{\mathrm{III}}^{(j-1)},\boldsymbol{u}_{\mathrm{II}}^{(j)})}{\boldsymbol{u}_{\mathrm{III}}^{(j-1)}-\boldsymbol{u}_{\mathrm{III}}^{(j)}}+\beta_1\frac{1}{\boldsymbol{u}_{\mathrm{III}}^{(j-1)}-\boldsymbol{u}_{\mathrm{II}}^{(j)}}\Bigg)E_{\bar{\imath}}^{(\hat{n})}\otimes E_{\bar{\jmath}}^{(\hat{n})}$$
$$+\Bigg(\beta_1\frac{f^+(\boldsymbol{u}_{\mathrm{III}}^{(j-1)},\boldsymbol{u}_{\mathrm{II}}^{(j)})}{\boldsymbol{u}_{\mathrm{III}}^{(j-1)}-\boldsymbol{u}_{\mathrm{III}}^{(j)}}+\beta_2\frac{1}{\boldsymbol{u}_{\mathrm{III}}^{(j-1)}-\boldsymbol{u}_{\mathrm{II}}^{(j)}}\Bigg)E_{\bar{\jmath}}^{(\hat{n})}\otimes E_{\bar{\imath}}^{(\hat{n})}\Bigg]\otimes\Psi(\boldsymbol{u}_{\mathrm{I}}^{(1\ldots n)}) \tag{107}$$

Combining (107) with (86) and acting with $\mathscr{b}^{(n)}_{\dot{a}_{m_n}\ddot{a}_{m_n}}(u^{(n)}_{m_n})$ gives the last term on the right hand side of (67). □

4.3 Proof of Lemma 3.7

The idea of the proof is to construct a certain Bethe vector and evaluate this vector in two different ways. Equating the resulting expressions will yield the claim of the Lemma.

We begin by rewriting the wanted relation in a more convenient way. From (17) and (35) we find that

$$\left\{\frac{p(v)}{u_i^{(n)}-v}\,\mathscr{b}^{(n)}_{\dot{a}_{m_n}\ddot{a}_{m_n}}(v)\right\}^{v}=\mathscr{b}^{(n)}_{\dot{a}_{m_n}\ddot{a}_{m_n}}(v)\left(\frac{f^+(u_i^{(n)},\tilde{v})}{u_i^{(n)}-v}+\frac{1}{u_i^{(n)}-\tilde{v}}\,P^{(\hat{n},\hat{n})}_{\dot{a}_{m_n}\ddot{a}_{m_n}}\right). \tag{108}$$

Repeating the steps used in deriving (83) and applying (108) we rewrite (50) as

$$s_{\hat{n}\hat{n}}(v)\,\Psi(\boldsymbol{u}^{(1\ldots n)})=\Gamma_{\hat{n}}(v,\boldsymbol{u}^{(1\ldots n)})\,\Psi(\boldsymbol{u}^{(1\ldots n)})$$
$$-\sum_i\mathscr{b}^{(n)}_{\dot{a}_{m_n}\ddot{a}_{m_n}}(v)\left(\frac{f^+(u_i^{(n)},\tilde{v})}{u_i^{(n)}-v}+\frac{1}{u_i^{(n)}-\tilde{v}}\,P^{(\hat{n},\hat{n})}_{\dot{a}_{m_n}\ddot{a}_{m_n}}\right)$$
$$\times\,\Gamma_{\hat{n}}(u_i^{(n)},\boldsymbol{u}^{(1\ldots n)}\backslash u_i^{(n)})\,\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u_i^{(n)})\,\dot{\Psi}^{(n-1)}_{2,1}(\boldsymbol{u}^{(1\ldots n-1)}\,|\,\boldsymbol{u}^{(n)}_{\sigma_i})$$
$$-\sum_{i\neq i'}\mathscr{b}^{(n)}_{\dot{a}_{m_n-1}\ddot{a}_{m_n-1}}(v)\left(\frac{f^+(u_i^{(n)},\tilde{v})}{u_i^{(n)}-v}+\frac{1}{u_i^{(n)}-\tilde{v}}\,P^{(\hat{n},\hat{n})}_{\dot{a}_{m_n-1}\ddot{a}_{m_n-1}}\right)$$
$$\times\,\Gamma_{\hat{n}}((u_i^{(n)},u_{i'}^{(n)});\boldsymbol{u}^{(1\ldots n)}\backslash(u_i^{(n)},u_{i'}^{(n)}))\,\frac{f^-(u_i^{(n)},u_{i'}^{(n)})\,f^+(u_i^{(n)},\tilde{u}_{i'}^{(n)})}{u_{i'}^{(n)}-\tilde{u}_i^{(n)}}$$
$$\times\,\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash(u_i^{(n)},u_{i'}^{(n)}))\,\dot{\Psi}^{(n-1)}_{2,2}(\boldsymbol{u}^{(1\ldots n-1)}\,|\,\boldsymbol{u}^{(n)}_{\sigma_i}\backslash u_{i'}^{(n)})\,. \tag{109}$$

Let $\Psi_{m_n+1}(\boldsymbol{u}^{(1\dots n)}\cup v)$ denote a Bethe vector with m_n+1 level-n excitations and the reference vector $\eta^{\boldsymbol{m}}_{m_n+1} := (E_{12}^{(\hat{n})})_{\dot{a}_{m_n+1}}\eta^{\boldsymbol{m}}$; here v denotes the $(m_n$+1)-st level-n Bethe root. Applying (71) and (83) to this Bethe vector we obtain

$$\begin{aligned}\Psi_{m_n+1}(\boldsymbol{u}^{(1\dots n)}\cup v) = \Gamma_{\hat{n}}(v,\boldsymbol{u}^{(1\dots n)})\,\Psi(\boldsymbol{u}^{(1\dots n)}) \\ -\sum_i \frac{f^+(u_i^{(n)},\tilde{v})}{u_i^{(n)}-v}\,\Gamma_{\hat{n}}(u_i^{(n)};\boldsymbol{u}^{(1\dots n)}\backslash u_i^{(n)}) \\ \times\,\mathscr{b}^{(n)}_{\dot{a}_{m_n}\ddot{a}_{m_n}}(v)\,\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u_i^{(n)})\,\dot{\Psi}^{(n-1)}_{2,1}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}\backslash u_i^{(n)}\cup v) \\ -\sum_{i'\neq i}\Gamma_{\hat{n}}((u_i^{(n)},u_{i'}^{(n)});\boldsymbol{u}^{(1\dots n)}\backslash(u_i^{(n)},u_{i'}^{(n)}))\,K(u_i^{(n)},u_{i'}^{(n)}\,|\,v,\tilde{v})\,f^+(u_i^{(n)},u_{i'}^{(n)}) \\ \times\,\mathscr{b}^{(n)}_{\dot{a}_{m_n-1}\ddot{a}_{m_n-1}}(v)\,\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash(u_i^{(n)},u_{i'}^{(n)})) \\ \times\,\dot{\Psi}^{(n-1)}_{2,2}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}\backslash(u_i^{(n)},u_{i'}^{(n)})\cup v)\,. \end{aligned} \tag{110}$$

Next, recall (78) and note that $P^{(\hat{n},\hat{n})}_{\dot{a}_i\ddot{a}_{m_n}}\eta^{\boldsymbol{m}}_{m_n} = P^{(\hat{n},\hat{n})}_{\dot{a}_{m_n}\ddot{a}_{m_n}}\eta^{\boldsymbol{m}}_i$ giving

$$\mathscr{R}^{(\hat{n})}(u_{m_n}^{(n)};\boldsymbol{u}^{(n)}\backslash u_{m_n}^{(n)})\cdot\eta^{\boldsymbol{m}}_{m_n} = \eta^{\boldsymbol{m}}_{m_n} + \sum_{i<m_n}\frac{\prod_{i<k<m_n} f^+(u_k^{(n)},\tilde{u}_{m_n}^{(n)})}{u_i^{(n)}-\tilde{u}_{m_n}^{(n)}}\,P^{(\hat{n},\hat{n})}_{\dot{a}_{m_n}\ddot{a}_{m_n}}\,\eta^{\boldsymbol{m}}_i\,. \tag{111}$$

This yields the analogue of (83) for $\Psi_{m_n+1}(\boldsymbol{u}^{(1\dots n)}\cup v)$:

$$\begin{aligned}\Psi_{m_n+1}(\boldsymbol{u}^{(1\dots n)}\cup v) = \mathscr{b}^{(n)}_{\dot{a}_{m_n+1}\ddot{a}_{m_n+1}}(v)\,\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)})\,\dot{\Psi}^{(n-1)}_{1,1}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}\cup v) \\ +\sum_i\frac{\Gamma_{\hat{n}}(u_i^{(n)};\boldsymbol{u}^{(1\dots n)}\backslash u_i^{(n)})}{u_i^{(n)}-\tilde{v}} \\ \times\,\mathscr{b}^{(n)}_{\dot{a}_{m_n}\ddot{a}_{m_n}}(v)\,\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u_i^{(n)})\,\dot{\Psi}^{(n-1)}_{1,2}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}\backslash u_i^{(n)}\cup v)\,. \end{aligned} \tag{112}$$

The next step is to evaluate products of creation operators $\mathscr{B}^{(n)}$ and the dotted Bethe vectors $\dot{\Psi}^{(n-1)}$. This is done applying the same techniques used in the proof of Proposition 4.4. Hence, we will skip the technical details and state the final expressions only.

Evaluating the named products in (110) and (112) gives

$$\begin{aligned}&\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u_i^{(n)})\,\dot{\Psi}^{(n-1)}_{2,1}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}\backslash u_i^{(n)}\cup v) \\ &= E_2^{(\hat{n})}\otimes E_1^{(\hat{n})}\otimes\Psi(\boldsymbol{u}^{(1\dots n)}\backslash u_i^{(n)}) \\ &\quad+\sum_j\frac{\Gamma_n(u_j^{(n-1)};\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_i^{(n)}))}{u_j^{(n-1)}-v} \\ &\quad\times\sum_{1\le k<n}E_{k+2}^{(\hat{n})}\otimes E_1^{(\hat{n})}\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}\,\Psi(\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_i^{(n)}|u_j^{(n-1)}) \\ &\quad+\sum_j\sum_{i'\neq i}\frac{\Gamma_n(u_j^{(n-1)};\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_i^{(n)},u_{i'}^{(n)}))\,\Gamma_{\hat{n}}(u_{i'}^{(n)};\boldsymbol{u}^{(1\dots n)}\backslash(u_i^{(n)},u_{i'}^{(n)}))}{(u_j^{(n-1)}-\tilde{v})(u_j^{(n-1)}-u_{i'}^{(n)})} \\ &\quad\times\sum_{1\le k<n}\left(\frac{1}{u_j^{(n-1)}-v}E_{k+2}^{(\hat{n})}\otimes E_2^{(\hat{n})}+E_2^{(\hat{n})}\otimes E_{k+2}^{(\hat{n})}\right)\otimes(E_k^{(n-1)})^*_{a_j^{n-1}} \\ &\qquad\times\Psi(\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_i^{(n)},u_{i'}^{(n)})\,|\,u_j^{(n-1)}) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j<j'}\sum_{i'\neq i}\Gamma_n((u_j^{(n-1)},u_{j'}^{(n-1)});\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_{j'}^{(n-1)},u_i^{(n)},u_{i'}^{(n)}))\,\Gamma_{\hat{n}}(u_{i'}^{(n)};\boldsymbol{u}^{(1\dots n)}\backslash(u_i^{(n)},u_{i'}^{(n)}))\\
&\quad\times\sum_{1\le k,l<n}\frac{1}{\gamma}\Big(\beta_1^{(21)}E_{k+2}^{(\hat{n})}\otimes E_{l+2}^{(\hat{n})}+\beta_2^{(21)}E_{l+2}^{(\hat{n})}\otimes E_{k+2}^{(\hat{n})}\Big)\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}(E_l^{(n-1)})^*_{a_{j'}^{n-1}}\\
&\qquad\qquad\times\Psi(\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_{j'}^{(n-1)},u_i^{(n)},u_{i'}^{(n)})\,|\,u_j^{(n-1)},u_{j'}^{(n-1)})
\end{aligned}\tag{113}
$$

422 and

$$
\begin{aligned}
&\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash u_i^{(n)})\,\dot{\Psi}_{1,2}^{(n-1)}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}\backslash u_i^{(n)}\cup v)\\
&=E_1^{(\hat{n})}\otimes E_2^{(\hat{n})}\otimes\Psi(\boldsymbol{u}^{(1\dots n)}\backslash u_i^{(n)})\\
&\quad+\sum_j\frac{\Gamma_n(u_j^{(n-1)};\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_i^{(n)}))}{u_j^{(n-1)}-\tilde{v}}\\
&\quad\times\sum_{1\le k<n}\left(\frac{1}{u_j^{(n-1)}-v}E_{k+2}^{(\hat{n})}\otimes E_1^{(\hat{n})}+E_1^{(\hat{n})}\otimes E_{k+2}^{(\hat{n})}\right)\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}\\
&\qquad\qquad\times\Psi(\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_i^{(n)})\,|\,u_j^{(n-1)})\\
&\quad+\sum_j\sum_{i'\neq i}\frac{\Gamma_n(u_j^{(n-1)};\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_i^{(n)},u_{i'}^{(n)}))\,\Gamma_{\hat{n}}(u_{i'}^{(n)};\boldsymbol{u}^{(1\dots n)}\backslash(u_i^{(n)},u_{i'}^{(n)}))}{(u_j^{(n-1)}-v)(u_j^{(n-1)}-u_{i'}^{(n)})}\\
&\quad\times\sum_{1\le k<n}E_{k+2}^{(\hat{n})}\otimes E_2^{(\hat{n})}\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}\,\Psi(\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_i^{(n)},u_{i'}^{(n)})\,|\,u_j^{(n-1)})\\
&\quad+\sum_{j<j'}\sum_{i'\neq i}\Gamma_n((u_j^{(n-1)},u_{j'}^{(n-1)});\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_{j'}^{(n-1)},u_i^{(n)},u_{i'}^{(n)}))\,\Gamma_{\hat{n}}(u_{i'}^{(n)};\boldsymbol{u}^{(1\dots n)}\backslash(u_i^{(n)},u_{i'}^{(n)}))\\
&\quad\times\sum_{1\le k,l<n}\frac{1}{\gamma}\Big(\beta_1^{(12)}E_{k+2}^{(\hat{n})}\otimes E_{l+2}^{(\hat{n})}+\beta_{12}^{(12)}E_{l+2}^{(\hat{n})}\otimes E_{k+2}^{(\hat{n})}\Big)\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}(E_l^{(n-1)})^*_{a_{j'}^{n-1}}\\
&\qquad\qquad\times\Psi(\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_{j'}^{(n-1)},u_i^{(n)},u_{i'}^{(n)})\,|\,u_j^{(n-1)},u_{j'}^{(n-1)})
\end{aligned}\tag{114}
$$

423 and

$$
\begin{aligned}
&\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)})\,\dot{\Psi}_{1,1}^{(n-1)}(\boldsymbol{u}^{(1\dots n-1)}\,|\,\boldsymbol{u}^{(n)}\cup v)\\
&=E_1^{(\hat{n})}\otimes E_1^{(\hat{n})}\otimes\Psi(\boldsymbol{u}^{(1\dots n)})\\
&\quad+\sum_j\sum_i\frac{\Gamma_n(u_j^{(n-1)};\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_i^{(n)}))\,\Gamma_{\hat{n}}(u_i^{(n)};\boldsymbol{u}^{(1\dots n)}\backslash u_i^{(n)})}{u_j^{(n-1)}-u_i^{(n)}}\\
&\quad\times\sum_{1\le k<n}\left(\frac{f^+(u_j^{(n-1)},\tilde{v})}{u_j^{(n-1)}-v}E_{k+2}^{(\hat{n})}\otimes E_1^{(\hat{n})}+\frac{1}{u_j^{(n-1)}-\tilde{v}}E_1^{(\hat{n})}\otimes E_{k+2}^{(\hat{n})}\right)\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}\\
&\qquad\qquad\times\Psi(\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_i^{(n)})\,|\,u_j^{(n-1)})\\
&\quad+\sum_{j<j'}\sum_{i<i'}\Gamma_n((u_j^{(n-1)},u_{j'}^{(n-1)});\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_{j'}^{(n-1)},u_i^{(n)},u_{i'}^{(n)}))\\
&\quad\times\Gamma_{\hat{n}}((u_i^{(n)},u_{i'}^{(n)});\boldsymbol{u}^{(1\dots n)}\backslash(u_i^{(n)},u_{i'}^{(n)}))\,K(u_j^{(n-1)},u_{j'}^{(n-1)}\,|\,u_i^{(n)},u_{i'}^{(n)})\,f^+(u_i^{(n)},\tilde{u}_{i'}^{(n)})\\
&\quad\times\sum_{1\le k,l<n}\Big(\beta_1^{(11)}E_{k+2}^{(\hat{n})}\otimes E_{l+2}^{(\hat{n})}+\beta_2^{(11)}E_{l+2}^{(\hat{n})}\otimes E_{k+2}^{(\hat{n})}\Big)\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}(E_l^{(n-1)})^*_{a_{j'}^{n-1}}\\
&\qquad\qquad\times\Psi(\boldsymbol{u}^{(1\dots n)}\backslash(u_j^{(n-1)},u_{j'}^{(n-1)},u_i^{(n)},u_{i'}^{(n)})\,|\,u_j^{(n-1)},u_{j'}^{(n-1)})
\end{aligned}\tag{115}
$$

and

$$
\begin{aligned}
&\mathscr{B}^{(n)}(\boldsymbol{u}^{(n)}\backslash(u_i^{(n)},u_{i'}^{(n)}))\,\dot{\Psi}_{2,2}^{(n-1)}(\boldsymbol{u}^{(1\ldots n-1)}\,|\,\boldsymbol{u}^{(n)}\backslash(u_i^{(n)},u_{i'}^{(n)})\cup v)\\
&\quad= E_2^{(\hat{n})}\otimes E_2^{(\hat{n})}\otimes\Psi(\boldsymbol{u}^{(1\ldots n)}\backslash(u_i^{(n)},u_{i'}^{(n)}))\\
&\qquad+\sum_j \Gamma_n(u_j^{(n-1)};\boldsymbol{u}^{(1\ldots n)}\backslash(u_j^{(n-1)},u_i^{(n)},u_{i'}^{(n)}))\\
&\qquad\quad\times\sum_{1\le k<n}\left(\frac{f^+(u_j^{(n-1)},\tilde{v})}{u_j^{(n-1)}-v}\,E_{k+2}^{(\hat{n})}\otimes E_2^{(\hat{n})}+\frac{1}{u_j^{(n-1)}-\tilde{v}}\,E_2^{(\hat{n})}\otimes E_{k+2}^{(\hat{n})}\right)\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\Psi(\boldsymbol{u}^{(1\ldots n)}\backslash(u_j^{(n-1)},u_i^{(n)},u_{i'}^{(n)})\,|\,u_j^{(n-1)})\\
&\qquad+\sum_{j<j'}\Gamma_n((u_j^{(n-1)},u_{j'}^{(n-1)});\boldsymbol{u}^{(1\ldots n)}\backslash(u_j^{(n-1)},u_{j'}^{(n-1)},u_i^{(n)},u_{i'}^{(n)}))\\
&\qquad\quad\times\sum_{1\le k,l<n}\left(\beta_1^{(11)}E_{k+2}^{(\hat{n})}\otimes E_{l+2}^{(\hat{n})}+\beta_2^{(11)}E_{l+2}^{(\hat{n})}\otimes E_{k+2}^{(\hat{n})}\right)\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}(E_l^{(n-1)})^*_{a_{j'}^{n-1}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\Psi(\boldsymbol{u}^{(1\ldots n)}\backslash(u_j^{(n-1)},u_{j'}^{(n-1)},u_i^{(n)},u_{i'}^{(n)})\,|\,u_j^{(n-1)},u_{j'}^{(n-1)},\boldsymbol{u}^{(n)})
\end{aligned}
\tag{116}
$$

where $\beta_1^{(21)}$, $\beta_2^{(21)}$ and γ are given by (102) and (91) except $u_{m_n}^{(n)}$ should be replaced by v, and

$$
\begin{aligned}
\beta_1^{(12)} &:= \frac{u_{j'}^{(n-1)}-v}{u_{j'}^{(n-1)}-u_{i'}^{(n)}}\left(f^+(u_j^{(n-1)},u_{i'}^{(n)})+\frac{(u_{j'}^{(n-1)}-u_{i'}^{(n)})(u_j^{(n-1)}-\tilde{v})}{u_j^{(n-1)}-u_{i'}^{(n)}}\right),\\
\beta_2^{(12)} &:= \frac{u_j^{(n-1)}-\tilde{v}}{u_j^{(n-1)}-u_{i'}^{(n)}}\,f^+(u_{j'}^{(n-1)},u_{i'}^{(n)})\,f^+(u_j^{(n-1)},u_{j'}^{(n-1)})\\
&\quad+\frac{u_{j'}^{(n-1)}-\tilde{v}}{u_{j'}^{(n-1)}-u_{i'}^{(n)}}\,f^+(u_j^{(n-1)},u_{i'}^{(n)})\left(u_j^{(n-1)}-v-\frac{1}{u_j^{(n-1)}-u_{j'}^{(n-1)}}\right),\\
\beta_1^{(11)} &:= \frac{f^+(u_j^{(n-1)},\tilde{v})}{(u_j^{(n-1)}-v)(u_{j'}^{(n-1)}-\tilde{v})},\qquad \beta_2^{(11)}:=\frac{1}{u_{j'}^{(n-1)}-v}\left(\beta_1^{(11)}+\frac{1}{u_j^{(n-1)}-\tilde{v}}\right).
\end{aligned}
\tag{117}
$$

Adapting (113) and (116) to the relevant producs in (109) allows us to rewrite the latter as

$$
\begin{aligned}
&\Gamma_{\hat{n}}(u_i^{(n)},\boldsymbol{u}^{(1\ldots n)}\backslash u_i^{(n)})\;E_2^{(\hat{n})}\otimes E_1^{(\hat{n})}\otimes\Psi(\boldsymbol{u}^{(1\ldots n)}\backslash u_i^{(n)})\\
&\quad+\sum_j\frac{\Gamma_{\hat{n}}(u_i^{(n)},\boldsymbol{u}^{(1\ldots n)}\backslash u_i^{(n)})\,\Gamma_n(u_j^{(n-1)};\boldsymbol{u}^{(1\ldots n)}\backslash(u_j^{(n-1)},u_i^{(n)}))}{u_j^{(n-1)}-u_i^{(n)}}\\
&\qquad\times\sum_{1\le k<n}E_{k+2}^{(\hat{n})}\otimes E_1^{(\hat{n})}\otimes(E_k^{(n-1)})^*_{a_j^{n-1}}\,\Psi(\boldsymbol{u}^{(1\ldots n)}\backslash(u_j^{(n-1)},u_i^{(n)})\,|\,u_j^{(n-1)})\\
&\quad+\sum_{i'\neq i}\Gamma_{\hat{n}}((u_i^{(n)},u_{i'}^{(n)});\boldsymbol{u}^{(1\ldots n)}\backslash(u_i^{(n)},u_{i'}^{(n)}))\,\frac{f^-(u_i^{(n)},u_{i'}^{(n)})\,f^+(u_i^{(n)},\tilde{u}_{i'}^{(n)})}{u_{i'}^{(n)}-\tilde{u}_i^{(n)}}\\
&\qquad\times\left(E_2^{(\hat{n})}\otimes E_2^{(\hat{n})}\otimes\Psi(\boldsymbol{u}^{(1\ldots n)}\backslash(u_i^{(n)},u_{i'}^{(n)}))+A\right)
\end{aligned}
\tag{118}
$$

where

$$
\begin{aligned}
A := \sum_j \Gamma_n(u_j^{(n-1)}; \boldsymbol{u}^{(1\dots n)} \backslash (u_j^{(n-1)}, u_i^{(n)}, u_{i'}^{(n)})) \\
\times \sum_{1\le k<n} \left(\frac{f^+(u_j^{(n-1)}, \tilde{u}_{i'}^{(n)})}{u_j^{(n-1)} - u_i^{(n)}} E_{k+2}^{(\hat{n})} \otimes E_2^{(\hat{n})} + \frac{1}{u_j^{(n-1)} - u_{i'}^{(n)}} E_2^{(\hat{n})} \otimes E_{k+2}^{(\hat{n})} \right) \otimes (E_k^{(n-1)})^*_{a_j^{n-1}} \\
\times \Psi^{(n-1)}(\boldsymbol{u}^{(1\dots n)} \backslash (u_j^{(n-1)}, u_i^{(n)}, u_{i'}^{(n)}) \,|\, u_j^{(n-1)}) \\
+ \sum_{j<j'} \frac{\Gamma_n((u_j^{(n-1)}, u_{j'}^{(n-1)}); \boldsymbol{u}^{(1\dots n)} \backslash (u_j^{(n-1)}, u_{j'}^{(n-1)}, u_i^{(n)}, u_{i'}^{(n)}))}{(u_j^{(n-1)} - u_i^{(n)})(u_{j'}^{(n-1)} - u_{i'}^{(n)})} \\
\times \sum_{1\le k,l<n} \left(f^+(u_j^{(n-1)}, u_i^{(n)})\, E_{k+2}^{(\hat{n})} \otimes E_{l+2}^{(\hat{n})} + \theta\, E_{l+2}^{(\hat{n})} \otimes E_{k+2}^{(\hat{n})} \right) \otimes (E_k^{(n-1)})^*_{a_j^{n-1}} (E_l^{(n-1)})^*_{a_{j'}^{n-1}} \\
\times \Psi(\boldsymbol{u}^{(1\dots n)} \backslash (u_j^{(n-1)}, u_{j'}^{(n-1)}, u_i^{(n)}, u_{i'}^{(n)}) \,|\, u_j^{(n-1)}, u_{j'}^{(n-1)})
\end{aligned}
$$

and

$$
\theta := \frac{(u_j^{(n-1)} - u_i^{(n)})(u_{j'}^{(n-1)} - u_{i'}^{(n)}) + u_j^{(n-1)} - u_{i'}^{(n)} + 1}{(u_j^{(n-1)} - u_{i'}^{(n)})(u_{j'}^{(n-1)} - u_i^{(n)})} .
$$

The final step is to substitute (113)–(116) into the difference of (112) and (110), and (118) into (109), and equate the resulting expressions.

5 Conclusion

This paper is a continuation [GMR19], where twisted Yangian based models, known as one-dimensional "soliton non-preserving" open spin chains, were studied by means of the algebraic bethe ansatz. The present paper extends the results of [GMR19] to the odd case, when the underlying Lie algebra is $\mathfrak{gl}_{2n+1}$, see Theorem 3.8. Additionally, in Proposition 3.11, we presented a more symmetric form of the trace formula for Bethe vectors. We also obtained recurrence relations for Bethe vectors. The latter are given by Propositions 4.2 and 4.4 for the even and odd cases, respectively.

The recurrence relations found in this paper provide elegant expressions when the rank is small, see Examples 4.3 and 4.5. However, they become rather complex otherwise. In general, they are much more involved than their periodic counterparts obtained in [HL+18a], especially in the odd case. This raises a natural question, if there exists an alternative (simpler) method of constructing Bethe vectors for open spin chains.

For closed spin chains the current ("Drinfeld New") presentation of Yangians and quantum loop algebras [Dri88] has played a significant role in obtaining not only recurrence relations, but also action relations, scalar products and norms of Bethe vectors, see [HL+17a, HL+17b, HL+18a, HL+18b, HL+20]. Thus, it is natural to expect that a current presentation of twisted Yangians could pave a fruitful path for open spin chains analysis.

A current presentation of twisted Yangians $Y^+(\mathfrak{gl}_N)$ was recently is obtained in [LWZ23]. (The rank 2 case was considered earlier in [Brw16].) However, in [LWZ23] a different, the so-called split, realisation of twisted Yangian is considered, which is not compatible (at least in a natural way) with the Bethe nested vacuum state. Nevertheless, we believe that the presentation obtained in [LWZ23] may have applications in open spin chain analysis and thus deserves to be investigated.

Overall, the approach presented in this paper does open a door to an exploration of scalar products and norms of Bethe vectors for twisted Yangian based models. However, ultimately, developing Bethe ansatz techniques in the current presentation of twisted Yangians should open a gateway to open spin chain analysis.

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