Enhancement of stability of metastable states in the presence of Lévy noise

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Abstract

The barrier crossing event for superdiffusion in the form of symmetric Lévy flights is investigated. We derive from the fractional Fokker-Planck equation a general differential equation with the corresponding conditions useful to calculate the mean residence time of a particle in a fixed interval for an arbitrary smooth potential profile, in particular metastable, with a sink and a Lévy noise with an arbitrary index α. A closed expression in quadrature of the nonlinear relaxation time for Lévy flights with the index $\alpha = 1$ in cubic metastable potential is obtained. Enhancement of the mean residence time in the metastable state, analytically derived, due to Lévy noise is found.

Contents

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1 Introduction

Anomalous diffusion, which is a deviation from *normal* Gaussian diffusion, has one of the manifestations in Lévy flights [1,2]. These are stochastic processes characterized by the occurrence of extremely long jumps, obeying the Lévy stable distribution. Lévy flights, characterized by a scale invariance property, are extensively observed in physics, chemistry, biology, ecological and financial sytems, see [3–5] and references therein. Furthermore, using the Markovianity of Lévy flights, the generalized Kolmogorov equation can be derived from the Lévy noise-driven Langevin equation [6].

Metastability, as well as the transition process between metastable states, is a ubiquitous phenomenon in nature affecting different fields of natural sciences and advancing in its understanding it is a key challenge in complex systems [7–20]. Experimental [16, 21–24] and theoretical [19, 25–27] results show that long-lived metastable states, even if observed in different areas of physics, were not fully explained.

Recently, the first passage time properties of Lévy flights for random search strategies have been investigated [29–31]. Furthermore, noise-induced escape from a metastable state in the presence of Lévy noise governs a plethora of transition phenomena in complex systems, of physical, chemical and biological nature, ranging from the motion of molecules to climate signals (see [3, 12, 15, 32–36] and references therein). The main focus in these papers is to understand how the barrier crossing in different potential profiles $V(x)$, is modified by the presence of the Lévy-stable noise $L_\alpha(t)$ with index α. This is achieved by particle displacement analysis, which obeys the following Langevin equation

$$\frac{dx}{dt} = -V'(x) + L_\alpha(t), \tag{1}$$

where α is the stability index of the Lévy distribution, with $0 < \alpha < 2$. The main tools to investigate the barrier crossing problem for Lévy flights in these above-mentioned papers are the first passage times and residence times. Concerning these time characteristics, there are a lot of numerical results and some analytical approximations [3, 32, 35]. However, exact analytical results for barrier crossing problems in metastable systems in the presence of Lévy noise are lacking. Obtaining exact analytical results remains an open problem. This paper aims to answer this question by investigating the barrier crossing process in a system with a metastable state driven by Lévy noise. Starting from the fractional Fokker-Planck equation corresponding to Eq. (1), we investigate the barrier crossing event, by focusing on the nonlinear relaxation time (NLRT). Specifically, we look at the average time spent by the particle into the potential well (see Fig. 1). This time is, in other words, the mean residence time in the metastable state, which characterizes its stability.

Here, we address the following open questions: (*i*) the exact results of the NLRT for a particle moving in an arbitrary smooth potential profile with a sink and in the presence of Lévy noise with arbitrary index α; (*ii*) a closed expression in quadrature of the NLRT for Lévy flights with the index $\alpha = 1$ (Cauchy noise) in cubic metastable potential; (*iii*) the enhancement of stability of metastable states, analytically derived, due to Lévy noise.

2 The model

The anomalous diffusion in the form of Lévy flights, for a particle moving in a potential profile $V(x)$, is described by the following Fokker-Planck equation with fractional space derivative

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left[V'(x)P\right] + D_\alpha \frac{\partial^\alpha P}{\partial |x|^\alpha}, \tag{2}$$

where $P(x, t|x_0, 0)$ is the transition probability density, and D_α is the noise intensity parameter, in the sense that the size of a cloud of particles undergoing Lévy motion increases with time as $(D_\alpha t)^{1/\alpha}$. Equation (2) can be easily obtained directly from Eq. (1) [6].

The NLRT τ_{NLR} for a diffusion in a potential profile with a sink, see Fig. 1, is defined as

$$\tau_{NLR}(x_0) = \int_0^\infty \Pr(t, x_0)\, dt, \tag{3}$$

where

$$\Pr(t, x_0) = \int_a^b P(x, t|x_0, 0)\, dx \tag{4}$$

represents the probability to find a particle in the interval (a, b) at the time t, if it starts from some internal point x_0. Substituting Eq. (4) into Eq. (3) and changing the order of integration, we arrive at

$$\tau_{NLR}(x_0) = \int_a^b Z(x, x_0)\, dx, \tag{5}$$

where

$$Z(x, x_0) = \int_0^\infty P(x, t|x_0, 0)\, dt. \tag{6}$$

Integrating Eq. (2) with respect to t from 0 to ∞ and taking into account the initial condition $P(x, 0|x_0, 0) = \delta(x - x_0)$ and the asymptotic condition $P(x, \infty|x_0, 0) = 0$ (for a potential with a sink), we obtain the following equation for the function $Z(x, x_0)$

$$\frac{d}{dx}\left[V'(x)Y\right] + D_\alpha \frac{d^\alpha Z}{d|x|^\alpha} = -\delta(x - x_0). \tag{7}$$

To solve Eq. (7) it is better to consider the Fourier transform of the function $Z(x, x_0)$, i.e.,

$$\widetilde{Z}(k, x_0) = \int_{-\infty}^\infty Z(x, x_0)\, e^{ikx}\, dx. \tag{8}$$

For a smooth potential profiles $V(x)$, after Fourier transform, Eq. (7) can be written in the differential form

$$ik\, V'\left(-i\frac{d}{dk}\right)\widetilde{Z} + D_\alpha |k|^\alpha \widetilde{Z} = e^{ikx_0}. \tag{9}$$

It is convenient to introduce a new function $G(k, x_0)$, namely, the derivative of the function $\widetilde{Z}(k, x_0)$ with respect to x_0

$$G(k, x_0) = \frac{\partial}{\partial x_0} \widetilde{Z}(k, x_0). \tag{10}$$

Differentiating both parts of Eq. (9) with respect to x_0 we find

$$V'\left(-i\frac{d}{dk}\right)G - iD_\alpha |k|^{\alpha-1} \operatorname{sgn} k\, G = e^{ikx_0}, \tag{11}$$

where $\operatorname{sgn} x$ is the sign function.

Substituting $Z(x, x_0)$ from the backward Fourier transformation into Eq. (5) and changing the order of integration, we have

$$\tau_{NLR}(x_0) = \frac{1}{2\pi} \int_{-\infty}^\infty \widetilde{Z}(k, x_0) \frac{e^{-ika} - e^{-ikb}}{ik}\, dk. \tag{12}$$

After differentiation of both sides of Eq. (12) with respect to x_0, in accordance with Eq. (10), we find

$$\tau'_{NLR}(x_0) = \frac{1}{2\pi} \int_{-\infty}^\infty G(k, x_0) \frac{e^{-ika} - e^{-ikb}}{ik}\, dk. \tag{13}$$

One can easily check that, after replacing k with $-k$, Eq. (11) coincides with the equation for the complex conjugate function $G^*(k,x_0)$, i.e. $G(-k,x_0) = G^*(k,x_0)$. As a result, Eq. (13) can be rearranged into a simpler form

$$\tau'_{NLR}(x_0) = \int_0^\infty \mathrm{Re}\left\{ G(k,x_0)\,\frac{e^{-ika} - e^{-ikb}}{\pi i k} \right\} dk, \tag{14}$$

where Re{...} denotes the real part of the expression.

If a sink of the potential profile $V(x)$ is located at the point $x = \infty$ we have

$$\lim_{x_0\to\infty} \tau_{NLR}(x_0) = 0. \tag{15}$$

After integrating Eq. (14) with respect to x_0 and taking into account the condition (15), we find

$$\tau_{NLR}(x_0) = \int_{x_0}^\infty \mathrm{Re}\left\{ \int_0^\infty G(k,z)\,\frac{e^{-ikb} - e^{-ika}}{\pi i k}\, dk \right\} dz. \tag{16}$$

Thus, it is sufficient to solve Eq. (11) only in the region $k > 0$, obtaining

$$V'\left(-i\frac{d}{dk}\right) G - i D_\alpha k^{\alpha-1} G = e^{ikx_0}. \tag{17}$$

Equations (16) and (17), which are among the main results of the paper, give the exact relations useful to calculate the NLRT of the symmetric Lévy flights with arbitrary index α in a smooth potential profile with a sink at $x = \infty$.

3 Results

Metastable state and noise enhanced stability Now, we focus on a particle moving in a metastable cubic potential profile (see Fig. 1)

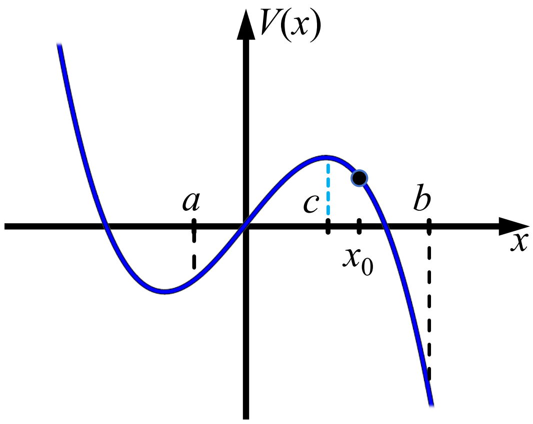

Figure 1: Cubic potential $V(x)$ with metastable state, archetype model for any metastable state. a and b are the interval boundaries, x_0 is the initial condition, c is a potential parameter.

$$V(x) = -\frac{x^3}{3} + c^2 x, \tag{18}$$

and driven by a Cauchy-stable noise with Lévy index $\alpha = 1$. Using Eq. (18) and placing $\alpha = 1$ in Eq. (11), we get

$$\frac{d^2G}{dk^2} + \left(c^2 - iD_1 \operatorname{sgn} k\right) G = e^{ikx_0}, \tag{19}$$

which for $k > 0$ becomes

$$\frac{d^2G}{dk^2} + \left(c^2 - iD_1\right) G = e^{ikx_0}. \tag{20}$$

The general solution of Eq. (20) is the sum of the general solution of the homogeneous equation and its particular solution. Under the condition of its limitation, not divergent for arbitrary k, it takes the form

$$G(k, x_0) = C\, e^{-\lambda k} + \frac{e^{ikx_0}}{c^2 - x_0^2 - iD_1}, \tag{21}$$

where C is an unknown complex constant and λ is one of the complex roots

$$z = \pm\sqrt{iD_1 - c^2},$$

having a positive real part, $\lambda = \lambda_1 + i\lambda_2$, where

$$\begin{aligned} \lambda_1 &= \left(c^4 + D_1^2\right)^{1/4} \sin\left[\frac{1}{2} \arctan\left(\frac{D_1}{c^2}\right)\right], \\ \lambda_2 &= \left(c^4 + D_1^2\right)^{1/4} \cos\left[\frac{1}{2} \arctan\left(\frac{D_1}{c^2}\right)\right]. \end{aligned} \tag{22}$$

To find the unknown constant C we use the continuity conditions for the function $G(k, x_0)$ and its first derivative at the point $k = 0$

$$\begin{aligned} \lim_{k\to 0^+} G(k, x_0) &= \lim_{k\to 0^-} G(k, x_0), \\ \lim_{k\to 0^+} \frac{dG(k, x_0)}{dk} &= \lim_{k\to 0^-} \frac{dG(k, x_0)}{dk}. \end{aligned} \tag{23}$$

For $k < 0$, Eq. (19) transforms into

$$\frac{d^2G}{dk^2} + \left(c^2 + iD_1\right) G = e^{ikx_0}, \tag{24}$$

and its solution under the condition of its limitation reads

$$G(k, x_0) = C^* e^{\lambda^* k} + \frac{e^{ikx_0}}{c^2 - x_0^2 + iD_1}. \tag{25}$$

Using Eqs. (21), (25) and conditions (23) we get

$$\begin{aligned} C + \frac{1}{c^2 - x_0^2 - iD_1} &= C^* + \frac{1}{c^2 - x_0^2 + iD_1} \\ -\lambda C + \frac{ix_0}{c^2 - x_0^2 - iD_1} &= \lambda^* C^* + \frac{ix_0}{c^2 - x_0^2 + iD_1}. \end{aligned} \tag{26}$$

The final expression for the constant C, obtained from the system (26), reads

$$C = -\frac{D_1 (x_0 + \lambda_2 + i\lambda_1)}{\lambda_1\left[\left(c^2 - x_0^2\right)^2 + D_1^2\right]}. \tag{27}$$

Substituting Eqs. (21) and (27) into Eq. (16) and calculating the internal integral we arrive at

$$\tau_{NLR}(x_0) = \frac{D_1}{\pi}\int_{x_0}^{\infty}\left(A\frac{z+\lambda_2}{\lambda_1}+B\right)\frac{dz}{(z^2-c^2)^2+D_1^2} + \frac{D_1}{\pi}\int_{x_0}^{\infty}\ln\left|\frac{z-a}{z-b}\right|\frac{dz}{(z^2-c^2)^2+D_1^2}$$
$$+ \int_{x_0}^{b}\frac{\left(z^2-c^2\right)dz}{(z^2-c^2)^2+D_1^2}, \tag{28}$$

where

$$A = \arctan\frac{\lambda_2+b}{\lambda_1} - \arctan\frac{\lambda_2+a}{\lambda_1},$$
$$B = \frac{1}{2}\ln\frac{\lambda_1^2+(b+\lambda_2)^2}{\lambda_1^2+(a+\lambda_2)^2}. \tag{29}$$

The exact quadrature formula of Eq. (28) is the other main result of the paper. For unstable initial conditions of the particle just beyond the potential barrier and within the interval $c < x_0 < b$, the normalized mean residence time in the metastable state $\langle\tau_{NLR}(x_0)\rangle/\tau_d(x_0)$ as a function of the noise intensity parameter D_1 has a nonmonotonic behaviour with a maximum (see curves in Figs. 2 and 3). This is the noise enhanced stability (NES) phenomenon, already investigated with Gaussian noise sources [7–20].

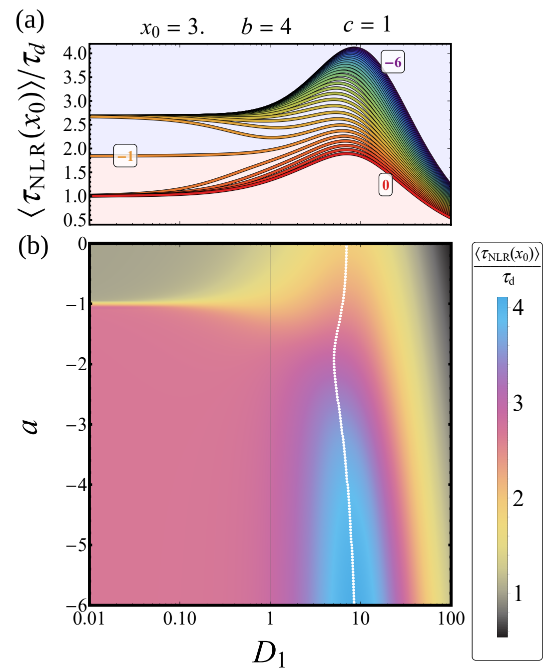

Figure 2: The results of analytical calculations according to Eq. (28) are shown. (a) Normalized NLRT $\langle\tau_{NLR}(x_0)\rangle/\tau_d(x_0)$ for a metastable cubic potential as a function of the noise intensity parameter D_1 for different positions a of the left boundary ranging from 0 to −6 with steps 0.2. (b) Density plot of $\langle\tau_{NLR}(x_0)\rangle/\tau_d(x_0)$ versus a and D_1. The white dotted line marks the position of the NES maxima. The parameter values are: $x_0 = 3.0$, $b = 4$, $c = 1$.

$\tau_d(x_0)$ is the dynamical time obtained by setting $D_\alpha = 0$ in Eq. (28)

$$\tau_d = \int_{x_0}^{b}\frac{dz}{z^2-c^2}. \tag{30}$$

For $x_0 < c < b$ the integral in Eq. (30) diverges, which means the impossibility for a particle to cross the potential barrier, located at the point $x = c$, in the absence of driving noise. For $c < x_0 < b$ we obtain the finite dynamical time

$$\tau_d(x_0) = \frac{1}{2c} \ln \frac{(b-c)(x_0+c)}{(b+c)(x_0-c)}. \tag{31}$$

The normalized NLRT $<\tau_{NLR}(x_0)>/\tau_d(x_0)$ for a metastable cubic potential as a function of the noise intensity parameter D_1 for different positions a of the left boundary and a fixed value of the right boundary $b = 4$ in a semilog plot is shown in Fig. 2. The different values of a range from 0 to −6 with steps 0.2. A nonmonotonic behavior of the normalized NRLT with a maximum as a function of the noise intensity is observed for all values of a analyzed, that is the particle is temporarily trapped in the metastable state.

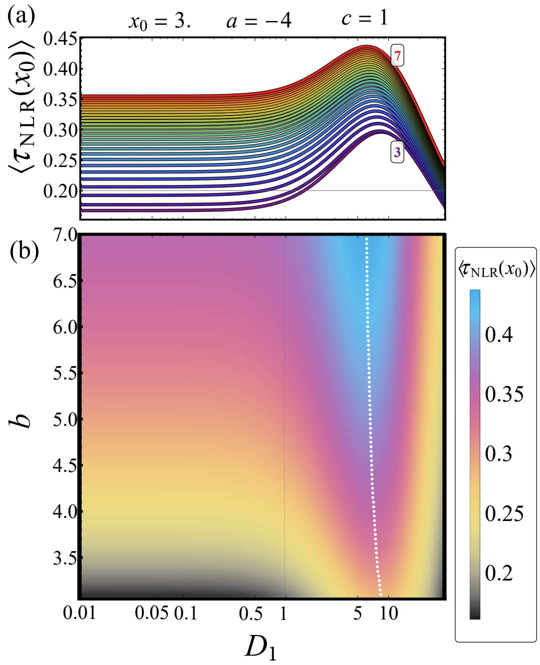

Figure 3: The results of analytical calculations according to Eq. (28) are shown. (a) NLRT $\langle\tau_{NLR}(x_0)\rangle$ for a metastable cubic potential as a function of the noise intensity parameter D_1 for different positions b of the right boundary ranging from 3 to 7 with steps 0.2. (b) Density plot of $\langle\tau_{NLR}(x_0)\rangle$ versus b and D_1. The white dotted line marks the position of the NES maxima. The parameter values are: $x_0 = 3.0$, $a = -4$, $c = 1$.

In Fig. 2b a density plot of $<\tau_{NLR}(x_0)>/\tau_d(x_0)$ versus a and D_1 is shown. The white dotted line marks the position of the NES maxima.

The maxima and all curves increase as the value of the left limit a decreases. This gives rise to an increasing size of the basin of attraction of the metastable state [12, 13], responsible for the increase in the normalized NRLT. Furthermore, in the limit $D_1 \to 0$ there are three different asymptotic values of the normalized NRLT, the value of which increases with the size of the basin of attraction when a varies from 0 to −6. We note that, in the limit $D_1 \to 0$ and for unstable initial conditions of the particle, there is a divergent behavior of $\tau_{NLR}(x_0)$ [7–20], with a Gaussian noise source. For Lévy flights, instead, we obtain for $\tau_{NLR}(x_0)$ a finite nonmonotonic behavior as a function of the noise intensity parameter D_1. Because of heavy tails, a particle spends a finite time in the metastability area even in the limit $D_1 \to 0$. For very large noise intensity, in the limit $D_1 \to \infty$, the normalized NRLT has a power law behavior as a function of the noise intensity [3,4].

In Fig. 3, the NRLT $<\tau_{NLR}(x_0)>$ versus D_1 in a semilog plot for different positions b of the right boundary at a fixed value of the left boundary $a = -4$ is shown. The different values of b range from 3 to 7 with steps 0.2. A nonmonotonic behavior of $<\tau_{NLR}(x_0)>$ versus D_1 for all values of b investigated is observed. In Fig. 3b a density plot of $<\tau_{NLR}(x_0)>$ versus b and D_1 is shown. The white dotted line marks the position of the NES maxima.

Asymptotic behaviors The asymptotic behaviors shown in Figs. 2 and 3 reproduce the asymptotic expressions of Eq. (28) in the limits $D_1 \to 0$ and $D_1 \to \infty$. In particular, for $D_1 \to 0$ we have (see Appendix A, paragraph 1)

$$\tau_{NLR}(x_0) \simeq C_1 \left[\frac{c}{x_0 - c} + \frac{1}{2} \ln \frac{x_0 - c}{x_0 + c} \right] + \tau_d(x_0), \tag{32}$$

where

$$C_1 \simeq \begin{cases} 1, & a < -c, \\ 1/2, & a = -c, \\ 0, & a > -c, \end{cases} \tag{33}$$

giving rise to three different asymptotic values of the NLRT for $D_1 \to 0$ in the case of fixed b (see Fig. 2) and different asymptotic values depending on the different values of b in the case of fixed a (see Fig. 3). And for $D_1 \to \infty$ (see Appendix A, paragraph 2) we have

$$\tau_{NLR}(x_0) \sim \frac{1}{D_1}, \tag{34}$$

that is a power law behavior of the NLRT as a function of the noise intensity (see Figs. 2 and 3).

4 Conclusions

We obtain the general equations useful to calculate the NLRT for superdiffusion in the form of symmetric Lévy flights, for an arbitrary Lévy index α and an arbitrary smooth potential profile with a sink. For a Cauchy driven noise ($\alpha = 1$) we find the closed expression in quadratures of the NLRT as a function of the noise intensity parameter, the initial position, and the parameters of the potential. The interplay between trapping in the metastable state, at small noise intensities, and long jumps of Lévy flights produces a finite nonmonotonic enhancement of the mean residence time in the metastable state. Our general equations provide a useful tool to describe different dynamical behaviors in complex systems, characterized by anomalous diffusion and non-exponential relaxation phenomena, such as spatial extended systems [36].

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A Investigation of the NRLT in the limits $D_1 \to 0$ and $D_1 \to \infty$

To investigate the NLRT in the asymptotic limits $D_1 \to 0$ and $D_1 \to \infty$, we start from the expressions of the parameters λ_1 and λ_2 of Eq. (22)

$$\lambda_1 = \left(c^4 + D_1^2\right)^{1/4} \sin\left[\frac{1}{2}\arctan\left(\frac{D_1}{c^2}\right)\right],$$
$$\lambda_2 = \left(c^4 + D_1^2\right)^{1/4} \cos\left[\frac{1}{2}\arctan\left(\frac{D_1}{c^2}\right)\right], \tag{35}$$

and, using trigonometry formulas, we rewrite Eq. (35) in a simpler form

$$\lambda_1 = \frac{c}{\sqrt{2}}\sqrt{\sqrt{1+\frac{D_1^2}{c^4}}-1},$$
$$\lambda_2 = \frac{c}{\sqrt{2}}\sqrt{\sqrt{1+\frac{D_1^2}{c^4}}+1}. \tag{36}$$

1. Asymptotics for $D_1 \to 0$

For small values of D_1, we can use the approximate expansion: $\sqrt{1+x} \simeq 1 + x/2 - x^2/8$, where $x = D_1^2/a^4 \ll 1$. As a result, we obtain

$$\lambda_1 \simeq \frac{D_1}{2c}\left(1-\frac{D_1^2}{8c^4}\right), \qquad \lambda_2 \simeq c\left(1+\frac{D_1^2}{8c^4}\right). \tag{37}$$

First of all, it is better to write the expression for A in another form. Using the well-known relation

$$\arctan\frac{1}{x} = \frac{\pi}{2} - \text{arccot}\frac{1}{x} = \frac{\pi}{2} - \arctan x,$$

we can rewrite the expressions (28) for A and B in the following form

$$A = \arctan\frac{\lambda_1}{\lambda_2+a} - \arctan\frac{\lambda_1}{\lambda_2+b} + \pi \cdot 1(-\lambda_2 - a),$$
$$B = \frac{1}{2}\ln\frac{\lambda_1^2+(b+\lambda_2)^2}{\lambda_1^2+(a+\lambda_2)^2}, \tag{38}$$

where $1(x)$ is the step function.

Substituting Eq. (37) into Eq. (38) for A and B we have

$$A \simeq \begin{cases} \pi + D_1(b-a)/[2c(c+a)(c+b)], & a < -c, \\ \pi/2 - D_1(3c+b)/\left[4c^2(c+b)\right], & a = -c, \\ D_1(b-a)/[2c(c+a)(c+b)], & a > -c. \end{cases} \tag{39}$$

$$B \simeq \begin{cases} \ln[(c+b)/|c+a|], & a \neq -c, \\ \ln[2c(c+b)/D_1], & a = -c. \end{cases} \tag{40}$$

As seen from Eq. (39), the value A does not go to zero in the limit $D_1 \to 0$ in the case $a \leq -c$.

Substitution of Eqs. (39) and (40) into Eq. 28 gives in the limit $D_1 \to 0$

$$\tau_{NLR}(x_0) \simeq \frac{2cC_1}{\pi} \int_{x_0}^{\infty} \frac{dz}{(z-c)^2(z+c)} + \tau_d(x_0), \tag{41}$$

where

$$C_1 \simeq \begin{cases} \pi, & a < -c, \\ \pi/2, & a = -c, \\ 0, & a > -c \end{cases} \tag{42}$$

and $\tau_d(x_0)$ is the dynamical time.

The integral in Eq. (41) can be calculated in the analytical form. As a result, we obtain in the limit $D_1 \to 0$

$$\tau_{NLR}(x_0) \simeq C_1 \left[\frac{c}{x_0 - c} + \frac{1}{2} \ln \frac{x_0 - c}{x_0 + c} \right] + \frac{1}{2c} \ln \frac{(b-c)(x_0+c)}{(b+c)(x_0-c)}. \tag{43}$$

This gives rise to three different asymptotic values of the NLRT for $D_1 \to 0$ in the case of fixed b (see Fig. 2) and different asymptotic values depending on the different values of b in the case of fixed a (see Fig. 3).

2. Asymptotics for $D_1 \to \infty$

Now we consider the case of very large D_1. From Eq. (36) we easily find

$$\lambda_1 \simeq \sqrt{\frac{D_1}{2}}, \quad \lambda_2 \simeq \sqrt{\frac{D_1}{2}}. \tag{44}$$

Substituting Eq. (44) into Eqs. (29) we obtain the following approximate expressions for the constants A and B

$$A = B \simeq \frac{b-a}{\sqrt{2D_1}}. \tag{45}$$

Substitution of Eqs. (44) and (45) into Eq. (28) gives

$$\tau_{NLR}(x_0) \simeq \frac{b-a}{\pi} \int_{x_0}^{\infty} \frac{(z+\sqrt{2D_1})\,dz}{(z^2-c^2)^2 + D_1^2} \tag{46}$$

$$+ \frac{D_1}{\pi} \int_{x_0}^{\infty} \ln\left|\frac{z-a}{z-b}\right| \frac{dz}{(z^2-c^2)^2 + D_1^2} + \int_{x_0}^{b} \frac{(z^2-c^2)\,dz}{(z^2-c^2)^2 + D_1^2}.$$

The first integral in Eq. (46) can be calculated in analytical form and for the large D_1 gives

$$\frac{b-a}{\pi} \int_{x_0}^{\infty} \frac{(z+\sqrt{2D_1})\,dz}{(z^2-c^2)^2 + D_1^2} \simeq \frac{3(b-a)}{4D_1}. \tag{47}$$

The second integral in Eq. (46) can be estimated for large D_1 using the mean value theorem for a definite integral, namely

$$\begin{aligned}&\frac{D_1}{\pi}\int_{x_0}^{\infty}\ln\left|\frac{z-a}{z-b}\right|\frac{dz}{(z^2-c^2)^2+D_1^2}\simeq\\&\frac{D_1}{\pi}\ln\left(\frac{\sqrt{D_1}-a}{\sqrt{D_1}-b}\right)\int_{x_0}^{\infty}\frac{dz}{(z^2-c^2)^2+D_1^2}\simeq\\&\frac{(b-a)\sqrt{D_1}}{\pi}\int_{x_0}^{\infty}\frac{dz}{(z^2-c^2)^2+D_1^2}\simeq\frac{(b-a)\sqrt{2}}{4D_1}\sim\frac{1}{D_1}\,.\end{aligned} \tag{48}$$

The last integral in Eq. (46), due to the finite limits, can be easily estimated

$$\int_{x_0}^{b}\frac{\left(z^2-c^2\right)dz}{(z^2-c^2)^2+D_1^2}\simeq\frac{1}{D_1^2}\left[\frac{b^3-x_0^3}{3}-c^2(b-x_0)\right]\sim\frac{1}{D_1^2}\,. \tag{49}$$

Taking into account Eqs. (47), (48), and (49), we find finally for large D_1

$$\tau_{NLR}(x_0)\sim\frac{1}{D_1}\,, \tag{50}$$

that is a power law behavior in agreement with previous investigations (see Refs. [2,3]).

References

[1] P. Lévy, *Theory de l'addition des variables Aléatoires* (Gauthier–Villars, Paris, 1937).

[2] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison–Wesley, Cambridge, 1954).

[3] A. V. Chechkin, V. Yu. Gonchar, J. Klafter, and R. Metzler, *Fundamental of Lévy flight processes*, in: W.T. Coffee, Yu.P. Kalmykov (editors), Fractals, Diffusion, and Relaxation in Disordered Complex Systems. A Special Volume of Advances in Chemical Physics **133B**, 439 (2006), ISBN-13: 978-0471725077.

[4] A. A. Dubkov, B. Spagnolo, and V. V. Uchaikin, *Lévy flights superdiffusion: an introduction*, Int. J. Bifur. Chaos **18**, 2649 (2008), https://doi.org/10.1142/S0218127408021877.

[5] A. Padash, A. V. Chechkin, B. Dybiec, I Pavlyukevich, B. Shokri, R. Metzler, *First-passage properties of asymmetric Lévy flights*, J. Phys. A: Math. Theor. **52**, 454004 (2019), https://doi.org/10.1088/1751-8121/ab493e.

[6] A. Dubkov and B. Spagnolo, *Generalized Wiener process and Kolmogorov's equation for diffusion induced by non-Gaussian noise source*, Fluct. Noise Lett. **5**, L267 (2005), https://dx.doi.org/10.1142/S0219477505002641.

[7] A. Fiasconaro, B. Spagnolo, and S. Boccaletti, *Signatures of noise-enhanced stability in metastable states*, Phys. Rev. E **72**, 061110 (2005), https://doi.org/10.1103/PhysRevE.72.061110.

[8] D. Valenti, A. Carollo, B. Spagnolo, *Stabilizing effect of volatility in financial markets*, Phys. Rev. A **97**, 042109 (2018), https://doi.org/10.1103/PhysRevE.97.062307.

[9] I. A. Surazhevsky, V. A. Demin, A. I. Ilyasov, A. V. Emelyanov, K. E. Nikiruy, V. V. Rylkov, S. A. Shchanikov, I. A. Bordanov, S. A. Gerasimova, D. V. Guseinov, N. V. Malekhonova, D. A. Pavlov, A. I. Belov, A. N. Mikhaylov, V. B. Kazantsev, D. Valenti, B. Spagnolo, M. V. Kovalchuk, *Noise-assisted persistence and recovery of memory state in a memristive spiking neuromorphic network*, Chaos, Soliton. Fract. **146**, 110890 (2021), https://doi.org/10.1016/j.chaos.2021.110890.

[10] C. Guarcello, *Lévy noise effects on Josephson junctions*, Chaos, Soliton. Fract. **153**, 111531 (2021), https://doi.org/10.1016/j.chaos.2021.111531.

[11] P. J. Menck, J. Heitzig, N. Marwan, J. Kurths, *How basin stability complements the linear stability paradigm*, Nat. Phys. **9**, 89 (2013), https://doi.org/10.1038/nphys2516.

[12] L. Serdukova, Y. Zheng, J. Duan, J. Kurths, *Stochastic basins of attraction for metastable states*, Chaos **26**, 073117 (2016), https://doi.org/10.1063/1.4959146.

[13] L. Serdukova, Y. Zheng, J. Duan, J. Kurths, *Metastability for discontinuous dynamical systems under Lévy noise: Case study on Amazonian Vegetation*, Sci. Rep. **7**, 9336 (2017), https://doi.org/10.1038/s41598-017-07686-8.

[14] D. K. Ghosh, A. Ranjan, *The metastable states of proteins*, Protein. Science **29**, 1559 (2020), https://doi.org/10.1002/pro.3859.

[15] V. Lucarini, L. Serdukova, G. Margazoglou, *Lévy noise versus Gaussian-noise-induced transitions in the Ghil–Sellers energy balance model*, Nonlinear Proc. Geoph. **29**, 183 (2022), https://doi.org/10.5194/npg-29-183-2022.

[16] L. Hruby, N. Dogra, M. Landini, T. Donner, T. Esslinger, *Metastability and avalanche dynamics in strongly correlated gases with long-range interactions*, Proc. Natl. Acad. Sci. USA **115**, 3279 (2018), https://doi.org/10.1073/pnas.1720415115.

[17] G. Zeng, J. Gao, L. Shekhtmand, S. Guoe, W. Lve, J. Wuf, H. Liug, O. Levyh, D. Lia, Z. Gaoi, H. E. Stanley, S. Havlin, *Multiple metastable network states in urban traffic*, Proc. Natl. Acad. Sci. USA **117**, 17528 (2020), https://doi.org/10.1073/pnas.1907493117.

[18] J. Ravnik, M. Diego, Y. Gerasimenko, Y. Vaskivskyi, I. Vaskivskyi, T. Mertelj, J. Vodeb, D. Mihailovic, *A time-domain phase diagram of metastable states in a charge ordered quantum material*, Nat. Commun. **12**, 2323 (2021), https://doi.org/10.1038/s41467-021-22646-7.

[19] T. Mori, *Metastability associated with many-body explosion of eigenmode expansion coefficients*, Phys. Rev. Res. **3**, 043137 (2021), https://doi.org/10.1103/PhysRevResearch.3.043137.

[20] X.-D. Zheng, C. Li, S. Lessard, Y. Tao, *Environmental noise could promote stochastic local stability of behavioral diversity evolution*, Phys. Rev. Lett. **120**, 218101 (2018), https://doi.org/10.1103/PhysRevLett.120.218101.

[21] V. Nosenko, S. K. Zhdanov, A. V. Ivlev, C. A. Knapek, G. E. Morfill, *2D melting of plasma crystals: equilibrium and nonequilibrium regimes*, Phys. Rev. Lett. **103**, 015001 (2009), https://doi.org/10.1103/PhysRevLett.103.015001.

[22] J. J. L. Morton, A. Tiwari, G. Dantelle, K. Porfyrakis, A. Ardavan, G. A. D. Briggs, *Switchable $ErSc_2N$ rotor within a C_{80} fullerene cage: an electron paramagnetic resonance and photoluminescence excitation study*, Phys. Rev. Lett. **101**, 013002 (2008), https://doi.org/10.1103/PhysRevLett.101.013002.

[23] S. Hayami, D. Urakami, Y. Kojima, Y. H. Lee, *Magnetic behavior and liquid-crystal properties in spin-crossover Cobalt(II) compounds with long Alkyl chains*, Inorg. Chem. **49**, 1428 (2010), https://doi.org/10.1002/ejic.201201107.

[24] B. Tu, R. Si, Y. Shen, *Experimental access to observing decay from extremely long-lived metastable electronic states via Penning trap spectrometry*, Phys. Rev. Res. **5**, 043014 (2023), https://doi.org/10.1103/PhysRevResearch.5.043014.

[25] I. Biazzo, F. Caltagirone, G. Parisi, F. Zamponi, *Theory of amorphous packings of binary mixtures of hard spheres*, Phys. Rev. Lett. **102**, 195701 (2009), https://doi.org/10.1103/PhysRevLett.102.195701.

[26] R. Giacchetti, E. Sorace, *States of the Dirac equation in confining potentials*, Phys. Rev. Lett. **101**, 190401 (2008), https://doi.org/10.1103/PhysRevLett.101.190401.

[27] L. M. Krauss, J. Dent, *Late time behavior of false vacuum decay: possible implications for cosmology and metastable inflating states*, Phys. Rev. Lett. **100**, 171301 (2008), https://doi.org/10.1103/PhysRevLett.100.171301.

[28] P. H. Chavanis, *On the lifetime of metastable states in self-gravitating systems*, Astron. Astrophys **432**, 117 (2005), https://doi.org/10.1051/0004-6361:20041114.

[29] V. V. Palyulin, G. Blackburn, *First passage and first hitting times of Lévy flights and Lévy walks*, New J. Phys.**21**, 103028 (2019), https://doi.org/10.1088/1367-2630/ab41bb.

[30] A. Padash, T. Sandev, H. Kantz, R. Metzler, A. V. Chechkin, *Asymmetric Lévy flights are more efficient in random search*, Fractal Fract. **6**, 260 (2022). https://doi.org/10.3390/fractalfract6050260.

[31] M. R. Evans, S. N. Majumdar, G. Schehr, *Stochastic resetting and applications*, J. Phys. A: Math. Theor. **53**, 193001 (2020).

[32] A. V. Chechkin, O. Yu. Sliusarenko, R. Metzler, and J. Klafter, *Barrier crossing driven by Lévy noise: Universality and the role of noise intensity*, Phys. Rev. E **75**, 041101 (2007), https://doi.org/10.1103/PhysRevE.75.041101.

[33] A. A. Dubkov, A. La Cognata, and B. Spagnolo, *The problem of analytical calculation of barrier crossing characteristics for Lévy flights*, J. Stat. Mech-Theory E., **01**, P01002 (2009), https://doi.org/10.1088/1742-5468/2009/01/P01002.

[34] A. A. Dubkov, B. Dybiec, B. Spagnolo, A. Kharcheva, C. Guarcello, D. Valenti, *Statistics of residence time for Lévy flights in unstable parabolic potential*, Phys. Rev. E **102**, 042142 (2020), https://doi.org/10.1103/PhyRE.1.042142.

[35] A. Baule, P. Sollich, *Exponential increase of transition rates in metastable systems driven by non-Gaussian noise*, Sci. Rep. **13**, 3853 (2023), https://doi.org/10.1038/s41598-023-30577-0.

[36] C. Jiao and G. A. Gottwald, *Lévy flights as an emergent phenomenon in a spatially extended system*, Proc. Roy. Soc. A-Math. Phys. **479**, 20230349 (2023), https://doi.org/10.1098/rspa.2023.0349.