Josephson current through the SYK model

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Abstract

We calculate the equilibrium Josephson current through a disordered interacting quantum dot described by a Sachdev-Ye-Kitaev model contacted by two BCS superconductors. We show that, at zero temperature and at the conformal limit, i.e. in the strong interacting limit, the Josephson current is strongly suppressed by U, the strength of the interaction, as $\ln(U)/U$ and becomes universal, namely it gets independent on the superconducting pairing. At finite temperature T, instead, it depends on the ratio between the gap and the temperature and vanishes as $1/T^2$ upon increasing the temperature. A proximity effect exists but the self-energy corrections induced by the coupling with the superconducting leads seem subleading as compared to the self-energy due to interaction for large number of particles.

Contents

1 Introduction

The Sachdev-Ye-Kitaev (SYK) model, a non-Fermi liquid describing fermions with infinite-range interactions, has attracted a lot of scientific interest in recent years [1, 2]. Compared to ordinary Fermi liquids, it displays highly unusual properties; for example, its resistivity is linear in temperature [3–5]. Moreover, it has been shown that the SYK model is dual to an anti-de Sitter space in two-dimensions [2, 6–8], opening an alternative way for investigating black holes. Regarding its experimental implementation, several proposals have already been formulated in solid state physics based on quantum dots coupled to topological superconducting wires [9], graphene flakes with irregular boundary [10, 11], by applying optical lattices [12], and in the field of cavity quantum electrodynamics [13].

A lot of theoretical studies have been devoted to study many peculiar properties of the SYK model, either at equilibrium and out-of equilibrium. Different investigations about many aspects of this model have been carried out, ranging from the dynamics triggered by a quantum quench [14], to realizing traversable wormholes [15], or about the Bekenstein-Hawking entropy [16, 17] and the existence of anomalous power laws in the temperature dependent conductance [18, 19]. Several studies have been also conducted investigating the mesoscopic physics by the SYK model, considering a lattice of SYK dots [20], analyzing the thermoelectric transport [21] and the charge transport by coupling with metallic leads [22, 23], characterizing the current and supercurrent driven by double contact setups [24], looking at the dynamics by coupling with Markovian reservoirs [25], and thermal baths, detecting some peculiar thermalization properties [26–28].

Despite this intense scientific activity done on the transport properties of the SYK model, one of the most interesting and yet-little studied topic is about the currents driven by superconducting leads. Non-equilibrium currents triggered by a voltage applied either through normal and superconducting leads have been investigated [24], however a study of the equilibrium Josephson current is still laking. The Josephson effect provides a fundamental signature of phase-coherent transport through mesoscopic sample [29]. We calculate the direct Josephson current obtained by contacting a SYK dot by two conventional Bardeen-Cooper-Schrieffer (BCS) superconductors. We show that a proximity effect is induced in the dot, which causes the tunneling current originated by a phase-difference without applying any voltage, however the self-energy of the dot is weakly affected by the coupling with the superconducting leads in the so-called conformal limit, namely for large interaction and in the limit of large number of particles. We found that, in this limit, the Josephson current is strongly suppressed by U, the strength of the interaction, as $\ln(U)/U$ and becomes universal, namely the current gets independent on the superconducting pairing. This means that the Josephson current, at zero temperature, and in the conformal limit, is the same for all BCS-like superconductors. At finite temperature T, instead, the dependence on the superconducting gap appears again. The current turns out to be dependent on the ratio between the gap and the temperature and vanishes as $1/T^2$ upon increasing the temperature.

2 Model

We study a system composed by a dot, modeled by a complex SYK Hamiltonian H_d, and two-superconducting leads described by H_0. The hybridization of the dot and the superconducting reservoirs takes place by the tunneling term H_T. The full Hamiltonian is, therefore,

$$H = H_0 + H_T + H_d \tag{1}$$

where the first term

$$H_0 = \sum_{p=L,R} \left(\sum_{k,\sigma} (\epsilon_k - \mu_p) c^\dagger_{p\sigma k} c_{p\sigma k} + \sum_k \Delta_p e^{i\phi_p} c^\dagger_{p\uparrow k} c^\dagger_{p\downarrow -k} \right) + h.c. \tag{2}$$

describes the two BCS-Hamiltonians, contacted to the left side ($p = L$) and to the right side ($p = R$) of the dot, $c_{L\sigma k}$, $c_{R\sigma k}$ the annihilation and $c^\dagger_{L\sigma k}$, $c^\dagger_{R\sigma k}$ creation fermionic operators, ϵ_k is

the single particle spectrum, μ_L and μ_R the chemical potentials, Δ_L and Δ_R the gaps with phases ϕ_L and ϕ_R, respectively.
The second term is the tunneling Hamiltonian

$$H_T = \frac{1}{\sqrt{N}} \sum_{p=L,R} \sum_{k,\sigma,n} t_{pn}\, c_{p\sigma k} d_n^\dagger + h.c. \tag{3}$$

where t_{Ln} and t_{Rn} are the probability amplitudes for a fermion to jump into or out of the dot. The fermionic operators d_n and $d_n^\dagger$ are defined in the dot. The last term of Eq. (1) is the following complex SYK Hamiltonian of the dot

$$H_d = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,l=1}^{N} U_{ijkl}\, d_i^\dagger d_j^\dagger d_k d_l - \mu \sum_i d_i^\dagger d_i \tag{4}$$

where N fermions have a disordered all-to-all four-body interaction U_{ijkl}, Gaussian distributed.

2.1 Tunneling term

Let us first consider the tunneling term and the lead We introduce the following Nambu-Jona-Lasinio spinors

$$\Psi_{pk} = \begin{pmatrix} c_{p\uparrow k} \\ c^\dagger_{p\downarrow -k} \end{pmatrix}, \qquad \bar{\Psi}_{pk} = \begin{pmatrix} c^\dagger_{p\uparrow k} & c_{p\downarrow -k} \end{pmatrix} \tag{5}$$

for the fermions in of the leads, and

$$D_n = \begin{pmatrix} d_n \\ d_n^\dagger \end{pmatrix}, \qquad \bar{D}_n = \begin{pmatrix} d_n^\dagger & d_n \end{pmatrix} \tag{6}$$

for the fermions of the dot. The Hamiltonian of the leads, H_0, in this representation, becomes

$$H_0 = \sum_{p,k} \bar{\Psi}_{pk} \left[(\epsilon_k - \mu_p)\tau_3 + \Delta_p \cos(\phi_p)\tau_1 - \Delta_p \sin(\phi_p)\tau_2 \right] \Psi_{pk} \tag{7}$$

and the tunneling Hamiltonian H_T reads

$$H_T = \frac{1}{\sqrt{N}} \sum_{p,k,n} \left\{ \mathrm{Re}(t_{pn})(\bar{\Psi}_{pk}\tau_3 D_n + \bar{D}_n \tau_3 \Psi_{pk}) - i\, \mathrm{Im}(t_{pn})(\bar{\Psi}_{pk} D_n - \bar{D}_n \Psi_{pk}) \right\} \tag{8}$$

where τ_1, τ_2, τ_3 are Pauli matrices. Let us take t_{pn} real, and $\mu_R = \mu_L = \mu$, i.e. at equilibrium. Defining

$$G_{kp}^{-1} = i\omega + \xi_k \tau_3 + \Delta_p \cos(\phi_p)\tau_1 - \Delta_p \sin(\phi_p)\tau_2 \tag{9}$$

where $\xi_k = \epsilon_k - \mu$, and integrating over Ψ,

$$e^{-S_c} = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp\left\{ -\sum_{p,k,\omega} \left[\bar{\Psi}_{pk} G_{pk}^{-1} \Psi_{pk} + \frac{1}{\sqrt{N}} \sum_n t_{pn} (\bar{\Psi}_{pk}\tau_3 D_n + \bar{D}_n \tau_3 \Psi_{pk}) \right] \right\} \tag{10}$$

we get the contribution to the action of the dot due to the coupling with the leads

$$S_c = \sum_{n,m,\omega} \bar{D}_n(\omega) \mathcal{T}_{nm}(\omega) D_m(\omega) \tag{11}$$

where

$$\mathcal{T}_{nm}(\omega) = \frac{1}{N} \sum_{kp} t_{pn} t_{pm} \frac{-i\omega + \xi_k \tau_3 - \Delta_p \cos(\phi_p)\tau_1 + \Delta_p \sin(\phi_p)\tau_2}{\xi_k^2 + \Delta_p^2 + \omega^2} \tag{12}$$

we can integrate over ξ_k, and introducing ν_0 the density of states at the Fermi energy, equal for both sides, we get

$$\mathcal{T}_{nm}(\omega) = -\frac{1}{N}\sum_p \frac{\pi\nu_0 t_{pn} t_{pm}}{\sqrt{\omega^2+\Delta^2}}\left(i\omega + \Delta_p\cos(\phi_p)\tau_1 - \Delta_p\sin(\phi_p)\tau_2\right) \tag{13}$$

Defining, in the symmetric case, $\Gamma_{p,nm} = \pi\nu_0 t_{pn}t_{pm} = \Gamma_{nm}/2$, $\phi_L = -\phi_R = \phi/2$ and $\Delta = \Delta_L = \Delta_R$, summing over $p = R, L$, namely summing the right and left terms, we get

$$\mathcal{T}_{nm}(\omega) = i\omega\frac{1}{N}\frac{\Gamma_{nm}}{\sqrt{\omega^2+\Delta^2}} + \frac{1}{N}\frac{\Gamma_{nm}\Delta\cos(\phi/2)}{\sqrt{\omega^2+\Delta^2}}\tau_1 \tag{14}$$

Let us now make the reasonable assumption that $t_{pn} = t_{pm}$ for any n and m, then $\Gamma_{nm} \equiv \Gamma J_{nm}$ where J is a $N\times N$ unit matrix, a matrix consisting of all 1s

$$J = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

We can write, therefore,

$$\mathcal{T}(\omega) = i\omega\frac{1}{N}\frac{\Gamma}{\sqrt{\omega^2+\Delta^2}}\tau_0 + \frac{1}{N}\frac{\Gamma\Delta\cos(\phi/2)}{\sqrt{\omega^2+\Delta^2}}\tau_1 \tag{15}$$

such that $\mathcal{T}_{nm}(\omega) = \mathcal{T}(\omega)J_{nm}$.

3 SYK Dot

The Hamiltonian of the dot is given by Eq. (4), where U_{ijkl} are complex, independent Gaussian random couplings with zero mean obeying $U_{ijkl} = -U_{jikl} = U_{ijlk}$, $U_{ijkl} = U^*_{klij}$ and mean value $\overline{U^2_{ijkl}} = U^2$. Introducing n replicas, $a = 1,\dots,n$, labeling the field as d_{na}, we can average over disorder so that the action of the uncoupled dot can be written as follows

$$S_d = \sum_{n,a}\int_0^\beta d\tau\, d^\dagger_{na}(\tau)\left(\partial_\tau - \mu\right)d_{na}(\tau) - \frac{U^2}{4N^3}\sum_{a,b}\int_0^\beta d\tau d\tau'\left(\sum_n d^\dagger_{na}(\tau)d_{nb}(\tau')\right)^4 + S_c \tag{16}$$

3.1 Effective action

We can decouple the interaction, in different channels, introducing a number of auxiliary fields, getting

$$\begin{aligned} S_d = &\sum_{na}\int_0^\beta d\tau\, d^\dagger_{na}(\tau)\left(\partial_\tau-\mu\right)d_{na}(\tau) + \sum_{ab}\int_0^\beta d\tau d\tau'\Big[\frac{N}{4c_0U^2}[Q^0_{ab}(\tau,\tau')]^2 + \frac{N^3}{4c_1U^2}|Q^P_{ab}(\tau,\tau')|^2 \\ &+\frac{N^3}{4c_2U^2}[Q^\Delta_{ab}(\tau,\tau')]^2 + \frac{N}{2}Q^0_{ab}(\tau,\tau')|P^{ab}_0(\tau,\tau')|^2 - Q^0_{ab}(\tau,\tau')P^{ab}_0(\tau,\tau')\sum_n d^\dagger_{na}(\tau)d_{nb}(\tau') \\ &+\frac{1}{4}Q^P_{ab}(\tau,\tau')\sum_{nm}P^{ab}_{nm}(\tau,\tau')P^{ab}_{mn}(\tau,\tau') - \frac{1}{2}Q^P_{ab}(\tau,\tau')\sum_{nm}d^\dagger_{na}(\tau)P^{ab}_{nm}(\tau,\tau')d_{mb}(\tau') \\ &-\frac{1}{2}Q^P_{ab}(\tau,\tau')\sum_{nm}d^\dagger_{ma}(\tau)P^{ab}_{mn}(\tau,\tau')d_{nb}(\tau') + \frac{1}{2}Q^\Delta_{ab}(\tau,\tau')\sum_{nm}|\Delta^{ab}_{nm}(\tau,\tau')|^2 \\ &-\frac{1}{2}Q^\Delta_{ab}(\tau,\tau')\sum_{nm}d^\dagger_{na}(\tau)\Delta^{ab}_{nm}(\tau,\tau')d^\dagger_{mb}(\tau') - \frac{1}{2}Q^\Delta_{ab}(\tau,\tau')\sum_{nm}d_{ma}(\tau)\Delta^{ab*}_{nm}(\tau,\tau')d_{nb}(\tau')\Big] + S_c \end{aligned} \tag{17}$$

where the weights c_0, c_1, c_2 are arbitrary positive real numbers such that $c_0 + c_1 + c_2 = 1$. The auxiliary fields are such that $Q^{\Delta}_{ab}(\tau,\tau')$ is real, $\Delta^{ab}_{nm}(\tau,\tau')$ is complex and $\Delta^{ab}_{nm}(\tau,\tau') = \Delta^{ab}_{mn}(\tau,\tau')$, while $Q^{P}_{ab}(\tau,\tau')$ is complex and $Q^{P}_{ab}(\tau,\tau') = Q^{P*}_{ba}(\tau',\tau)$ and $P^{ab}_{nm}(\tau,\tau') = P^{ab}_{mn}(\tau,\tau')$ can be taken real (it can be complex but only the real part matters), while $Q^{0}_{ab}(\tau,\tau') = Q^{0}_{ba}(\tau',\tau)$ is real and $P^{ab}_{0}(\tau,\tau') = P^{ba*}_{0}(\tau',\tau)$ complex. Using the representation in Eq. (6), with replica indices, namely $\bar{D}^a_n = \begin{pmatrix} d^\dagger_{na} & d_{na} \end{pmatrix}$ and $D^b_m = \begin{pmatrix} d_{mb} \\ d^\dagger_{mb} \end{pmatrix}$, we can write

$$
\begin{aligned}
S_d = {} & \frac{1}{2}\sum_{nmab}\int_0^\beta d\tau d\tau' \Big\{ \bar{D}^a_n(\tau)\Big[\delta_{\tau\tau'}\delta_{ab}\delta_{nm}\left(\tau_0\partial_\tau - \tau_3\mu\right) - \frac{1}{2}Q^0_{ab}(\tau,\tau')\delta_{nm}\left(P^{ab}_0(\tau,\tau')(\tau_3+\tau_0)\right. \\
& \left. + P^{ab*}_0(\tau,\tau')(\tau_3-\tau_0)\right) - \frac{1}{2}\left(Q^P_{ab}(\tau,\tau')P^{ab}_{nm}(\tau,\tau')(\tau_3+\tau_0) + Q^P_{ba}(\tau',\tau)P^{ba}_{nm}(\tau',\tau)(\tau_3-\tau_0)\right) \\
& -\frac{1}{2}Q^\Delta_{ab}(\tau,\tau')\left(\Delta^{ab}_{nm}(\tau,\tau')(\tau_1+i\tau_2) + \Delta^{ab*}_{nm}(\tau,\tau')(\tau_1-i\tau_2)\right)\Big] D^b_m(\tau')\Big\} \\
& + \sum_{ab}\int_0^\beta d\tau d\tau' \Big[\frac{N}{4U^2}\Big(\frac{1}{c_0}[Q^0_{ab}(\tau,\tau')]^2 + \frac{N^2}{c_1}|Q^P_{ab}(\tau,\tau')|^2 + \frac{N^2}{c_2}[Q^\Delta_{ab}(\tau,\tau')]^2\Big) \\
& + \frac{N}{2}Q^0_{ab}(\tau,\tau')|P^{ab}_0(\tau,\tau')|^2 + \frac{1}{4}Q^P_{ab}(\tau,\tau')\sum_{nm}(P^{ab}_{nm}(\tau,\tau'))^2 + \frac{1}{2}Q^\Delta_{ab}(\tau,\tau')\sum_{nm}|\Delta^{ab}_{nm}(\tau,\tau')|^2\Big] + S_c
\end{aligned} \tag{18}
$$

Let us now calculate the main contributions to the partition function, deriving the saddle point equations.

3.2 Saddle point equations

Imposing $\delta S_d = 0$ under varying the auxiliary fields we derive the following saddle point equations

$$P^{ab}_0(\tau,\tau') = -\frac{1}{2N}\sum_n \mathrm{Tr}\left(\langle D^b_n(\tau')\bar{D}^a_n(\tau)\rangle(\tau_3-\tau_0)\right) \tag{19}$$

$$P^{ba}_0(\tau',\tau) = -\frac{1}{2N}\sum_n \mathrm{Tr}\left(\langle D^b_n(\tau')\bar{D}^a_n(\tau)\rangle(\tau_3+\tau_0)\right) \tag{20}$$

$$P^{ab}_{nm}(\tau,\tau') = -\frac{1}{2}\mathrm{Tr}\left(\langle D^b_m(\tau')\bar{D}^a_n(\tau)\rangle(\tau_3+\tau_0)\right) \tag{21}$$

$$P^{ba}_{mn}(\tau',\tau) = -\frac{1}{2}\mathrm{Tr}\left(\langle D^b_m(\tau')\bar{D}^a_n(\tau)\rangle(\tau_3-\tau_0)\right) \tag{22}$$

$$\Delta^{ab}_{nm}(\tau,\tau') = -\frac{1}{2}\mathrm{Tr}\left(\langle D^b_m(\tau')\bar{D}^a_n(\tau)\rangle(\tau_1-i\tau_2)\right) \tag{23}$$

$$\Delta^{ab*}_{nm}(\tau,\tau') = -\frac{1}{2}\mathrm{Tr}\left(\langle D^b_m(\tau')\bar{D}^a_n(\tau)\rangle(\tau_1+i\tau_2)\right) \tag{24}$$

$$
\begin{aligned}
Q^0_{ab}(\tau,\tau') = {} & -c_0U^2\Big\{|P^{ba}_0(\tau',\tau)|^2 + \frac{1}{2N}\sum_n \mathrm{Tr}\Big[\langle D^b_n(\tau')\bar{D}^a_n(\tau)\rangle\Big((\tau_3+\tau_0)P^{ab}_0(\tau,\tau') \\
& + (\tau_3-\tau_0)P^{ab*}_0(\tau,\tau')\Big)\Big]\Big\} = c_0U^2|P^{ba}_0(\tau',\tau)|^2
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
Q^P_{ab}(\tau,\tau') &= -\frac{c_1U^2}{N^3}\Big\{\sum_{nm}(P^{ba}_{nm}(\tau',\tau))^2 + \sum_{nm}\mathrm{Tr}\left(\langle D^b_m(\tau')\bar{D}^a_n(\tau)\rangle(\tau_3-\tau_0)\right)P^{ba}_{nm}(\tau',\tau)\Big\} \\
&= \frac{c_1U^2}{N^3}\sum_{nm}(P^{ba}_{nm}(\tau',\tau))^2
\end{aligned} \tag{26}
$$

$$Q^P_{ba}(\tau',\tau) = -\frac{c_1U^2}{N^3}\Big\{\sum_{nm}(P^{ab}_{nm}(\tau,\tau'))^2 + \sum_{nm}\mathrm{Tr}\left(\langle D^b_m(\tau')\bar{D}^a_n(\tau)\rangle(\tau_3+\tau_0)\right)P^{ab}_{nm}(\tau,\tau')\Big\}$$
$$= \frac{c_1U^2}{N^3}\sum_{nm}(P^{ab}_{nm}(\tau,\tau'))^2 \tag{27}$$

$$Q^\Delta_{ab}(\tau,\tau') = -\frac{c_2U^2}{N^3}\Big\{\sum_{nm}|\Delta^{ba}_{nm}(\tau',\tau)|^2 + \frac{1}{2}\sum_{nm}\mathrm{Tr}\Big[\langle D^b_m(\tau')\bar{D}^a_n(\tau)\rangle\Big((\tau_1+i\tau_2)\Delta^{ab}_{nm}(\tau,\tau')$$
$$+(\tau_1-i\tau_2)\Delta^{ab*}_{nm}(\tau,\tau')\Big)\Big]\Big\} = \frac{c_2U^2}{N^3}\sum_{nm}|\Delta^{ba}_{nm}(\tau',\tau)|^2 \tag{28}$$

We restrict our attention to replica diagonal solutions, $P^{ab}_0(\tau,\tau') = \delta_{ab}G_0(\tau,\tau')$, $P^{ab}_{nm}(\tau,\tau') = \delta_{ab}G_{nm}(\tau,\tau')$ and $\Delta^{ab}_{nm}(\tau,\tau') = \delta_{ab}F_{nm}(\tau,\tau') = \delta_{ab}F^*_{nm}(\tau'-\tau)$. We define

$$\mathbf{G}_{nm}(\tau,\tau') = \begin{pmatrix} G_0(\tau,\tau')\delta_{nm} + G_{nm}(\tau,\tau') & F^*_{nm}(\tau,\tau') \\ F_{nm}(\tau,\tau') & -G_0(\tau',\tau)\delta_{nm} - G_{mn}(\tau',\tau)\end{pmatrix} \tag{29}$$

and the self-energies

$$\Sigma(\tau,\tau') = -\begin{pmatrix} Q^0(\tau,\tau')G_0(\tau,\tau') & 0 \\ 0 & -Q^0(\tau',\tau)G_0(\tau',\tau)\end{pmatrix} \tag{30}$$

$$L_{nm}(\tau,\tau') = -\begin{pmatrix} Q^P(\tau,\tau')G_{nm}(\tau,\tau') & 0 \\ 0 & -Q^P(\tau',\tau)G_{mn}(\tau',\tau)\end{pmatrix} \tag{31}$$

$$A_{nm}(\tau,\tau') = -\begin{pmatrix} 0 & Q^\Delta(\tau,\tau')F_{nm}(\tau,\tau') \\ Q^\Delta(\tau,\tau')F^*_{nm}(\tau,\tau') & 0\end{pmatrix} \tag{32}$$

We can define

$$\mathcal{G}_0(\tau,\tau') = \begin{pmatrix} G_0(\tau,\tau') & 0 \\ 0 & -G_0(\tau',\tau)\end{pmatrix}, \tag{33}$$
$$\mathcal{G}_{nm}(\tau,\tau') = \begin{pmatrix} G_{nm}(\tau,\tau') & 0 \\ 0 & -G_{mn}(\tau',\tau)\end{pmatrix}, \tag{34}$$
$$\mathcal{F}_{nm}(\tau,\tau') = \begin{pmatrix} 0 & F^*_{nm}(\tau,\tau') \\ F_{nm}(\tau,\tau') & 0\end{pmatrix} \tag{35}$$

At the saddle point, from Eqs. (19)-(24), we have

$$\mathbf{G}_{nm}(\tau,\tau') = \mathcal{G}_0(\tau,\tau')\delta_{nm} + \mathcal{G}_{nm}(\tau,\tau') + \mathcal{F}_{nm}(\tau,\tau') = -\langle D_m(\tau')\bar{D}_n(\tau)\rangle \tag{36}$$

which depends on the time difference $\bar{\tau} = \tau' - \tau \in [-\beta,\beta]$, namely $\mathbf{G}_{nm}(\tau,\tau') = \mathbf{G}_{nm}(\bar{\tau})$. In Fourier space the full matrix $\hat{\mathbf{G}}(\bar{\tau})$ in spinorial and in the multimodal spaces, including the tunneling contribution $\hat{\mathcal{T}}(\omega) = \mathcal{T}(\omega)J$, reads

$$\hat{\mathbf{G}}(\omega) = \left[\left(i\omega\tau_0 + \mu\tau_3 - \Sigma(\omega)\right)\hat{\mathbb{I}} + \hat{\mathcal{T}}(\omega) - \left(\hat{L}(\omega) + \hat{A}(\omega)\right)\right]^{-1} \tag{37}$$

where, from Eqs. (25)-(28), the self-energies $\Sigma(\omega)$, $\hat{L}(\omega)$ and $\hat{A}(\omega)$ are the Fourier transforms of

$$\Sigma(\bar{\tau}) = -c_0U^2\mathcal{G}_0(\bar{\tau})^2\mathcal{G}_0(-\bar{\tau}) = -c_0U^2\begin{pmatrix} G_0(\bar{\tau})^2\,G_0(-\bar{\tau}) & 0 \\ 0 & -G_0(-\bar{\tau})^2\,G_0(\bar{\tau})\end{pmatrix} \tag{38}$$

and of $\hat{L}(\bar{\tau})$ and $\hat{A}(\bar{\tau})$, whose elements are

$$L_{nm}(\bar{\tau}) = -\frac{c_1 U^2}{N^3}\sum_{kl}\mathcal{G}_{kl}(\bar{\tau})^2\mathcal{G}_{nm}(-\bar{\tau}) = -\frac{c_1 U^2}{N^3}\sum_{kl}\begin{pmatrix} G_{kl}(\bar{\tau})^2\, G_{nm}(-\bar{\tau}) & 0 \\ 0 & -G_{kl}(-\bar{\tau})^2\, G_{mn}(\bar{\tau})\end{pmatrix} \tag{39}$$

$$A_{nm}(\bar{\tau}) = -\frac{c_2 U^2}{N^3}\sum_{kl}\mathcal{F}_{kl}(\bar{\tau})^2\mathcal{F}_{nm}(\bar{\tau}) = -\frac{c_2 U^2}{N^3}\sum_{kl}\begin{pmatrix} 0 & |F_{kl}(\bar{\tau})|^2 F_{nm}(\bar{\tau}) \\ |F_{kl}(\bar{\tau})|^2 F^*_{nm}(\bar{\tau}) & 0\end{pmatrix} \tag{40}$$

One has to solve self-consistently Eqs. (37)-(40), fixing then c_0, c_1, c_2, with constraint $c_0 + c_1 + c_2 = 1$, by minimizing the action at the saddle point. However what we found is that, if G_{nm} and $F_{mn} \sim 1/N^\delta$ with $\delta > 0$, the self-energies $\hat{L}$ and $\hat{A}$ can be neglected in the large N limit. As we will see in the last section, this seems to be the case.

4 Josephson current

As shown in the previous section, the self energies induced by he coupling can be neglected in the large N limit. In such approximation the Green's function of the dot can be written as

$$\mathcal{G}^{-1}_{nm} = \mathcal{G}_0^{-1}\delta_{nm} + \mathcal{T} J_{nm} \tag{41}$$

where $\mathcal{G}_0$ is the Green's function of the uncoupled dot, solution of the equations

$$\mathcal{G}_0^{-1}(\omega) = i\omega\tau_0 + \mu\tau_3 - \Sigma(\omega) \tag{42}$$

$$\Sigma(\tau) = -U^2\mathcal{G}_0(\tau)^2\mathcal{G}_0(-\tau) \tag{43}$$

Let us write the self-energy in the following form

$$\Sigma(\omega) = \Sigma_0(\omega)\tau_0 + \Sigma_3(\omega)\tau_3 \tag{44}$$

so that we can write

$$\mathcal{G}_0^{-1}(\omega) = \tilde{G}_0^{-1}(\omega)\tau_0 + \tilde{G}_3^{-1}(\omega)\tau_3 \equiv \Big(i\omega - \Sigma_0(\omega)\Big)\tau_0 + \Big(\mu - \Sigma_3(\omega)\Big)\tau_3 \tag{45}$$

Actually, from Eq. (33), defining

$$G_0(\omega) = \frac{1}{2}\int d\tau\, e^{i\omega\tau}\Big(G_0(\tau) - G_0(-\tau)\Big), \qquad G_3(\omega) = \frac{1}{2}\int d\tau\, e^{i\omega\tau}\Big(G_0(\tau) + G_0(-\tau)\Big) \tag{46}$$

we have that

$$\tilde{G}_0^{-1}(\omega) = \frac{G_0(\omega)}{G_0(\omega)^2 - G_3(\omega)^2}, \qquad \tilde{G}_3^{-1}(\omega) = \frac{G_3(\omega)}{G_3(\omega)^2 - G_0(\omega)^2} \tag{47}$$

The Josephson current can be obtain from the phase derivative of the free energy

$$I = -\frac{1}{\beta}\,\partial_\phi \sum_\omega \ln\Big(\det[\mathcal{G}^{-1}(\omega)]\Big) \tag{48}$$

where $\beta = 1/T$ is the inverse of the temperature and the determinant of $\mathcal{G}^{-1}(\omega)$, from Eq. (41), is given by

$$\det[\mathcal{G}^{-1}] = \Big(\det[\mathcal{G}_0^{-1}]\Big)^N\left(1 + N\,\mathrm{Tr}[\mathcal{T}\mathcal{G}_0] + N^2\frac{\det[\mathcal{T}]}{\det[\mathcal{G}_0^{-1}]}\right) \tag{49}$$

from which, using $\det[\mathcal{G}_0^{-1}] = (\tilde{G}_0^{-1})^2 - (\tilde{G}_3^{-1})^2$, we get the following expression

$$\det[\mathcal{G}^{-1}] = \left(\det[\mathcal{G}_0^{-1}]\right)^{N-1}\left[\left(\tilde{G}_0^{-1}(\omega) + i\omega\frac{\Gamma}{\sqrt{\omega^2+\Delta^2}}\right)^2 - \frac{\Gamma^2\Delta^2\cos^2(\phi/2)}{\omega^2+\Delta^2} - \left(\tilde{G}_3^{-1}(\omega)\right)^2\right] \tag{50}$$

From Eq. (48) we finally obtain the Josephson current

$$I = \frac{\Gamma^2\Delta^2}{\beta}\sin(\phi)\sum_\omega \frac{1}{\Gamma^2\Delta^2\cos^2(\phi/2) - \left(\tilde{G}_0^{-1}(\omega)\sqrt{\omega^2+\Delta^2} + i\omega\Gamma\right)^2 + \left(\tilde{G}_3^{-1}(\omega)\sqrt{\omega^2+\Delta^2}\right)^2} \tag{51}$$

By numerically solving of Eqs. (42), (43), using Eqs. (44), (45), one gets the Josephson current for the SYK dot from Eq. (51).

4.1 Large interaction limit

In the so-called conformal limit, namely for very large U, i.e. for $|\omega| \ll U$, the analytical solution of Eqs. (42) and (43), obtained for $\Sigma_3(0) = \mu$, implying $G_3(0) = 0$ and $\tilde{G}_0 = G_0$, and for $T \to 0$, is given by [1,5,16]

$$G_0^{-1}(\omega) = iC\,\mathrm{sgn}(\omega)|\omega|^{1/2} \tag{52}$$

with $C \sim U^{1/2}$. The Josephson current, Eq. (51), for $T \to 0$, in the continuum, becomes

$$I = \frac{\Gamma^2\Delta^2}{\pi}\sin(\phi)\int_0^\infty \frac{d\omega}{\Gamma^2\Delta^2\cos^2(\phi/2) + \left(C\sqrt{\omega(\omega^2+\Delta^2)} + \omega\Gamma\right)^2} \tag{53}$$

This equation can be well approximated by

$$I \simeq \frac{\Gamma^2}{\pi}\sin(\phi)\int_0^\Delta \frac{d\omega}{\Gamma^2\cos^2(\phi/2) + C^2\omega} \tag{54}$$

getting the following analytical result

$$I \simeq \frac{\Gamma^2}{\pi C^2}\sin(\phi)\ln\left(1 + \frac{C^2\Delta}{\Gamma^2\cos^2(\phi/2)}\right) \tag{55}$$

For $U\Delta \gg \Gamma^2$, the current I drops the dependence on Δ, except for logarithmic corrections, so that

$$I \sim \frac{\Gamma^2}{U}\sin(\phi)\ln\left(\frac{U}{\Gamma\cos^2(\phi/2)}\right) \tag{56}$$

namely, we get a universal behavior since it is valid for all BCS-like superconductors.

4.2 Finite temperature

At finite temperature, the analytical solution Eq. (52) becomes [1,5,16]

$$G_0^{-1}(\omega) \sim iC\sqrt{2\pi T}\,e^{i\theta}\frac{\Gamma(3/4 + \omega/2\pi T + i\epsilon)}{\Gamma(1/4 + \omega/2\pi T + i\epsilon)} \tag{57}$$

where $\Gamma(x)$ is the Gamma function, $C = (U^2\cos(2\theta)/\pi)^{1/4}$, while θ and ϵ are linked by $e^{2\pi\epsilon} = \sin(\pi/4+\theta)/\sin(\pi/4-\theta)$, and $G_0(\tau = 0^-) = 1/2 - \theta/\pi - \sin(2\theta)/4$. Let us fix the density of particles at half-filling, $\theta = 0$, $\epsilon = 0$. Defining

$$g_\omega = i\Gamma G_0(\omega) \tag{58}$$

from Eq. (51), in the case $UT \gg \Gamma^2$, the Josephson current becomes

$$I \simeq \frac{\Delta^2}{\beta}\sin(\phi)\sum_\omega \frac{g_\omega^2}{\omega^2 + g_\omega^2\Delta^2\cos^2(\phi/2)+\Delta^2} \tag{59}$$

The Green's function $G_0(\omega)$ is cut-offed by $1/\sqrt{T}$ at low frequency. We approximate, therefore, $g_\omega \approx g_0$ in Eq. (59) and, after summing over the Matsubara frequencies, we get

$$I \simeq \frac{\Delta}{2\alpha}\sin(\phi)\, g_0^2\, \frac{\tanh\left(\frac{\beta}{2}\Delta\sqrt{1+g_0^2\cos^2(\phi/2)}\right)}{\sqrt{1+g_0^2\cos^2(\phi/2)}} \tag{60}$$

which is a function of the temperature $T = 1/\beta$, and of the interaction U since $g_0 = r\Gamma/\sqrt{UT}$, with r a numerical coefficient, $r = \Gamma(1/4)/(\sqrt{2}\pi^{1/4}\Gamma(3/4))$. We find numerically that Eq. (59) is better approximated by the same expression where g_ω is replaced by g_0 if we include an overall factor $\alpha \approx 5.6$. Since $g_0^2 \ll 1$, calling $c = r^2/(2\alpha)$ the numerical coefficient, we have

$$I \simeq c\frac{\Gamma^2\Delta}{UT}\sin(\phi)\tanh\left(\frac{\Delta}{2T}\right) \tag{61}$$

therefore, for large temperature, $T \gg \Delta$, it vanishes as $1/T^2$,

$$I \simeq \frac{c}{2}\frac{\Gamma^2\Delta^2}{U\,T^2}\sin(\phi) \tag{62}$$

On the contrary, in the intermediate regime with small enough temperatures, specifically for $\Delta \gg T \gg \Gamma^2/U$, we can approximate the hyperbolic tangent by one, getting a $1/T$ decay

$$I \simeq \frac{\Delta}{2\alpha}\sin(\phi)\frac{g_0^2}{\sqrt{1+g_0^2\cos^2(\phi/2)}} \simeq c\frac{\Gamma^2\Delta}{U\,T}\sin(\phi) \tag{63}$$

For $UT \ll \Gamma^2$ ($g_0 \gg 1$), instead, we have to distinguish two regions in frequency space, with $|\omega| < \Lambda_T$ and $|\omega| > \Lambda_T$, where $\Lambda_T \sim T$ is an energy cut-off below which $g_\omega \sim g_0$ while above $g_\omega \sim C^{-1}\mathrm{sgn}(\omega)|\omega|^{-1/2}$, as for the zero temperature limit. We have, therefore, the following expression

$$I \simeq \frac{1}{\beta}\sin(\phi)\left\{\sum_{|\omega|<\Lambda_T}\frac{\Delta^2}{\omega^2+\Delta^2\cos^2(\phi/2)+g_0^{-2}\Delta^2} + \sum_{\Delta>|\omega|>\Lambda_T}\frac{\Gamma^2}{\Gamma^2\cos^2(\phi/2)+g_\omega^{-2}}\right\} \tag{64}$$

Since $T \ll 1$ we can use the integrals, $\frac{1}{\beta}\sum_\omega \to \int\frac{d\omega}{2\pi}$, getting

$$I \simeq \frac{\Delta}{\pi}\sin(\phi)\, g_0 \frac{\arctan\left(\frac{g_0\Lambda_T}{\sqrt{1+g_0^2\cos^2(\phi/2)}}\right)}{\sqrt{1+g_0^2\cos^2(\phi/2)}} + \frac{\Gamma^2}{\pi C^2}\sin(\phi)\ln\left(\frac{\Gamma^2\cos^2(\phi/2)+C^2\Delta}{\Gamma^2\cos^2(\phi/2)+C^2\Lambda_T}\right) \tag{65}$$

4.3 Zero interaction limit

For $U = 0$ we have $\Sigma = 0$, therefore $\tilde{G}_0^{-1} = i\omega$ and $\tilde{G}_3^{-1} = \mu$, therefore the Josephson current, Eq. (51), becomes the same as that for a single-level dot

$$I = \frac{\Gamma^2\Delta^2}{\beta}\sin(\phi)\sum_\omega\frac{1}{\Gamma^2\Delta^2\cos^2(\phi/2)+\left(\omega\sqrt{\omega^2+\Delta^2}+\omega\Gamma\right)^2+\mu^2\left(\omega^2+\Delta^2\right)} \tag{66}$$

which for $\Gamma \gg \Delta$ has an analytical form

$$I = \frac{\Delta}{2}\sin(\phi)\frac{\mathrm{t}_o \tanh\left(\frac{\beta}{2}\Delta\sqrt{1-\mathrm{t}_o\sin^2(\phi/2)}\right)}{\sqrt{1-\mathrm{t}_o\sin^2(\phi/2)}} \tag{67}$$

which, for $T \to 0$, becomes simply

$$I = \frac{\Delta}{2}\sin(\phi)\frac{\mathrm{t}_o}{\sqrt{1-\mathrm{t}_o\sin^2(\phi/2)}} \tag{68}$$

where t_o is the transmission coefficient, $0 \le \mathrm{t}_o \le 1$,

$$\mathrm{t}_o = \frac{\Gamma^2}{\Gamma^2+\mu^2} \tag{69}$$

For large temperature, $T \gg \Delta$, the current in Eq. (67) becomes

$$I \simeq \frac{\Delta^2\mathrm{t}_o}{4T}\sin(\phi) \tag{70}$$

namely, it decays as $1/T$. This result has to be contrasted with Eq. (62) obtained for large interaction.

4.4 Proximity effect

Let us discuss, now, how the dot is affected by the presence of the superconducting leads and check whether we can neglect the self-energy corrections in the large N limit. We will focus in particular on the hybridization of the dot due to the superconducting pairing, considering the following tunneling matrix, neglecting, for simplicity, the term proportional to τ_0,

$$\hat{\mathcal{T}}(\omega) \simeq \mathcal{T}_1(\omega)\,\tau_1\,J \equiv \frac{\Gamma\Delta\cos(\phi/2)}{N\sqrt{\omega^2+\Delta^2}}\,\tau_1\,J \tag{71}$$

We make the following ansatz for the anomalous contribution to the self-energy: $A\,\tau_1 J$. The Green's function, then, reads

$$\mathcal{G}^{-1}_{nm} \simeq \mathcal{G}_0^{-1}\delta_{nm} + (\mathcal{T}_1(\omega)-A)\,\tau_1\,J_{nm} \tag{72}$$

From Eq. (23), we have the following effective equal-time pairing between two generic modes $n \neq m$

$$F \equiv F_{nm}(\tau,\tau) = \frac{1}{\beta}\sum_\omega \mathrm{Tr}\Big(\mathcal{G}(\omega)\tau_1\Big)_{nm} \tag{73}$$

and, therefore, from Eq. (40), we will have, consistently,

$$A = -\frac{U^2F^3}{N} \tag{74}$$

At low temperature the sum in Eq. (73) becomes an integral, which reads

$$F = \int_{-\Lambda}^{\Lambda}\frac{d\omega}{2\pi}\,\frac{\mathcal{T}_1(\omega)-A}{N^2(\mathcal{T}_1(\omega)-A)^2-G_0^{-2}} = \frac{1}{N}\int_{-\Lambda}^{\Lambda}\frac{d\omega}{2\pi}\,\frac{\Gamma\Delta\cos(\phi/2)\sqrt{\omega^2+\Delta^2}+U^2F^3(\omega^2+\Delta^2)}{\left(\Gamma\Delta\cos(\phi/2)+U^2F^3\sqrt{\omega^2+\Delta^2}\right)^2+C^2|\omega|(\omega^2+\Delta^2)} \tag{75}$$

where we introduced a cut-off since $\omega \ll U$ for the expression of G_0 to be valid, therefore we can take $\Lambda \sim U$.
For large U and for large but still finite N such that $U^2F^3 \gg \Gamma$, we can approximate Eq. (75) getting

$$F \simeq \frac{1}{N}\int_0^{\Lambda}\frac{d\omega}{\pi}\frac{U^2F^3}{(U^4F^6+C^2\omega)} = \frac{U^2F^3}{\pi NC^2}\ln\left(1+\frac{C^2\Lambda}{U^4F^6}\right) \tag{76}$$

which has to be solved in terms of F. For $U^4F^6 \gg C^2\Lambda \sim U^2$, we get, for F and A, the following results

$$F \approx \left(\frac{\Lambda}{N\pi U^2}\right)^{1/4}, \qquad A \approx -\frac{U^{2/3}}{N}\left(\frac{\Lambda}{N\pi}\right)^{3/4} \tag{77}$$

We found that the pairing is super-extensive, meaning that a single particle in the dot is paired with all the other particles in such a way that NF is not $O(1)$ but $O(N^{3/4})$.
We expect that U^2F^3 becomes irrelevant upon further increasing N, therefore Eq. (75) becomes

$$F = \frac{1}{N}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\Gamma\Delta\cos(\phi/2)\sqrt{\omega^2+\Delta^2}}{\Gamma^2\Delta^2\cos^2(\phi/2)+C^2|\omega|(\omega^2+\Delta^2)} \tag{78}$$

which can be approximated by

$$F \simeq \frac{\Gamma\Delta^2}{N\pi}\cos(\phi/2)\int_0^{\Lambda}\frac{d\omega}{\Gamma^2\Delta^2\cos^2(\phi/2)+C^2\Delta^2\omega} \simeq \frac{1}{N}\frac{\Gamma}{\pi C^2}\cos(\phi/2)\ln\left(1+\frac{C^2\Lambda}{\Gamma^2\cos^2(\phi/2)}\right) \tag{79}$$

where now $\Lambda \sim \max(\Delta, \Gamma\cos(\phi/2))$. Therefore we have

$$F \sim \frac{1}{N}\frac{\Gamma}{U}\cos(\phi/2)\ln\left(\frac{U}{\Gamma\cos^2(\phi/2)}\right), \qquad A = -\frac{U^2F^3}{N} \sim -\frac{\left(\Gamma\ln(U)\right)^3}{UN^4} \tag{80}$$

This result implies that, even if the pairing is a sparse matrix whose elements are $\propto \frac{1}{N}$, the corresponding self-energy decays much faster upon increasing N, validating the approach used for calculating the Josephson current.

5 Conclusions

We studied the Josephson effect obtained by contacting a SYK dot by two superconducting leads. We showed that a proximity effect is induced in the dot, however the self-energy is weakly affected by the coupling with the leads in the so-called conformal limit, namely for large interaction and large number of particles. We found that, in this limit, the Josephson current is strongly suppressed by U, the strength of the interaction, as $\ln(U)/U$ and becomes universal, since the current turns out to be independent on the superconducting pairing. This result implies that the Josephson current, at zero temperature, and in the conformal limit, is the same for all BCS-like superconductors. At finite temperature T, instead, the dependence on the superconducting gap is restored. The current becomes dependent on the ratio between the gap and the temperature and vanishes as $1/T^2$ upon increasing the temperature.

Acknowledgements
The author acknowledges financial support from the project BIRD 2021 ”Correlations, dynamics and topology in long-range quantum systems” of the Department of Physics and Astronomy, University of Padova, from the European Union - Next Generation EU within the National Center for HPC, Big Data and Quantum Computing (Project No. CN00000013, CN1 Spoke 10 Quantum Computing) and from the Project ”Frontiere Quantistiche” (Dipartimenti di Eccellenza) of the Italian Ministry for Universities and Research.

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