Two-dimensional topological paramagnets protected by $\mathbb{Z}_3$ symmetry: Properties of the boundary Hamiltonian

Hrant Topchyan[1], Vasilii Iugov[2,3], Mkhitar Mirumyan[1], Tigran Hakobyan[1,4], Tigran Sedrakyan[5⋆] and Ara Sedrakyan[1]

1 Alikhanyan National Science Laboratory, Yerevan Physics Institute, Yerevan, Armenia
2 Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, NY, 11794-3636, USA
3 C. N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, NY, 11794-3636, USA
4 Institute of Physics, Yerevan State University, Yerevan, Armenia
5 Department of Physics, University of Massachusetts, Amherst, Massachusetts 01003, USA

⋆ tsedrakyan@umass.edu

Abstract

We systematically study gapless edge modes corresponding to $\mathbb{Z}_3$ symmetry-protected topological (SPT) phases of two-dimensional three-state Potts paramagnets on a triangular lattice. First, we derive microscopic lattice models for the gapless edge and, using the density-matrix renormalization group (DMRG) approach, investigate the finite size scaling of the low-lying excitation spectrum and the entanglement entropy. Based on the obtained results, we identify the universality class of the critical edge, namely the corresponding conformal field theory and the central charge. Finally, we discuss the inherent symmetries of the edge models and the emergent winding symmetry distinguishing between two possible $\mathbb{Z}_3$ SPT phases. As a result, the two topologically non-trivial and one trivial phases define a general one-dimensional chain supporting a tricriticality, which we argue supports a gapless SPT order in one dimension. Numerically, we show that low energy states in the continuous limit of the edge model can be described by conformal field theory (CFT) with central charge $c = 1$, given by the coset $SU_k(3)/SU_k(2)$ CFT at level k=1.

1 Introduction

Symmetry-protected topological (SPT) phases [1–10] are a relatively new concept in condensed matter physics. Over recent years, there has been a notable research activity in that direction [11–41]. These phases fundamentally differ from the conventional phases defined by local order parameters while simultaneously possessing a topological nature. The notion of protection by symmetry means the impossibility of a smooth transition between phases without breaking the symmetry. This protection is what gives SPT phases their distinctive and robust topological properties. SPT phases usually emerge in non-degenerate gapped quantum systems with some symmetry S at zero temperature and are beyond the classical theory of phase transitions. Two states belong to different phases if they cannot be connected adiabatically (without closing the gap) while preserving the symmetry S on the whole path. The distinction between phases becomes apparent once considered on a system with a boundary. In this case, the "non-trivial" phases exhibit gapless excitations, and the ground state of the "trivial" phase can be given as a tensor product of local states. In d-dimensional space, SPT phases are classified by the non-trivial elements of the $H^{d+1}(S, U(1))$ cohomology group.

Commencing with the conventional understanding of phases described by the Landau theory [42, 43], a cornerstone for the emergence of distinct phases and phase transitions is the symmetry group associated with the order parameter and its corresponding finite-size scaling universality classes. Spontaneously or explicitly broken symmetry is the key feature in this conventional notion of phases. The conventional states of the matter, as well as the SPT states, demonstrate short-range entanglement [37], unlike the topologically ordered states, which are long-range entangled [44–58]. Generally speaking, phases with short-range entangled states can be classified as either symmetry-broken (within the scope of the conventional theory), SPT, or capable of simultaneously hosting symmetry-breaking and SPT orders. A distinctive feature of SPT states is their role in enabling the emergence of symmetry-protected gapless boundary excitations. Those excitations frequently belong to non-standard statistics and are important as key elements in the foundation of topological quantum computation.

The concept describing the SPT phases allows their classification based on cohomology classes of corresponding discrete symmetry groups. It was introduced and developed by X.G. Wen and coauthors [1, 6–8, 13, 14]. As a particular case of the cohomology classification, the problem of the complete classification of SPT phases in one spatial dimension was solved in, e.g., Refs. [6, 7, 59, 60]. This formalism allows an intuitive understanding of the variety and properties of SPT phases. However, it lacks an explicit presentation of specific models or a precise way of manipulations to construct models of SPT phases based on a desirable symmetry and a known topologically trivial mode. The situation changed with the work of Levin and Gu [12], where the authors show an SPT modification of $\mathbb{Z}_2$ paramagnetic quantum

Ising model with gapless $\mathbb{Z}_2$ symmetry-protected edge states.

Recently, the approach of [12] was extended to the $(\times\mathbb{Z}_3)^3$ symmetric Potts model, and the corresponding critical boundary model was derived [61]. For that case, SPT phases are classified by the cohomology group $H^3((\times\mathbb{Z}_3)^3, U(1)) = (\times\mathbb{Z}_3)^7$. A study was performed on one of the boundary models belonging to a seven-generator group of phases. It was suggested that the low energy effective conformal field theory is equivalent to the coset model $SU_k(3)/SU_k(2)$ at the level $k = 2$.

Based on the conventional definition, SPT models possess the following properties: firstly, the system must exhibit a global symmetry S, which remains unbroken in all phases. The so-called "trivial" phase of the system typically features a gapped spectrum and has the simplest form of the Hamiltonian, with the ground state usually expressed as a direct product of different subsystem states. The non-trivial phases of SPT models generally feature gapless edge modes. The fact of phases being symmetry protected is implemented as the non-existence of a symmetric series of transformations U^α continuous on $\alpha \in [0, 1]$, with $U^0 = 1$ and U^1 being the transformation connecting the two states in different phases. An alternative description of various phases is related to the 't Hooft anomaly [62]. It is the obstruction to gauging the system's symmetry S. In one dimension, cohomology classes describing the phases can be identified with the emergence of a projective representation of the symmetry group. This concept can be generalized for higher dimensions. The system can be gauged with the modified representation. In this article, we study SPT phases of the pure $\mathbb{Z}_3$ paramagnetic Potts model, in contrary to [61], where $(\times\mathbb{Z}_3)^3$ was considered, and only a single edge state of the phase with the diagonal $\mathbb{Z}_3$ symmetry was investigated. In this case, SPT phases are classified by cohomology group $H^3(Z_3, U(1)) = Z_3$. The objective is to systematically construct a model that features SPT characteristics, starting with the $\mathbb{Z}_3$ Potts paramagnet as its trivial phase. We explicitly define the one-dimensional edge Hamiltonian and study its properties, including spectrum, symmetries, and low-energy modes. We also show the model has hidden $U(1)$ "winding number" anomalous symmetry. Through the further numerical study of finite size scaling behavior of excitation gap, entanglement entropy, and Kac-Moody currents on edge states, we argue that the effective low energy theory of the edge model is the coset CFT $SU_k(3)/SU_k(2)$ at level $k = 1$. The present work also enables the study of gauged $\mathbb{Z}_3$ SPT models, disclosing anyonic properties of the excitations in the topologically ordered phase.

2 Symmetry protected topological phase and the edge Hamiltonian

We start with the three-state paramagnetic Potts model defined on a triangular lattice. The Hamiltonian is given by

$$H_0 = -\sum_p (X_p + X_p^\dagger) \tag{1}$$

with the sum taken over all sites of the triangular lattice and X_p being the $\mathbb{Z}_3$ generators on the sites [61]. We also introduce on-cite $\mathbb{Z}_3$ operators Z_p which obey commutativity relations $X_p Z_p = \varepsilon Z_p X_p$ with $\varepsilon = e^{\frac{2\pi i}{3}}$. As $\mathbb{Z}_3$ generators, they have eigenvalues $1, \varepsilon, \varepsilon^*$. The matrices for X_p and Z_p are explicitly given as

$$X_p = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad Z_p = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon^* \end{pmatrix}. \tag{2}$$

Z_p can also be represented as $Z_p = \varepsilon^{n_p}$, with n_p having eigenvalues $0, \pm 1$. Commutativity relations take the form $X_p n_p = (n_p + 1)X_p$, where addition to n_p should always be understood as by mod 3. The Hamiltonian Eq.1 has a global $\mathbb{Z}_3$ symmetry, given by operator

$$S = \prod_p X_p \; . \tag{3}$$

It is known, that non-trivial SPT phases are defined by cohomology classes of corresponding symmetry group, that is $H^3(\mathbb{Z}_3, U(1)) = \mathbb{Z}_3$ in our case [6–8, 13]. The transformations of a topologically trivial Hamiltonian to Hamiltonians of non-trivial SPT phases are given by a unitary transformation generated by non-exact closed 3-forms (cocycles) of the symmetry group. The cocycle for $\mathbb{Z}_3$ symmetry was constructed in [61] and has form $\psi_3(n_1, n_2, n_3) = n_1 n_2 n_3 (n_1 + n_2)$. The corresponding unitary transformation is written as

$$U = \prod_{\langle abc \rangle} (U_\triangle)^{\epsilon_\triangle} = \prod_{\langle abc \rangle} \varepsilon^{\epsilon_\triangle n_a n_b (n_c - n_b)(n_b - n_a)} \tag{4}$$

where $\langle abc \rangle$ denotes all the triangles in the lattice with a, b, and c always being nodes of a specific corresponding color, and $U_\triangle$ is the portion of the unitary operator corresponding to the given $\langle abc \rangle$ triangle. The exponent is induced from ψ_3 as $\psi_3(n_a, n_b - n_a, n_c - n_b)$ [61]. In this formula, $\epsilon_\triangle = \pm 1$ indicates the orientation of the triangle, $\triangle$.

The fact of ψ_3 being a closed form guarantees that U is symmetric under S in the bulk, however the non-exactness ensures it is not symmetric on the boundary. We can write the U-transformed Hamiltonian as

$$H_a = U H_0 U^{-1} = -\sum_p (\bar{X}_p + \bar{X}_p^\dagger), \quad \bar{X}_p = U_p X_p U_p^\dagger \tag{5}$$

where U_p is the part of U that was generated by triangles containing the site p. In other words, it is the part of U that is not commutative with X_p. We can define $\bar{n}_p = p_p$ and this way, the barred operators, $\bar{X}_p$, satisfy the same algebra relations as the initial ones. The Hamiltonian operator mentioned above can be split into bulk and boundary parts, which are still commutative. The symmetry of the boundary part, $H_{\partial,a}$, with respect to S, was broken by the proposed unitary transformation, but it can be restored straightforwardly by substituting

$$H_{\partial,a} \to H_\partial = \frac{1}{3}(H_{\partial,a} + S H_{\partial,a} S^\dagger + S^\dagger H_{\partial,a} S) \tag{6}$$

and the bulk part $H_{\partial*} = H_{\partial*,a}$ stays the same. Notice that the commutativity $[H_\partial, H_{\partial*}] = 0$ holds. By denoting $V = SUS^\dagger U^\dagger$ and $V' = S^\dagger U S U^\dagger = SVS^\dagger V$, we can rewrite the edge Hamiltonian Eq.6 as

$$H_\partial = -\frac{1}{3} \sum_{p \in \partial} (\bar{X}_p + V \bar{X}_p V^\dagger + V' \bar{X}_p V'^\dagger) + \text{H.c.} \; . \tag{7}$$

By using the relations $X_p \varepsilon^{k n_p} X_p^\dagger = \varepsilon^{k(n_p + 1)}$ for $\forall k \in \mathbb{Z}$ one can obtain the expressions for V and V'. Particularly,

$$V = \prod_{\langle abc \rangle} \varepsilon^{\epsilon_\triangle(-n_b^3 - n_b^2 + n_a n_b + n_b n_c - n_a n_c + n_a^2 n_b + n_b^2 n_c - n_a^2 n_c)} \; . \tag{8}$$

Now we will focus on bringing V to a simpler form, and then calculate the expression for V' accordingly.

We note that $U_\triangle$ is not uniquely defined, and it can be changed without changing the SPT phase thus allowing us to tweak V. This can be done by adding any 2-vertex terms (leading

to a trivially identical transformation) or adding terms created by exact 3-forms to U_Δ. The different 2-vertex terms are $f_{\alpha\beta}(n_p, n_q) = \varepsilon^{n_p^\alpha n_q^\beta}$ with $\alpha, \beta \in \{0, 1, 2\}$. It is straightforward to see that adding any of those terms to U_Δ creates an additional term $g_{\alpha\beta}^{pq} = S f_{\alpha\beta}^{pq} S^\dagger f_{\alpha\beta}^{pq\dagger}$ to V (we indicate the arguments of the functions as upper indices for more compact writing). The situation is simple because all $g_{\alpha\beta}^{pq}$ are commutative with initial U_Δ and with each other. The expressions of $g_{\alpha\beta}^{pq}$ are

	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$
$\beta = 0$	1	ε	ε^{1-n_p}
$\beta = 1$	ε	$\varepsilon^{1+n_p+n_q}$	$\varepsilon^{1+n_p-n_q-n_p n_q+n_q^2}$
$\beta = 2$	ε^{1-n_q}	$\varepsilon^{1+n_q-n_p-n_p n_q+n_p^2}$	$\varepsilon^{1-n_p-n_q+n_p n_q-n_p n_q^2-n_p^2 n_q+n_p^2+n_q^2}$

Adding different combinations of $f_{\alpha\beta}^{pq}$ terms to U_Δ can lead to the creation of basic elements in the exponent of V. Namely

$$\begin{aligned}
f_{(1)}^{pq} &= f_{10}^{pq} \to 1 \,, \\
f_{(2)}^{pq} &= f_{(1)}^{pq}/f_{20}^{pq} \to n_p \,, \\
f_{(3)}^{pq} &= f_{(1)}^{pq} f_{(2)}^{pq}/f_{(2)}^{qp} f_{21}^{pq} \to n_p n_q - n_q^2 \,, \\
f_{(4)}^{pq} &= f_{(1)}^{pq}/f_{(2)}^{pq} f_{(2)}^{qp} f_{(3)}^{pq} f_{(3)}^{qp} f_{22}^{pq} \to n_p^2 n_q + n_p n_q^2 \,.
\end{aligned} \tag{9}$$

Using these simpler expressions, it is easy to see that multiplying $f_{(2)}^{bc} f_{(3)}^{ab} f_{(3)}^{cb} f_{(4)}^{ac}$ to U_Δ will bring V to a simpler, color-rotation-invariant form:

$$V \to \prod_{\langle abc \rangle} \varepsilon^{\epsilon_\Delta (n_a n_b (n_a-1) + n_b n_c (n_b-1) + n_c n_a (n_c-1))} = \prod_{\Delta,\gamma} \varepsilon^{\epsilon_\Delta n_\gamma^\Delta n_{\gamma+1}^\Delta (n_\gamma^\Delta - 1)}, \tag{10}$$

where n_γ^Δ is the spin of the vertex with color γ in triangle Δ. γ runs through the colors, and adding 1 to γ changes it as the color changes when passing through points $a \to b \to c \to a$. Further addition of $f_{(3)}^{ca} f_{(3)}^{cb}/f_{(3)}^{ba} f_{(3)}^{ab}$ eliminates the (-1) in the braces and another $f_{(4)}^{ab} f_{(4)}^{bc} f_{(4)}^{ca}$ leaves us with the final form

$$V = \prod_{\Delta,\gamma} \varepsilon^{-\epsilon_\Delta n_\gamma^\Delta n_{\gamma+1}^\Delta (n_\gamma^\Delta - n_{\gamma+1}^\Delta)} \tag{11}$$

which will be used in Eq.7 for the edge Hamiltonian. This expression has a mixed symmetry with regards to color permutations (it is symmetric under rotation and anti-symmetric under reflection). We will later see the usefulness of such property when writing the edge Hamiltonian.

The exact 3-forms aren't particularly helpful for our case, nevertheless it is useful to see the calculation of their investment directly. The generators of exact 3-forms are written as $\delta\psi_2^{\alpha\beta}(n_1, n_2, n_3)$ where $\psi_2^{\alpha\beta}(n_1, n_2) = n_1^\alpha n_2^\beta$ and δ is the co-boundary operator, which is explicitly known. The 3-forms themselves that should be written in U_Δ are $f'_{\alpha\beta}(n_a, n_b, n_c) = \varepsilon^{\delta\psi_2^{\alpha\beta}(n_a, n_b-n_a, n_c-n_b)}$. The multiplier added to V as a result is $g'_{\alpha\beta} = S f'_{\alpha\beta} S^\dagger f'^\dagger_{\alpha\beta}$. Here again, we use the commutativity relations of g' operators. The results for g' are given in the following table:

	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$
$\beta = 0$	1	ε^*	ε^{n_b-1}
$\beta = 1$	1	1	$\varepsilon^{n_a n_b + n_b n_c - n_a n_c - n_b^2}$
$\beta = 2$	1	$\varepsilon^{-n_a n_b - n_b n_c + n_a n_c + n_b^2}$	$\varepsilon^{n_a^2(n_b-n_c)+n_b^2(n_a+n_b+n_c)-n_a n_b - n_b n_c + n_a n_c + n_b^2}$

This result provides no additional freedom of transformation selection. However, some otherwise complicated transformations are in plain view here. Notice that we do not have the freedom to choose p and q here. After fixing the transformation V, the explicit expression of the aforementioned $V' = SVSV^\dagger$ should be calculated accordingly.

What remains is to find the result of the action of V and V' on $\bar{X}_p$ of the edge. All terms in V-s depend on the spins of two neighboring sites. Although there were 3-spin dependent terms in U, there was no way they could have made their way up to here, as it would mean that those factors would not cancel out in bulk. There were also single-spin-dependent terms in our intermediate calculation, which could not have vanished. We will discuss how to handle terms like that in a moment. Once again, we should note that only the parts of V and V' generated by the triangles that contain p are important. It is also easy to see that 2-spin terms that depend on a vertex in bulk will cancel out, as they exist in two neighboring triangles with opposite signs. So V-s reduces to border link terms. Let us denote the neighboring edge vertices of p by $p \pm 1$ with a specific positive traversal orientation (clockwise). We choose $\epsilon_\triangle = 1$ for the triangles which have ascending γ along the traversal with that orientation, and $\epsilon_\triangle = -1$ for the others. So $\gamma_{p+1} = \gamma_p \pm 1$ implies $\epsilon_{\triangle_{p,p+1}} = \pm 1$. Using denotation $v(\alpha, \beta) = \varepsilon^{\alpha\beta(\beta-\alpha)}$, V can be rewritten as

$$V_\partial = \prod_{p \in \partial} v(n_p, n_{p+1}) \,. \tag{12}$$

The 1-spin terms can be handled in the final transformation. One can see that terms with n_p cancel out if there is an even number of triangles containing p, which is always the case for p-s in bulk. In case of an odd number of containing triangles, the sign of the term will depend on $\epsilon_{\triangle_{p,p-1}} = \epsilon_{\triangle_{p,p+1}}$. So, for our v to handle such terms, v should be split into two v_+ and v_-, depending on the orientation of the triangle. To satisfy the above-mentioned conditions, the 1-spin term in v_+ should be the conjugate of one in v_-. Notice that the n_p term will have come only from one triangle, as the rest will cancel out each other, so if we intend to include it in our new $v_\pm$, we should do it with an opposite sign than in V, as there will be two terms of that kind for the same p. The factorization for V' corresponding to Eq.12 will be

$$V'_\partial = \prod_{p \in \partial} v'(n_p, n_{p+1}) = \prod_{p \in \partial} v^\dagger(n_p, n_{p+1}) \tag{13}$$

with $v'(n_p, n_{p+1}) = \varepsilon^{-(\alpha\beta-\alpha-\beta-1)(\beta-\alpha)}$. Products over v' and $v^\dagger$ are equal, as $v'(\alpha, \beta)$ and $v^\dagger(\alpha, \beta)$ differ only by terms $\propto \alpha - \beta$ and $\propto \alpha^2 - \beta^2$, and those contributions from neighbour links cancel out each other. These lead to the following expressions for the action on $\bar{X}_p$-s:

$$\begin{aligned}
V\bar{X}_p V^\dagger &= \varepsilon^{n_{p-1}(n_{p-1}+n_p-1)} \bar{X}_p \varepsilon^{-n_{p+1}(n_{p+1}+n_p+1)} \,, \\
V'\bar{X}_p V'^\dagger &= \varepsilon^{-n_{p-1}(n_{p-1}+n_p-1)} \bar{X}_p \varepsilon^{n_{p+1}(n_{p+1}+n_p+1)} \,.
\end{aligned} \tag{14}$$

As a result, we get a Hamiltonian H_∂ that is translation invariant and does not depend on the shape of the edge:

$$H_\partial = -\frac{1}{3} \sum_{p \in \partial} \bar{X}_p \left(\frac{1}{2} + \varepsilon^{(n_{p-1}-n_{p+1})(n_{p-1}+n_p+n_{p+1}+1)} + \text{H.c.} \right) + \text{H.c.} \,. \tag{15}$$

An alternative way to write H_∂ is

$$H_\partial = -\sum_{p \in \partial} \bar{X}_p \, \delta_{(n_{p-1}-n_{p+1})(n_{p-1}+n_p+n_{p+1}+1)} + \text{H.c.} \,. \tag{16}$$

Here δ_n is a function that is equal to 1 if n is divisible by 3 and is equal to 0 otherwise. This becomes apparent after the observation that δ_n can be written as $3\delta_n = 1 + \varepsilon^n + \varepsilon^{-n}$.

3 Symmetries of edge Hamiltonian

Besides the translation and reflection symmetries, our Hamiltonian might have some additional symmetries. To discuss them, let us parametrize the Hamiltonian as

$$
\begin{aligned}
H_\partial(\lambda_1, \lambda_2, \lambda_3) &= -\frac{1}{3} \sum_{p \in \partial} \bar{X}_p \left[\lambda_1 + \lambda_2 \varepsilon^{(n_{p-1}-n_{p+1})(n_{p-1}+n_p+n_{p+1}+1)} \right. \\
&\quad + \left. \lambda_3 \varepsilon^{-(n_{p-1}-n_{p+1})(n_{p-1}+n_p+n_{p+1}+1)} \right] + \text{H.c.},
\end{aligned}
\tag{17}
$$

with $H_\partial(1,1,1)$ being the boundary Hamiltonian we are dealing with.

From Eq.6 one can derive that the three parameters λ_i, $i = 1, 2, 3$ will permute cyclically under action of opertor S:

$$
SH_\partial(\lambda_1, \lambda_2, \lambda_3)S^\dagger = H_\partial(\lambda_3, \lambda_1, \lambda_2) .
\tag{18}
$$

There is yet another global symmetry. Let us define the operator χ_p to be

$$
\chi_p = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ in the basis, where } \bar{X}_p = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } n_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} .
$$

It can be seen that $\chi_p \bar{X}_p^\pm \chi_p^\dagger = \bar{X}_p^\mp$ and $\chi_p n_p \chi_p^\dagger = -n_p$. We can define the product of all χ_p-s in the system as an operator, P. Using the commutation relation between $\bar{X}_p$ and n_p, one can show that

$$
PH_\partial(\lambda_1, \lambda_2, \lambda_3)P^\dagger = H_\partial(\lambda_1, \lambda_3, \lambda_2) .
\tag{19}
$$

Now it is clear that the Hamiltonian has a global $\mathbb{S}_3$ symmetry with generators S and P.

3.1 Winding number

Here we identify an additional "winding" symmetry of the edge Hamiltonian. To this end, we write it using a different parametrization of the basis states. Namely, we will parametrize them by the last spin n_N and a set of differences w_i defined by

$$
\varepsilon^{w_i} = \varepsilon^{n_i - n_{i-1}}, \quad w_i \in \{-1, 0, 1\}, \quad i = 1, ..., N, \quad n_0 \equiv n_N ; .
\tag{20}
$$

Alternatively,

$$
w_i = \delta_{n_i - n_{i-1} - 1} - \delta_{n_i - n_{i-1} + 1}, \quad i = 1, ..., N, \quad n_0 \equiv n_N .
\tag{21}
$$

This set of variables is not completely independent because $\varepsilon^{w_1 + \cdots + w_N} = 1$. The Hamiltonian can be written in terms of w_i as (see Eq.16)

$$
H_\partial = -\sum_{p \in \partial} \bar{X}_p \delta_{(w_p + w_{p+1})(w_{p+1} - w_p + 1)} + \text{H.c.} ,
\tag{22}
$$

$$
H_\partial = -\sum_{p \in \partial} \bar{X}_p (\delta_{w_p - 1} \delta_{w_{p+1} + 1} + (1 - \delta_{w_p - 1})(1 - \delta_{w_{p+1} + 1})) + \text{H.c.} .
\tag{23}
$$

Now, we introduce a new operator, $\mathcal{W}$, which, in terms of $w_p \in \{-1, 0, 1\}$ can be expressed as

$$
\mathcal{W} = \frac{1}{3} \sum_{p \in \partial} w_p = \sum_{p \in \partial} (\delta_{w_p - 1} - \delta_{w_p + 1}) .
\tag{24}
$$

The operator $\mathcal{W}$ counts the full number of turns that ε^{n_p} makes around the unit circle as we move around the boundary ∂. Thus we will refer to $\mathcal{W}$ as to the the winding-number operator.

When $n_{p+1} = n_p + 1$, we count that as a rotation by $\frac{2\pi}{3}$ and when $n_{p+1} = n_p - 1$, we count that as a rotation by $-\frac{2\pi}{3}$. One can check that the winding-number operator $\mathcal{W}$ commutes with the Hamiltonian. The nontrivial part of that commutator calculation reduces to the observation that

$$
\begin{aligned}
\bar{X}_p(\delta_{w_p-1} - \delta_{w_p+1} + \delta_{w_{p+1}-1} - \delta_{w_{p+1}+1})(\delta_{w_p-1}\delta_{w_{p+1}+1} + (1-\delta_{w_p-1})(1-\delta_{w_{p+1}+1})) = \\
= \bar{X}_p(\delta_{w_p+1} - \delta_{w_p} + \delta_{w_{p+1}} - \delta_{w_{p+1}-1})(\delta_{w_p-1}\delta_{w_{p+1}+1} + (1-\delta_{w_p-1})(1-\delta_{w_{p+1}+1})) = \\
= (\delta_{w_p-1} - \delta_{w_p+1} + \delta_{w_{p+1}-1} - \delta_{w_{p+1}+1})\bar{X}_p(\delta_{w_p-1}\delta_{m_{p+1}+1} + (1-\delta_{w_p-1})(1-\delta_{w_{p+1}+1})),
\end{aligned}
\tag{25}
$$

which one can confirm by substituting all 9 possible combinations of w_p and w_{p+1}. Thus, the winding-number operator $\mathcal{W}$ is conserved by the Hamiltonian. The winding-number symmetry generated by $\mathcal{W}$ is a topological symmetry of the boundary model, as it takes values in $\mathbb{Z}$ instead of $\mathbb{Z}_3$. It distinguishes the non-trivial phase from the trivial one, as the latter has no conserved topological quantities.

The symmetry can be reformulated in terms of a conserved current $j_p^\mu = (q_p, m_p)$ as

$$
\partial_\mu j_p^\mu = \partial_t q_p - \nabla_p m_p = i[H_\partial, q_p] - (m_{p+1} - m_{p-1}) = 0,
\tag{26}
$$

where ∇_p is the discrete derivative in real space. As the charge of the symmetry is given by $\mathcal{W}$, q_i should be given as $q_i = (w_i + w_{i+1})/3$. The sum is taken to ensure space reflection symmetry of q_i, which is broken for w_i. Using the original form of edge Hamiltonian, one can check that $i[H_\partial, w_p/3] = m_p - m_{p-1}$ if

$$
m_p = i\bar{X}_p\left(\frac{1}{3}\delta_{(n_{p-1}-n_{p+1})(n_{p-1}+n_p+n_{p+1}+1)} - \delta_{n_{p-1}-n_{p+1}}\delta_{n_{p-1}+n_p+n_{p+1}+1}\right) + \text{H.c.}
\tag{27}
$$

and Eq.26 follows immediately. In the process of derivation we have used identities $\delta_{ab} = \delta_a + \delta_b - \delta_a\delta_b$, $\delta_a\delta_b = \delta_a\delta_{ka+b}$ and $\delta_n + \delta_{n+1} + \delta_{n-1} = 1$ for any $a, b, k, n \in \mathbb{Z}$.

3.2 Boundary 't Hooft anomaly for the SPT Hamiltonian

In the nontrivial SPT phase, the Hamiltonian we found at the boundary for the non-trivial edge mode possesses the anomalous $\mathbb{Z}_3$ symmetry itself called 't Hooft anomaly [62]. Generally speaking, the 't Hooft anomaly is the obstruction to introducing a gauge symmetry with a given discrete group into the system, which results in irreducible topological terms. In our case, it is the initial $\mathbb{Z}_3$ symmetry. Let us consider the uniform symmetry operator S to see the anomaly. The transformation of Eq.5 affects the symmetry group $\{\mathbb{I}, S, S^\dagger\}$ as well: $\mathbb{I} \to \mathbb{I}$, $S \to V^\dagger S$, $S^\dagger \to S^\dagger V$. Note that in this section, we only consider the boundary Hamiltonian, so the subscript "∂" is dropped everywhere.

The presence of an anomaly indicates that we should pass to the projective representation of the group based on the modification of the associativity condition of group multiplication. The physical picture for non-trivial SPT phases looks as follows [63]. The ground state of the system consists of different regions separated by domain walls, representing defect lines of the discrete symmetry realizations. Moreover, the states on defect lines themselves are the realization of SPT phases in one lower dimension.

As a result of our symmetrization procedure, the factor $S = \prod_p X_p$ commutes trivially with the Hamiltonian. One can check that V also commutes with the Hamiltonian and S. The exponent of V (12) tracks the passes between sites with states $n = 1$ and $n = 2$ throughout the boundary, so $V = \varepsilon^{\mathcal{W}}$ for the closed interval, where $\mathcal{W}$ is the winding number symmetry that

we have identified earlier. In this sense S and V form a $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry group. Summing up, the elements of the transformed symmetry group can be given by

$$\mathcal{S}(g) = V^{-g} S^g = \left(\prod_p \varepsilon^{-g n_p n_{p+1}(n_{i+1} - n_i)} \right) \cdot \left(\prod_p X_p^g \right) \tag{28}$$

with $g \in \{0, 1, 2\}$. Now let us identify the anomaly of this symmetry. Once the symmetry is considered on a finite section of the edge, the anomaly manifests as broken associativity at the endpoints of this section. The implied locality of $\mathcal{S}$ allows us to consider a single endpoint at a time [64,65]. First, we restrict the symmetry operators to a half-infinite interval $p \in (0, 1, \dots)$

$$\mathcal{S}_r(g) = \left(\prod_{p=0}^{\infty} \varepsilon^{-g n_p n_{p+1}(n_{p+1} - n_p)} \right) \cdot \left(\prod_{p=0}^{\infty} X_p^g \right). \tag{29}$$

These operators satisfy

$$\mathcal{S}_r(g)\mathcal{S}_r(h) = A(g,h)\mathcal{S}_r(gh), \quad A(g,h) = \varepsilon^{hg n_0^2 + h g^2 n_0}. \tag{30}$$

The anomaly is then given by the extra phase factor, ω_3, in the associativity condition

$$\mathcal{S}_r(g)(\mathcal{S}_r(h)\mathcal{S}_r(k)) = \omega_3(g,h,k)(\mathcal{S}_r(g)\mathcal{S}_r(h))\mathcal{S}_r(k), \tag{31}$$

which indicates that $\mathcal{S}_r$ is a projective representation of the $\mathbb{Z}_3$ symmetry group. The 3-cocycle ω_3 can be calculated explicitly as

$$\omega_3(g,h,k) = \frac{\mathcal{S}_r(g)A(h,k)\mathcal{S}_r^{-1}(g)A(g,hk)}{A(g,h)A(gh,k)} = \varepsilon^{ghk(g+h)}. \tag{32}$$

This is precisely the non-trivial element of cohomology group $H^3(\mathbb{Z}_3, U(1))$ that we started with.

4   Alternate forms of the Hamiltonian

The boundary Hamiltonian can be further simplified upon introducing the "wall operators," $\hat{n}_p$, and the corresponding $\hat{X}_p$ to satisfy

$$\begin{aligned} \hat{n}_p &= n_{p+1} - n_p \quad \text{mod } 3 \\ \bar{X}_p &= \hat{X}_{p-1}\hat{X}_p^{\dagger}. \end{aligned} \tag{33}$$

These operators straightforwardly obey the initial algebra of operators n_p and X_p. In terms of operators in Eq.33, H_∂ can be written as

$$H_\partial = -\sum_{p \in \partial} \hat{X}_{p-1}\hat{X}_p^{\dagger} \delta_{\hat{n}_{p-1} + \hat{n}_p} + \delta_{\hat{n}_{p-1}} \hat{X}_{p-1}\hat{X}_p^{\dagger} \delta_{\hat{n}_p} + \delta_{\hat{n}_p} \hat{X}_{p-1}\hat{X}_p^{\dagger} \delta_{\hat{n}_{p-1}} + \text{H.c.} \tag{34}$$

which is the easiest to verify by directly observing the action of different terms of Eq.16 on possible configurations $\{n_p\}$. This can be further transformed into the following form

$$H_\partial = -\sum_{p \in \partial} (\hat{X}_{p-1}\hat{X}_p^{\dagger} + \text{H.c.})\delta_{\hat{n}_{p-1} + \hat{n}_p} - (\delta_{\hat{n}_p} - \delta_{\hat{n}_{p-1}})(\hat{X}_{p-1}\hat{X}_p^{\dagger} + \text{H.c.})(\delta_{\hat{n}_p} - \delta_{\hat{n}_{p-1}}). \tag{35}$$

Here we used the fact that $\delta_{n_p} X_p \delta_{n_p} = 0$. The expression can be written in a more compact form by introducing notation for a discrete transfer operation: $(1 + \Delta)A_p = A_{p+1}$, for any

operator A_p. Note that Δ is the discrete derivative to the right, and $(1+\Delta)^{-1} \neq (1-\Delta)$, as it would be in the infinitesimal continuous case. Then the boundaru Hamiltonian acquires the following form:

$$H_{\partial} = -\sum_{p \in \partial} (2 - \hat{X}_p \overleftrightarrow{\Delta} \hat{X}_p^{\dagger}) \delta_{(\Delta-1)\hat{n}_p} + \Delta \delta_{\hat{n}_p} (2 - \hat{X}_p \overleftrightarrow{\Delta} \hat{X}_p^{\dagger}) \Delta \delta_{\hat{n}_p}. \tag{36}$$

This can be used in studies of the ground state properties and the excited states of the boundary model.

4.1   From $\mathbb{Z}_3^N$ to $\mathbb{Z}^N/3\mathbb{Z}$

The Hilbert space of our boundary model, $\mathcal{H}_{\partial}$, has a basis labeled with strings of N numbers $n_i \in \{-1, 0, 1\}$. We could expand this model to one set in a bigger Hilbert space, $\tilde{\mathcal{H}}_{\partial}$, where basis elements are strings of N arbitrary integers, up to a total shift by 3:

$$\mathcal{H}_{\partial} = L^2(\{|n_1, \ldots n_N\rangle : n \in \mathbb{Z}_3^N\}) \hookrightarrow \tilde{\mathcal{H}}_{\partial} = \frac{L^2(\text{span}\{|n_1, \ldots, n_N\rangle : n \in \mathbb{Z}\})}{S^3}. \tag{37}$$

Now, One can define operators X_p, $X_p^{\dagger}$ on $\tilde{\mathcal{H}}_{\partial}$ similarly to what was done for $\mathcal{H}_{\partial}$. Namely, they act on the states on p by increasing or decreasing n_p by 1. We already used this redefinition of X_p in Eq.37, as a part of $S = \prod_p X_p$. In the expanded Hilbert space $\tilde{H}_{\partial}$, operators X_p and n_p form another representation of the ladder algebra, because the relation $X_p^3 = 1$ is lost.

There is an immersion $\mathcal{I} : \mathcal{H}_{\partial} \hookrightarrow \tilde{\mathcal{H}}_{\partial}$, which maps the basis elements of $\mathcal{H}_{\partial}$ to the basis elements of $\tilde{\mathcal{H}}_{\partial}$ in the following manner. Let $\mathcal{I}(|n_1, \ldots, n_N\rangle) = |\tilde{n}_1, \ldots, \tilde{n}_N\rangle$, where

$$\begin{aligned} \tilde{n}_1 &= n_1, \\ \tilde{n}_k &= \tilde{n}_{k-1} + w_k, \quad k \geq 2. \end{aligned} \tag{38}$$

Reversing the procedure, we note that there is a surjective map $\mathcal{S} : \tilde{\mathcal{H}}_{\partial} \twoheadrightarrow \mathcal{H}_{\partial}$ that takes $|\tilde{n}_i\rangle$ to $|\tilde{n}_i \bmod 3\rangle$. An operator $\mathcal{P} = \mathcal{I}\mathcal{S}$ is a projection operator on $\mathbf{Im}\,\mathcal{I}$ and $\mathcal{S}\mathcal{I} = 1$.

To this end, one can define a Hamiltonian acting on the larger Hilbert space $\tilde{\mathcal{H}}_{\partial}$ as follows:

$$\tilde{H}_{\partial} = -\sum_{p \in \partial} \left[X_p^2 \delta_{n_p - n_{p-1} - 1} \delta_{n_{p+1} - n_p + 1} + X_p (1 - \delta_{n_p - n_{p-1} + 1})(1 - \delta_{n_{p+1} - n_p - 1}) \right] + \text{H.c.} \,. \tag{39}$$

It has a few special properties. First, the subspace $\mathbf{Im}\,\mathcal{P} \approx \mathcal{H}_{\partial}$ is invariant under $\tilde{H}_{\partial}$

$$[\mathcal{P}, \tilde{H}_{\partial}] = 0 \tag{40}$$

and the restriction of $\tilde{H}_{\partial}$ to $\mathcal{H}_{\partial}$ is H_{∂}:

$$H_{\partial} = \mathcal{S}\tilde{H}_{\partial}\mathcal{I}. \tag{41}$$

Second, it is defined in terms of ladder operators that act on the space of states labeled by $\mathbb{Z}$-numbers instead of $\mathbb{Z}_3$.

As the next step, let us look closely at what basis states in $\mathbf{Im}\,\mathcal{P}$ look like. Those are labeled by sets of numbers $n_1, \ldots, n_N$ that satisfy

$$|n_p - n_{p-1}| \leq 1, \quad 2 \leq p \leq N \tag{42}$$

up to a total shift by 3. The fact that $\mathbf{Im}\,\mathcal{P}$ is conserved by the Hamiltonian means that if we start with a state that satisfies the condition Eq.42, we will obtain a linear combination of states that also satisfy this condition. For those states, the winding number is just

$$\mathcal{W} = \frac{n_N - n_1 + \delta_{n_1 - n_N - 1} - \delta_{n_1 - n_N + 1}}{3} = \left[\frac{n_N - n_1 + 1}{3} \right], \tag{43}$$
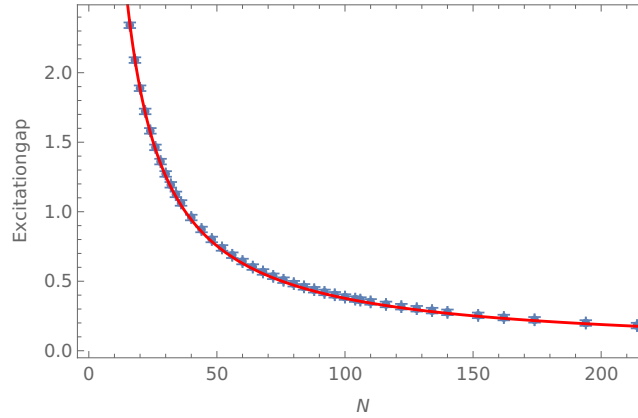
Figure 1: Results for the first excitation gap of the edge modes in open chains given by simulation based on density matrix renormalization group (DMRG) plotted versus the boundary length up to N = 214. The error bars demonstrate the precision of the calculation and are of the order 10^{-2}. The red line is the $\propto N^{-1}$ fitting curve.

where [ ] denotes the integer part. We can make this statement even stronger. The Hamiltonian acting on a basis state $|n_i\rangle$ that satisfies the condition Eq.42 produces a linear combination of all states that differ from $|n_i\rangle$ in exactly one spin, satisfy Eq.42 and have the same winding number. This proves that there are no diagonal symmetry operators other than $\mathcal{W}$, as any two states $|\Psi_1\rangle, |\Psi_2\rangle \in \mathcal{H}_\partial \approx \text{Im}\,\mathcal{P}$ with the same winding number produce a nonzero matrix element of the evolution operator:

$$\langle \Psi_2 | \exp(-iH_\partial t) |\Psi_1\rangle \neq 0 . \tag{44}$$

This shows, that the only symmetry of our Hamiltonian which is diagonal in basis of n_i operators is $\mathcal{W}$.

5 Conformal properties of the edge model

As was done in our previous study, Ref. [61], one can detect conformal properties of the edge modes by analyzing the finite-size effects of the gap and entanglement entropy of the boundary model (Eq.15). For numerical calculations, we use the transformed form, (Eq.A.3), of the Hamiltonian (Eq.15). We use DMRG to calculate the excitation gap in our model for various system sizes, N. According to [66,67], the finite size behavior of the excitation gap in a CFT reads

$$\Delta_N = \frac{2\pi x_N}{N} + \mathcal{O}(10^{-2}). \tag{45}$$

Here we found the number $x_N \simeq 2.01$, which within precision of $O(10)^{-2}$ fits with the conformal dimension $x_N = 2$ of the scaling operator that is concerned with the correlation length. The numerical data of the gap are presented in Ref. Fig.1. Calculations have been made for system sizes from $N = 20 \sim 214$. To find out the central charge of massless edge excitations, we should analyze the entanglement entropy of the ground state. We performed calculations of the entanglement entropy for open and closed chains. The results of the numerical calculations for open chains of lengths $N = 56, 84, 112, 156, 200, 256, 300$ are presented in Fig.2.

As shown in Ref. [68], the entanglement entropy in the conformal field theory with a central charge c on the open strip of length N can be determined by the following formula:

$$S_N(l) = a + \frac{c}{6} \log\left(\frac{N}{\pi} \sin\left[\frac{\pi l}{N}\right]\right). \tag{46}$$
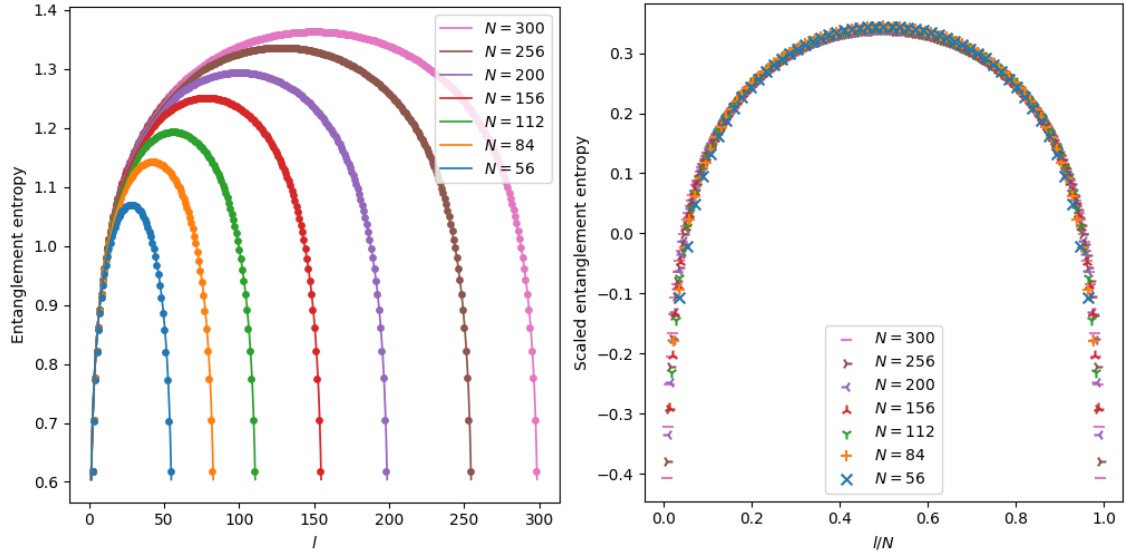
Figure 2: Points represent numerical values of the unscaled (left) and scaled (Eq.47) (right) entanglement entropy for finite open chains with lengths $N = 56, 84, 112, 156, 200, 256, 300$. Curves are defined by Eq.46 with $c = 1.06$ and $a = 0.557$.

Here a is a constant while l is a length of entangled spins. For a closed chain, the proportionality coefficient should be set to $c/3$ instead of $c/6$. This equation characterizes how the entanglement entropy of the spin chain model at criticality changes with the finite system size. Numerical values for the entanglement entropy of open chains at various values of N is shown in Fig.2. It also demonstrates the fitting of the analytic formula Eq.46 with parameters $c = 1.064$ and $a = 0.557$ as well as the collection of all numerical points of open chains on a single scaled function.

$$S_{scaled}(x) = S_N(l) - \frac{c}{6}\log[N] = a + \frac{c}{6}\log\left(\frac{1}{\pi}\sin[\pi x]\right), \quad x = \frac{l}{N}, \quad l = 2, \cdots N - 1 \, . \quad (47)$$

Similar results for closed chains are presented in Fig.3. Here, we have analyzed smaller chain lengths because the calculations take longer in this case. Fitting the results to Eq.47 but with coefficients $\frac{c}{3}$ in front of **log** we have obtained $c = 1.01$ and $a = 0.991$. Both results unambiguously show that our edge model in che continuum limit is described by a CFT with central charge $c = 1$. Thus it may be tempting to conclude that the edge model for $\mathbb{Z}_3$ paramagnets coincides with edge states of the Levin-Gu model with $\mathbb{Z}_2$ symmetry, which is defined by the XX model of free fermions [12]. This, however, is not the case, as we will show below.

Another piece of information about the low energy effective CFT can be obtained from the study of Kac-Moody currents, which can appear in the model. The winding number symmetry $\mathcal{W}$ forms a $U(1)$ current $j_p^\mu = (q_p, m_p)$ and it is necessary to check if it can be holomorphically factorised into chiral $U(1)$ currents, $j_p^\pm = q_p \pm m_p$, forming a Kac-Moody algebra. For the latter to form, the currents must satisfy commutativity relations,

$$[j_p^+, j_{p'}^-] = 0, \quad (48)$$

in thermodynamic limit. This is necessary for the currents to be independent. Using the exact diagonalization approach, we numerically calculate matrix elements of the commutators mentioned above between low energy states for relatively short system sizes. Our results for
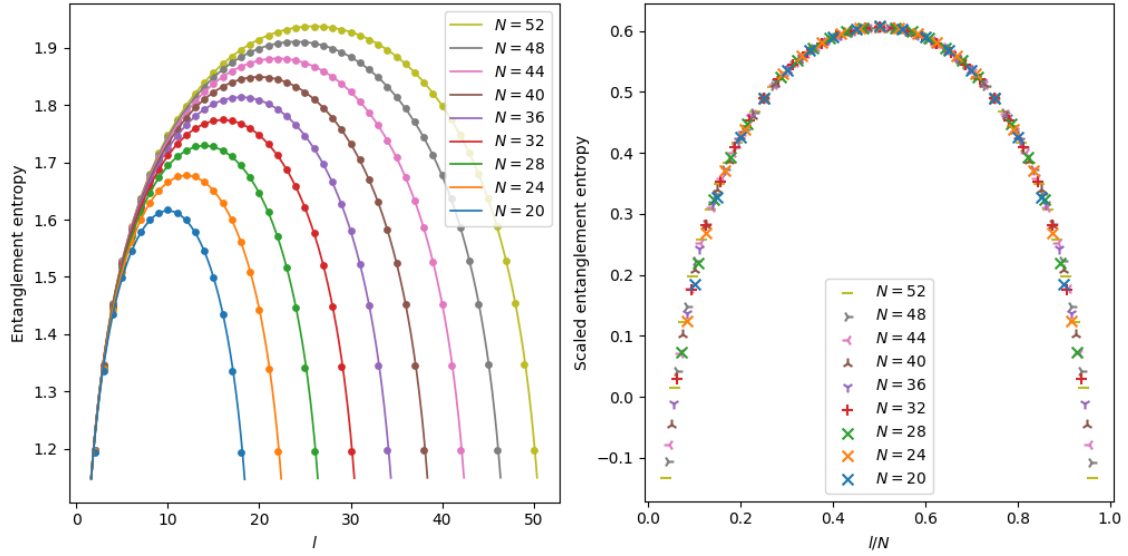
Figure 3: Points represent numerical values of the unscaled (left) and scaled (right) entanglement entropy for finite closed chains with lengths $N = 20, 24, ..., 52$. Curves are defined by Eq.46 but with coefficient $\frac{c}{3}$ instead of $\frac{c}{6}$, with $c = 1.01$ and $a = 0.991$.

systems of sizes $N = 4, 6, ..., 14$ show that commutators with $p' = p$ are numerically 0. In the $p' = p + 1$ case, they decrease as $\sim N^{-1.35}$, and are exactly 0 otherwise. Calculations are presented in Fig.4

The next step is to check the holomorphicity condition:

$$\partial_- j^+ = \partial_+ j^- = 0, \tag{49}$$

with $\partial_\pm = \partial_t \pm \nabla_p$. After excluding the analytically known relation $\partial_- j^+ + \partial_+ j^- = 2\partial_\mu j^\mu = 0$ we are left with equation $\partial_- j^+ - \partial_+ j^- = 0$. The latter can be written as

$$i[H, m_p] - (q_{p+1} - q_{p-1}) = 0 \tag{50}$$

in our discrete case. We use the same numerical approach to test the validity of the conjecture in the *thermodynamic* limit. The dependence of matrix elements for both the commutator and the difference terms on system size N for $N \in \{3, 4, ..., 15\}$ is presented in Fig.4. It has some instabilities, but it is clear that their upper limit exponentially decreases as $e^{-N/6}$, implying the same decay for the overall expression.

And at last, after verifying the existence of Kac-Moody currents, we have to check the corresponding anomaly, given by the momentum space current commutators $[j_n^\pm, j_{-n}^\pm] = nk$ with $j_n^\pm = \frac{1}{2\pi} \sum_{p=1,N} e^{\frac{\pi i n p}{N}} j_p^\pm$. Here k denotes the level of the Kac-Moody algebra and defines the anomaly. The commutator can be reduced to a simpler form using the fact that $[j_p^\pm, j_{p'}^\pm] \neq 0$ if and only if $p' = p \pm 1$. Thus

$$[j_n^+, j_{-n}^+] = \frac{i}{2\pi^2} \sin\left[\frac{2\pi n}{N}\right] \sum_p [j_p^+, j_{p+1}^+] \xrightarrow[n \ll N]{} n \frac{i \sum_p [j_p^+, j_{p+1}^+]}{\pi N} . \tag{51}$$

Resorting again to numerics, we are able to determine the Kac-Moody algebra level. Computations similar to the above produce $k \simeq 0.95$, which reproduces $k = 1$ within precision of $O(10)^{-2}$ as shown in Fig.5.

Based on the values of c, Δ and k, it can be conjectured that the corresponding edge theory can be given by the $SU_1(3)/SU_1(2)$ coset CFT model of current algebras at level k=1. $SU_1(3)$
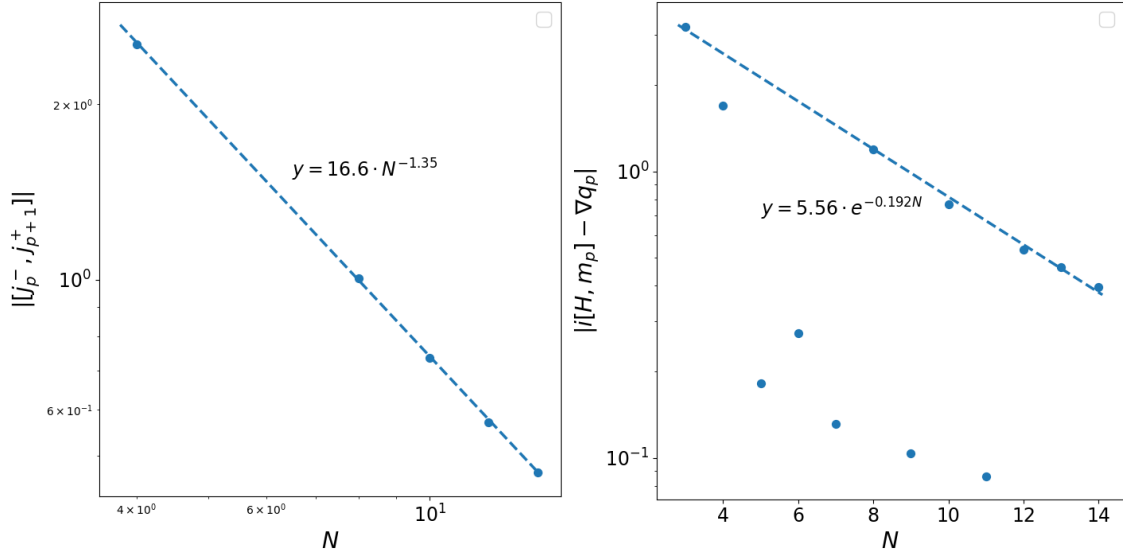
Figure 4: The finite size behavior of expressions indicating the existence of anomalous $U(1)$ symmetry of the boundary Hamiltonian. Left panel shows a power low decrease of the commutator, $[j_p^+, j_{p+1}^-]$. Right panel shows an exponential decrease of $i[H, m_p] - \nabla q_p$.
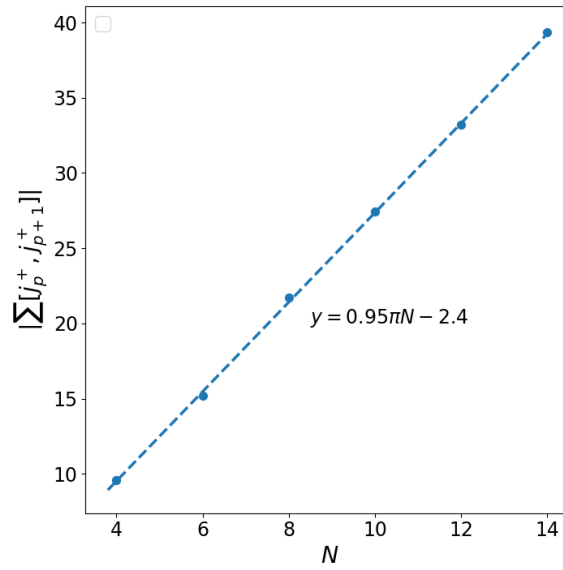
Figure 5: $|\sum_p [j_p^+, j_{p+1}^+]|$
versus system size N. The slope of the line is 0.95π, indicating that current anomaly $k \approx 1$.

contains two $U_1(1)$ subgroups at level k=1, one of which will be gauged out as a subgroup of denominator group $SU_1(2)$, while the next will remain. It is clear that this $U(1)$ is the maximal remaining subgroup in the factor, and it is linked to anomalous winding number symmetry observed above as $U(1)$ Kac-Moody at level k=1. One can expect that 't Hooft anomalous $\mathbb{Z}_3$, presented in our boundary model (16), can be a subgroup of this remained $U_1(1)$ Kac-Moody, since, as it is well known, central extensions in current algebra can be regarded as projective representations of $U(1)$ symmetry. This means that a phase factor will appear in the partition function of our boundary model after $U(1)$ symmetry transformation, as 't Hooft $\mathbb{Z}_3$ anomaly should appear. Of course, precise observation of this phase factor is necessary, which is a tusk for subsequent studies. The 't Hooft anomaly in the low energy model can also be detected in other ways, presented in Refs. [64, 69].

6   Concluding remarks

We have studied the three-state paramagnetic Potts model with $\mathbb{Z}_3$ symmetry on a 2D triangular lattice. According to expectations, different SPT phases are classified by the cohomology group $H^3(\mathbb{Z}_3, U(1)) = \mathbb{Z}_3$. We have constructed the unitary operator responsible for this phase and found the Hamiltonian of edge states. Numerical study of finite size effects of the edge Hamiltonian showed that it is gapless and the conformal dimension of the scaling operator, which defines the correlation length, is $\Delta_N = 2$. Calculation of the entanglement entropy of low-lying excitation shows that it has a central charge $c = 1$. We also find hidden $U(1)$ symmetry of the Hamiltonian corresponding to the winding number, which, as it appeared, leads to anomalous $U(1)$ Kac-Moody current algebra with level k=1. This analysis points out that the effective theory of low energy excitations of our edge Hamiltonian is the coset CFT $SU_k(3)/SU_k(2)$ with k=1. This coset contains $U(1)$ anomalous subalgebra.

The incorporation of SPT degrees of freedom into the three-state Potts paramagnet opens new avenues for understanding phase transitions, anyonic excitations, and the interplay between topology and quantum information. Experimental realizations of the SPT Potts paramagnet, such as in cold atom systems or magnetic materials, are still an open problem. The present work opens up a possibility for developing experimental platforms for probing and manipulating the topological aspects of $\mathbb{Z}_3$ SPT paramagnets.

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A   Identities

One can check that the following identities hold for $n_1, n_2, n_3 \in \{-1, 0, 1\}$ and $\varepsilon = e^{2\pi i/3}$

$$\varepsilon^{n_1 n_2(n_2-n_1)} = \frac{1}{3}\left(\varepsilon^{n_1-n_2} - \varepsilon^{n_1+n_2} - \varepsilon^{-n_1-n_2} + \varepsilon^{-n_1} + \varepsilon^{n_2} + 2\right), \quad \text{(A.1)}$$

$$\varepsilon^{n_1 n_2} = \frac{1}{3}\left(1 + \varepsilon^{n_1} + \varepsilon^{-n_1} + \varepsilon^{n_2} + \varepsilon^{-n_2} + \varepsilon^{n_1} + \varepsilon^{n_1+n_2-1} + \right.$$
$$\left. + \varepsilon^{-n_1-n_2-1} + \varepsilon^{1+n_1-n_2} + \varepsilon^{1-n_1+n_2}\right), \quad \text{(A.2)}$$

$$1 + \varepsilon^{(n_1-n_3)(1+n_1+n_2+n_3)} + \varepsilon^{-(n_1-n_3)(1+n_1+n_2+n_3)} =$$
$$= \frac{1}{3}\left[\frac{5}{2} - \varepsilon^{1-n_1+n_2} - \varepsilon^{1-n_3+n_2} + 2\varepsilon^{n_1-n_3} + 2\varepsilon^{1+n_1+n_2+n_3}\right] + c.c. . \quad \text{(A.3)}$$

These identities can be used to reduce two-site interactions of given forms to the sum of single-site operator products. It proves useful for computational purposes.

References

[1] Z.-C. Gu and X.-G. Wen, *Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order,* Phys. Rev. B **80**, 155131 (2009), doi:10.1103/PhysRevB.80.155131.

[2] T. Senthil, *Symmetry-protected topological phases of quantum matter,* Annual Review of Condensed Matter Physics **6**(1), 299 (2015), doi:10.1146/annurev-conmatphys-031214-014740.

[3] X.-G. Wen, *Colloquium: Zoo of quantum-topological phases of matter,* Rev. Mod. Phys. **89**, 041004 (2017), doi:10.1103/RevModPhys.89.041004.

[4] J. Li, A. Chan and T. B. Wahl, *Classification of symmetry-protected topological phases in two-dimensional many-body localized systems,* Phys. Rev. B **102**, 014205 (2020), doi:10.1103/PhysRevB.102.014205.

[5] H. Song, S.-J. Huang, L. Fu and M. Hermele, *Topological phases protected by point group symmetry,* Phys. Rev. X **7**, 011020 (2017), doi:10.1103/PhysRevX.7.011020.

[6] X. Chen, Z.-C. Gu and X.-G. Wen, *Classification of gapped symmetric phases in one-dimensional spin systems,* Phys. Rev. B **83**, 035107 (2011), doi:10.1103/PhysRevB.83.035107.

[7] X. Chen, Z.-C. Gu and X.-G. Wen, *Complete classification of one-dimensional gapped quantum phases in interacting spin systems,* Phys. Rev. B **84**, 235128 (2011), doi:10.1103/PhysRevB.84.235128.

[8] X. Chen, Z.-C. Gu and X.-G. Wen, *Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order,* Phys. Rev. B **82**, 155138 (2010), doi:10.1103/PhysRevB.82.155138.

[9] F. Pollmann, E. Berg, A. M. Turner and M. Oshikawa, *Symmetry protection of topological phases in one-dimensional quantum spin systems,* Phys. Rev. B **85**, 075125 (2012), doi:10.1103/PhysRevB.85.075125.

[10] F. Pollmann, A. M. Turner, E. Berg and M. Oshikawa, *Entanglement spectrum of a topological phase in one dimension*, Phys. Rev. B **81**, 064439 (2010), doi:10.1103/PhysRevB.81.064439.

[11] X. Chen, Z.-C. Gu, Z.-X. Liu and X.-G. Wen, *Symmetry-protected topological orders in interacting bosonic systems*, Science **338**(6114), 1604 (2012), doi:10.1126/science.1227224.

[12] M. Levin and Z.-C. Gu, *Braiding statistics approach to symmetry-protected topological phases*, Phys. Rev. B **86**, 115109 (2012), doi:10.1103/PhysRevB.86.115109.

[13] X. Chen, Z.-C. Gu, Z.-X. Liu and X.-G. Wen, *Symmetry protected topological orders and the group cohomology of their symmetry group*, Phys. Rev. B **87**, 155114 (2013), doi:10.1103/PhysRevB.87.155114.

[14] Z.-C. Gu and X.-G. Wen, *Symmetry-protected topological orders for interacting fermions: Fermionic topological nonlinear σ models and a special group supercohomology theory*, Phys. Rev. B **90**, 115141 (2014), doi:10.1103/PhysRevB.90.115141.

[15] A. Kapustin, R. Thorngren, A. Turzillo and Z. Wang, *Fermionic symmetry protected topological phases and cobordisms*, Journal of High Energy Physics **2015**(12), 1 (2015), doi:10.1007/JHEP12(2015)052.

[16] A. Kapustin and R. Thorngren, *Fermionic spt phases in higher dimensions and bosonization*, Journal of High Energy Physics **2017**(10), 80 (2017), doi:10.1007/JHEP10(2017)080.

[17] D. Gaiotto and T. Johnson-Freyd, *Symmetry protected topological phases and generalized cohomology*, Journal of High Energy Physics **2019**(5), 7 (2019), doi:10.1007/JHEP05(2019)007.

[18] J. C. Wang, Z.-C. Gu and X.-G. Wen, *Field-theory representation of gauge-gravity symmetry-protected topological invariants, group cohomology, and beyond*, Phys. Rev. Lett. **114**, 031601 (2015), doi:10.1103/PhysRevLett.114.031601.

[19] T. Morimoto, H. Ueda, T. Momoi and A. Furusaki, *$\digamma_3$ symmetry-protected topological phases in the su(3) aklt model*, Phys. Rev. B **90**, 235111 (2014), doi:10.1103/PhysRevB.90.235111.

[20] L. H. Santos and J. Wang, *Symmetry-protected many-body aharonov-bohm effect*, Phys. Rev. B **89**, 195122 (2014), doi:10.1103/PhysRevB.89.195122.

[21] J. C. Wang, L. H. Santos and X.-G. Wen, *Bosonic anomalies, induced fractional quantum numbers, and degenerate zero modes: The anomalous edge physics of symmetry-protected topological states*, Phys. Rev. B **91**, 195134 (2015), doi:10.1103/PhysRevB.91.195134.

[22] M. Cheng and D. J. Williamson, *Relative anomaly in $(1+1)d$ rational conformal field theory*, Phys. Rev. Res. **2**, 043044 (2020), doi:10.1103/PhysRevResearch.2.043044.

[23] Y. Alavirad and M. Barkeshli, *Anomalies and unusual stability of multicomponent luttinger liquids in $\digamma_n \times \digamma_n$ spin chains*, Phys. Rev. B **104**, 045151 (2021), doi:10.1103/PhysRevB.104.045151.

[24] R. A. Lanzetta and L. Fidkowski, *Bootstrapping lieb-schultz-mattis anomalies*, Phys. Rev. B **107**, 205137 (2023), doi:10.1103/PhysRevB.107.205137.

[25] B. Yoshida, *Topological color code and symmetry-protected topological phases*, Phys. Rev. B **91**, 245131 (2015), doi:10.1103/PhysRevB.91.245131.

[26] B. Yoshida, *Topological phases with generalized global symmetries*, Phys. Rev. B **93**, 155131 (2016), doi:10.1103/PhysRevB.93.155131.

[27] B. Yoshida, *Gapped boundaries, group cohomology and fault-tolerant logical gates*, Annals of Physics **377**, 387 (2017), doi:10.1016/j.aop.2016.12.014.

[28] Y. Hu, Y. Wan and Y.-S. Wu, *Twisted quantum double model of topological phases in two dimensions*, Phys. Rev. B **87**, 125114 (2013), doi:10.1103/PhysRevB.87.125114.

[29] L. Tsui, Y.-T. Huang, H.-C. Jiang and D.-H. Lee, *The phase transitions between zn×zn bosonic topological phases in 1+1d, and a constraint on the central charge for the critical points between bosonic symmetry protected topological phases*, Nuclear Physics B **919**, 470 (2017), doi:10.1016/j.nuclphysb.2017.03.021.

[30] N. Tantivasadakarn, *Dimensional reduction and topological invariants of symmetry-protected topological phases*, Phys. Rev. B **96**, 195101 (2017), doi:10.1103/PhysRevB.96.195101.

[31] T. D. Ellison, K. Kato, Z.-W. Liu and T. H. Hsieh, *Symmetry-protected sign problem and magic in quantum phases of matter*, Quantum **5**, 612 (2021), doi:10.22331/q-2021-12-28-612.

[32] Y. Chen, A. Prakash and T.-C. Wei, *Universal quantum computing using $(\mathbb{Z}_d)^3$ symmetry-protected topologically ordered states*, Phys. Rev. A **97**, 022305 (2018), doi:10.1103/PhysRevA.97.022305.

[33] S. D. Geraedts and O. I. Motrunich, *Exact models for symmetry-protected topological phases in one dimension* (2014), arXiv:1410.1580.

[34] C. Wang, C.-H. Lin and M. Levin, *Bulk-boundary correspondence for three-dimensional symmetry-protected topological phases*, Phys. Rev. X **6**, 021015 (2016), doi:10.1103/PhysRevX.6.021015.

[35] Q.-R. Wang and Z.-C. Gu, *Towards a complete classification of symmetry-protected topological phases for interacting fermions in three dimensions and a general group supercohomology theory*, Phys. Rev. X **8**, 011055 (2018), doi:10.1103/PhysRevX.8.011055.

[36] E. O'Brien, E. Vernier and P. Fendley, *"not-a", representation symmetry-protected topological, and potts phases in an S_3-invariant chain*, Phys. Rev. B **101**, 235108 (2020), doi:10.1103/PhysRevB.101.235108.

[37] N. Tantivasadakarn, R. Thorngren, A. Vishwanath and R. Verresen, *Long-range entanglement from measuring symmetry-protected topological phases* (2022), arXiv:2112.01519.

[38] T. A. Sedrakyan, V. M. Galitski and A. Kamenev, *Topological spin ordering via chern-simons superconductivity*, Phys. Rev. B **95**, 094511 (2017), doi:10.1103/PhysRevB.95.094511.

[39] K. Wang and T. A. Sedrakyan, *Universal finite-size scaling around tricriticality between topologically ordered, symmetry-protected topological, and trivial phases*, Phys. Rev. B **101**, 035410 (2020), doi:10.1103/PhysRevB.101.035410.

[40] K. Wang and T. A. Sedrakyan, *Universal finite-size amplitude and anomalous entanglement entropy of $z = 2$ quantum Lifshitz criticalities in topological chains*, SciPost Phys. **12**, 134 (2022), doi:10.21468/SciPostPhys.12.4.134.

[41] R. Wang, B. Wang and T. Sedrakyan, *Chern-simons superconductors and their instabilities*, Phys. Rev. B **105**, 054404 (2022), doi:10.1103/PhysRevB.105.054404.

[42] L. D. Landau, *On the theory of phase transitions*, Zh. Eksp. Teor. Fiz. **7**, 19 (1937), doi:10.1016/B978-0-08-010586-4.50034-1.

[43] L. Landau and L. E.M., *Statistical Physics. Vol. 5*, Elsevier Science, ISBN 9780080570464 (2013).

[44] X.-G. Wen, *Topological order: From long-range entangled quantum matter to a unified origin of light and electrons*, ISRN Condensed Matter Physics **2013**, 198710 (2013), doi:10.1155/2013/198710.

[45] M. Levin and X.-G. Wen, *Detecting topological order in a ground state wave function*, Phys. Rev. Lett. **96**, 110405 (2006), doi:10.1103/PhysRevLett.96.110405.

[46] M. Oshikawa and T. Senthil, *Fractionalization, topological order, and quasiparticle statistics*, Phys. Rev. Lett. **96**, 060601 (2006), doi:10.1103/PhysRevLett.96.060601.

[47] T. A. Sedrakyan, V. M. Galitski and A. Kamenev, *Statistical transmutation in floquet driven optical lattices*, Phys. Rev. Lett. **115**, 195301 (2015), doi:10.1103/PhysRevLett.115.195301.

[48] T. A. Sedrakyan, L. I. Glazman and A. Kamenev, *Spontaneous formation of a nonuniform chiral spin liquid in a moat-band lattice*, Phys. Rev. Lett. **114**, 037203 (2015), doi:10.1103/PhysRevLett.114.037203.

[49] S. Maiti and T. Sedrakyan, *Fermionization of bosons in a flat band*, Phys. Rev. B **99**, 174418 (2019), doi:10.1103/PhysRevB.99.174418.

[50] R. Wang, Z. Y. Xie, B. Wang and T. Sedrakyan, *Emergent topological orders and phase transitions in lattice chern-simons theory of quantum magnets*, Phys. Rev. B **106**, L121117 (2022), doi:10.1103/PhysRevB.106.L121117.

[51] E. Fradkin and S. H. Shenker, *Phase diagrams of lattice gauge theories with higgs fields*, Phys. Rev. D **19**, 3682 (1979), doi:10.1103/PhysRevD.19.3682.

[52] E. Ardonne, P. Fendley and E. Fradkin, *Topological order and conformal quantum critical points*, Annals of Physics **310**(2), 493–551 (2004), doi:10.1016/j.aop.2004.01.004.

[53] S. Sachdev, *Topological order, emergent gauge fields, and fermi surface reconstruction*, Reports on Progress in Physics **82**(1), 014001 (2018), doi:10.1088/1361-6633/aae110.

[54] L. Kong and H. Zheng, *Categories of quantum liquids i*, Journal of High Energy Physics **2022**(8), 70 (2022), doi:10.1007/JHEP08(2022)070.

[55] T. A. Sedrakyan, A. Kamenev and L. I. Glazman, *Composite fermion state of spin-orbit-coupled bosons*, Phys. Rev. A **86**, 063639 (2012), doi:10.1103/PhysRevA.86.063639.

[56] T. A. Sedrakyan, L. I. Glazman and A. Kamenev, *Absence of bose condensation on lattices with moat bands*, Phys. Rev. B **89**, 201112 (2014), doi:10.1103/PhysRevB.89.201112.

[57] C. Wei and T. A. Sedrakyan, *Chiral spin liquid state of strongly interacting bosons with a moat dispersion: A monte carlo simulation*, Annals of Physics **456**, 169354 (2023), doi:10.1016/j.aop.2023.169354.

[58] R. Wang, T. A. Sedrakyan, B. Wang, L. Du and R.-R. Du, *Excitonic topological order in imbalanced electron–hole bilayers*, Nature **619**(7968), 57 (2023), doi:10.1038/s41586-023-06065-w.

[59] N. Schuch, D. Pérez-García and I. Cirac, *Classifying quantum phases using matrix product states and projected entangled pair states*, Phys. Rev. B **84**, 165139 (2011), doi:10.1103/PhysRevB.84.165139.

[60] L. Fidkowski and A. Kitaev, *Topological phases of fermions in one dimension*, Phys. Rev. B **83**, 075103 (2011), doi:10.1103/PhysRevB.83.075103.

[61] H. Topchyan, V. Iugov, M. Mirumyan, S. A. Khachatryan, T. S. Hakobyan and T. A. Sedrakyan, z_3 *and* $(\times z_3)^3$ *symmetry protected topological paramagnets*, Journal of High Energy Physics (in press) (2023), arXiv:2210.01187.

[62] G. 't Hooft, *Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking*, NATO Sci. Ser. B **59**, 135 (1980), doi:10.1007/978-1-4684-7571-5_9.

[63] X. Chen, Y.-M. Lu and A. Vishwanath, *Symmetry-protected topological phases from decorated domain walls*, Nature Communications **5**(1), 3507 (2014), doi:10.1038/ncomms4507.

[64] D. V. Else and C. Nayak, *Classifying symmetry-protected topological phases through the anomalous action of the symmetry on the edge*, Phys. Rev. B **90**, 235137 (2014), doi:10.1103/PhysRevB.90.235137.

[65] L. Li, M. Oshikawa and Y. Zheng, *Decorated defect construction of gapless-spt states* (2023), arXiv:2204.03131.

[66] J. L. Cardy, *Conformal invariance and universality in finite-size scaling*, Journal of Physics A: Mathematical and General **17**(7), L385 (1984), doi:10.1088/0305-4470/17/7/003.

[67] J. L. Cardy, *Operator content of two-dimensional conformally invariant theories*, Nuclear Physics B **270**, 186 (1986), doi:10.1016/0550-3213(86)90552-3.

[68] P. Calabrese and J. Cardy, *Entanglement entropy and conformal field theory*, Journal of Physics A: Mathematical and Theoretical **42**(50), 504005 (2009), doi:10.1088/1751-8113/42/50/504005.

[69] K. Kawagoe and M. Levin, *Anomalies in bosonic symmetry-protected topological edge theories: Connection to f symbols and a method of calculation*, Phys. Rev. B **104**, 115156 (2021), doi:10.1103/PhysRevB.104.115156.