

The role of atomic interactions in cavity-induced continuous time crystals

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1 Abstract

2 We consider continuous time-crystalline phases in dissipative many-body sys-
 3 tems of atoms in cavities, focusing on the role of short-range interatomic in-
 4 teractions. First, we show that the latter can alter the nature of the time
 5 crystal by changing the type of the underlying critical bifurcation. Second, we
 6 characterize the heating mechanism and dynamics resulting from the short-
 7 range interactions and demonstrate that they make the time crystal inherently
 8 metastable. We argue this is generic for the broader class of dissipative time
 9 crystals in atom-cavity systems whenever the cavity loss rate is comparable
 10 to the atomic recoil energy. We observe that such a scenario for heating
 11 has several similarities to the one proposed for preheating in the early uni-
 12 verse, where the oscillating coherent inflation field decays into a cascade of
 13 exponentially growing fluctuations. By extending approaches for dissipative
 14 dynamical systems to our many-body problem, we obtain analytical predic-
 15 tions for the parameters describing the phase transition and the heating rate
 16 inside the time-crystalline phase. We underpin and extend the analytical pre-
 17 dictions of the heating rates with numerical simulations, which also show that
 18 the metastable regime exists when the inherent stochastic nature is taken into
 19 account.

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21 Contents

22	1 Introduction	2
23	2 Model	3
24	3 Nature of the time crystal	4
25	4 Energy redistribution	6
26	5 Metastable nature of time crystal	8
27	5.1 Finite size of atom clouds	11
28	5.2 Cavity parameters	11

29	6 Conclusion	12
30	A Center manifold coefficients	13
31	A.1 The critical eigenvector	14
32	A.2 Computing g^r and γ	15
33	B Including fluctuations outside the center manifold	17
34	References	19

35

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37 1 Introduction

38 Following the first conceptualization of time-crystalline phases of matter [1, 2], it was
 39 quickly proven that such phases cannot appear in thermal equilibrium [3–5]. However, it
 40 is possible to realize such phases in periodically driven systems, both closed [6–11] and
 41 dissipative [12, 13].

42 Among the latter, systems of atoms in optical cavities have emerged as an ideal
 43 platform to realize continuous time-crystalline phases [14–16], where an effectively time-
 44 independent drive of the atomic system is counterbalanced by the loss of photons out of the
 45 cavity mirrors. In these phases, continuous time-translation invariance is spontaneously
 46 broken, and oscillations persist even though the system possesses a macroscopic number
 47 of degrees of freedom, among which energy can be redistributed via interactions.

48 Since the phase space of scattering by cavity-mediated interactions between atoms
 49 is limited due to their long range, redistribution of energy through these processes is
 50 inefficient [17–19]. However, the intrinsic atomic short-range interactions allow for efficient
 51 redistribution of energy among the atoms. Indeed, experiments show strong indications
 52 that these interactions are one of the main fundamental limiting factors to the measured
 53 lifetime of the time crystal [12].

54 Despite their crucial role, short-range atomic interactions have not been theoretically
 55 investigated so far in a systematic way for continuous time crystals in atom-cavity setups.
 56 In this work, we undertake this task. We not only provide a complete picture of the
 57 possible destabilization processes but also show that short-range interactions can alter the
 58 nature of the time crystal itself.

59 In Section 2 we describe the model under consideration, which has a simple and ex-
 60 perimentally realizable mechanism for the appearance of time-crystalline phases for an
 61 interacting BEC coupled to two cavity modes [20]. In Section 3 approaches for classical
 62 non-linear dissipative systems is extended to our many-body problem and we obtain an
 63 analytic description of the time crystal in terms of cavity-induced critical bifurcations and
 64 show how inter-atomic interactions can modify the nature of the latter. These results are
 65 used in Section 4, where we identify the dominating scattering processes responsible for
 66 energy redistribution among the atoms. Using these scattering processes we compute the
 67 dependence of the energy-redistribution rates on external parameters identify the corre-
 68 sponding time scales. Lastly, in Section 5, we show numerically that for realistic noise
 69 levels, the metastability of the time crystal is fully determined by the redistribution rates
 70 due to inter-atomic interactions.

71 The analytic understanding of the results, underpinned with numerical analysis, allows
 72 for a deep insight into the generic features of the phenomenology beyond the specific

73 model considered and provides orientation for future investigations both in theory and
74 experiment.

75 2 Model

76 The system considered is an ultracold gas of bosonic atoms in a BEC state, dispersively
77 coupled with equal strength to two modes of an optical cavity. In this regime, a photon
78 imparts a recoil momentum of $Q = 2\pi/\lambda$ to an atom, with λ being the wavelength of
79 the photon in a given mode. In the thermodynamic limit, the atomic BEC at momentum
80 k is described by a complex field ψ_k satisfying the Gross-Pitaevski mean-field equations.
81 Furthermore, in the limit of a small transverse extend of the BEC compared to the cavity
82 waist we can simplify the model to one spatial dimension [19, 20]

$$i\partial_t\psi_k = k^2\psi_k + U \sum_{q,q'} \psi_q\psi_{q'}\bar{\psi}_{q+q'-k} + \frac{\tilde{\eta}}{\sqrt{2}} \sum_{j=1,2} \text{Re}(\phi_j) (\psi_{k+Q} + \psi_{k-Q}), \quad (1)$$

83 where the bar denotes complex conjugation. This equation has been written in units of the
84 recoil energy $E_R = \hbar^2 Q^2/2m$ and in the rotating frame of the laser. The time-dependence
85 of the fields is kept implicit and the atom field has been normalized to 1. The cavity-mode
86 wavelengths have been chosen to be equal, as we assume the modes differ in the transverse
87 direction [20]. The coupling strength $\tilde{\eta}$ can experimentally be tuned by the strength of
88 the transverse pump while the atoms are interacting with each other through a contact
89 interaction of strength U . The complex field ϕ_j corresponds to the coherent cavity-field
90 amplitude which satisfies the equation

$$i\partial_t\phi_j = (\Delta_j - i\kappa)\phi_j + \frac{\tilde{\eta}}{2\sqrt{2}} \sum_{k=-\infty}^{\infty} \bar{\psi}_k (\psi_{k+Q} + \psi_{k-Q}), \quad (2)$$

91 where the cavity field has been normalized by the square root of the atom number. The
92 cavity linewidths, κ , have been assumed to be identical for both modes. In the following we
93 will consider κ on an energy scale similar to the recoil energy, as realized for instance in [21].
94 In the actual implementation of the dispersive atom-cavity coupling, the characteristic
95 frequency of each cavity mode Δ_j corresponds to the detuning of the mode frequency
96 with respect to laser-driven two-photon transitions [20]. The steady state of this model
97 can break time-translation invariance when the two detunings have opposite signs. With
98 this in mind the detunings are parametrized as $\Delta_1 = -(\Delta - \frac{\delta}{2})$ and $\Delta_2 = \Delta + \frac{\delta}{2}$. By
99 choosing $0 < \delta < 2\Delta$ the negative detuning has the smallest amplitude $|\Delta_1| < |\Delta_2|$.

100 The mean-field description of this system becomes exact in the thermodynamic limit
101 [22]. It, however, does not give rise to a unique steady state of the atomic system. To
102 determine the latter, one needs to include quantum fluctuations [23]. The time scale
103 needed to reach this steady state does grow inversely with the system size. For the large
104 atomic clouds considered here, the relaxation to this steady state is thus irrelevant on the
105 experimental time scale. Additionally, for a system with a finite number of atoms, the
106 openness of the cavity gives rise to an additional stochastic term in the cavity equation.
107 The strength of the stochastic term scales with κ/\sqrt{N} , with N being the number of atoms.
108 We will initially consider the limit of $N \rightarrow \infty$ and in Sec. 5 show that the conclusions
109 remain valid also for finite system sizes.

110 3 Nature of the time crystal

111 Below a critical coupling strength η_c , all atoms are in the homogeneous state ψ_0 , and the
 112 coherent part of the cavity fields is empty. This configuration is denoted as the normal
 113 phase (NP) and it is always a fixed point of the equations of motion Eqs. (1) and (2).
 114 As $\tilde{\eta}$ is increased beyond η_c the NP fixed point becomes unstable and the system enters
 115 a state where a fraction of the atom population is transferred to $\psi_{\pm Q}$ and the coherent
 116 fields of the cavity becomes finite. This symmetry-broken state is often referred to as
 117 the superradiant (SR) or self-organized state [24, 25]. The frequency ω_c of the excitation
 118 becoming undamped above η_c , can be derived through a linear expansion around the NP
 119 fixed point [26] (see [27] for an alternative approach). This is done by considering a small
 120 perturbation to the state NP state and only keeping the terms linear in the perturbation.
 121 One can then derive an equation for when the perturbation becomes unstable, indicated
 122 by a zero real part of the eigenvalue of the resulting Jacobian of the linear system.

123 The frequency of the unstable perturbation is determined by the imaginary part of the
 124 eigenvalue. It is found that an unstable mode can appear for three different frequencies
 125 of the critical mode. The determination of which the three possible modes that ends up
 126 manifesting in the system is set identifying which of the three modes require the smallest
 127 corresponding values of η_c . The first of the three potential instabilities is a static solution
 128 with $\omega_c = \omega_{c,s} = 0$. The second frequency at which the system can become critical is a
 129 resonance at the energy of the Bogoliubov excitation of the BEC at the recoil momentum
 130 with frequency $\omega_c = \omega_a = \sqrt{E_R(E_R + 2U)}$ and lastly, there is a possibility of an instability
 131 at a frequency given by

$$\omega_c = \sqrt{\frac{\delta^2}{4} + \sqrt{(4\Delta^2 - \delta^2)(\Delta^2 + \kappa^2)} - \Delta^2 - \kappa^2}, \quad (3)$$

132 which is solely determined by cavity parameters, that is, it does not depend on U and E_R .
 133 This feature, which can be attributed to the fact that the cavity is the only dissipation
 134 channel, implies a robustness of this self-sustained periodic signal to perturbations of the
 135 nonlinear medium that causes this signal to appear in the first place. To determine which
 136 mode that becomes unstable one identifies which mode leads to the smallest real critical
 137 coupling given by

$$\eta_c = \lim_{\kappa_a \rightarrow 0} \sqrt{\frac{\omega_a^2 + \kappa_a^2 - \omega_c^2}{E_R \sum_{j=1,2} \frac{\Delta_j (\Delta_j^2 + \kappa^2 - \omega_c^2)}{\omega_c^4 + 2\omega_c^2 (\kappa^2 - \Delta_j^2) + (\Delta_j^2 + \kappa^2)^2}}}, \quad (4)$$

138 where κ_a is the atom lifetime which we, for simplicity consider infinitely long. always
 139 depends on both cavity and atom parameters [26]. The phase diagram will therefore
 140 depend on all parameters of the model.

141 In Fig. 1(a) the frequency of the critical mode at η_c is plotted as a function of κ and Δ
 142 and is a good order parameter for distinguishing the three different phases of the system.
 143 For $\Delta < \delta/2$ both cavity modes have a positive detuning and the system always exhibits
 144 static superradiance (SSR), characterized by a critical mode with zero frequency. SSR
 145 requires a finite critical atom-cavity coupling such that the critical mode is a polariton.
 146 For $\Delta > \delta/2$ one of the modes acquires a negative detuning. Differently from a positively-
 147 detuned mode, a negatively-detuned one disfavors a superradiant density modulation.
 148 The competition between the two cavity modes induces an oscillating superradiant phase
 149 (OSR) [20, 28], which also requires a finite coupling strength such that the critical mode is
 150 again a polariton. Instead, when the solution with $\omega_c = \omega_a$ has a real critical coupling, η_c

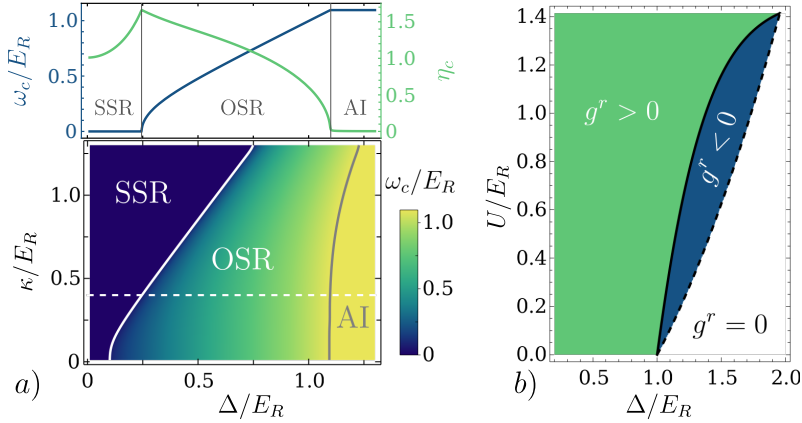


Figure 1: The critical frequency of the instability is shown in the lower plot of a) as a function of Δ and κ . By tuning Δ the critical mode change from exhibiting static to oscillating superradiance and a purely atomic instability over a large range of cavity loss rates. The upper plot shows the critical frequency and coupling along the dashed line in the lower plot. In b) the sign of the cubic interaction as a function of Δ and U is plotted for $\kappa = 0.4E_R$. This determines the stability of the symmetry-broken state beyond the linear analysis. For the entire figure $\delta = 0.2E_R$.

151 in Eq. (4) vanishes, making the critical mode purely atomic. We refer to this instability
 152 as the atomic instability (AI).

153 Both the OSR and AI critical modes break continuous time-translation invariance and
 154 can thus potentially signal a continuous time-crystal phase. However, whether the latter
 155 is stable is determined by non-linear effects not included so far. In order to capture
 156 these in the present interacting many-body system, we perform a systematic perturbative
 157 expansion in the relative distance from the critical point $\eta = (\tilde{\eta} - \eta_c) / \eta_c$. The resulting
 158 effective non-linear equation is of the Stuart-Landau form (see e.g. [29]), and is an equation
 159 of motion for the collective degrees of freedom which are excited in the SR phases. These
 160 degrees of freedom constitute the so-called center manifold and are defined by the critical
 161 mode, which is composed of both cavity modes as well as of the zero and recoil momentum
 162 components of the BEC, ψ_0 and $\psi_{\pm Q}$. Within the center manifold and to leading order in
 163 η the recoil momentum component is given by

$$\psi_{\pm Q}(t) = \sqrt{\eta}R (c_+e^{i\omega_c t} + c_-e^{-i\omega_c t}), \quad (5)$$

164 with c_{\pm} being the atomic components of the critical-mode eigenvector obtained from the
 165 linear analysis [26]. The cavity fields have the same form with c_{\pm} replaced by the cavity
 166 components of the critical mode. Finally, since to leading order the only occupied atom
 167 components are ψ_0 and $\psi_{\pm Q}$, these are linked by normalization such that

$$\psi_0 = \sqrt{1 - |\psi_Q|^2 - |\psi_{-Q}|^2} \sim b_0 + b_+e^{i2\omega_c t} + \bar{b}_+e^{-i2\omega_c t}, \quad (6)$$

168 with $b_0 = 1 - \eta R^2 (|c_+|^2 + |c_-|^2)$ and $b_+ = -\eta R^2 c_+ \bar{c}_-$. The perturbative approach yields
 169 an equation of motion for the amplitude, R , in the symmetry-broken phase:

$$\dot{R} = \gamma R - g^r R^3, \quad (7)$$

170 where γ is the exponential growth rate of the critical mode obtained from the linear
 171 analysis, which in this case can be shown to be positive. The non-linearity of the center

172 manifold or in other words, the strength of the self-interaction of the excitations present
 173 in the critical mode, is quantified by g^r (see Appendix A.2 for closed expressions for these
 174 quantities and the proof of the sign of γ). For stable time-crystalline and static solutions,
 175 R must be time-independent, real, and positive:

$$R = \sqrt{\frac{\gamma}{g^r}} > 0. \quad (8)$$

176 As $\gamma > 0$, our analytic solutions can only be stable if $g^r > 0$. This is physically clear since
 177 otherwise the attractive self-interaction would lead to a first order transition into a phase
 178 that requires higher-order non-linearities for stabilization.

179 The sign of g^r is shown in Fig. 1(b). If ω_c is pushed to ω_a , $g_r = 0$ i.e. the self-interaction
 180 vanishes as the critical mode is purely atomic, which corresponds to the white region in
 181 Fig. 1(b). As the fraction γ/g^r goes to zero as ω_c approaches ω_a (see Eq. (33)), the AI
 182 phase has no stable time-crystalline solution.

183 Short-range interactions between the atoms qualitatively modify g^r and lead to two
 184 separatrices in Fig. 1(b). The expression for the separatrix $U_{c2}(\Delta)$, drawn with a solid line
 185 is given in Eq. (29), while the separatrix $U_{c1}(\Delta)$ between the white and the blue region,
 186 is defined by the condition that the energy cost of a Bogoliubov excitation, ω_a , equals Δ .
 187 When $U > U_{c1}$ the self-interactions of the critical mode become finite and repulsive as
 188 $\omega_c < \omega_a$, leading to a finite cavity component of the critical mode.

189 It is further remarkable that the sign of the self-interactions can be changed via U .
 190 Indeed, within the blue region in Fig. 1(b), that is, for $U_{c1} < U < U_{c2}$, the self-interactions
 191 of the critical mode are attractive: $g_r < 0$. This is due to the fact the short-range
 192 repulsion U , which penalizes density modulations and in particular excitation of the recoil
 193 component $\psi_{\pm Q}$, is not sufficient to counteract the decrease of energy due to coupling to the
 194 negatively detuned cavity mode. The resulting instability of the OSR solution corresponds
 195 to a subcritical Hopf bifurcation [30] of Eq. (6). On the other hand, when $U > U_{c2}$ (green
 196 region in the figure), the short-range repulsion penalizes density-modulations enough to
 197 change the sign of the self-interaction of the critical mode and thus stabilize the OSR phase.
 198 This corresponds to a transition from a subcritical to a supercritical Hopf bifurcation.

199 4 Energy redistribution

200 Up to this point, the OSR time crystal is found to exist in a stable fashion as a supercritical
 201 Hopf bifurcation. Its infinite lifetime is a consequence of the coherent scattering between
 202 the atomic modes with momentum $q = 0$ and $q = \pm nQ$, with n being a positive integer.
 203 Only the $n = 1$ modes belong to the CM but the higher-order integer modes only lead to
 204 a small renormalization of the amplitude of the OSR phase. This happens because the
 205 nQ modes all scatter effectively with the cavity such that a steady state is found with an
 206 occupation of the $n > 1$ modes that is exponentially decreasing with n . For this reason we
 207 refer to the manifold with all the nQ modes as the extended center manifold. However, the
 208 atomic interactions allow for scattering of occupation into modes where $q \neq nQ$. We will
 209 refer to those as the not-center-manifold (NCM) modes. As these modes are not coupled
 210 effectively to the CM modes by the cavity they can exhaust the occupation in the nQ
 211 modes and thereby destroy the coherent nature of the OSR phase. To fully assess the
 212 stability of the OSR phase, one has to allow for the redistribution of energy including the
 213 NCM modes. Hence, one needs to treat the many-body problem of scattering between
 214 quasi-particles and a time-dependent coherent field.

215 Let us first predict which NCM modes initially participate in the scattering process,
 216 assuming we are only slightly into the OSR phase. If one also considers the regime of small

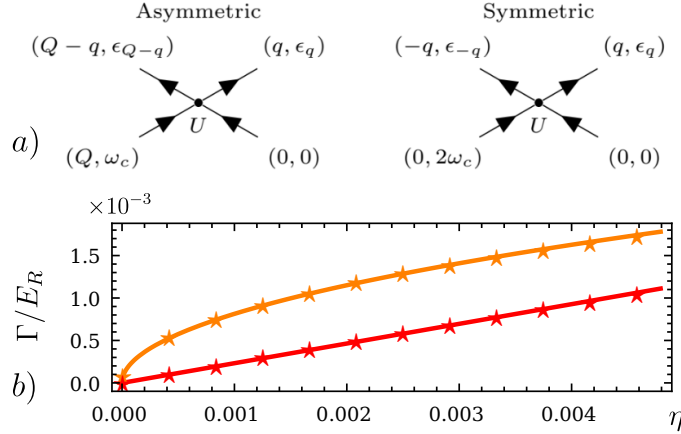


Figure 2: a) The dynamic nature of the OSR phases combined with finite atom-interaction leads to occupation of atom modes out of the center manifold, through the symmetric and asymmetric process illustrated here. b) The scaling of the growth rates, computed from the Floquet quasi-energies of the linearized equations, for the asymmetric channel marked with orange stars, with a square-root fit (orange line) and the scaling of symmetric channel marked with red stars, with a linear fit (red line). The same parameters as in Fig. 3 have been used.

217 U , which is the experimentally relevant one [21], then the occupation of an NCM mode is
 218 described by a collision that is first order in U . If all NCM modes are initially unoccupied,
 219 then the incoming modes must belong to the center manifold. From Eqs. (5) and (6) it
 220 is seen that there are essentially five occupied components in the stable OSR phase. The
 221 dominant component is a zero frequency and zero momentum component with weight b_0 .
 222 Then there are components carrying momentum $\pm Q$ at the frequency ω_c with weight c_{\pm}
 223 and finally the smallest contribution comes from the zero momentum modes with frequency
 224 $2\omega_c$ which are weighted by b_+ . As the scattering is momentum and energy conserving the
 225 fastest-growing NCM mode results from scattering between the atomic components b_0 and
 226 c_{\pm} of the center manifold, as illustrated in Fig. 2(a). For this process, the outgoing NCM
 227 modes with energies $\epsilon_q, \epsilon_{q'}$ have to satisfy $q + q' = Q$, $\epsilon_q + \epsilon_{q'} = \omega_c$. Since here $q \neq -q'$,
 228 we call this the asymmetric channel. Near the critical point, we can approximate ϵ_q with
 229 the Bogoliubov dispersion of the BEC excitations in the absence of the cavity field, which
 230 for small U reads $\omega_B(k) \approx E_R k^2 + U$. This yields $q = Q/2 + \sqrt{\omega_c - E_R/2 - 2U}/\sqrt{2}$ and
 231 $q' = Q - q$. From the approximation of the CM components in Eqs. (5) and (6), we predict
 232 an exponential growth of these two Bogoliubov modes with a rate $\Gamma = \Gamma_{asym} \propto U\sqrt{\eta}$. This
 233 asymmetric channel can be closed off if $\omega_c < E_R/2 + 2U$, which leaves us with a different
 234 channel where the component b_0 scatters with b_+ , or c_+ with c_- . Both these processes
 235 produce a symmetric NCM pair with $q = -q' = \sqrt{\omega_c - U}$. One representative process of
 236 this symmetric channel is shown in Fig. 2(a). In contrast to the asymmetric counterpart,
 237 we predict $\Gamma = \Gamma_{sym} \propto U\eta$ for the symmetric scattering processes.

238 In order to further verify the above predictions, we have linearized Eqs. (1) and (2)
 239 around the OSR phase and extracted the rate by computing the Floquet quasi-energies.
 240 The details of these calculations can be found in Appendix B and the resulting growth rates
 241 are shown in the lower panel of Fig. 3. It is seen that the predicted momentum (orange
 242 marks for the asymmetric channel and red mark for the symmetric channel) is only reliable
 243 close to the phase transition as the dispersion of the NCM mode is quickly modified due
 244 to the growing oscillating density modulation. We also find an additional momentum

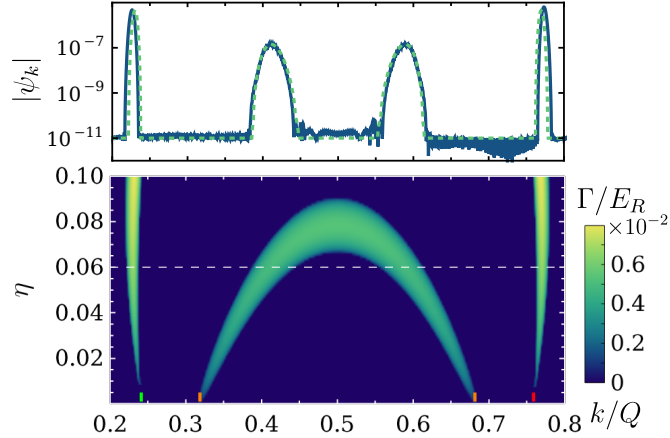


Figure 3: The lower plot shows the exponential growth rates of the atomic modes outside of the center manifold. The parameters are equivalent to those in Fig. 1 with $\Delta = 0.6E_R$ resulting in $\omega_c = 0.586E_R$, and we choose $U = 0.01E_R$. The orange ticks indicate the predicted momentum based on the asymmetric channel, while the red tick signifies the symmetric channel momentum. The green tick is the atom mode coupled to the symmetric channel through the cavity. The upper plot shows the resulting atom distribution after 200 periods at the dashed line in the lower plot, both with numerical integration of Eqs. (1) and (2) in blue and from the linearized prediction with the dashed green line.

245 component that grows (marked in green), which arises from the scattering between a
 246 negative momentum NCM mode in the symmetric channel and the recoil component of the
 247 center manifold. The computed growth rates for the symmetric and asymmetric modes are
 248 shown in Fig. 2(b), and in both cases, an excellent agreement with our simple predictions
 249 based on Fig. 2(a) is demonstrated. Finally, in order to fully confirm our predictions,
 250 we performed a full numerical integration using a Runge-Kutta-4 routine, starting from
 251 the OSR phase at $\eta = 0.06$, corresponding to the white dashed line in the lower panel of
 252 Fig. 3. After evolving the system for 200 periods we compared the momentum distribution
 253 with the predictions based on the Floquet quasi-energies and find excellent agreement, as
 254 shown in the upper panel of Fig. 3.

255 As long as the NCM modes with the largest occupation are still small compared to
 256 unity, i.e. the time crystal has not melted yet, the CM modes still satisfy

$$\sum_{q=0,\pm Q} |\psi_q|^2 \approx 1.$$

257 The rate of the fastest-growing NCM modes therefore sets the time scale on which the
 258 cascade becomes relevant. This time scale is explicitly given by $\tau = \ln a_0/\Gamma$, where a_0 is
 259 the initial seed in the NCM modes. This initial seed is determined by the temperature of
 260 the system and energy of the fastest growing NCM mode.

261 5 Metastable nature of time crystal

262 In the previous section, we have established that atom interactions lead to scattering from
 263 the CM to NCM modes with $q \neq nQ$, with $n \in \mathbb{Z}^+$. We have identified the fastest-
 264 growing NCM modes and that they grow exponentially. As these NCM modes are not

265 coupled via the cavity to the CM modes, one expects them to evolve incoherently, such
 266 that the coherent nature of the OSR will be destroyed once the occupation of the NCM
 267 modes becomes non-negligible. To verify this, in this section, we will analyze the full
 268 numerical solution of Eqs. (1) and (2) in more detail.

269 Before discussing this we note that the numerical calculations require very dense mo-
 270 mentum grids. This is because the NCM modes have to be numerically broadened such
 271 that they can be sampled on a finite grid. The artificial linewidth corresponds to a time
 272 scale, which has to be much smaller than the simulated time. Here we use a broaden-
 273 ing of $2 \times 10^{-6} E_R$, allowing us to reach simulation times of order $10^5/E_R$. As quantum
 274 fluctuations have been neglected a small finite seed is needed for all the NCM modes
 275 and we initialize all the NCM modes with a constant magnitude of 10^{-11} , corresponding
 276 to a_0 discussed at the end of the previous section. Additionally, it is also necessary to
 277 go to large momentum values to ensure the results are fully converged (we truncate at
 278 $|k_{\max}| = 5.4Q$). The resulting numerical momentum grids are thus very long $\mathcal{O}(4 \times 10^4)$.
 279 To fully understand the stability of the OSR phase it is also necessary to include the
 280 fact that a finite number of atoms gives a stochastic nature to the equations due to the
 281 openness of the cavity. This limits the magnitude of the numerical time step as the noise
 282 appears correlated on a time scale of Δt , with Δt being the numerical time step. We
 283 therefore require that $\Delta t \ll 1/E_{\max}$ where $E_{\max} \sim k_{\max}^2$ is the largest energy scale in our
 284 simulation. The simulations therefore require a small time step (we use $\Delta t = 5 \times 10^{-4}$
 285 and a Runge-Kutta-2 routine for the deterministic part [31]). All parameters apart from
 286 U will be kept the same as those used in Fig. 3 such that $\omega_c = 0.586 E_R$ leading to a
 287 characteristic periode of $T = 2\pi/\omega_c = 10.735 E_R^{-1}$. To verify that the asymmetric channel
 288 is closed when $\omega_c < E_R/2 + 2U$, while simultaneously decreasing the computation time,
 289 we consider $U = 0.1 E_R$. As the symmetric channel grows with a rate proportional to η ,
 290 the increase of U by a factor of 10 ultimately results in an increase of the NCM growth
 291 rate by ≈ 2.45 . Consequently, the necessary simulation time is more than halved even
 292 though the asymmetric scattering channel is closed.

293 The resulting effective cavity potential felt by atoms defined as

$$V_c(t) = \eta_c \frac{\eta + 1}{\sqrt{2}} \sum_{j=1,2} \text{Re}(\phi_j(t)), \quad (9)$$

294 is plotted in Fig. 4(a). $V_c(t)$ is good observable as it not only describes the optical potential
 295 felt by the atoms but also, up to a constant prefactor, corresponds to measuring the x-
 296 quadrature of the cavity field. For early times one observes perfectly coherent oscillations
 297 expected from the stable OSR phase. After the blue shaded region, which stops at $t =$
 298 $600 E_R^{-1} \approx 56T$ the amplitude of V_c starts being perturbed as the occupation of the NCM
 299 modes becomes relevant. The exponential growth and following cascade leads to a sharp
 300 decrease in the amplitude of optical potential and a loss of its temporal coherence.

301 The loss of coherence is more clearly seen in Fig. 4(b) where the power spectrum of
 302 the Fourier transform of the two shaded regions in Fig. 4(a) is shown. For early time the
 303 spectrum is clearly dominated by a peak at the predicted frequency (marked in red) and
 304 a small contribution from a higher harmonic at $3\omega_c$. The finite width for early times is
 305 a consequence of the finite interval of the Fourier transform combined with the use of a
 306 window function to avoid aliasing. The spectrum of the long-time region clearly shows that
 307 no coherent oscillation is present in the cavity fields. At late times, there is a competition
 308 between synchronization via the cavity that favors the OSR phase, and decoherence caused
 309 by scattering involving NCM modes. The amplitude of the incoherent oscillations in this
 310 region is therefore related to the magnitude of U . To highlight the fact that the loss of
 311 coherence is an effect of having lost occupation in the CM modes, Fig. 4(c) shows the

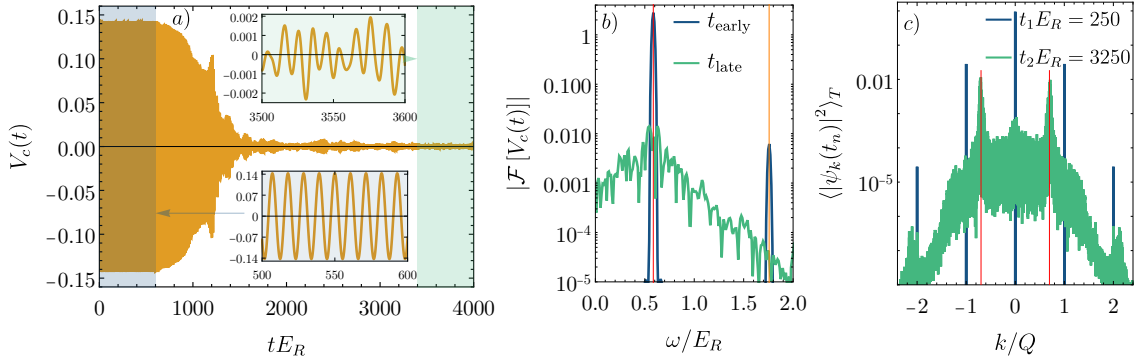


Figure 4: a) The effective cavity potential felt by the atoms resulting from the full numerical solution of Eqs. (1) and (2) for the same parameters as Fig. 3 apart from $U = 0.1E_R$. The insets show a zoomed-in view of $V_c(t)$ from the early and late time regions respectively (the arrows indicate the regions the insets are parts of). b) The power spectrum of the Fourier transform performed on the finite interval marked by the blue ($t_{\text{early}} \in \{0, 600\}E_R^{-1}$) and green ($t_{\text{late}} \in \{3400, 4000\}E_R^{-1}$) shaded regions in plot (a). The red line indicates the predicted ω_c while the orange line is at $3\omega_c$. c) The corresponding momentum distribution of the atomic modes within the stable OSR region in blue ($t_1 = 250E_R^{-1}$) and after its destruction in green ($t_2 = 3250E_R^{-1}$). The red lines indicate the momentum of the fastest-growing modes predicted by symmetric channel scattering predictions.

312 occupation of the atomic modes, averaged over one period T , for a time within the early
 313 and late time regions respectively. Here the early time occupation shows all occupation
 314 is basically in the CM modes ($q = \{0, \pm Q\}$). For late times this occupation has been
 315 completely redistributed to a continuum of momentum modes with the most occupied
 316 mode being the symmetric modes predicted at $q = \pm\sqrt{\omega_c - U}$ which are marked with red
 317 lines. It is worth pointing out that, as predicted, the asymmetric momentum modes seen
 318 in Fig. 3 are absent. The broad nature of the atomic distribution at late times is a result of
 319 the exponential growth of the initial NCM modes together with the fact that the occupied
 320 NCM modes act as sources for occupying new NCM modes, leading to an exponentially
 321 growing cascade of modes out of CM and into the NCM.

322 From the perspective of the growing NCM modes, this cascade can be understood
 323 as the unstable regime of parametric resonance with the center manifold modes acting
 324 as parametric drive via the short-range interactions. This is analogous to preheating in
 325 the early universe [32, 33], where the weakly interacting and oscillating, coherent inflation
 326 field decays into a cascade of exponentially growing fluctuations, leading to extreme non-
 327 equilibrium conditions inaccessible to perturbative methods and finally to thermalization
 328 [34]. Explicitly, the CM modes take the role of the inflaton field and the growing NCM
 329 modes resemble the particles created by the oscillating field. The analytic discussion we
 330 have presented corresponds to the linearized classical regime [35], which at later times will
 331 be superseded by increasingly non-linear effects leading to a cascade of even more quickly
 332 growing fluctuations that eventually thermalize [36]. At this point, a conceptual difference
 333 between the closed evolution considered for the early universe, and the driven open cavity
 334 model we discuss here emerges. Our model never fully thermalizes. Instead, it reaches a
 335 highly non-trivial state that corresponds to a compromise between thermalization due to
 336 short-ranged interactions and the flow equilibrium imposed by drive and dissipation.

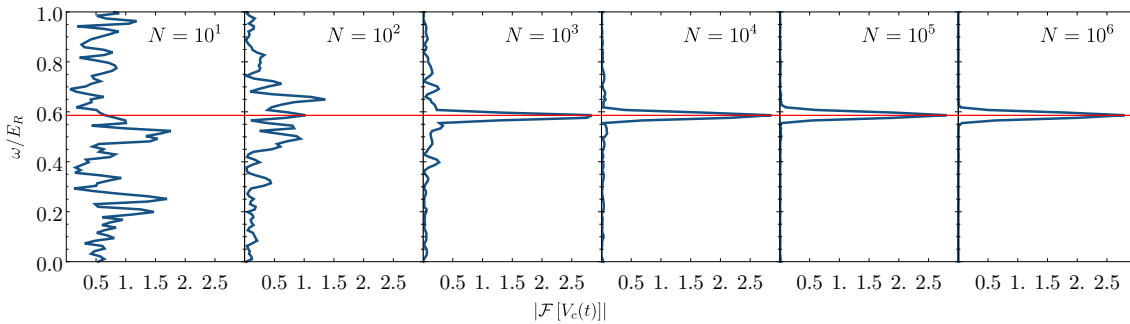


Figure 5: Plot of the Fourier spectrum of $V_c(t)$ for different atom numbers which is inversely related to the noise level. The parameters are kept equivalent to those in Fig. 4. As the time crystalline phase is only metastable the Fourier transform is performed on finite region with $t \in \{0, 600E_R^{-1}\}$ as the early time result in Fig. 4(b). The red line indicates the predicted ω_c .

337 5.1 Finite size of atom clouds

338 Any experiment will necessarily be performed with a finite number of atoms, N , in the
 339 cloud. As mentioned, in Section 2, the openness of the cavity then gives rise to a stochastic
 340 term in the cavity equations such that they take the form

$$i\partial_t\phi_j = (\Delta_j - i\kappa)\phi_j + \frac{\tilde{\eta}}{2\sqrt{2}} \sum_{k=-\infty}^{\infty} \bar{\psi}_k (\psi_{k+Q} + \psi_{k-Q}) + \sqrt{\frac{\kappa}{N}}\Delta W(t), \quad (10)$$

341 where $\Delta W(t)$ is the Wiener increment [31]. To verify that the metastable OSR phase
 342 is not just robust towards atomic perturbation but also cavity noise we have performed
 343 the numerical calculations with different atom numbers (effectively changing the noise
 344 level). In principle one should compute several trajectories. Here, however, we are not
 345 interested in the noise-averaged state at late times but merely in the characteristic lifetime
 346 of the OSR phase. As the latter is much longer than any microscopic time scale of the
 347 system, it is faithfully obtained already from a single trajectory. To this extent, the Fourier
 348 spectrum for different levels of noise has been plotted in Fig. 5. Here it is seen that the
 349 time-crystalline order emerges once the cloud contains 10^3 atoms or more. It should be
 350 pointed out that we are considering a setup with a subrecoil linewidth of $\kappa = 0.4E_R$ which
 351 is comparable to relevant experimental setups [21] in which $N = 10^5$. We thus predict
 352 that the metastable region of the OSR phase is stable against realistic noise levels.

353 5.2 Cavity parameters

354 Of particular experimental relevance is the question regarding the dependence of the life-
 355 time of the OSR phase on the choice of cavity parameters. So far, we have investigated
 356 the dependence of the lifetime on the frequency of the CM components. However, the
 357 same frequency can be generated with an infinite number of different cavity parameters.
 358 This is because one can depart from the simple fully symmetric cases discussed here and
 359 include different loss rates for the different cavity modes and different coupling to the
 360 atomic cloud. Experimentally the latter can be easily controlled by changing the power
 361 in the corresponding pump sideband. So the question then becomes, given that the CM
 362 oscillates at a set frequency does it matter how the cavity is configured to achieve this fre-
 363 quency? Our analysis shows that, for small U , the scattering out of the OSR phase, which
 364 sets the lifetime, is contained in the first-order scattering processes between the atomic

365 CM components of the atoms illustrated in Fig. 2(a). The magnitude of the atomic CM
 366 components therefore have to depend on cavity parameters directly (and not just through
 367 ω_c) to impact the lifetime of the OSR phase. Using the full analytic description of the OSR
 368 phase derived in Appendix A.2, one finds that the time-averaged magnitude of the recoil
 369 components in CM, given in Eq. (37) is only a function of η , U , ω_c and E_R . In this model
 370 the lifetime of the OSR phase is thus independent of the specific cavity configuration.
 371 This should be contrasted with the previous results for the frequency of the OSR phase in
 372 Eq. (3), which was shown to be independent of U and as such stable against interatomic
 373 interactions. Apart from this intriguing duality, the fact that the lifetime is not explicitly
 374 depending on the cavity configuration gives a large amount of freedom for experimental
 375 realizations.

376 6 Conclusion

377 We have analyzed the role of short-range interactions on the nature and stability of contin-
 378 uous time crystals in dissipative many-body systems of ultracold bosonic atoms in cavities.

379 First, we have shown that short-range interatomic interactions can alter the nature of
 380 the time crystal by transforming the underlying classical bifurcation from sub- to super-
 381 critical.

382 Second, we have studied the effect of interatomic interactions on the time crystal. Our
 383 analysis shows that when the system can only dissipate energy through the cavity, the
 384 interatomic interactions makes the time crystal inherently metastable. This can be un-
 385 derstood from the fact that the time crystalline phase requires superradiance and thus
 386 coherent scattering of cavity photons between the atomic states. The interatomic interac-
 387 tions deplete the superradiant mode by populating atomic states that scatter incoherently.
 388 In the absence of an ordering principle, like a low temperature in thermal equilibrium, the
 389 depletion proceeds until the time crystalline phase ceases to exist.

390 We showed that near the transition, the amplitude of atomic component of the time
 391 crystalline phase is independent of the details of the underlying critical polaritonic mode.
 392 Instead it only depends on the frequency of the oscillations and the proper dimensionless
 393 distance from the critical point. The cavity dissipation is therefore not able to efficiently
 394 cool the system [37–39] (NCM modes can be de-excited only at higher order in our expan-
 395 sion, see Appendix B). This suggests that the metastable nature we identified is generic for
 396 these cavity systems [40], as long as the cavity linewidth is comparable to the recoil energy.
 397 We note that time-dependent Hartree-Fock approximations would miss this heating [41],
 398 as they lack collisions, and thus redistribution [42].

399 While we that the time crystalline phase is only metastable, we have shown that
 400 its lifetime increases significantly by considering $\omega_c < E_R/2 + 2U$ which prohibits the
 401 asymmetric scattering processes that lead to much higher growth rates. As interatomic
 402 interactions can be weak in dilute systems, the time-crystalline phases can appear stable
 403 on a time scale of the order of hundreds of periods, consistent with numerical predictions
 404 in related models [43, 44]. We numerically show this statement to hold true, also in the
 405 presence of realistic noise levels.

406 Finally, we pointed out that cascading effect responsible for the destruction of the time
 407 crystalline phase is analogous to preheating in the early universe. It will be interesting to
 408 pursue this analogy deeper into the highly excited regime using appropriate atom-photon
 409 diagrammatic approaches [45, 46].

410 Acknowledgements

411 CHJ would like to thank Johnathan Dubois for many helpful and insightful discussions.

412 A Center manifold coefficients

413 The two-cavity mode system considered is a simplification of the N cavity mode system
 414 discussed in [26]. In this reference the standard methods of stability analysis and bifur-
 415 cation theory [29, 30] is applied to understand the origin of the limit cycle, which we use
 416 as the foundation for our exploration. We refer the interested reader to reference [26]
 417 for a more detailed derivation of the critical frequency and coupling. In this appendix,
 418 we instead focus on a self-contained derivation of the analytical description of the center
 419 manifold dynamics.

420 The starting point is the equations of motion in Eqs. (1) and (2), which we use to
 421 define the autonomous system of non-linear first order ODE's

$$\dot{\mathbf{v}} = F(\mathbf{v}), \quad (11)$$

422 Here \mathbf{v} is a vector containing the two complex cavity fields and all the complex atom
 423 fields with the different discretized momenta. As both cavity modes transfer the same
 424 longitudinal momentum (Q) and the BEC is initially homogeneous, the emerging critical
 425 mode only contains the cavity fields, the homogeneous atom state and the $\pm Q$ atom modes.
 426 These modes constitute the center manifold

$$\mathbf{v}_{cm} = (\phi_0, \phi_1, \psi_0, \psi_Q, \psi_{-Q})^T. \quad (12)$$

427 The normal phase $\mathbf{X}_0 = (0, 0, 1, 0, 0)^T$ is a fixed point of F . When $\tilde{\eta} < \eta_c$ this fixed point
 428 is stable but becomes unstable for $\tilde{\eta} \geq \eta_c$.

429 As stated in the main text, slightly past the critical point the symmetry-broken state
 430 can be approximated as

$$\mathbf{u} = \sqrt{\mu} R \mathbf{v}^R e^{i\omega_c t} + c.c., \quad (13)$$

431 where $\mu = \tilde{\eta} - \eta_c$ is the absolute distance to the critical point and ω_c is the frequency of
 432 the unstable right eigenvector \mathbf{v}^R . We will write the equations of motion in terms of the
 433 real (x_α) and imaginary part (p_α) of the complex fields in which the center manifold is
 434 spanned by vectors of the form

$$\mathbf{v}^R = (\mathbf{v}_{c1}^R, \mathbf{v}_{c2}^R, \mathbf{v}_Q^R, \mathbf{v}_{-Q}^R)^T, \quad (14)$$

435 with $\mathbf{v}_\alpha^R = (x_\alpha, p_\alpha)^T$. The approximation in Eq. (13) only describes the new fixed point
 436 well if the bifurcation is of the supercritical form, which means that the self-interaction of
 437 the critical mode is repulsive. The linear coefficient in the amplitude equation Eq. (7) is
 438 given by the real part of

$$\lambda = \sum_{i,j} \mathbf{v}_i^L \left(\frac{\partial L}{\partial \mu} \right)_{i,j} \mathbf{v}_j^R, \quad (15)$$

439 where the latin indices run over all components in the center manifold $i, j \in \{x_{c1}, p_{c1}, x_{c2}$
 440 $, p_{c2}, x_Q, p_Q, x_{-Q}, p_{-Q}\}$. The Jacobian matrix $L = \nabla F|_{\mathbf{X}_0}$ evaluated at the normal-phase
 441 fixed point \mathbf{X}_0 and \mathbf{v}^L (\mathbf{v}^R) is the left (right) critical eigenvector. We define the linear
 442 coefficient as $\gamma = \text{Re}(\lambda)$. The cubic coefficient of Eq. (7) is given by the real part of

$$g = -\frac{1}{2} \sum_{i,j,k,q} \mathbf{v}_i^L \frac{\partial^3 F_i}{\partial \mathbf{X}^j \partial \mathbf{X}^k \partial \mathbf{X}^q} \Big|_{\mathbf{X}_0, \mu=0} \mathbf{v}_j^R \mathbf{v}_k^R \mathbf{v}_q^R = -\mathbf{v}_i^L (N_0)_i^{j,k,q} \mathbf{v}_j^R \mathbf{v}_k^R \mathbf{v}_q^R, \quad (16)$$

443 where the cubic coefficient in the main text is $g^r = \text{Re}(g)$. Within the center manifold
 444 there is no contribution to quadratic contribution from $\partial^2 F$ because the center manifold
 445 obeys a reflection symmetry, which originates from the fact that the coupled Eqs. (1)
 446 and (2) posses a \mathcal{Z}_2 -symmetry. This symmetry is that the equations are invariant under the
 447 simultaneous phase shift of the atoms by $\psi_k \rightarrow e^{i\pi k/Q} \psi_k$ and the cavity fields $\phi_j \rightarrow -\phi_j$.

448 A.1 The critical eigenvector

449 To compute λ and g we use that the symmetry results in the right and left eigenvector
 450 components for each mode $\alpha \in \{c_1, c_2, Q, -Q\}$ are related by

$$\mathbf{v}_\alpha^L = \pm \sigma_x \mathbf{v}_\alpha^R, \quad (17)$$

451 where σ_x is the first Pauli spin-1/2 matrix. Furthermore, the eigenvectors are normalized
 452 such that Eq. (17) is realized with the upper sign. The two effective interaction parameters
 453 can now be written solely in terms of the right eigenvectors

$$\begin{aligned} \lambda &= \sum_{i,j} ((1_4 \otimes \sigma_x) \mathbf{v}^R)_i \left(\frac{\partial L}{\partial \mu} \right)_{i,j} \mathbf{v}_j^R, \\ g &= - \sum_{i,j,k,q} ((1_4 \otimes \sigma_x) \mathbf{v}^R)_i (N_0)_i^{j,k,q} \mathbf{v}_j^R \mathbf{v}_k^R \bar{\mathbf{v}}_q^R. \end{aligned} \quad (18)$$

454 The cavity eigenvector components are connected to the atomic eigenvector components
 455 through the definition of the critical eigenvector

$$(L_0 - i\omega_c) \mathbf{v}^R = \mathbf{0}, \quad (19)$$

456 which leads to the relation

$$\mathbf{v}_{c_j}^R = -\sqrt{2} \eta_c \beta_j \begin{pmatrix} 1 & 0 & 1 & 0 \\ \frac{\kappa+i\omega_c}{\Delta_j} & 0 & \frac{\kappa+i\omega_c}{\Delta_j} & 0 \end{pmatrix} \begin{pmatrix} x_Q \\ p_Q \\ x_{-Q} \\ p_{-Q} \end{pmatrix}, \quad (20)$$

457 where

$$\beta_j = \frac{\Delta_j}{2} \frac{\Delta_j^2 + \kappa^2 - \omega_c^2 - 2i\omega_c \kappa}{\Delta_j^4 + 2\Delta_j^2(\kappa^2 - \omega_c^2) + (\kappa^2 + \omega_c^2)^2}. \quad (21)$$

458 From the critical eigenvalue condition $\det(L_0 - I\omega_c) = 0$ one finds

$$\eta_c^2 \beta = \eta_c^2 \sum_{j=1,2} \text{Re} \beta_j = \frac{\omega_a^2 - \omega_c^2}{2E_R}, \quad (22)$$

$$\text{Im} \beta_j = 0.$$

459 Linearizing around the normal phase means that the short-range interaction only couples
 460 the modes Q and $-Q$ in a symmetric manner. As the cavity also couples identically to
 461 these two modes, the components of the critical eigenvector obeys $x_Q = x_{-Q} = x_a$ and
 462 $p_Q = p_{-Q} = p_a$. Using this symmetry x_a and p_a can be connected through Eq. (19) and
 463 one finds

$$p_a = \frac{i\omega_c}{E_R} x_a = i\tilde{\omega} x_a, \quad (23)$$

464 where the dimensionless frequency $\tilde{\omega} = \omega_c/E_R$ has been introduced for later convenience.
 465 Now \mathbf{v}^R can be fully expressed through the parameters of our theory and x_a . A closed-form
 466 expression for x_a can be found through the normalization condition

$$\sum_i \mathbf{v}_i^L \mathbf{v}_i^R = 1 \rightarrow x_a = \frac{1}{2} \left(4\eta_c^2 \sum_j \left[\beta_j^2 \frac{\kappa + i\omega_c}{\Delta_j} \right] + \frac{i\omega_c}{E_R} \right)^{-1/2}. \quad (24)$$

467 This form of x_a guarantees the upper sign in Eq. (17).

468 A.2 Computing g^r and γ

469 By substituting the critical eigenvector into Eq. (15) one finds the expression

$$\begin{aligned} \lambda &= \frac{8}{\eta_c} x_a^2 \eta_c^2 \beta = \frac{2\eta_c^2 \beta}{\eta_c \left(4\eta_c^2 \sum_j \left[\beta_j^2 \frac{\kappa + i\omega_c}{\Delta_j} \right] + \frac{i\omega_c}{E_R} \right)} \\ &= \frac{\omega_a^2 - \omega_c^2}{E_R} \frac{1}{\eta_c \left(4\eta_c^2 \sum_j \left[\beta_j^2 \frac{\kappa + i\omega_c}{\Delta_j} \right] + \frac{i\omega_c}{E_R} \right)}. \end{aligned} \quad (25)$$

470 The expression for g is

$$\begin{aligned} g &= x_a^2 |x_a|^2 E^R \left(\tilde{U} (3 + 2\tilde{\omega}^2 + 3\tilde{\omega}^4) + 4(1 - \tilde{\omega}^2)(3 + \tilde{\omega}^2) \right) \\ &= x_a^2 |x_a|^2 E^R W_a(\tilde{U}, \tilde{\omega}) \end{aligned} \quad (26)$$

471 where the dimensionless interaction is defined as $\tilde{U} = U/E_R$.

472 It is clear that the only part that makes both λ and g complex is in x_a^2 . As the
 473 coefficients for our theory are related to the real part of λ and g , it is relevant to extract
 474 the real part of x_a^2

$$\begin{aligned} \text{Re}(x_a^2) &= \eta_c^2 \kappa \frac{\sum_j \frac{\beta_j^2}{\Delta_j}}{\left| \left(4\eta_c^2 \sum_j \left[\beta_j^2 \frac{\kappa + i\omega_c}{\Delta_j} \right] + \frac{i\omega_c}{E_R} \right) \right|^2} \\ &= \frac{\eta_c^2 \kappa}{2 \left| \left(4\eta_c^2 \sum_j \left[\beta_j^2 \frac{\kappa + i\omega_c}{\Delta_j} \right] + \frac{i\omega_c}{E_R} \right) \right|^2} \sum_j \Delta_j \frac{\left(\Delta_j^2 + \kappa^2 - \omega_c^2 \right)^2 + 4\omega_c^2 \left(\Delta_j^2 - \omega_c^2 \right)}{\left(\Delta_j^4 + 2\Delta_j^2 (\kappa^2 - \omega_c^2) + (\kappa^2 + \omega_c^2)^2 \right)^2}. \end{aligned} \quad (27)$$

475 This directly shows that the only dependence on U in x_a is through η_c in Eq. (4). Due
 476 to the complexity of the full closed form expression of g it is insightful to consider the
 477 behavior of $\text{Re}(x_a^2)$ and W_a separately.

478 First considering W_a

$$W_a(\tilde{U}, \tilde{\omega}) = \tilde{U} (3 + 2\tilde{\omega}^2 + 3\tilde{\omega}^4) + 4(1 - \tilde{\omega}^2)(3 + \tilde{\omega}^2). \quad (28)$$

479 The interesting feature of W_a is the fact that it has a sign change through a zero-crossing
 480 at a critical frequency $\tilde{\omega}_0$ such that $W_a(\tilde{U}, \tilde{\omega}_0) = 0$. The closed form expression for $\tilde{\omega}_0$ is

$$\tilde{\omega}_0 = \sqrt{\frac{\tilde{U} - 4 + 2\sqrt{2}\sqrt{8 - 4\tilde{U} - \tilde{U}^2}}{4 - 3\tilde{U}}} = \sqrt{1 + \frac{\tilde{U}}{2} + \mathcal{O}(\tilde{U}^2)}. \quad (29)$$

481 This exactly defines the separatrix U_{c2} shown as a black line in Fig. 1(b). Fig. 1(b) is
 482 plotted as a function of Δ and not ω_c because of the atom instability. In the regime where
 483 $\kappa < E_R$, the equations simplify because near $\Delta \sim E_R$ one has that $\omega_c \approx \Delta$. This means
 484 that one can replace $\tilde{\omega}_0$ with Δ/E_R instead of substituting in the full expression in Eq. (3).
 485 The relevant quantity that one should compare $\tilde{\omega}_0$ to is the dimensionless frequency of the
 486 bare atomic instability, which happens at

$$\tilde{\omega}_a = \sqrt{1 + 2\tilde{U}}, \quad (30)$$

487 and which sets the dashed separatrix U_{c1} in Fig. 1(b). For $\tilde{U} = 0$ the frequencies $\tilde{\omega}_0$ and
 488 $\tilde{\omega}_a$ coincide, which means that there will be no cubic interactions for the atomic instability
 489 without short-range interactions. As \tilde{U} is made finite we see from the expansion in Eq. (29)
 490 that $\tilde{\omega}_a > \tilde{\omega}_0$ for small $\tilde{U} < 1$. By keeping the full expression for $\tilde{\omega}_0$, one finds that the
 491 critical \tilde{U}_c where $\tilde{\omega}_a = \tilde{\omega}_0$ is

$$\tilde{U}_c = \sqrt{2}, \quad (31)$$

492 which is the intersection point of the separatrices at finite U with $\Delta = \sqrt{1 + 2\sqrt{2}}E_R$.
 493 Below this interaction strength, $\tilde{\omega}_0$ is smaller than $\tilde{\omega}_a$. The effect is that $W_a(\tilde{U}, \tilde{\omega}_a) < 0$
 494 for all $\tilde{U} < \tilde{U}_c$.

495 To determine the nature of the interactions one has to determine the sign of $\text{Re}(x_a^2)$.
 496 This sign is fixed by the numerator of Eq. (27) and using the parametrization discussed
 497 in the main text one finds

$$\sum_{j=1,2} \Delta_j \left((\Delta_j^2 + \kappa^2 - \omega_c^2)^2 + 4\omega_c^2 (\Delta_j^2 - \omega_c^2) \right) = \frac{\delta}{2} \left(\frac{(\delta^2 + 4\kappa^2)^2}{16} + \Delta^2 (3\delta^2 + 4\kappa^2) + 8\Delta^4 \right). \quad (32)$$

498 So for any values of κ , Δ and $|\delta|$, the sign of $\text{Re}(x_a^2)$ is set by the sign of δ . This means that
 499 for a chosen sign of δ the sign of W_a determines whether the non-linear self-interactions
 500 are repulsive or attractive. Additionally this also means that $\gamma > 0$. If $\delta > 0$ then $\omega_c < \omega_a$
 501 and both $\eta_c^2\beta$ in Eq. (22) and $\text{Re}(x_a^2)$ are greater than zero. If $\delta < 0$ then $\omega_c > \omega_a$ and
 502 both $\eta_c^2\beta$ and $\text{Re}(x_a^2)$ are negative such that γ is again positive.

503 Next consider the fraction γ/g^r which determines the magnitude of the stable time-
 504 crystalline phase. By using the above derived relations one can show that it scales with
 505 $\sqrt{\epsilon}$ in the limit where $\omega_c^2 \rightarrow \omega_a^2 - \epsilon$ with $\epsilon \ll \{E_R, \Delta_{1/2}, \kappa\}$

$$\begin{aligned} \lim_{\omega_c^2 \rightarrow \omega_a^2 - \epsilon} \frac{\gamma}{g^r} &= \lim_{\omega_c^2 \rightarrow \omega_a^2 - \epsilon} \frac{8 \omega_a^2 - \omega_c^2}{\eta_c} \frac{\text{Re}(x_a^2)}{2E_R \text{Re}(x_a^2) |x_a|^2 E_R W_a(U/E_R, \omega/E_R)} \\ &= \lim_{\omega_c \rightarrow \omega_a} \frac{32}{\eta_c} \frac{\omega_a^2 - \omega_c^2}{2E_R^2 W_a(U/E_R, \omega/E_R)} \left| 4\eta_c^2 \sum_j \left[\beta_j^2 \frac{\kappa + i\omega_c}{\Delta_j} \right] + \frac{i\omega_c}{E_R} \right| \\ &= \lim_{\omega_c^2 \rightarrow \omega_a^2 - \epsilon} \frac{32}{\sqrt{\frac{\omega_a^2 - \omega_c^2}{E_R \sum_{j=1,2} \frac{\Delta_j (\Delta_j^2 + \kappa^2 - \omega_c^2)}{\omega_c^4 + 2\omega_c^2 (\kappa^2 - \Delta_j^2) + (\Delta_j^2 + \kappa^2)^2}}} \frac{\omega_a^2 - \omega_c^2}{2E_R^2 W_a(U/E_R, \omega/E_R)} \\ &\quad \times \left| 4\eta_c^2 \sum_j \left[\beta_j^2 \frac{\kappa + i\omega_c}{\Delta_j} \right] + \frac{i\omega_c}{E_R} \right| \\ &= 32\sqrt{\epsilon} \frac{\sqrt{E_R \sum_{j=1,2} \frac{\Delta_j (\Delta_j^2 + \kappa^2 - \omega_a^2)}{\omega_a^4 + 2\omega_a^2 (\kappa^2 - \Delta_j^2) + (\Delta_j^2 + \kappa^2)^2}}}{2E_R^2 W_a(U/E_R, \omega/E_R)} \frac{\omega_a}{E_R} + \mathcal{O}(\epsilon^{3/2}). \end{aligned} \quad (33)$$

506 This is important as it proves that the AI region does not possess a stable time-crystalline
 507 solution, to leading order in μ , as $\epsilon \rightarrow 0$ for the AI.

508 The fact that we have analytical expressions for all the important quantities also
 509 allows us to show some intriguing features of the time-crystalline phase within the center
 510 manifold. The first important feature was discussed in the main text, namely that the
 511 frequency of OSR phase is independent of the atom parameters. The second important
 512 feature we will show now is that the time-averaged occupation in the recoil field is only
 513 indirectly depending on the cavity parameters. As stated in the conclusions, this means
 514 that our heating discussion is more generic, as it does not depend on the specific cavity
 515 configuration. If we write Eq. (5) using x_a and p_a the occupation in the recoil mode is
 516 given by

$$\begin{aligned} |\psi_Q(t)|^2 &= \frac{1}{2} \left(|x_a|^2 + |p_a|^2 \right) \\ &= \frac{\mu}{4} R^2 \left(2|x_a|^2 + x_a^2 \exp(i2\omega_c t) + \bar{x}_a^2 \exp(-i2\omega_c t) \right. \\ &\quad \left. + 2|p_a|^2 + p_a^2 \exp(i2\omega_c t) + \bar{p}_a^2 \exp(-i2\omega_c t) \right). \end{aligned} \quad (34)$$

517 Due to the periodicity of the system the time average is given by

$$\langle |\psi_Q|^2 \rangle_T = \int_0^{2\pi/\omega_c} |\psi_Q(t)|^2 dt = \frac{\mu}{2} R^2 \left(|x_a|^2 + |p_a|^2 \right) = \frac{\mu}{2} R^2 |x_a|^2 (1 + \tilde{\omega}^2), \quad (35)$$

518 where p_a have been eliminated through Eq. (23). Using the results from Eqs. (25) and (26)
 519 we find

$$R^2 = \frac{\text{Re}(\lambda)}{\text{Re}(g)} = \frac{4(\tilde{\omega}_a^2 - \tilde{\omega}^2)}{\eta_c |x_a|^2 W_a(\tilde{U}, \tilde{\omega})}. \quad (36)$$

520 Inserting this into Eq. (35) we find

$$\langle |\psi_Q|^2 \rangle_T = \frac{2\eta(\tilde{\omega}_a^2 - \tilde{\omega}^2)(1 + \tilde{\omega}^2)}{W_a(\tilde{U}, \tilde{\omega})}, \quad (37)$$

521 which only depends on the atom parameters, the OSR frequency, and the relative depth
 522 into the OSR phase, η . The same $\tilde{\omega}$ can be generated with many different cavity con-
 523 figurations, for example by changing δ and having a small κ or even more generally by
 524 departing from the fully symmetric case presented here.

525 B Including fluctuations outside the center manifold

526 While our theory within the center manifold predicts that the time crystal is stable also
 527 with finite interactions U , it does not capture atom modes outside of the center manifold.
 528 The contact interaction allows occupation in the center manifold to scatter to the other
 529 atom modes with momenta different from $\pm Q$ and 0. Inside the OSR phase the NCM
 530 modes can be occupied due to the presence of the OSR. This leads to heating and poten-
 531 tially also the destruction of the time crystal in the long time limit. One way to describe
 532 this is to linearize around the OSR solution $\mathbf{v}_{\text{osr}}(t)$

$$\mathbf{v}(t) = \mathbf{v}_{\text{osr}}(t) + \delta\mathbf{v}(t). \quad (38)$$

533 This leads to an equation for the fluctuations

$$\begin{aligned} \dot{\mathbf{v}} &= \dot{\mathbf{v}}_{\text{OSR}} + \delta\dot{\mathbf{v}} = F(\mathbf{v}_{\text{OSR}} + \delta\mathbf{v}) = F(\mathbf{v}_{\text{OSR}}) + \nabla F|_{\mathbf{v}=\mathbf{v}_{\text{OSR}}} \delta\mathbf{v} + \mathcal{O}(\delta\mathbf{v}^2) \\ &\rightarrow \delta\dot{\mathbf{v}} = \nabla F|_{\mathbf{v}=\mathbf{v}_{\text{OSR}}} \delta\mathbf{v} + \mathcal{O}(\delta\mathbf{v}^2) \approx J_{\text{OSR}}(t)\delta\mathbf{v}, \end{aligned} \quad (39)$$

534 where $J_{\text{OSR}}(t)$ is a time-dependent matrix-valued function. Using the approximate fixed
535 point from the analytical OSR solution we can derive an approximate form of $J_{\text{OSR}}(t)$. Due
536 to the periodicity of the OSR solution $J_{\text{OSR}}(t)$ can be expanded in a discrete Fourier series
537 of the form

$$J_{\text{OSR}}(t) = \sum_{n=-4}^4 M_n e^{in\omega_c t}. \quad (40)$$

538 The coupling to the cavity in Eq. (1) is proportional to a product of a NCM mode and
539 a cavity field. To first order in fluctuations there is therefore no coupling between cavity
540 fluctuations and the NCM modes. For the leading-order heating mechanism $\delta\mathbf{v}$ only
541 includes the atom modes with momentum $k \notin \{0, Q, -Q\}$ and is therefore solely described
542 by Eq. (1) with the cavity fields replaced by the OSR solution. It is for this reason that
543 we are able to use the simple scattering description, discussed in the main text, to predict
544 the momentum of the growing modes.

545 Because the cavity loss has already been used to stabilize the OSR phase within the
546 center manifold this means that the cavity is not able to cool down the NCM modes
547 at the linear level. As one includes higher orders in fluctuations the cavity fluctuations
548 can potentially start cooling down the NCM but as this is a higher-order effect, fine
549 tuning would be needed to make it overcome the first-order heating before the system has
550 thermalized and the OSR phase is destroyed.

551 To verify our simple scattering predictions we derive $J_{\text{OSR}}(t)$ from Eq. (1). The linearized
552 equation for the NCM mode with momentum k is

$$\begin{aligned} i\partial_t \psi_k &= \left(-\dot{\Omega} + k^2 - U|\hat{\psi}_0|^2 + 2U \left(|\hat{\psi}_0|^2 + |\hat{\psi}_Q|^2 + |\hat{\psi}_{-Q}|^2 \right) \right) \psi_k + U \left(\hat{\psi}_0^2 + 2\hat{\psi}_{-Q}\psi_Q \right) \bar{\psi}_{-k} \\ &\quad + \frac{\tilde{\eta}}{\sqrt{2}} \sum_j \text{Re}(\hat{\psi}_j) (\psi_{k+Q} + \psi_{k-Q}), \end{aligned} \quad (41)$$

553 where the hat has been used to identify the OSR components that are approximated as
554 unchanged within the linearization. The overall phase of the atoms is set by $\dot{\Omega}$ and chosen
555 such that $\text{Im}\{\psi_0\} = 0$ within the center manifold [26]. Within the linearization the value
556 is

$$\begin{aligned} \dot{\Omega} &= \frac{\tilde{\eta}}{\sqrt{2}} \sum_j \text{Re}(\phi_j) \frac{\psi_Q + \bar{\psi}_Q + \psi_{-Q} + \bar{\psi}_{-Q}}{2\psi_0} \\ &\quad + U \left(2 - \psi_0^2 + \frac{1}{2} [(\psi_Q + \bar{\psi}_Q)(\psi_{-Q} + \bar{\psi}_{-Q}) + (\psi_Q - \bar{\psi}_Q)(\psi_{-Q} - \bar{\psi}_{-Q})] \right). \end{aligned} \quad (42)$$

557 From Eq. (41) we see that the finite occupation of the cavity field leads to coupling of
558 the k NCM mode with the NCM mode at $k \pm Q$. As the occupation of the NCM fields
559 are small and we consider $\omega_c < E_R$, one can truncate after one recoil kick such that
560 $|k| < Q$. This is confirmed by the full numerical solution of Eqs. (1) and (2) shown in
561 the main text. With this truncation each NCM, ψ_k , couples to the seven other fields
562 $\{\bar{\psi}_k, \psi_{-k}, \bar{\psi}_{-k}, \psi_{k-Q}, \bar{\psi}_{k-Q}, \psi_{-k+Q}, \bar{\psi}_{-k+Q}\}$. For each value of k we therefore find a

563 $J_{\text{OSR}}(t)$ given by

$$J_{\text{OSR}}(t) = i \begin{pmatrix} -m_k & 0 & 0 & -g_k & -m_Q & -g_Q & 0 & 0 \\ 0 & \bar{m}_k & \bar{g}_k & 0 & \bar{g}_Q & \bar{m}_Q & 0 & 0 \\ 0 & -g_k & -m_k & 0 & 0 & 0 & -m_Q & -g_Q \\ \bar{g}_k & 0 & 0 & \bar{m}_k & 0 & 0 & \bar{g}_Q & \bar{m}_Q \\ -m_Q & -g_Q & 0 & 0 & -m_{k-Q} & 0 & 0 & -g_k \\ \bar{g}_Q & \bar{m}_Q & 0 & 0 & 0 & \bar{m}_{k-Q} & \bar{g}_k & 0 \\ 0 & 0 & -m_Q & -g_Q & 0 & -g_k & -m_{k-Q} & 0 \\ 0 & 0 & \bar{g}_Q & \bar{m}_Q & \bar{g}_k & 0 & 0 & \bar{m}_{k-Q} \end{pmatrix}, \quad (43)$$

564 with the vector $\delta \mathbf{v}^T = (\psi_k, \bar{\psi}_k, \psi_{-k}, \bar{\psi}_{-k}, \psi_{k-Q}, \bar{\psi}_{k-Q}, \psi_{-k+Q}, \bar{\psi}_{-k+Q})^T$ and the five
565 different entries being

$$\begin{aligned} m_k &= k^2 + U - 2U \left(\hat{\psi}_Q^2 + \bar{\hat{\psi}}_Q^2 \right) - \frac{\tilde{\eta}}{\sqrt{2}} \sum_j \text{Re}(\hat{\phi}_j) \frac{\hat{\psi}_Q + \bar{\hat{\psi}}_Q}{\hat{\psi}_0}, \\ m_{k-Q} &= m_{k \rightarrow k-Q}, \\ m_Q &= \frac{\tilde{\eta}}{\sqrt{2}} \sum_j \text{Re}(\hat{\phi}_j) + 2U \hat{\psi}_0 \left(\bar{\hat{\psi}}_Q + \hat{\psi}_Q \right), \\ g_k &= U \left(2\hat{\psi}_Q^2 + \hat{\psi}_0^2 \right), \\ g_Q &= 2U \hat{\psi}_Q \hat{\psi}_0. \end{aligned} \quad (44)$$

566 Inserting the OSR solutions into Eq. (44) one finds an analytical expression for $J_{\text{OSR}}(t)$
567 which is periodic such that $J_{\text{OSR}}(t) = J_{\text{OSR}}(t + T)$ with $T = 2\pi/\omega_c$. We then employ
568 standard Floquet theory by numerically time-evolving the eight equations over one period
569 T . This allows us to find the fundamental matrix $\Phi(t)$ which is defined as the solution to

$$\partial_t \Phi(t) = J_{\text{OSR}}(t) \Phi(t), \quad (45)$$

570 with the initial condition $\Phi(0) = 1_8$. The eigenvalues λ_i of the monodromy matrix $M =$
571 $\Phi(T)$ determines the growth rates of the NCM modes $\Gamma_i = \text{Re}(\log(\lambda_i)/T)$. To understand
572 the initial heating effects we only need to investigate the eigenmode with the largest growth
573 rate $\Gamma = \max(\Gamma_i)$. By Computing Γ as a function of k we are able to compute the growth
574 rates of the different channels as plotted in Fig. 3.

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