Space-time first-order correlations of an open Bose Hubbard model with incoherent pump and loss

Martina Zündel[1*], Leonardo Mazza [2], Léonie Canet[1,3], Anna Minguzzi[1]

1 Univ. Grenoble Alpes, CNRS, LPMMC, 38000 Grenoble, France
2 Université Paris-Saclay, CNRS, LPTMS, 91405, Orsay, France
3 Institut Universitaire de France, 5 rue Descartes, 75005 Paris
*martina.zuendel@lpmmc.cnrs.fr

Abstract

We investigate the correlation properties in the steady state of driven-dissipative interacting bosonic systems in the quantum regime, as for example non-linear photonic cavities. Specifically, we consider the Bose-Hubbard model on a periodic chain and with spatially homogeneous one-body loss and pump within the Markovian approximation. The steady state corresponds to an infinite temperature state at finite chemical potential with diagonal spatial correlations. Nonetheless, we observe a nontrivial behaviour of the space-time two-point correlation function in the steady state, obtained by exact diagonalisation. In particular, we find that the decay width of the propagator is not only renormalised at increasing interactions, as it is the case of a single non-linear resonator, but also at increasing hopping strength. We then compute the full spectral function, finding that it contains both a dispersive free-particle like dispersion at low energy and a doublon branch at energy corresponding to the on-site interactions. We compare with the corresponding calculation for the ground state of a closed quantum system and show that the driven-dissipative nature – determining both the steady state and the dynamical evolution – changes the low-lying part of the spectrum, where noticeably, the dispersion is quadratic instead of linear at small wavevectors.

1   Introduction

Closed quantum systems under a unitary time evolution have been largely studied in the last two decades, with particular impetus after the experimental reach of the strongly interacting regime both in three-dimensional optical lattices, as well as in lower-dimensional setups [1–4]. One-dimensional interacting quantum systems have attracted much attention since a wealth of theoretical techniques are available, including in particular exact solutions by Bethe Ansatz for homogeneous systems with contact interactions for bosonic or fermionic gases, or mixtures thereof [5–9], and for trapped systems in the limit of infinitely repulsive interactions [10–14]. While lattice fermions are integrable [15], lattice bosons are not, except for the case of the hard-core bosons (see e.g. [16–18]), and the Bose-Hubbard model for two particles [19, 20]. A manifold of attempts to effectively

describe non-integrable or 'weakly' non-integrable systems have been put forward, such as the generalised Gibbs ensemble [21], generalised hydrodynamics [22, 23], macroscopic fluctuation theory [24], and these fields are under active investigation.

While all these studies concern closed quantum systems, in several experimental situations quantum systems are subjected to external pump and/or losses, or put in contact with some type of bath. Examples of bosonic open systems are Josephson junction arrays [25], superconducting microwave circuits [26], polariton chains [27], atoms or ions in optical cavities [28, 29], optomechanical systems [30], and lossy quantum gases [31–41]. In contrast to the equilibrium case, much less is known for interacting open quantum systems, and several questions arise, ranging from the properties of the steady state and its phase diagram to its coherence, its excitations and its dynamical properties. It is also very interesting to investigate which properties known for unitary systems are still present in open quantum systems, and under which conditions or parameters.

A way to model an open system is to describe it as a subsystem embedded in a larger system acting as a bath and with which it interacts. If one requires the time evolution of the subsystem to be completely-positive and trace-preserving (CPTP map), this evolution has to be of the Lindblad form [42], and described by a Markovian quantum master equation. This evolution follows the composition law for universal dynamical maps which is the quantum analogue to the classical differential Chapman-Kolmogorov equation [43]. Despite its simplicity, the Lindblad master equation suffices to display rich physical behaviour, as it has been demonstrated for the case of the steady state of interacting driven-dissipative bosons [44, 45]. Quadratic Lindbladians, meaning Lindbladians with quadratic Hamiltonian and linear jump operators in the bosonic (fermionic) creation and annihilation operators, are known to be exactly solvable by the method of third quantisation, presented by Prosen [46] and Prosen and Seligman [47]. Within the Keldysh formalism these systems correspond to a quadratic action. Also instances of quasi-free Lindbladians for fermions and bosons with quadratic hermitean jump operators have been solved [48, 49]. Recently, the connection between both formalisms and the phase-space formulation has been pioneered [50]. Non-quadratic Lindbladians require in most cases a numerical solution. We refer to [51] for a comparative analysis of the state-of-art numerical methods, specifically tailored for non-equal time correlation functions. This method was used to obtain the second-order correlation function $g^{(2)}$ [52].

An especially important quantity to characterise a quantum system is its two-point correlation function, corresponding to first-order correlations $g^{(1)}$, see e.g. [53]. At equal times, it contains information about the spatial coherence and the presence of (quasi)off-diagonal long-range order, i.e. the existence of Bose-Einstein condensation according to the Penrose Onsager criterion [54]. Its Fourier transform yields the momentum distribution function, giving information on the velocity of the quantum particles in a given quantum state. The full space-time correlation function describes the evolution of the quantum system when removing or adding a particle to it. In space-time, it allows one to study the spread of correlations, and to test the Lieb-Robinson bound [55], and its generalisation to open quantum systems [56]. Its Fourier transform in frequency-wavevector space provides the spectral function, whose poles give the dispersion of single-particle excitations. The spectral function is also needed to describe the transport properties among two reservoirs across a quantum channel within linear response [57].

In this work, we study the two-point correlation function of an interacting bosonic open quantum system described by the Bose-Hubbard model subjected to incoherent drive and losses. At non-zero interactions, similarly to its closed-system counterpart, this model is not integrable. We choose the case of spatially uniform pump and losses, where the non-equilibrium steady-state (NESS) density matrix is known exactly [58], and it is closely
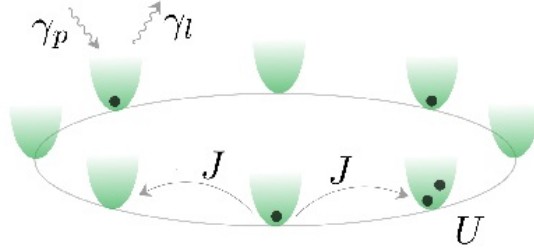
Figure 1: Sketch of the Bose-Hubbard model with periodic boundary conditions, nearest-neighbour hopping amplitude J and on-site interaction strength U, coupled to spatially homogeneous Markovian baths with pump rate γ_p and loss rate γ_l. The illustrated couplings apply to all sites.

related to an infinite-temperature state at fixed chemical potential, independently of the interaction strength. Despite the steady state taking a simple form, we find that the two-point correlation function shows a non-trivial dependence on the interaction strength, displaying in particular typical temporal oscillations with frequency related to the interaction strength as well as an interaction-dependent exponential decay. We compare these results to the exactly known non-interacting limit as well as to the strongly interacting limit where the dynamics of the different sites decouples and can be described using the exact results for the dissipative Kerr resonator [59]. We then proceed in obtaining the full spectral function, where in particular we find a doublon-like branch at excitation energy U as well as a cosine-like excitation branch in the single-particle-hole sector. The features of the spectral function are a clear signature of the non-equilibrium nature of the quantum system, further supported by a Keldysh action analysis, and by the comparison with the closed-quantum system spectral function [60].

The paper is organised as follows. We start by presenting the model and the characterisation of the non-equilibrium steady state in Sec. 2, which includes the Lindblad equation in Sec. 2.1 and the Keldysh action in Sec. 2.3. We first provide our predictions for the particle density as a function of time in Sec. 3. We then proceed to present our results for the space-time correlations in Sec. 4, where we display the time-resolved correlations and analyse their decay. The spectral function is shown in Sec. 5. Finally, Sec. 6 presents our concluding remarks.

2 Model and observables

2.1 Lindblad equation

We consider a lattice of L sites occupied by bosonic particles with on-site repulsive interactions, where each site is homogeneously coupled to a bath allowing for one-body losses and pump. The corresponding unitary evolution is described by the Bose-Hubbard Hamiltonian:

$$H = \sum_{i=1}^{L} \left[- J\left(b_i^\dagger b_{i+1} + \text{h.c.}\right) + \frac{U}{2} b_i^\dagger b_i^\dagger b_i b_i \right] , \tag{1}$$

where J is the hopping amplitude and U the interaction strength. The bosonic operators b_i and $b_i^\dagger$ fulfill the usual commutation relations $[b_i, b_j^\dagger] = \delta_{ij}$, $[b_i, b_j] = 0 = [b_i^\dagger, b_j^\dagger]$, and we take periodic boundary conditions, i.e. we set $L + l \equiv l$.

The full temporal evolution is described by the Lindblad – Gorini-Kossakowski-Sudarsahan equation for the system density matrix ρ [42, 61]:

$$\partial_t\rho = \mathcal{L}[\rho] = -\mathrm{i}[H,\rho] + \gamma_l \sum_{i=1}^{L} \left(b_i\rho b_i^\dagger - \frac{1}{2}\{b_i^\dagger b_i, \rho\}\right) + \gamma_p \sum_{i=1}^{L}\left(b_i^\dagger\rho b_i - \frac{1}{2}\{b_i b_i^\dagger, \rho\}\right). \qquad (2)$$

The master equation (2) can be obtained from a microscopic model where the system is weakly coupled to a bath, under the assumption that time scales of system and bath are well separated and that the correlations built in the bath do not affect the system (see e.g. [43] for a detailed derivation).

2.2   Correlation functions

The two-point correlation function of interest in this work is the retarded Green's function

$$G_{j\ell}^{R}(t,t') = -\mathrm{i}\theta(t-t')\Big\langle [b_j(t), b_\ell^\dagger(t')]\Big\rangle, \qquad (3)$$

where $\langle ... \rangle$ indicates the trace weighted with the respective density matrix, $\theta(t)$ is the Heavyside step function, and $b_j(t)$ ($b_j^\dagger(t)$) is the bosonic annihilation (creation) field operator in the Heisenberg picture, time-evolved according to Lindbladian dynamics[1] (in the open case), or Hamiltonian one (in the closed case).

In order to obtain the retarded Green's function for the non-equilibrium steady state (NESS) of the driven-dissipative system, we evaluate unequal time correlation functions, such as

$$\Big\langle b_j^\dagger(t)b_0\Big\rangle_{\mathrm{NESS}}^{(\mathcal{L})} \equiv \mathrm{Tr}\Big\{b_j^\dagger \mathrm{e}^{\mathcal{L}t}\Big[b_0\rho_{\mathrm{NESS}}\Big]\Big\}. \qquad (4)$$

It will be useful to compare it with the correlation function obtained following a unitary evolution starting from the same NESS density matrix:

$$\Big\langle b_j^\dagger(t)b_0\Big\rangle_{\mathrm{NESS}}^{(H)} \equiv \mathrm{Tr}\Big\{\mathrm{e}^{\mathrm{i}Ht}\,b_j^\dagger \mathrm{e}^{-\mathrm{i}Ht}\,b_0\rho_{\mathrm{NESS}}\Big\}. \qquad (5)$$

For a closed system the corresponding correlation function evaluated on the ground state $|\Psi_0\rangle$ reads

$$\Big\langle b_j^\dagger(t)b_0\Big\rangle_{\mathrm{GS}} \equiv \langle\Psi_0|\mathrm{e}^{\mathrm{i}Ht}\,b_j^\dagger\mathrm{e}^{-\mathrm{i}Ht}\,b_0|\Psi_0\rangle. \qquad (6)$$

We finally consider the spectral function, whose poles provide the excitation branches of the system, and which is defined as

$$A(\omega,k) = -\frac{1}{\pi}\mathrm{Im}\,G^R(\omega,k) \qquad (7)$$

for a homogeneous system, and with the Fourier conventions

$$f(p,\omega) = \sum_{j\in\mathbb{Z}}\int_{-\infty}^{\infty} dt\, f_j(t)\, e^{ipj}e^{-i\omega t};$$

$$f_j(t) = \frac{1}{L}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, f(p,\omega)\, e^{-ipj}e^{i\omega t}. \qquad (8)$$

---

[1]In the case of open quantum systems, a way of defining the time evolution of an operator A is given by the adjoint superoperator $\bar{\mathcal{L}}$, according to $A(t) = e^{\bar{\mathcal{L}}t}[A]$ where $\bar{\mathcal{L}}[A] = +\mathrm{i}[H,A] + \gamma_l\sum_{i=1}^{L}\big(b_i^\dagger A b_i - \frac{1}{2}\{b_i^\dagger b_i, A\}\big) + \gamma_p\sum_{i=1}^{L}\big(b_i A b_i^\dagger - \frac{1}{2}\{b_i b_i^\dagger, A\}\big)$. Notice that $\mathrm{Tr}\{A(t)\rho(0)\} = \mathrm{Tr}\{A\rho(t)\}$.

2.3 Keldysh field theory

It is useful to formulate the model in terms of the Keldysh formalism, following the derivation of [45]. Starting from the Lindblad equation (2), and expanding the system density matrix onto two coherent-state bases for each space-time point identified by the fields $\Psi_j^+(t)$ ($\Psi_j^-(t)$) for upper (lower) time contours, one obtains the time evolution of the density matrix in terms of a path integral. Following the usual notations, the Keldysh action in the rotated basis $(\Psi_{cj}(t), \Psi_{qj}(t)) = \frac{1}{\sqrt{2}}(\Psi_j^+(t) + \Psi_j^-(t), \Psi_j^+(t) - \Psi_j^-(t))$ reads

$$
S = \int dt \sum_j \left[\Psi_{cj}^* \, i \left(\partial_t - \frac{\gamma_l - \gamma_p}{2} \right) \Psi_{qj} + \Psi_{qj}^* \, i \left(\partial_t + \frac{\gamma_l - \gamma_p}{2} \right) \Psi_{cj} + i(\gamma_l + \gamma_p)|\Psi_{qj}|^2 \right.
$$
$$
+ J \left(\Psi_{cj+1}^* \Psi_{qj} + \Psi_{qj+1}^* \Psi_{cj} + \Psi_{cj}^* \Psi_{qj+1} + \Psi_{qj}^* \Psi_{cj+1} \right)
$$
$$
\left. - \frac{U}{2} \left((\Psi_{cj}^2 + \Psi_{qj}^2)\Psi_{cj}^*\Psi_{qj}^* + (\Psi_{cj}^{*2} + \Psi_{qj}^{*2})\Psi_{cj}\Psi_{qj} \right) \right] . \tag{9}
$$

In the same notation, the retarded Green's function (3) reads

$$
G_{j\ell}^R(t, t') = -i\left\langle \Psi_{cj}(t)\Psi_{q\ell}^*(t') \right\rangle_c . \tag{10}
$$

The saddle-point approximation in the path integral with the action (9) allows one to study the existence and stability of a condensate. For a static homogeneous field solution, if $\gamma_p < \gamma_l$, the imaginary part of the Keldysh potential has a global minimum for $\Psi_{cj} = \Psi_{qj} = 0$, which implies that the system is in the symmetric phase without breaking of the U(1) symmetry. Hence, there is no finite order parameter. As we shall see below, this is in agreement with the exact NESS solution, which rather corresponds to an infinite-temperature state for the bosons. Notice that this is at variance with the case where both one- and two-body losses are present, where a U(1) symmetry-broken phase emerges for $\gamma_p > \gamma_l$ and a (quasi-)condensate forms at weak interactions [62].

Expanding the action around this solution to second order in the fluctuations, we obtain the inverse of the retarded sector of the propagator as

$$
[G_0^R]_{ml}^{-1}(t', t) \equiv \begin{pmatrix} \frac{\delta^2 S}{\delta\Psi_{qm}^*(t')\delta\Psi_{cl}(t)} & \frac{\delta^2 S}{\delta\Psi_{qm}^*(t')\delta\Psi_{cl}^*(t)} \\ \frac{\delta^2 S}{\delta\Psi_{qm}(t')\delta\Psi_{cl}(t)} & \frac{\delta^2 S}{\delta\Psi_{qm}(t')\delta\Psi_{cl}^*(t)} \end{pmatrix} \Bigg|_{\Psi_c = \Psi_q = 0}
$$
$$
= \begin{pmatrix} (i\partial_t + i\kappa_0)\delta_{m,l} + J(\delta_{m,l+1} + \delta_{m,l-1}) & 0 \\ 0 & -(i\partial_t + i\kappa_0)\delta_{m,l} + J(\delta_{m,l+1} + \delta_{m,l-1}) \end{pmatrix}
$$
$$
\times \delta(t - t') , \tag{11}
$$

where we defined

$$
\kappa_0 = (\gamma_l - \gamma_p)/2, \tag{12}
$$

and where the 0 index indicates that this is the 'bare' propagator, obtained within the quadratic (non-interacting) approximation. In order to calculate the excitation branches $\Omega(k)$ in this approximation, we search for the solutions implicitly given by [45, 63]

$$
[G_0^R]^{-1}(k, \Omega(k)) \begin{pmatrix} \Psi_c \\ \Psi_c^* \end{pmatrix} = 0 . \tag{13}
$$

By Fourier transforming Eq. (11), we find

$$
\Omega_\pm(k) = -i\kappa_0 \mp 2J\cos(k) . \tag{14}
$$

This approach allows us to identify κ_0 with the decay rate at weak interactions corresponding to the broadening of the spectral line, as well as the excitation dispersion Ω_+, corresponding to the free-particle dispersion in a lattice. Inverting the matrix (11) in Fourier space and selecting the relevant element (10), we obtain the retarded Green's function

$$G_0^R(k,\omega) = \frac{1}{\omega + 2J\cos k + \mathrm{i}\kappa_0} \ , \tag{15}$$

which has poles for $\omega = \Omega_+(k)$. Notice that the second branch $\Omega_-(k)$ has zero spectral weight in this approximation. Hence, in the open system under investigation, there is no ghost branch predicted at quadratic order in the fluctuations, due to the absence of a quasi Bose-Einstein condensate. To include the effect of interactions, we turn to numerical simulations in Secs. 4 and 5 below. Prior to this, we present the steady-state density matrix.

3 Exact properties of the non-equilibrium steady state

3.1 Steady state density matrix

It has been shown in Ref. [58] that the ansatz

$$\rho_{\mathrm{NESS}} = \frac{1}{\mathcal{N}} \sum_{N=0}^{\infty} z^N \mathbb{1}_N \ , \qquad z = \frac{\gamma_p}{\gamma_l} \ , \tag{16}$$

is a NESS of the Lindblad equation (2) for $z < 1$. Notice that this state corresponds to an infinite-temperature grand-canonical partition function with fugacity z. The normalisation $\mathcal{N}$ in Eq. (16) is given by

$$\mathcal{N} = \sum_{N=0}^{\infty} D_{L,N}^B z^N \ , \qquad D_{L,N}^B = \mathrm{Tr}\, \mathbb{1}_N = \left(\binom{L}{N} \right) \equiv \binom{L-1+N}{N} \ , \tag{17}$$

where we used the fact that the dimension of the Hilbert space for N bosons in a system of length L is given by the number of L-tuples of total length N. This is mathematically equivalent to the 'star and bar' problem of identifying in how many ways one can place $L - 1$ bars (separating sites) between N stars (occupied sites). The normalisation can be further simplified to $\mathcal{N} = (1 - z)^{-L}$. A generalised formula for the normalisation in a truncated Hilbert space can be found in Appendix B.

We remark that, in this idealised model, there is no high-energy cutoff on the pump intensity. We refer to Ref. [58] for a model with frequency dependent effective jump operators. The infinite dimensional Hilbert space of the model as written above renders the Hamiltonian unbounded. We point out that in the numerical calculations, we have to introduce a cutoff for the local occupation, which gives an upper bound for the Hamiltonian and ensures that the time evolution is a CPT map as well as the existence of a NESS [43]. We have generalised the solution (16) for ρ_{NESS} to the case of driven-dissipative free fermions and hard-core bosons on the lattice. The results are given in Appendix C.

3.2 Equal-time correlations in the steady state

The simple form of the steady state solution (16) allows one to exactly calculate the equal-time correlation functions of the bosonic field operators in the NESS. We here present the solution for the one-body density matrix $\langle b_i^\dagger b_j \rangle$ and for the variance of the local particle

occupation number $\text{VAR}(n_i)$, as they give insights in the occupation of the system and fluctuations around it. The higher moments can be easily obtained following the same method. The equal-time correlation function in the NESS is given by

$$\langle b_i^\dagger b_j \rangle_{\text{NESS}} = \text{Tr}\{b_i^\dagger b_j \rho_{\text{NESS}}\}$$

$$= (1-z)^L \sum_{N=0}^{\infty} z^N \sum_{\{m_r\}_{r=1,\dots,L}}' \langle \{m_r\}|b_i^\dagger b_j|\{m_r\}\rangle \, , \tag{18}$$

where we inserted a complete basis in the Fock space $|\{m_r\}\rangle$ for N particles on L sites, with $b_r^\dagger b_r|\{m_r\}\rangle = m_r|\{m_r\}\rangle$, and where $\sum_{\{m_r\}_{r=1,\dots,L}}'$ runs over the sets $\{m_r\}$ such that $\sum_r m_r = N$. We note that, due to the diagonal form of ρ_{NESS}, the matrix elements $\langle \{m_r\}|b_i^\dagger b_j|\{m_r\}\rangle$ are non-vanishing only if $i = j$. After some combinatorial manipulations (see analogous derivation in Appendix A.3), we find that

$$\langle b_i^\dagger b_j \rangle_{\text{NESS}} = \delta_{ij} \frac{z}{1-z} \, , \tag{19}$$

where one must assume that $z < 1$, i.e. $\gamma_l > \gamma_p$. The loss rate must be larger than the pump rate, in agreement with the Keldysh analysis of Sec. 2.3. Note that since $\langle b_i^{(\dagger)}\rangle_{\text{NESS}} = 0$, the correlation function (19) corresponds also to the connected correlation function.

With a similar method, we next calculate the fluctuations of the local particle number density. This quantity is useful in order to assess the effect of the truncation of the local Hilbert space in the numerical calculations. The variance is given by

$$\text{VAR}(n_i) := \langle n_i^2 \rangle - \langle n_i \rangle^2 = \frac{z}{(1-z)^2} \, . \tag{20}$$

We note that at vanishing pump rate $z \to 0$ the local number fluctuations vanish, while they diverge for $z \to 1$. In the latter regime the system is close to the instability point given by the change of curvature of the Keldysh potential. The ratio of the number fluctuation $\Delta(n_i) = \sqrt{\text{VAR}(n_i)}$ to the average occupancy $\langle n_i \rangle$ is given by $\frac{1}{\sqrt{z}}$ and decreases for large occupation. This calculation provides a good estimate on the error made by truncating the Hilbert space to N_s states per lattice site. For example, choosing $z = 1/10$ ($z = 1/5$) gives an average occupation of $1/9$ ($1/4$) with the number fluctuation of $\sqrt{10}/9$ ($\sqrt{5}/4$). Hence, for these parameters, a truncation of the Hilbert space to $N_s = 3$ allowing only for the states $\{|0\rangle, |1\rangle, |2\rangle\}$ on each site is justified. This will be confirmed by the numerics in the following.

For the total particle number $N = \langle \sum_{j=1}^{L} b_j^\dagger b_j \rangle$ in the system, one can solve the full time evolution analytically. The result reads

$$\langle N(t) \rangle = (N_0 - N_{\text{NESS}})\, e^{-(\gamma_l - \gamma_p)t} + N_{\text{NESS}} \, , \tag{21}$$

where $N_0 = N(t=0)$ and $N_{\text{NESS}} = \frac{z}{1-z}L$ is the total steady state occupation. The details can be found in Appendix A.2. Notice that there exists a NESS if and only if the pump rate is chosen strictly smaller than the loss rate, i.e. $z < 1$, consistently with the previous considerations.

4 Correlation function in real time

We report in this section our results for the non-equal time two-point correlation function.

4.1 Analysis of the parameter space

The original model (2) depends on four couplings: $\{J, U, \gamma_l, \gamma_p\}$. By rescaling time by the typical loss time $\tau_l = 1/\gamma_l$, we are left with the three independent parameters $\{J/\gamma_l, U/\gamma_l, z\}$, with $z = \gamma_p/\gamma_l$ for which $J \in \mathbb{R}_0^+$, $U \in \mathbb{R}_0^+$ and $z \in [0, 1)$ for a NESS to exist. The parameter space and known limiting cases are illustrated in Fig. 2. The plane $z = 0$ corresponds to a purely lossy system with empty steady state. The plane $J = 0$ corresponds to the case of absence of tunneling among the sites – in this case the problem maps to a set of single-site dissipative Kerr resonator which has an exact solution [59]. The plane $U = 0$ corresponds to the non-interacting limit which can be solved exactly using the non-interacting (Gaussian) Keldysh action. In the following, we fix a value for z and explore numerically the two-point correlation function in the interaction/tunnel energy plane.
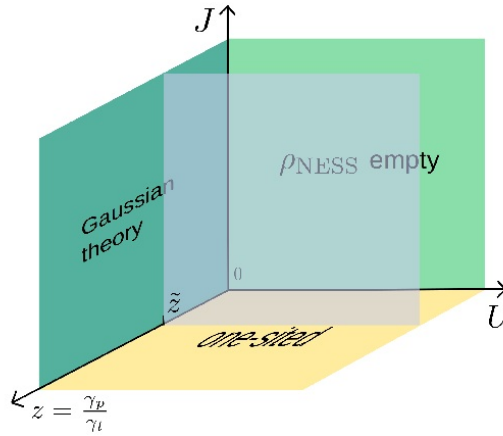
Figure 2: Sketch of the parameter space under investigation: $J/\gamma_l \in \mathbb{R}_0^+$, $U/\gamma_l \in \mathbb{R}_0^+$ and $z \in [0, 1)$; the plane $z = 0$ (light green) describes a purely lossy system with empty steady state; the plane $J = 0$ (yellow) corresponds to a local problem: the solution to L independent dissipative Kerr resonators is known exactly; the plane $U = 0$ is the non-interacting limit which can also be solved exactly. The inserted plane for finite $\tilde{z}$ indicates the numerically investigated phase space.

4.2 Time correlations

Using the analytical expression for the NESS density matrix, we have performed numerical calculations using exact diagonalisation (see details in Appendix F.1) to obtain the retarded Green's function in space-time. Fig. 3 displays an example of such a Green's function, which displays the emergence of two main features. First, the Green's function decays exponentially in time, and second, it oscillates at well-defined frequencies. Quite remarkably, both the decay rate and oscillation frequencies are affected by the interactions. We next focus on the properties of the decay rate κ. The frequencies are discussed in Sec. 5 and Appendix A.4.

4.3 Renormalisation of the temporal decay

In order to capture the main features of the non-static quantities in the NESS, we determine the inverse life time, i.e. the decay κ of the equal-space retarded Green's function. We investigate how the decay $\kappa(U, J)$ changes for fixed ratio of pump-to-loss rate z. As
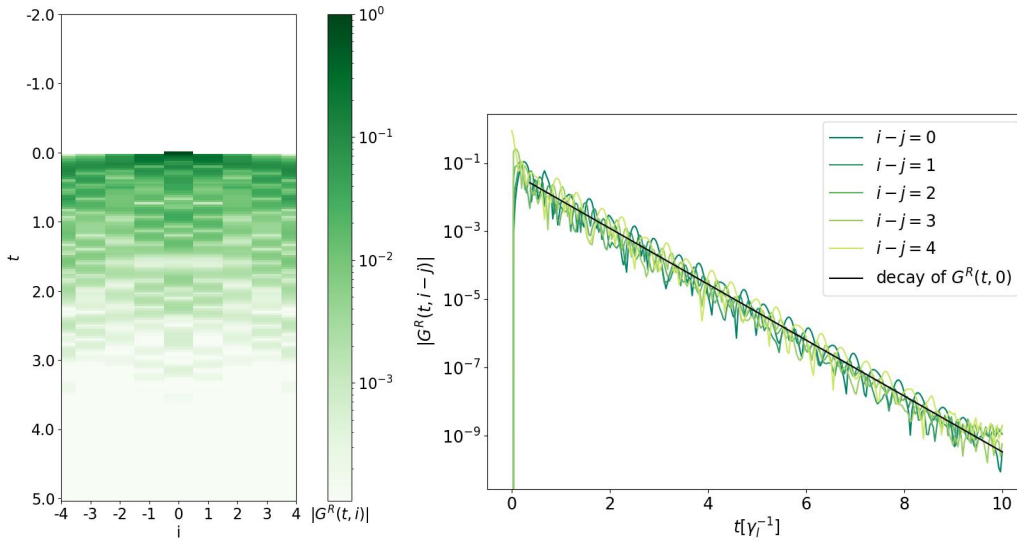
Figure 3: Left panel: space-time evolution of the real part of the retarded Green's function (dimensionless). Right panel: cuts for various spatial differences as indicated on the legend as a function of time (in units of $\tau_l = 1/\gamma_l$) and extracted exponential decay rate (solid black line). All results are obtained by exact diagonalisation with parameters $U = 50$, $J = 10$, $z = 0.2$, $L = 8$, $N_s = 3$.

a benchmark of our numerical procedure, we recover the known exact results, both in the limit of weak interaction and in the limit of weak hopping.

To extract the decay rate from the numerical data, we assume that the retarded Green's function $G_{00}^R(t)$ endows at coincident spatial points a single-pole form corresponding to

$$G_{\mathrm{sp}}^R(t) = -\mathrm{i}\theta(t)\mathrm{e}^{-\mathrm{i}\Omega t - \kappa t} \quad . \tag{22}$$

This leads us to define the decay rate as $\kappa = -\Re\left[\ln \mathrm{i}G_{\mathrm{sp}}^R(t)\right]/t$ for $t > 0$. We thus represent the numerical data as $\mathcal{G}(t) = -\Re\left[\ln \mathrm{i}G_{00}^R(t)\right]$ and determine the slope of $\mathcal{G}$ to obtain an estimate of the decay rate (see Appendix F.2 for details). The resulting decay rate is depicted in Fig. 4 in the $\{U, J\}$ plane. At weak interactions, i.e. for $U/\gamma_l \lesssim 1$, we recover the prediction from the free, quadratic theory of Sec. 2.3, given by Eq. (12). At strong interactions, when tunneling is negligible with respect to both interaction energy $U/J \gg 1$ and pump/loss rates $J \ll \gamma_l, \gamma_p$, the decay rate tends to a plateau independent of the interaction strength which is to the best of our knowledge unknown for extended systems. For the single-site case, the value of this plateau can be extracted from the exact solution of Ref. [59] for a single dissipative Kerr resonator as

$$\kappa_\infty = \frac{\gamma_l + 3\gamma_p}{2} = \kappa_0 + 2\gamma_p \tag{23}$$

(see Appendix E for details). In between these two limits, we observe a renormalisation of the decay smoothly connecting weak and strong interactions, and weak and large decay rate, similarly to the predictions for the single-site case [59]. We have checked that our numerical simulation for $L = 1$ agrees well with the exact solution all through the crossover from weak to strong interactions.

We notice that the decay rate increases both when the interaction strength increases at fixed J, as well as when the tunnel amplitude increases at fixed (large) U. We understand this latter effect as being due to increased decay possibilities upon allowing for tunneling
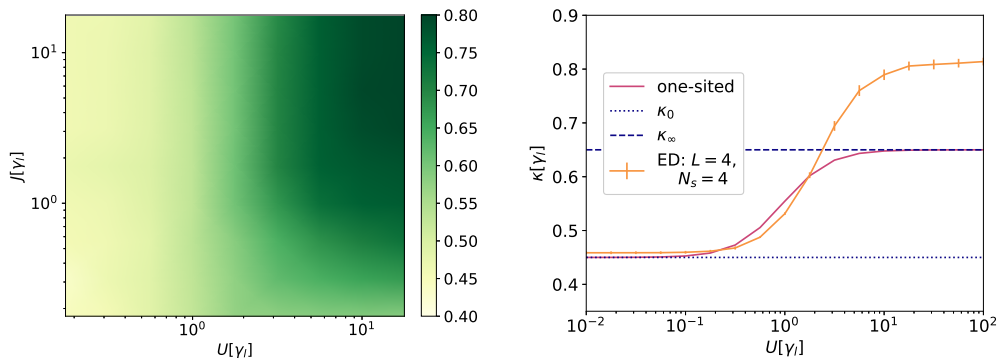
Figure 4: Left panel: decay rate κ of the retarded Green's function for $z = 0.1$ in the $\{U, J\}$ plane with U, J in units of γ_l. Right panel: cut of the decay rate as a function of U/γ_l at fixed $J/\gamma_l = 10$. The analytical solution for a single site Eq. (58) is also shown, as well as the two limits $\kappa_0 = \kappa(U = 0)$ from Eq. (12) and $\kappa_\infty = \kappa(U \to \infty)$ from Eq. (23). Calculations are done by exact diagonalisation with $L = 4$, $N_s = 4$.

among particles. For the parameter regime we could access numerically, we find that the value of the decay rate for large tunneling amplitude and large interaction strengths depends on the system size (see Appendix F.4).

5   The spectrum

We now focus on the properties of the single-particle excitations on top of the NESS by analysing the spectral function $A(k, \omega)$ defined in Eq. (7). For interacting 1D bosons in closed systems, this function displays several noticeable features as broad spectra with power-law singularities in correspondence of the Lieb-I and Lieb-II excitation branches [64]. At low energy, the dispersion of the Goldstone branch is linear, confirming the $z = 1$ dynamical critical exponent of the equilibrium model. On the lattice, the spectral function also displays a third excitation branch associated with the change of curvature of the single-particle dispersion [17], as well as a doublon branch at energy close to U [65]. These results are used as a reference to analyse and discuss the spectral function of the open system.

5.1   Spectrum of the open system

Our exact diagonalisation results for the spectral function are shown in Fig. 5. At weak interaction, we find the free spectral line with the Lorenzian shape in frequency predicted from the field theory calculation in Sec. 2.3. Two main features emerge at strong interactions. First, differently from the closed-system case, the low-energy branch is quadratic rather than linear, with an emerging dispersion branch corresponding to the free particle one. The background of excitations behind this main branch is due to the $N > 1$ particle sectors in ρ_{NESS}. They provide additional contributions to the spectral function whose precise shape is difficult to resolve given the system size. Second, we identify an additional excitation branch centered around energy U, which is the analogue of the doublon branch of the closed-system case. This branch provides a clear signature of interactions. This is remarkable as ρ_{NESS} does not contain information on the interaction itself and does not depend explicitly on U. We provide an estimate of the position of this excitation branch using a strong-coupling approach in Sec. 5.2.

The spectral function here obtained has limited resolution due to the size of the system accessible in numerical diagonalisation, but displays nevertheless emergent features characteristic also of larger system sizes. This is illustrated in Sec. 5.3, where we present the calculation of the spectral function for the closed-system case for two choices of system size.
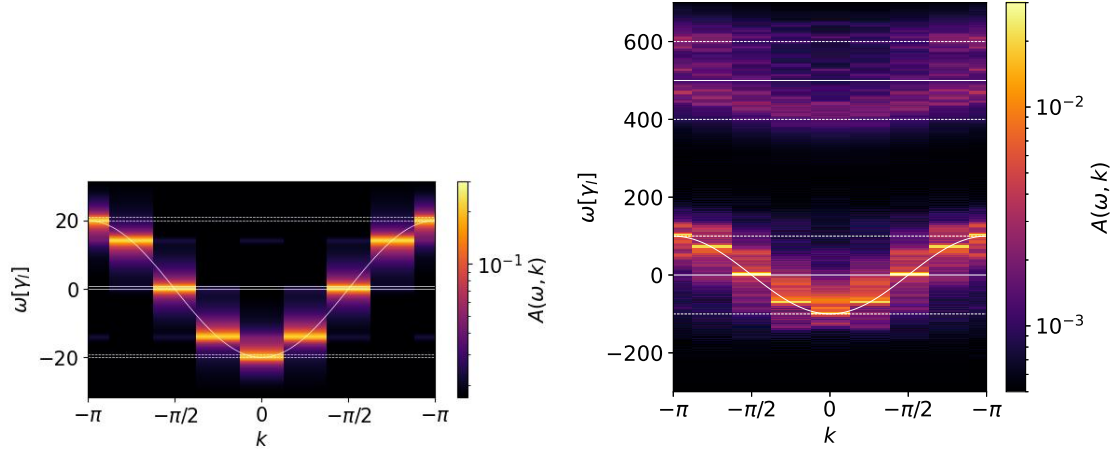
Figure 5: Spectral function $A(\omega, k)$ for the NESS of a driven-dissipative Bose-Hubbard model in the weakly (left panel, $U/\gamma_l = 1$, $J/\gamma_l = 10$) and strongly (right panel $U/\gamma_l = 500$, $J/\gamma_l = 50$) interacting regimes. The other parameters are: $z = 0.2$, $L = 8$, $N_s = 3$. The horizontal white lines are set at $\omega_b := kU$, $k \in \{0, ... N_s - 1\}$ (positions of the branches for closed system and $J = 0$, compare with (27)), and dotted lines at $\omega = \omega_b \pm 2J$.

5.2 Position of the excitation spectral lines for small hopping

In order to estimate the position of the excitation branches in the interacting case, we analytically calculate the oscillation frequency of the doublon branch of the NESS under unitary time evolution. As shown in Sec. 5.4, the unitary time evolution well accounts for the position of the peaks of the spectral function.

We use the one-site model, i.e. we set $J = 0$. In this case, the Hamiltonian becomes local in position space

$$H = \frac{U}{2} \sum_j n_j(n_j - 1) \, . \tag{24}$$

Upon evaluating the two-point correlation function in the NESS,

$$\left\langle b_j^\dagger(t) b_0 \right\rangle_{\text{NESS}} = (1 - z)^L \sum_N z^N \sum_{\{m\}} \langle \{m\} | e^{iHt} b_j^\dagger e^{-iHt} b_0 | \{m\} \rangle \, , \tag{25}$$

we obtain that the only non-vanishing term is the one with $j = 0$ due to the locality of the strong-coupling Hamiltonian. This term reads

$$\langle m_0 | e^{i\frac{U}{2} n_0(n_0-1)t} b_0^\dagger e^{i\frac{U}{2} n_0(n_0-1)t} b_0 | m_0 \rangle = m_0 e^{iU(m_0-1)t} \, . \tag{26}$$

This shows that there is an oscillation with period depending on the interaction strength U. Altogether, we find

$$\left\langle b_j^\dagger(t) b_0 \right\rangle_{\text{NESS}} = \delta_{0j} z(1 - z) \sum_{m=0}^\infty z^m (m + 1) e^{iUmt} \, , \tag{27}$$

with $m + 1$ corresponding to the occupation number of the site $j = 0$. Further details of the calculation can be found in Appendix A.4. We notice from Eq. (27) that the position of the first excitation peak for vanishing J and large U corresponds to the contribution with $m = 1$, i.e. $\omega_d(m = 1) = U$. This is at the origin of the doublon branch in Fig. 5. Furthermore, we observe that the number of dominant peaks in the excitation spectrum at $k = 0$ is exactly $N_s - 1$, as predicted by (27).

5.3 Spectral function for the ground state of the Bose-Hubbard model

In order to assess the role of drive and dissipation on the spectral function, we show here the results for the spectral function of a closed Bose-Hubbard system at small filling for the same values of parameters, namely interactions and hopping strength, as well as the same system size. In the closed system, numerical calculations allow one to tackle larger system sizes, and to explore the interplay between interaction and finite-size effects in the spectral function. As shown in Fig. 6, features clearly identified for large system size are also found at smaller size and serve as guideline to interpret our results for the open case.
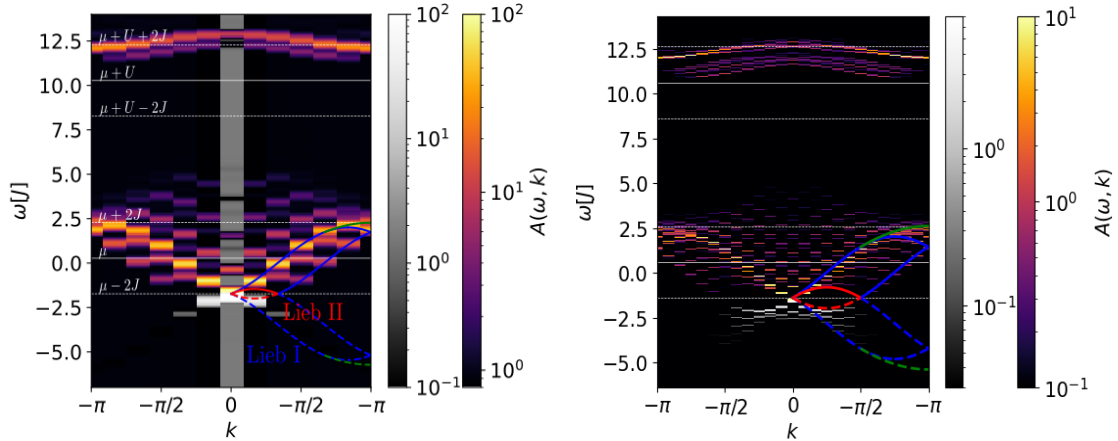
Figure 6: Spectral function $A(k, \omega)$ for the ground state of the Bose-Hubbard model under unitary evolution (negative values in grey). Left panel: Exact solution without truncation at finite interaction ($L = 12$, $N = 2$ particles, $U/J = 10$). Right panel: larger system size ($L = 24$, $N = 6$, $U/J = 10$, with $N_s = 3$ obtained with DMRG).

Let us comment on the results for the closed system. In the lowest-energy excitation manifold, we clearly identify the Lieb-I and Lieb-II branches, as well as the third branch predicted for the lattice case [17]. The predictions for the dispersion branches in the infinite interaction limit is also shown in Fig. 6. Notice that in this limit all the branches are slightly shifted upwards with respect to the finite-U spectrum, since they are centered at $\omega = \mu - 2J$ and the value of the chemical potential μ at infinite interactions is larger than the one at finite interactions. At frequencies $\omega \simeq \mu + U + 2J$, we also see in the figure a dispersive doublon branch.

We stress that this result is very different from the one obtained by the Lindblad evolution presented in the previous section. In the latter case, the spectral function displays a quadratic behaviour at low momenta, while in the current case the dispersion at low momenta is linear. This is an illustration of the fact that the nature of the low-energy excitations involved is different depending on the state of the system considered. For the ground state, low-energy excitations are particle-hole excitations on top of an effective
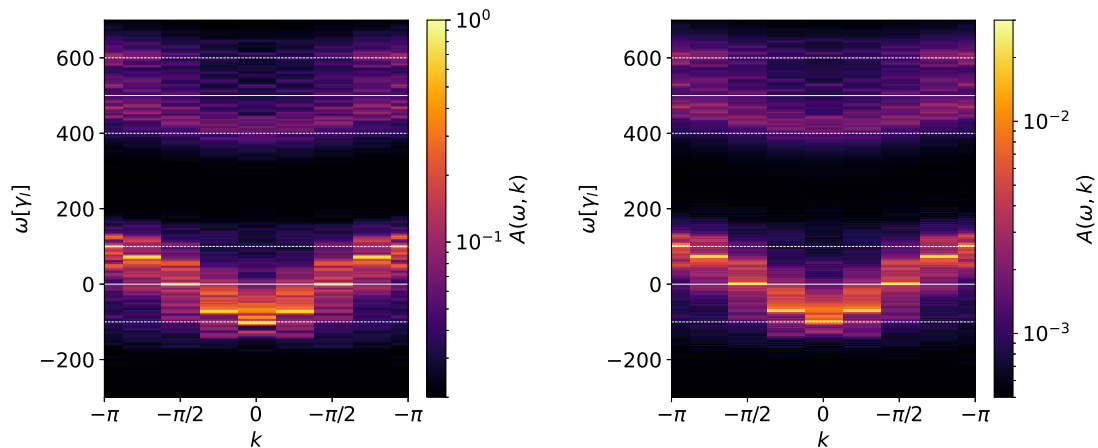
Figure 7: Spectral function $A(\omega, k)$ of the Markovian open system dynamics in the NESS. Left: Under unitary dynamics; Right: Under Lindbladian dynamics. The parameters are: $U = 500$, $J = 50$, $z = 0.2$, $L = 8$, $N_s = 3$. The horizontal white lines are as in Fig. 5.

Fermi sphere, while for the NESS state, all excitations are possible, and single particle excitations dominate the intensity of the spectral function.

5.4 Comparison of unitary evolution and Lindblad evolution of the NESS

Finally, to get further insights on the spectral function of the driven-dissipative system, we compare the results obtained with the Lindblad evolution (4) to the ones obtained starting from the same ρ_{NESS}, but with unitary evolution (5), in the regime where the coherent parameters are much larger then the incoherent drive and dissipation $J, U \gg 1, z$. The latter can be interpreted as a quenched system: The system is initially open and in the steady state and then, drive and dissipation are turned off and the temporal evolution considered is the unitary one. As shown in Fig. 7 for strong interactions, the spectral functions in both cases closely resemble each other, as one would expect in the limit of vanishing incoherent parameters. The broadening of the lines is very small comparatively to the typical features (depth, width) of the spectrum which are of the order of the coherent parameters.

6 Conclusions and perspectives

The study of the bosonic quantum gas coupled to an environment is a present problem of particular interest. Open quantum systems play a crucial role in the current noisy intermediate-scale quantum (NISQ) era [66], with quantum science and quantum engineering bringing forth a variety of platforms for quantum simulations of emergent collective many-body phenomena. Notwithstanding the exceptional technological advances, photonic platforms remain intrinsically dissipative and we foresee as particularly important for the near future the characterisation of the interplay of the open-system dynamics of these setups with the coherent Hamiltonian dynamics, which will be characterised by increasingly strong Kerr non-linearities.

In this article we have studied the space-time behaviour of two-point correlation functions and the spectral function for a driven-dissipative system of interacting bosons on a lattice, subjected to uniform incoherent pump and one-body losses. Our starting point

has been the analytical expression for the NESS density matrix of the system derived in Ref. [58]. We have used several techniques, namely numerical exact diagonalisation, Keldysh formalism and strong coupling approaches, to provide a complete understanding of the excitation spectrum on top of the non-equilibrium steady state. We have found that the decay of the retarded Green's function is exponential in time, and the decay rate is renormalised by interactions and increases at both increasing interactions and tunnel amplitude. The spectral function displays at low frequency a quadratic-like branch, very different from the linear branch found for the excitations on top of the ground state of the closed system, for which we have calculated the spectral function at finite interactions. At large interactions, we have identified, both in the open and in the closed system, a dispersive doublon branch at energy of order U. By performing calculations for various system sizes in the closed system case, we have shown that specific features of the spectrum clearly emerging at large system sizes are also recognisable in smaller systems, of typical size accessible via numerical diagonalisation in the open case.

In outlook, on a technical level, it would be interesting to push the calculations for the spectral function of the open system to larger system sizes using techniques based on matrix-product states such as those detailed in Refs. [67, 68]. This would provide conclusive evidence about the presence of excitations at negative energies (the 'ghost' branches) which, according to Bogoliubov theory, are predicted to be more elusive in the driven-dissipative case than in the closed one [69–71] and could not be resolved in our current numerical calculations. On a fundamental level, our work opens very interesting perspectives for the possibility of observing the effects of interactions in out-of-equilibrium systems within actual experiments, where short one-dimensional lattices of small photonic resonators can be engineered.

Acknowledgements

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A   Detailed calculations

A.1   Bosonic NESS density matrix

We show in this section that the ansatz (16) is a NESS of the Lindblad evolution (2). The Hamiltonian conserves the particle number and hence commutes with each identity operator in each particle number subspace. Furthermore, the non-hermitian Hamiltonian

terms in (2) can be written as

$$\rho_{\text{NESS}} \sum_i b_i^\dagger b_i = \rho_{\text{NESS}} N = N \rho_{\text{NESS}} = \sum_i b_i^\dagger b_i \rho_{\text{NESS}} \ , \tag{28}$$

$$\rho_{\text{NESS}} \sum_i b_i b_i^\dagger = \rho_{\text{NESS}}(N + L) = \sum_i b_i b_i^\dagger \rho_{\text{NESS}} \ , \tag{29}$$

such that we only have to compute the remaining terms

$$
\begin{aligned}
\sum_i b_i^\dagger \rho_{\text{NESS}} b_i &= \sum_i \sum_N \omega_{N-1,L} \sum_{\substack{\{l_r\}_{r=1,\ldots,L;N} \\ \{n_r\}_{r=1,\ldots,L;N}}} |\{l_p\}\rangle\langle\{n_s\}| \sum_{\{m_r\}_{r=1,\ldots,L;N-1}} \langle\{l_p\}|b_i^\dagger|\{m_r\}\rangle\langle\{m_r\}|b_i|\{n_s\}\rangle \\
&= \sum_N \omega_{N-1,L} \sum_{\substack{\{l_r\}_{r=1,\ldots,L;N} \\ \{n_r\}_{r=1,\ldots,L;N}}} |\{l_p\}\rangle\langle\{n_s\}|\langle\{l_p\}| \sum_i b_i^\dagger b_i |\{n_s\}\rangle \\
&= \sum_N N \omega_{N-1,L} \mathbb{1}_N \ ,
\end{aligned}
\tag{30}
$$

and similarly,

$$\sum_i b_i \rho_{\text{NESS}} b_i^\dagger = \sum_N (N + L) \omega_{N+1,L} \mathbb{1}_N \ . \tag{31}$$

The sum of all these contributions vanishes

$$
\begin{aligned}
\mathcal{L}_{\text{loss}}[\rho_{\text{NESS}}] + \mathcal{L}_{\text{gain}}[\rho_{\text{NESS}}] &= \sum_N \mathbb{1}_N \Big[\gamma_l \Big\{ (N + L)\omega_{N+1,L} - N\omega_{N,L} \Big\} \\
&\qquad\qquad + \gamma_p \Big\{ N\omega_{N-1,L} - (N + L)\omega_{N,L} \Big\} \Big] \\
&= \sum_N \mathbb{1}_N \omega_{N-1,L} \Big[\gamma_l \Big\{ z^2(N + L) - zN \Big\} + \gamma_p \Big\{ N - z(N + L) \Big\} \Big] \\
&= 0 \ ,
\end{aligned}
\tag{32}
$$

which shows that the ansatz (16) is indeed a steady state.

A.2   Approach to the NESS

The time evolution of the total particle number $N = \sum_{j=0}^L b_j^\dagger b_j$ is given by the Lindblad master equation (2). Since the total particle number commutes with the Hamiltonian $[H, N] = 0$, its evolution is purely generated by the dissipative and pump parts, which yield

$$\langle \dot{N} \rangle = (-\gamma_l + \gamma_p)\langle N \rangle + \gamma_p L \ . \tag{33}$$

Starting with a total occupation of $N_0 := N(t = 0)$ and depending on the value of γ_l w.r.t γ_p, we find the following solutions:

- $\gamma_p = \gamma_l$: The particle number grows linearly, there is no steady state. It can be interpreted as a bosonic enhancement, following

$$\langle N \rangle(t) = \gamma_p L t + N_0 \ . \tag{34}$$

- $\gamma_p > \gamma_l$: The particle number grows exponentially, there is no steady state.

- $\gamma_p < \gamma_l$: The particle number changes according to

$$\langle N \rangle(t) = (N_0 - N_{\text{NESS}})e^{(-\gamma_l + \gamma_p)t} + N_{\text{NESS}} \, , \tag{35}$$

where the steady state is given by

$$N_{\text{NESS}} = \frac{z}{1-z}L \, . \tag{36}$$

A.3   Equal-time correlation functions

We start from equation (18). If the number of particles in position j is k, then all other $N - k$ particles are distributed over the remaining $L - 1$ sites. Hence, we have

$$\sum_{\{m_r\}_{r=1,\dots,N}} \langle \{m_r\}|b_i^\dagger b_j|\{m_r\}\rangle = \delta_{ij} \sum_{k=1}^{N} k \left(\binom{L-1}{N-k} \right)$$

$$= \delta_{ij} \sum_{k=0}^{N-1} (k+1) \left(\binom{L-1}{N-1-k} \right) \, . \tag{37}$$

Summing over N, we get

$$\sum_{N=1}^{\infty} z^N \sum_{k=0}^{N-1} (k+1) \left(\binom{L-1}{N-1-k} \right) = z \sum_{N=0}^{\infty} z^N \sum_{k=0}^{N} (k+1) \left(\binom{L-1}{N-k} \right)$$

$$= z \sum_{l=0}^{\infty} (l+1)z^l \sum_{M=0}^{\infty} z^M \left(\binom{L-1}{M} \right)$$

$$= z \frac{1}{(1-z)^2}(1-z)^{-L+1}$$

$$= \frac{z}{(1-z)^{L+1}} \, , \tag{38}$$

where from the first to the second line, we used

$$\sum_{p=0}^{\infty} \sum_{q=0}^{p} f(q, p-q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m, n) \, . \tag{39}$$

The geometrical series in the second line only converges if $|z| < 1$. All together, we find the result (19) as expected from the calculation of the total particle number in the NESS.

For the variance, we need to calculate the average over quadratic terms in the local particle number density. Performing similar steps, we obtain

$$\langle n_i^2 \rangle_{\text{NESS}} = \sum_N \frac{z^N}{(1-z)^{-L}} \sum_{\{m_r\}_{r=1,\dots,N}} \langle \{m_r\}|b_i^\dagger b_i b_i^\dagger b_i|\{m_r\}\rangle$$

$$= \frac{z}{(1-z)^{-L}} \sum_{l=0}^{\infty} (l+1)^2 z^l \sum_{M=0}^{\infty} z^M \left(\binom{L-1}{M} \right)$$

$$= \frac{z(1+z)}{(1-z)^2} \, . \tag{40}$$

A.4   Oscillation frequency

The dependence of the oscillation frequency on the interaction U can be best understood for the single-particle Green's function under the unitary dynamics in the limit of strong interactions $U \gg J$. We hence consider the evolution under

$$H = \frac{U}{2} \sum_j n_j(n_j - 1) \,. \tag{41}$$

We evaluate the two-point correlation function in the NESS as

$$\left\langle b_j^\dagger(t)b_0 \right\rangle_{\text{NESS}} = (1-z)^L \sum_N z^N \sum_{\{m\}} \langle \{m\}| e^{iHt} \, b_j^\dagger \, e^{-iHt} \, b_0 |\{m\}\rangle \tag{42}$$

$$= (1-z)^L \sum_N z^N \sum_{\{m\}} \langle \{m\}| \Big(\prod_k e^{i\frac{U}{2} n_k(n_k-1)t} \Big) b_j^\dagger \Big(\prod_l e^{-i\frac{U}{2} n_l(n_l-1)t} \Big) b_0 |\{m\}\rangle \,.$$

The only non-vanishing contribution is the $j = 0$ one, which is just a consequence of the density matrix being diagonal in the Fock basis. (We can show the same oscillating behaviour for $j \neq 0$ for a more generic density matrix). The matrix element factorises and we find

$$\langle m_0| e^{i\frac{U}{2} n_0(n_0-1)t} \, b_0^\dagger \, e^{i\frac{U}{2} n_0(n_0-1)t} \, b_0 |m_0\rangle = m_0 e^{i\frac{U}{2} m_0(m_0-1)t} \, e^{-i\frac{U}{2}(m_0-1)(m_0-2)t}$$

$$= m_0 e^{iU(m_0-1)t} \,, \tag{43}$$

which indicates an overall oscillation with a period depending on the interaction strength U. We finally obtain

$$\left\langle b_0^\dagger(t)b_0 \right\rangle_{\text{NESS}} = (1-z)^L \sum_{N=0}^{\infty} z^N \sum_{m_0=0}^{N} m_0 e^{iU(m_0-1)t} \left(\binom{L-1}{N-m_0} \right)$$

$$= z(1-z)^L \sum_{N=0} z^N \sum_{m_0=0}^{N} (m_0+1) e^{iUm_0 t} \left(\binom{L-1}{N-m_0} \right)$$

$$= z(1-z)^L \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} z^l z^k (k+1) e^{iUkt} \left(\binom{L-1}{l} \right)$$

$$= z(1-z) \sum_{k=0}^{\infty} z^k (k+1) e^{iUkt} \,, \tag{44}$$

with k being the occupation minus one of site zero (relabeled), where we used the infinite double sum rule (39).

B   Normalisation with truncation of the local Hilbert space

This section gives the normalisation of the NESS density matrix in each N-particle sector depending on the system size L and on the local Hilbert space dimension d_l. For this, one has to solve a restricted star (balls/particles) and bar (box separations) problem which is well known in combinatorics

$$\sum_{i=1}^{L} x_i = N \,, \tag{45}$$

and $x_i \in 0, ..., N$. The latter condition is equivalent to the existence of an upper bound for the integers $x_i < d_l \quad \forall i$. This problem can be solved by using the inclusion-exclusion principle on the lower-bound integer sum. The latter is given by the previous sum with the constraint $x_i \geq a_i$. We substitute $x_i' := x_i - a_i$, such that the modified problem is the unbounded one with $x_i' \geq 0$:

$$\sum_i x_i' = N - \sum_i a_i \ . \tag{46}$$

The number of all possible solutions (as for the bosons) is given by

$$\binom{L + N - 1}{L - 1} \ , \tag{47}$$

then we subtract all the cases where there is at least one box with $x_i \geq d_l$ (solving the lower-bound integer sum):

$$\binom{L}{1} \binom{L + N - 1 - d_l}{L - 1} \ , \tag{48}$$

and finally add the two set intersections, such that

$$D_{L,N}^{d_l} := \#\text{states}(d_l, L, N) = \sum_{k=0}^{k(d_l-1) \leq N} (-1)^k \binom{L}{k} \binom{L + N - 1 - kd_l}{N - 1} \ . \tag{49}$$

C   Steady state solution for fermions

We consider the Lindblad master equation (2) where now the annihilation and creation operators are fermionic ones $b_i := c_i, b_i^\dagger := c_i^\dagger$, obeying the anti-commutation relations $\{c_i, c_j^\dagger\} = \delta_{ij}$ and $\{c_i, c_j\} = 0 = \{c_i^\dagger, c_j^\dagger\}$. The unitary evolution is given by the Fermi-Hubbard model.

C.1   NESS density matrix

In finite dimensional Hilbert spaces, there always exists a steady state solution of the Lindbladian superoperator [43]. Starting from the same ansatz (16) as in the main text, we obtain from the normalisation of the state that

$$\mathcal{N}_L^F = \frac{1}{(1 + \gamma)^L} \ , \tag{50}$$

where we used that the dimension of the Hilbert space for N fermions in a system of length L corresponds to $D_{L,N}^F = \binom{L}{N}$. This is exactly the size of the unit matrix $\text{Tr} \, \mathbb{1}_N = D_{L,N}^F$. This provides the NESS density matrix for fermions.

C.2   Equal-time two-point correlation functions

We assume that the sums for fermions run only over non-equal elements. The correlation function is given by

$$
\begin{aligned}
\langle c_i^\dagger c_j \rangle_{\text{NESS}} &= \sum_{N=0}^{L} \frac{z^N}{(1+z)^L} \sum_{\{m_r\}_{r=1,\dots,N}} \langle \{m_r\} | c_i^\dagger c_j | \{m_r\} \rangle \\
&= \delta_{i,j} \sum_{N=0}^{L} \frac{z^N}{(1+z)^L} \binom{L-1}{N-1} \\
&= \delta_{i,j} \frac{z}{1+z} \ ,
\end{aligned}
\tag{51}
$$

where we introduced a complete basis in position space $\{m_r\}$ for N particles and evaluated the sum as

$$
\begin{aligned}
\sum_{\{m_r\}_{r=1,\dots,N}} \langle \{m_r\} | c_i^\dagger c_j | \{m_r\} \rangle &= \sum_{i,\{m_r\}_{r=1,\dots,N-1} \in \{1,\dots,L\}/\{i\}} \langle \{m_r\} | c_i^\dagger c_j | \{m_r\} \rangle \delta_{i,j} \\
&= \binom{L-1}{N-1} \delta_{i,j} \ .
\end{aligned}
\tag{52}
$$

The variance is given by

$$
\begin{aligned}
\langle n_i^2 \rangle_{\text{NESS}} &= \sum_N \frac{z^N}{(1+z)^L} \sum_{\{m_r\}_{r=1,\dots,N}} \langle \{m_r\} | c_i^\dagger c_i c_i^\dagger c_i | \{m_r\} \rangle \\
&= \frac{z}{1+z} \ .
\end{aligned}
\tag{53}
$$

Since the number of fermions in state i is the same as its squared value, the result of this operator is the same as the first moment. This leads to

$$
\text{VAR}(n_i) = \langle n_i^2 \rangle - \langle n_i \rangle^2 = \frac{z}{(1+z)^2} \ .
\tag{54}
$$

This function has a maximum variance of $1/4$ for $\gamma_p = \gamma_l$. For either limit of pump much bigger/smaller than the loss, the variance tends to zero.

D   Steady-state solution for hard-core bosons

We consider a periodic chain of hard-core bosons, which can be mapped by the transformation due to Holstein and Primakoff [72] to the XX spin-$\frac{1}{2}$-chain given by the Hamiltonian

$$
H = -2J \sum_{j=1}^{L} (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) \ ,
\tag{55}
$$

with $S^\mu = \frac{1}{2} \sigma^\mu$ where $\{\sigma^\mu\}_{\mu=x,y,z}$ are the Pauli matrices. We define the first state as the spin-up state $|\uparrow\rangle = |+\rangle = |0\rangle$ and note that the operator $S_j^+ S_j^-$ acts as a local density operator for the spin-up state at site j, where $S^+ = S^x + iS^y$ and $S^- = S^x - iS^y$, which act as spin-flip operators.

The pump and loss of hard-core bosons hence correspond to spin flips in the XX-model, implying that the resulting Lindbladian is of the form

$$\mathcal{L}[\rho] = -\mathrm{i}[H, \rho] + \gamma_l \sum_{i=1}^{L} \left(S^-{}_i \rho S^-{}_i{}^\dagger - \frac{1}{2}\{S^-{}_i{}^\dagger S^-{}_i, \rho\} \right) + \gamma_p \sum_{i=1}^{L} \left(S_i^+ \rho S^+{}_i{}^\dagger - \frac{1}{2}\{S^+{}_i{}^\dagger S^+{}_i, \rho\} \right). \tag{56}$$

We make the following ansatz for the non-equilibrium steady state

$$\rho_{\mathrm{NESS}} = \frac{1}{\mathcal{N}_F} \sum_{N=0}^{L} \left(\frac{\gamma_p}{\gamma_l} \right)^N \mathbb{1}_N \,, \tag{57}$$

where the identity operator is the sum over the complete Fock basis for the N-particle sector. The number of states in the system of hard-core bosons is identical to the one of the fermionic system, which explains the identical normalisation $\mathcal{N}_F$ and structure of the solution. One can show that the Hamiltonian conserves the particle number, or equivalently the number of up-spins, and hence it commutes with the steady state density matrix. The dissipative part of the Lindbladian evolution vanishes with the same reasoning as for the bosons and fermions.

E   Decay rate at large interactions

For $J = 0$, each site in the Keldysh action (9) decouples. Hence, in the strong interaction limit $U \to \infty$, one is left with solving the problem of L independent dissipative Kerr resonators. This is a well-known exactly solvable problem [59]. One can exploit a weak symmetry to block-diagonalise the Lindbladian and then introduce an operator which renders the Lindbladian quadratic in the superoperators. Equivalently, one can introduce a time-dependent gauge transformation to obtain a quadratic action as shown in [50]. The Kerr non-linearity then transforms into a fluctuating frequency depending on the number density which in turn leads to dephasing. The result for the retarded Green's function is given by [59]

$$G^R(t) = -\mathrm{i}\theta(t) \frac{\mathrm{e}^{\mathrm{i}Ut + \frac{\gamma_l - \gamma_p}{2} t}}{\left(\cosh \frac{\Gamma}{2} t + R_1 \sinh \frac{\Gamma}{2} t \right)^2} \,, \tag{58}$$

where

$$\Gamma := \sqrt{(\gamma_l - \gamma_p)^2 - U^2 + 2\mathrm{i}U(\gamma_l + \gamma_p)} \,, \qquad R_1 := \frac{1}{\Gamma} \left[(\gamma_l - \gamma_p) + \mathrm{i}\frac{U(\gamma_l + \gamma_p)}{\gamma_l - \gamma_p} \right]. \tag{59}$$

We can expand the term (assuming $t > 0$)

$$
\begin{aligned}
\kappa_\infty t &= -\Re \left[\ln \mathrm{i} G^R(t) \right] \\
&= -\Re \left[\mathrm{i}Ut + \frac{\gamma_l - \gamma_p}{2} t \right] + 2\Re \ln \left[\cosh \frac{\Gamma}{2} t + R_1 \sinh \frac{\Gamma}{2} t \right] \\
&= -\frac{\gamma_l - \gamma_p}{2} t + t\Re\Gamma + 2\Re \ln \left[1 + R_1 + \mathrm{e}^{-\Gamma t}(1 - R_1) \right].
\end{aligned} \tag{60}
$$

The last term is suppressed in the large t limit. Expanding the second term for $U \gg J$, we find that

$$\Re\,\Gamma = \gamma_l + \gamma_p + \mathcal{O}(U^{-2}) \,. \tag{61}$$

such that the renormalised decay for $U \gg J$ reads

$$\kappa_\infty = \frac{\gamma_l + 3\gamma_p}{2} = \kappa_0 + 2\gamma_p \ . \tag{62}$$

F   Details on the numerical calculations

F.1   Exact diagonalisation

Numerical results were obtained by exact diagonalisation of the Hamiltonian or Lindbladian in the Fock basis of particle occupation numbers per site and the time evolution implemented in the qutip library [73, 74]. In the case of the closed-system evolution, we additionally use the conservation of particle number, as explained in section F.5.

F.2   Fit of the decay

In order to extract the decay of the retarded Green's function (3) in the NESS, we assume that it is of the form given by the single-pole ansatz (22) and define

$$\mathcal{G}(t) \coloneqq -\Re \ln \mathrm{i} G_{00}^R(t) \ . \tag{63}$$

In the non-interacting regime, the single-pole ansatz is exact and one has $\mathcal{G}(t) = \kappa t$. In the interacting case, the decay is renormalised and we determine it from the linear trend of the oscillating curve $\mathcal{G}(t)$.

Restricting the dimension of the local Hilbert space, we obtain results for all interaction strength by exact diagonalisation. Using this method, we extract the slope of $\mathcal{G}(t)$ by fitting a family of linear curves over different intervals. The multiple scale dependence of the oscillation of the two-point function makes this fitting procedure necessary. As the result depends on the chosen interval of the fit, we fit a family of 20 curves in the interval accessible in order to reach sufficient numerical precision. The average and variance are shown as an example in Fig. 8.
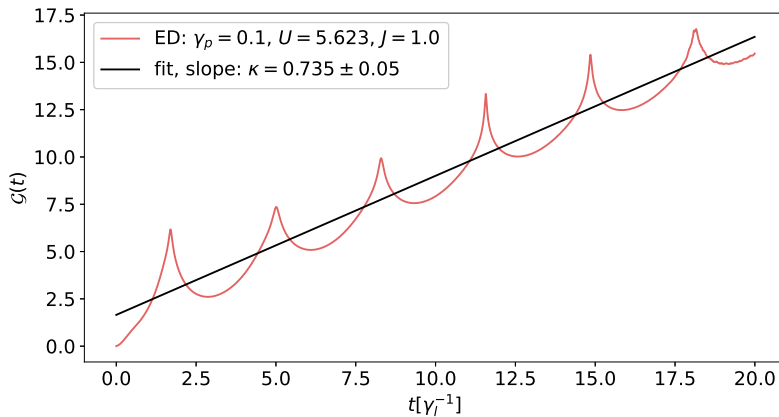
Figure 8: Logarithm of the retarded Green's function $\mathcal{G}(t) = -\Re\left[\ln \mathrm{i} G_{00}^R(0t)\right]$ (dimensionless) obtained from exact diagonalisation (red) together with the single-pole approximation (black, solid) as a function of time (in units of τ_l) for the site $j = 0$. The corresponding fitted decay rate is indicated by the black solid line. The parameters are $U = 5.623$, $J = 1.0$, $z = 0.1$, $L = 4$, $N_s = 4$.

F.3 Effect of the Hilbert space truncation

We showed in the main text that we recover the analytically predicted renormalisation of the decay $\kappa(U)$ for small hopping J. In order to benchmark the numerical truncation over the local Hilbert space, we here investigate the dependence of $\kappa(U)$ for large $J = 10$. We show in Fig. 9 the analytical (one-sited) prediction for small J and the numerical results for $N_s \in \{3, 4, 5\}$. For large U, we expect a deviation from the one-sited result and we observe that the reached plateau is independent of the local Hilbert space truncation. In the weak interaction limit, we interpret the closeness of the results for $N_s = 4$, $N_s = 5$ with the exact prediction as an indication of numerical convergence.
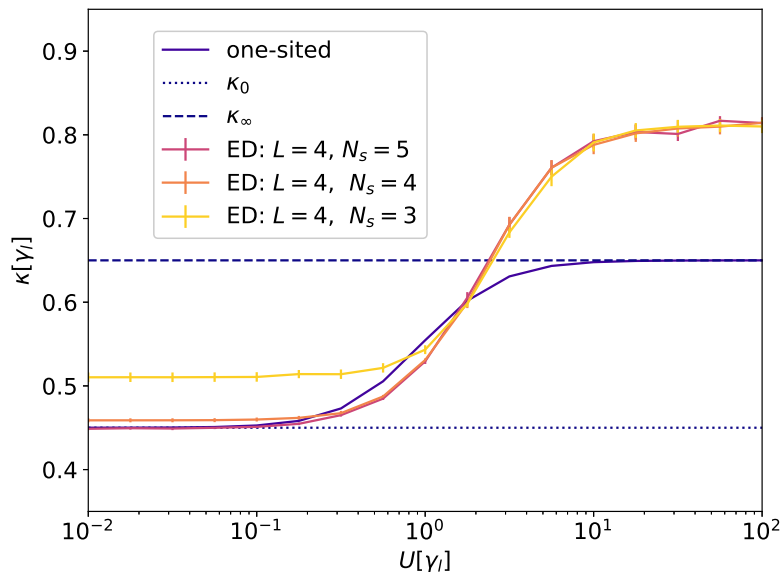
Figure 9: Decay $\kappa(U)$ for $J = 10$, $z = 0.1$ and $L = 4$, and for different local truncation of the Hilbert space N_s in the numerics. We expect to recover the non-interacting solution for small U and a deviation from the one-sited solution in the limit of large interaction.

F.4 System size dependence

We further investigate the dependence of the reached plateau value of the decay for large U and J on the system size. From Fig. 9, we conclude that a truncation of $N_s = 3$ is sufficient to reach the converged plateau value. We hence fix $N_s = 3$ and vary the system size L. The dependence of the renormalised decay on L is shown in Fig. 10, in a limit of large hopping and interaction ($J = 10$ and $U = 100$). We observe that the effective decay increases monotonously with the system size. Further analytical insight could be obtained from the perturbative calculation of corrections in J in the strong interaction limit around the exact result (58).

F.5 Exact diagonalisation for the unitary system

We calculate the Green's function as the overlap of two time-evolved states. Since we are interested in the unitary time evolution, the dimensionality of the problem greatly reduces, and is further diminished by the symmetries present in the problem (particle number conservation, momentum conservation and parity conservation). We calculate the
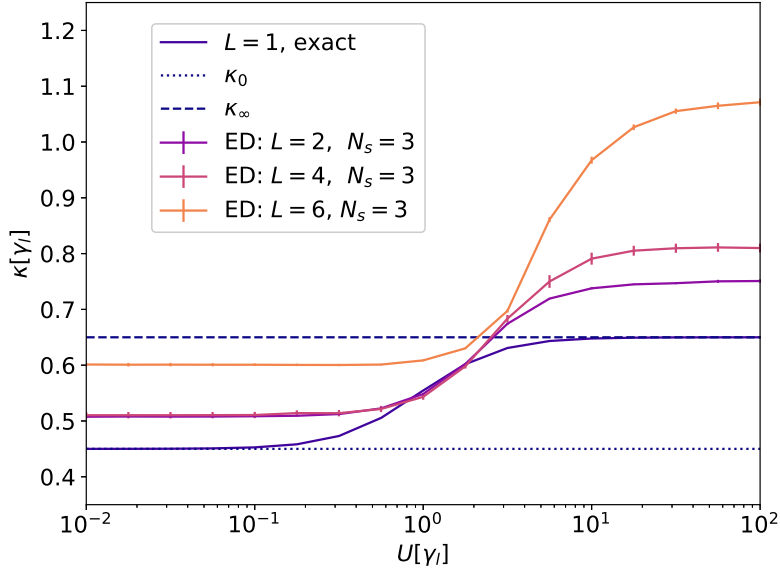
Figure 10: Decay $\kappa(U)$ for different system sizes L. The parameters are $J = 10$, $z = 0.1$ and $N_s = 3$.

Green's function as

$$G^R_{j0}(t, 0) = -\mathrm{i}\Theta(t)\mathrm{Tr}\left\{\left[b_j(t), b_0^\dagger(0)\right]\rho_{\mathrm{NESS}}\right\} \tag{64}$$

$$= -\mathrm{i}\Theta(t)\left(G^{(1)}_j(t) - G^{(2)}_j(t)\right) \tag{65}$$

with

$$
\begin{aligned}
G^{(1)}_j(t) &:= \mathrm{Tr}\left\{\mathrm{e}^{\mathrm{i}Ht}\, b_j \mathrm{e}^{-\mathrm{i}Ht}\, b_0^\dagger \rho_{\mathrm{NESS}}\right\} \\
&= \mathcal{N}^{-1}\sum_{N=0} z^N \sum_{\{m_i\}_N} \mathrm{Tr}\left\{b_j \left|\mathrm{e}^{-\mathrm{i}H_{N+1}t}\, b_0^\dagger \{m_i\}_N\right\rangle \otimes \left|\mathrm{e}^{-\mathrm{i}H_N t}\, \{m_i\}_N\right\rangle^\dagger\right\},
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
G^{(2)}_j(t) &:= \mathrm{Tr}\left\{b_0^\dagger \mathrm{e}^{\mathrm{i}Ht}\, b_j \mathrm{e}^{-\mathrm{i}Ht}\, \rho_{\mathrm{NESS}}\right\} \\
&= \mathcal{N}^{-1}\sum_{N=0} z^N \sum_{\{m_i\}_N} \mathrm{Tr}\left\{b_j \left|\mathrm{e}^{-\mathrm{i}H_N t}\, \{m_i\}_N\right\rangle \otimes \left|\mathrm{e}^{-\mathrm{i}H_{N-1}t}\, b_0 \{m_i\}_N\right\rangle^\dagger\right\}.
\end{aligned} \tag{67}
$$

In order to reduce the Hilbert space dimension further, we construct the Hamiltonian in each particle number sector, since $[N, H] = 0$. Then, we draw states from the distribution ρ_{NESS}, which constitute a sequence of numbers allowed by the local Hilbert space cutoff, and evolve them with the block Hamiltonian constructed in the reduced basis. This is performed using qutip [73, 74], which optimises the temporal evolution. We add and subtract particles (before/ after the evolution) by action on the sequence directly.

F.6   Simulations with matrix product states

In order to produce the plots in Fig. 6 we employ a representation of the many-body state as a matrix-product state [75] and code our algorithm using the Itensors library [76, 77]. We prepare the ground state considering a maximum number of bosons per site

equal to 2 on a lattice with periodic boundary condition and at density $N/L = 1/4$. We subsequently compute the lesser and greater Green's functions $G_{0j}^<(t) = -i\langle b_j^\dagger(t)b_0\rangle$ and $G_{0j}^>(t) = -i\langle b_0 b_j^\dagger(t)\rangle$ and using these data we reconstruct the spectral functions plotted in the figure. For the time evolution, we fix $J = 1$ and use a time step of $\delta t = 0.01$.

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