

Localization engineering by resonant driving in dissipative polariton arrays

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Abstract

Arrays of microcavity polaritons are very versatile systems that allow for broad possibilities for the engineering of multi-orbital lattice geometries using different state preparation schemes. One of these schemes, spatially modulated resonant driving, can be used, for instance, to selectively localize the polariton field within the particular region of the lattice enclosed by the driving laser. Both the frequency and the spatial amplitude distribution (module and phase) of the driven laser field are important and serve as a knob to control the leakage outside that region and hence the extend of the spatial localization. Here, we analyse both the linear and nonlinear regimes using the lattice Green function formalism that is particularly suitable for the case of polariton arrays described in a tight-binding approximation. We identify the conditions for the laser induced localization to occur on arbitrary lattice's geometries and discuss some experimentally relevant cases. We find that the polariton-polariton interaction leads to a frequency shift of the optimal localization condition that could be used to further control it.

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15 **1 Introduction**

16 Microcavity polaritons, composed quasiparticles formed by the coherent coupling of quan-
17 tum well excitons and cavity photons, offer an unique opportunity for the study of driven-
18 dissipative interacting bosonic systems [1, 2]. One particularity of these systems, in contrast
19 to other bosonic systems in condensed matter, is that they are inherently out of equilibrium
20 as they need to be externally pumped, either resonantly or non-resonantly, in order for the
21 polaritons population to be created on the first place. Due to that, it is also possible to control
22 the relevance of the interactions as the system can be taken from a linear regime (low power
23 excitation), where interactions can be neglected, to a nonlinear regime (high power excita-
24 tion) where the polariton-polariton interaction and/or the polariton-reservoir interactions (in
25 the non-resonant case) are important and lead to several interesting non-trivial effects. This
26 versatility, together with the available experimental accessibility and design capabilities has al-
27 lowed the observation of many interesting phenomena such as superfluidity [3, 4], lasing from
28 edge states [5], topological effects [6, 7], non-trivial spin effects [8], vortex quantization [9],
29 optomechanical effects [10–13], or the recent observation of Kardar–Parisi–Zhang universal
30 behaviour of the polariton field phase [14], among many others [15–19]. All these advances
31 required an adequate tailoring of the polariton’s arrays, such as its geometry or multi-orbital
32 character, as well as the use of different techniques to control the polariton dynamics, such
33 as properly engineered non-resonant reservoirs or suitable resonant driving schemes. In this
34 work we will focus on the later case.

35 Properly designed patterns of resonant driving lasers have been used recently to create
36 specific configurations of localized polaritons in 1D and 2D arrays [20–22] where the polariton
37 field is essentially confined to the region enclosed by the driving lasers. This has been done
38 both in the linear and nonlinear regimes. In the latter case the formation of an in-gap soliton in
39 a Su–Schrieffer–Heeger (SSH) chain was also reported [20]. The results were interpreted by
40 comparison with the numerical solution of a generalized Gross-Pitaeskkii equation (gGPE) [23].

41 In this work we discuss a related alternative approach based on lattice Green functions
42 that allows for a generalization to the case of arbitrary lattices and driving patterns. Using
43 this we are able to determine the optimal conditions for the localization (confinement) of the
44 polariton field inside the region enclosed by the driving lasers in a generic scenario in any di-
45 mension, hence providing a simple description of the experimental results of Ref. [21] while at
46 the same time offering a conceptual basis for the design of future experiments. Our approach
47 also allows, under certain conditions, to include the nonlinear effects mentioned earlier. We
48 apply it to 1D arrays and present both numerical and analytical results that illustrate the ba-
49 sic underlying phenomena for localization and the role of the interactions. We find that the
50 polariton-polariton interaction introduces a (mode dependent) frequency shift of the localiza-
51 tion condition that can be tuned with the amplitude of the driving [24]. In the particular case
52 of the SSH chain, we provide a clear description for the formation of the in-gap soliton and its
53 hysteric behavior [20].

54 2 Tight-binding model for polaritons

55 We consider an arbitrary array of polariton cavities. For simplicity, only a single mode is con-
 56 sidered on each cavity—a generalization to many modes/orbitals is straightforward. Assuming
 57 that the modes of each cavity are strongly bound, one can construct from the gGPE a simpli-
 58 fied tight binding Hamiltonian \mathbf{H} , as it is usually done for Bloch electrons. We only consider
 59 the coupling between nearest neighbor cavities, and ignore the effect of non-orthogonality be-
 60 tween different cavity modes [25]. We restrict ourselves to the semi-classical approximation,
 61 where the bosonic operators are replaced by complex functions, which implies that the solu-
 62 tion of interest contains a large number of polaritons so quantum fluctuation can be ignored.
 63 The dynamics of the system is then given by the following set of coupled nonlinear equations,

$$i\hbar \frac{d\phi_j}{dt} = \sum_{k=1}^M H_{jk} \phi_k - i \frac{\hbar\gamma}{2} \phi_j + \hbar F_j e^{-i\omega_d t}, \quad (1)$$

64 with $j = 1 \dots M$, M being the total number of sites on the lattice (which could be finite or
 65 not). Here we added the term $\hbar F_j e^{-i\omega_d t}$ to describe the resonant driving, with frequency ω_d ,
 66 and included the losses on each site, characterized by the rate γ (different rates could also be
 67 considered). In a mean field approximation, intra-cavity polariton-polariton interactions can
 68 be included by replacing $\hbar\omega_0 \rightarrow \hbar\omega_0 + \hbar U |\phi_j|^2$, where $\hbar\omega_0 = H_{jj}$ is the bare energy of the
 69 cavity mode and $\hbar U$ the polariton-polariton interaction strength. In the following sections we
 70 analyse the stationary solutions of Eq. (1) for different experimentally relevant cases using the
 71 lattice Green function of the system both for $U = 0$ and $U \neq 0$.

72 3 Linear regime

73 Let us first consider the case where the driving power (proportional to $|F_j|^2$) is small so that
 74 the nonlinear terms can be safely ignored ($U = 0$). Since we look for stationary solutions, all
 75 the modes must evolve at the frequency of the drive. Then, by moving to the rotating frame
 76 of the drive, $\phi_j(t) = \bar{\phi}_j e^{-i\omega_d t}$, we get

$$\hbar\omega_d \bar{\phi}_j = \sum_{k=1}^M H_{jk} \bar{\phi}_k - i \frac{\hbar\gamma}{2} \bar{\phi}_j + \hbar F_j, \quad (2)$$

77 whose general solution is

$$\bar{\boldsymbol{\phi}} = \mathbf{G}(\tilde{\omega}_d) \cdot \mathbf{F}, \quad (3)$$

78 with $\bar{\boldsymbol{\phi}} = (\bar{\phi}_1, \bar{\phi}_2, \dots, \bar{\phi}_M)$, $\mathbf{F} = (F_1, F_2, \dots, F_M)$, and $\tilde{\omega}_d = \omega_d + i\gamma/2$. Here $\mathbf{G}(\omega) = (\omega \mathbf{I} - \mathbf{H})^{-1}$
 79 is the Green function of the tight-binding lattice described by $(\mathbf{H})_{ij} = H_{ij}/\hbar$. Since there are
 80 numerous numerical methods to obtain $\mathbf{G}(\omega)$ for any lattice or arbitrary array of cavities, it is
 81 rather straightforward to get $\bar{\boldsymbol{\phi}}$ for any driving profile. Furthermore, in some specially simple
 82 cases even analytical solutions are possible. We note in passing that a similar approach was
 83 used in Ref. [26] to describe bound-states-in-the-continuum and its relation to localization in
 84 photonic lattices.

85 In addition, Eq. (3) leads to a general sum rule satisfied by the sum of the amplitudes of
 86 the modes, $I = \sum_j |\bar{\phi}_j|^2$, that reflects the balance between driving and dissipation. Namely,

$$\begin{aligned} I &= \sum_{j,\alpha,\beta} (G_{j\alpha}(\tilde{\omega}_d))^* F_\alpha^* G_{j\beta}(\tilde{\omega}_d) F_\beta, \\ &= \left(\frac{1}{\gamma}\right) \mathbf{F}^* \cdot \mathbf{A}(\tilde{\omega}_d) \cdot \mathbf{F}, \end{aligned} \quad (4)$$

87 where we have introduced the spectral function, $A(\tilde{\omega}_d) = i(\mathbf{G}(\tilde{\omega}_d) - \mathbf{G}(\tilde{\omega}_d)^\dagger)$, and used that it
 88 satisfies $A(\tilde{\omega}_d) = \gamma \mathbf{G}(\tilde{\omega}_d)^\dagger \cdot \mathbf{G}(\tilde{\omega}_d)$. Note that $A_{jj}(\omega) = -2\text{Im}(G_{jj}(\omega))$ relates directly to the
 89 local density of states $D_j(\omega) = -(1/\pi)\text{Im}(G_{jj}(\omega))$.

90 3.1 Linear array

91 Let us now consider an infinite linear array with a nearest neighbors hopping described by the
 92 coupling $J = H_{j(j+1)}/\hbar = H_{(j+1)j}/\hbar$.

93 *Single site driving*-. We assume, without any loss of generality, that site 0 is the one being
 94 driven, $F_j = F\delta_{j0}$. Hence, $\bar{\phi}_j = FG_{j0}(\tilde{\omega}_d)$. Using the recursion relations satisfied by the Green
 95 function we can write, for an infinite linear array,

$$G_{j0}(\omega) = [Jg(\omega)]^{|j|}G_{00}(\omega), \quad (5)$$

96 where, for $|\omega - \omega_0| \leq 2|J|$, we have

$$\begin{aligned} g(\omega) &= \frac{(\omega - \omega_0) - i\sqrt{4J^2 - (\omega - \omega_0)^2}}{2J^2}, \\ G_{00}(\omega) &= \frac{-i}{\sqrt{4J^2 - (\omega - \omega_0)^2}}. \end{aligned} \quad (6)$$

97 Here, $g(\omega)$ is the Green function of the surface site of a semi-infinite array, while $G_{00}(\omega)$ is
 98 the bulk Green function of an infinite array. Surface and bulk density of states are given by
 99 $-(1/\pi)\text{Im}(g(\omega))$ and $-(1/\pi)\text{Im}(G_{00}(\omega))$, respectively. Clearly, the amplitudes $|\bar{\phi}_j|^2 \propto e^{-\lambda|j|}$
 100 decay exponentially with $\lambda = -\ln(|Jg(\tilde{\omega}_d)|)$, which is proportional γ .

101 *Two sites driving*-. We now consider the case where $F_j = F_0\delta_{j0} + F_N\delta_{jN}$, with $N \geq 2$. The
 102 solution is obtained from the previous case by linear superposition,

$$\bar{\phi}_j = G_{j0}(\tilde{\omega}_d)F_0 + G_{jN}(\tilde{\omega}_d)F_N. \quad (7)$$

103 These amplitudes are easily evaluated from Eqs. (5) and (6) and the translation invariance of
 104 the array. For the sites that are not in between the driven sites, we have

$$\begin{aligned} \bar{\phi}_{N+n} &= G_{(N+n)0}(\tilde{\omega}_d)F_0 + G_{(N+n)N}(\tilde{\omega}_d)F_N, \\ &= [Jg(\tilde{\omega}_d)]^n G_{00}(\tilde{\omega}_d)(F_0[Jg(\tilde{\omega}_d)]^N + F_N), \\ &= [Jg(\tilde{\omega}_d)]^n \bar{\phi}_N, \end{aligned} \quad (8)$$

105 and

$$\begin{aligned} \bar{\phi}_{-n} &= G_{-n0}(\tilde{\omega}_d)F_0 + G_{-nN}(\tilde{\omega}_d)F_N, \\ &= [Jg(\tilde{\omega}_d)]^n G_{00}(\tilde{\omega}_d)(F_0 + F_N[Jg(\tilde{\omega}_d)]^N), \\ &= [Jg(\tilde{\omega}_d)]^n \bar{\phi}_0, \end{aligned} \quad (9)$$

106 with $n \geq 0$. This shows, as before, that the amplitudes of the modes far from the pumped
 107 region are exponentially small as $|Jg(\tilde{\omega}_d)| < 1$ for $\gamma \neq 0$ regardless the value of ω_d . It is
 108 evident that we could excite *only* the modes between the driven sites (maximum localization
 109 or confinement), so that $\bar{\phi}_{N+n} = \bar{\phi}_{-n} = 0 \forall n \geq 0$, if we can find a solution of

$$F_0[Jg(\tilde{\omega}_d)]^N + F_N = F_0 + F_N[Jg(\tilde{\omega}_d)]^N = 0. \quad (10)$$

110 This equation implies that $[Jg(\tilde{\omega}_d)]^N = \pm 1$ and that $F_N = \mp F_0$. However, as mentioned above,
 111 because $\gamma \neq 0$ this condition cannot be strictly satisfied. Despite of that, the numerical solution
 112 obtained from Eq. (3) shows a significant localization of the polariton field within sites 0 and

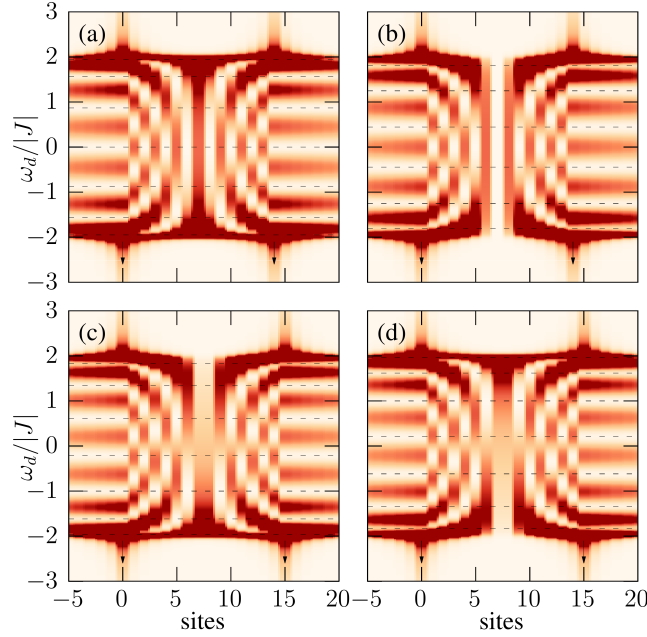


Figure 1: Colormap of $|\phi_j|^2$ as a function of the driving frequency for the case of a linear array with two driven sites (indicated by the black arrows): (a) and (b) correspond to $F_{14} = F_0$ (even excitation) and $F_{14} = -F_0$ (odd excitation), respectively, while (c) and (d) to $F_{15} = F_0$ and $F_{15} = -F_0$. The horizontal dashed lines correspond to $\omega_d = \Omega_m = 2J \cos(m\pi/N)$, with $m = 1, \dots, N-1$ and $N = 14, 15$. Here $\gamma = 0.05J$ and $\omega_0 = 0$. The color scale is saturated to enhance the contrast due to the presence of the band edge singularities. Dark (light) color reflects a large (small) amplitude.

113 N provided γ/J is small (Fig. 1). Therefore, for the sake of argument for finding approximate
 114 conditions for an optimal localization, let us take $\gamma = 0$. In such a case, $Jg(\omega_d) = e^{i|k|}$ with
 115 $\omega_d = \omega_0 + 2J \cos k$ defining k . The condition $e^{i|k|N} = \pm 1$ requires that $k = m\pi/N$ with
 116 $m = 1, \dots, N-1$ —the values $m = 0$ and $m = N$ are excluded due to the divergence of $G_{00}(\omega)$
 117 at the band edge. This has the clear interpretation that the stationary modes excited by the
 118 driving are the modes corresponding to the effectively ‘finite’ chain of $N-1$ sites formed
 119 between the two driven cavities. By fixing the relative phase of the driving, $F_N = \mp F_0$, one can
 120 excite either even or odd modes if ω_d is chosen to satisfy $\omega_d = \Omega_m = \omega_0 + 2J \cos(m\pi/N)$ with
 121 the appropriate choice for m . Indeed, this result could have been anticipated from Eq. (10) as
 122 in such a case Eq. (1) for ϕ_j with $1 \leq j \leq N-1$ corresponds to the isolated chain.

123 Figure 1 shows a colormap of $|\bar{\phi}_j|^2$ obtained numerically from Eq. (3) for two different
 124 values of N : $N = 14$ (top panels) and $N = 15$ (bottom panels). Maximum localization occurs
 125 whenever $\omega_d \sim \Omega_m$ for some of the allowed values for m , indicated by the dashed horizontal
 126 lines, as expected (here we choose $\omega_0 = 0$). Interestingly, this corresponds to a minimum value
 127 of the intensity I , as can be directly confirmed from Eq. (4). This can also be understood if
 128 one notice that I can be written as $I = (2/\gamma)\text{Im}(\mathbf{F} \cdot \bar{\phi}^*)$. Then, since the maximum localization
 129 condition requires the amplitude of the driven sites to be small, of the order of γ/J , the factor
 130 $1/\gamma$ cancels out. Any other condition with a finite value on the driven sites (not proportional
 131 to γ) will lead to a higher value of I [24, 26].

132 From the above considerations, it is clear that when $N = 2$, so that there is a single site
 133 ($j = 1$) between the driven cavities, the only possible solution that leads to the maximum
 134 localization is $\omega_d = \omega_0$ (corresponding to $k = \pm\pi/2$) and $F_2 = F_0$ as found in Ref. [21].

135 *Molecule*-. Another simple case corresponds to a finite chain of two sites, or a ‘polariton

136 molecule' [22]. A straightforward calculation gives for this system

$$G(\omega) = \frac{1}{\Delta\omega_1\Delta\omega_2 - J^2} \begin{pmatrix} \Delta\omega_2 & J \\ J & \Delta\omega_1 \end{pmatrix}, \quad (11)$$

137 with $\Delta\omega_{1,2} = \omega - \omega_{1,2}$, and $\hbar\omega_1$ and $\hbar\omega_2$ the energy of site 1 and 2, respectively. Evidently, if
 138 we drive, say, site 1 at frequency $\omega_d = \omega_2$ then $\bar{\phi}_1 \sim (i\gamma/2J)F/J$ (assuming $|\omega_1 - \omega_2| \ll |J|$)
 139 while the other site has $\bar{\phi}_2 \sim F/J$ and so the polaritons are localized on the undriven site,
 140 $|\bar{\phi}_1| \ll |\bar{\phi}_2|$, as observed in Ref. [22]. When both sites are driven it is possible to find a
 141 condition for one of the sites to have exactly zero amplitude. For instance $\bar{\phi}_1 = 0$ requires
 142 $F_1/F_2 = -J/(\omega_d - \omega_2 + i\gamma/2)$, that is, an specific amplitude/phase relation between the two
 143 driving lasers. In that case it is straightforward to verify that $\bar{\phi}_2 = -F_1/J$.

144 *Su-Schrieffer-Heeger (SSH) chain*-. Yet another interesting linear geometry is the case
 145 of an SSH chain (linear lattice with alternating hoppings J and J'). Since Eq. (3) is valid
 146 for arbitrary lattices in any dimension (see next section), one only needs to specify $G(\omega)$ for
 147 the SSH model. For simplicity, here we address only the situation where a single site (the
 148 0 site) is driven with frequency $\omega_d = \omega_0$, which corresponds to a driving frequency inside
 149 the gap of the SSH chain—more details on this case are presented in the nonlinear section.
 150 It follows from simple considerations that both the real and the imaginary part of $G_{00}(\tilde{\omega}_d)$,
 151 neglecting corrections of order $\gamma/|J - J'| \ll 1$, are zero due to the chiral symmetry of the
 152 lattice and the fact that there is a gap at ω_0 —if one moves away from the center of the gap,
 153 the real part becomes different from zero while the imaginary part is always proportional to
 154 γ . Therefore, the driven site has nearly zero amplitude, $\bar{\phi}_0 = G_{00}(\tilde{\omega}_d)F \sim 0$. This implies that
 155 the 1D lattice is effectively split in two independent parts. As such, one of the sides has the
 156 proper termination to host an edge mode (so called zero energy mode, meaning energy $\hbar\omega_0$)
 157 while the other does not. Thus, only one side is excited with an amplitude profile decaying
 158 exponentially away from it (the decay length being determined by the inverse of the gap, under
 159 our assumption that $\gamma/|J - J'| \ll 1$), and with nonzero amplitudes (up to corrections of order
 160 $\gamma/|J - J'|$) only on a given sub-lattice. Indeed,

$$\bar{\phi}_{\pm j} = G_{\pm j,0}(\tilde{\omega}_d)F = J_{\pm}g_{\pm j,\pm 1}(\tilde{\omega}_d)\bar{\phi}_0, \quad (12)$$

161 where $j > 0$, J_{\pm} is the hopping between site 0 and site ± 1 (so it takes either the value J or J')
 162 and $g_{\pm j,\pm 1}(\omega)$ is the surface Green function connecting site ± 1 and $\pm j$ of a semi-infinite SSH
 163 chain ending on site ± 1 . Because $\bar{\phi}_0 \sim 0$, or more precisely $\sim \gamma/|J - J'|$, in order for $\bar{\phi}_{\pm j}$ to
 164 have a finite value $g_{\pm j,\pm 1}(\omega)$ must have a pole at $\omega = \omega_0$. This is always the case for either
 165 $g_{j,1}(\omega)$ or $g_{-j,-1}(\omega)$ but not both so only one side of the chain is excited.

166 3.2 Arbitrary lattice and driving geometries

167 In this section we shall generalize the previous results to the case of an arbitrary lattice. For
 168 that, let us consider the case where M sites of the lattice are driven. The array of these sites
 169 encloses a region \mathcal{R} , with $N - 1$ sites inside it, so that they separate \mathcal{R} from the rest of the
 170 lattice, that is, each one of the sites in \mathcal{R} are only connected to another site on \mathcal{R} or to some of
 171 the driven sites. Note that it is not necessary for \mathcal{R} to be a single connected region—it could be
 172 a ring shaped region or two separated ones [see Fig. (2)]. If we label the M sites with Greek
 173 indices α, β, \dots and the rest of the sites outside \mathcal{R} with a, b, \dots it is then straightforward to
 174 show that $G_{\alpha\beta}(\omega) = g_{ac}(\omega)V_{ca}G_{\alpha\beta}(\omega)$ —sum over repeated indices is assumed from hereon
 175 unless stated otherwise—where $g_{ac}(\omega)$ is the surface Green function of the external lattice
 176 sites surrounding the driven sites (ie. the one obtained by decoupling or removing the M sites
 177 from the lattice) and $V_{c\alpha} (\equiv H_{c\alpha}/\hbar)$ the hopping matrix element connecting the α -site (driven)

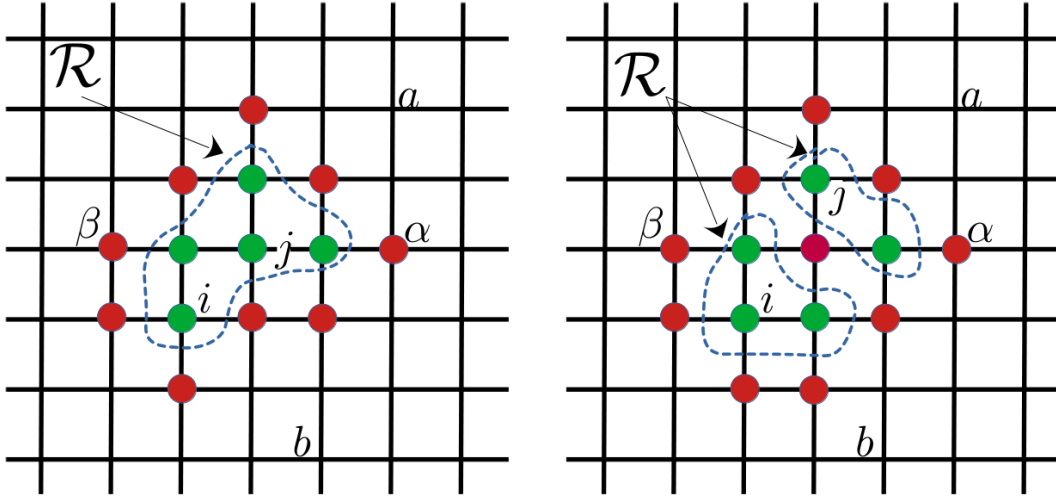


Figure 2: Scheme of different driving configuration on an arbitrary lattice. Driven sites are indicated by red dots and labeled by Greek letters (α, β, \dots). The region \mathcal{R} enclosed by the driven sites is indicated by the green dots and labeled by i, j, k, \dots . Notice we do not require for it to be a single connected region nor a single one as far as it is surrounded by the driven sites. The rest of the lattice sites are labelled as a, b, c, \dots .

178 with the c -site (external). Hence

$$\begin{aligned}\bar{\phi}_a &= G_{a\beta}(\tilde{\omega}_d)F_\beta, \\ &= g_{ac}(\tilde{\omega}_d)V_{c\alpha}\bar{\phi}_\alpha,\end{aligned}\tag{13}$$

179 where we used the fact that $\bar{\phi}_\alpha = G_{\alpha\beta}(\tilde{\omega}_d)F_\beta$. This shows that if, under the appropriate
180 conditions described below, $|\bar{\phi}_\alpha| \sim (\gamma/|J|)|F/J| \ll 1$, where $|J|$ is some typical value for the
181 hopping matrix elements, then the external sites are also $|\bar{\phi}_a| \ll 1$ (further suppressed by
182 the exponential decay of $g_{ac}(\tilde{\omega}_d)$ as a function of the distance $|r_a - r_c|$ induced by the finite
183 linewidth γ or a bulk gap, if present). Note here that in order to guarantee that $|\bar{\phi}_\alpha|$ is small it
184 is important that $g_{ac}(\omega)$ has no singularities at $\omega = \omega_d$, which is usually the case unless there
185 is a bound state at the surface (defined by removing the M sites). The latter can happen only
186 for special lattices and terminations or inside topological gaps hosting edge modes.

187 To calculate the amplitude $\bar{\phi}_j$ of the sites inside \mathcal{R} it is helpful to change to the basis
188 that diagonalizes the Hamiltonian $H_{\mathcal{R}}$ for the isolated region \mathcal{R} , $H_{\mathcal{R}}|\varphi_n\rangle = \hbar\Omega_n|\varphi_n\rangle$, so that
189 $\bar{\phi}_n = \sum_{i \in \mathcal{R}} \langle \varphi_n | i \rangle \bar{\phi}_i$. Then

$$\begin{aligned}\bar{\phi}_n &= G_{n\beta}(\tilde{\omega}_d)F_\beta, \\ &= g_{nn}^{\mathcal{R}}(\tilde{\omega}_d)V_{n\alpha}\bar{\phi}_\alpha,\end{aligned}\tag{14}$$

190 where $\mathbf{g}^{\mathcal{R}}(\omega) = (\omega\mathbf{I} - \mathbf{H}_{\mathcal{R}})^{-1}$ is the Green function of the isolated region described by $(\mathbf{H}_{\mathcal{R}})_{ij} =$
191 $(H_{\mathcal{R}})_{ij}/\hbar = H_{ij}/\hbar$ with $i, j \in \mathcal{R}$ and $V_{n\alpha} = \sum_{i \in \mathcal{R}} \langle \varphi_n | i \rangle V_{i\alpha}$ is the hopping matrix element
192 connecting $|\varphi_n\rangle$ to the α -site in the boundary. To make the analysis simpler, we assume that
193 $\mathbf{g}^{\mathcal{R}}(\omega)$ possesses well resolved poles, that is $\gamma \ll |\Omega_n - \Omega_m| \forall n \neq m$, which is always the case for a
194 sufficiently small region \mathcal{R} —band edges, which have a large density of states impose a stronger
195 condition on γ . We look for finite solutions for $\bar{\phi}_n$ under the condition that $\bar{\phi}_\alpha \sim \gamma/J \rightarrow 0$ (this
196 defines the maximum localization/confinement). Hence, it is clear that the driving frequency
197 must match one of the isolated poles of $\mathbf{g}^{\mathcal{R}}(\omega)$, say

$$\omega_d = \Omega_m,\tag{15}$$

198 in which case only the m -th mode is excited (describe by the normalized mode amplitude
199 $\varphi_j^{(m)} = \langle j | \varphi_m \rangle$).

200 Once the frequency of the driving is fixed, we still need to find the appropriate condi-
201 tions for the amplitudes F_β to guarantee that $\bar{\phi}_\alpha \rightarrow 0$. In order to do that, we first notice
202 that $G_{\alpha\beta}(\omega) = \tilde{g}_{\alpha\beta}(\omega) + \tilde{g}_{\alpha\delta}(\omega)\Sigma_{\delta\rho}(\omega)G_{\rho\beta}(\omega)$ with $\tilde{g}_{\alpha\beta}(\omega)$ the surface Green function as-
203 sociated to the M driven sites (that is, excluding (only) the region \mathcal{R} from the lattice) and
204 $\Sigma_{\alpha\beta}(\omega) = V_{\alpha n}g_{nn}^{\mathcal{R}}(\omega)V_{n\beta}$ the self-energy due to \mathcal{R} . In particular, $\Sigma_{\alpha\beta}(\tilde{\omega}_d) \simeq -i(2/\gamma)V_{\alpha m}V_{m\beta}$
205 (without the implicit sum on m) is an anti-hermitian matrix with a single nonzero eigenvalue
206 given by $\Sigma_{\lambda\lambda} = -i(2/\gamma)\sum_\alpha |V_{\alpha m}|^2 = \text{Tr}(\Sigma(\tilde{\omega}_d))$. On its eigenvector basis, labelled by $\tilde{\alpha}$ and
207 $\tilde{\beta}$, where the index $\tilde{\alpha} = \lambda$ corresponds to the singular eigenvector $\boldsymbol{\lambda}$ with eigenvalue $\Sigma_{\lambda\lambda} \neq 0$,
208 we can write

$$G_{\tilde{\alpha}\tilde{\beta}}(\tilde{\omega}_d) = \tilde{g}_{\tilde{\alpha}\tilde{\beta}}(\tilde{\omega}_d) + \Sigma_{\lambda\lambda} \frac{\tilde{g}_{\tilde{\alpha}\lambda}(\tilde{\omega}_d)\tilde{g}_{\lambda\tilde{\beta}}(\tilde{\omega}_d)}{1 - \tilde{g}_{\lambda\lambda}(\tilde{\omega}_d)\Sigma_{\lambda\lambda}}, \quad (16)$$

209 from where we see that only the elements $G_{\lambda\tilde{\beta}}$ and $G_{\tilde{\alpha}\lambda}$ are proportional to γ and so the
210 smallest—here we assume that $\tilde{\mathbf{g}}(\tilde{\omega}_d)$ has no singularities or edge modes. Hence, if we choose
211 \mathbf{F} to be proportional to the singular eigenvector, that is $\mathbf{F} = \xi\boldsymbol{\lambda}$ or, in terms of components,

$$F_\beta = \xi \frac{V_{\beta m}}{(\sum_\alpha |V_{\alpha m}|^2)^{1/2}}, \quad (17)$$

212 we obtain

$$\begin{aligned} \bar{\phi}_{\tilde{\alpha}} &= \frac{\mathbf{F} \cdot \boldsymbol{\lambda}^* \tilde{g}_{\tilde{\alpha}\lambda}(\tilde{\omega}_d)}{1 - \tilde{g}_{\lambda\lambda}(\tilde{\omega}_d)\Sigma_{\lambda\lambda}}, \\ &\simeq \frac{-i\gamma \mathbf{F} \cdot \boldsymbol{\lambda}^* \tilde{g}_{\tilde{\alpha}\lambda}(\tilde{\omega}_d)}{2\sum_\alpha |V_{\alpha m}|^2 \tilde{g}_{\lambda\lambda}(\tilde{\omega}_d)}, \end{aligned} \quad (18)$$

213 where we assumed $|\tilde{g}_{\lambda\lambda}(\tilde{\omega}_d)\Sigma_{\lambda\lambda}| \gg 1$. $\bar{\phi}_\alpha$ can be easily obtained after a change of basis.
214 From Eq. (14) we then get

$$\bar{\phi}_j \simeq -\frac{\mathbf{F} \cdot \boldsymbol{\lambda}^* \sum_\alpha V_{\alpha j} \tilde{g}_{\alpha\lambda}(\tilde{\omega}_d)}{\tilde{g}_{\lambda\lambda}(\tilde{\omega}_d) \sum_\alpha |V_{\alpha m}|^2} \varphi_j^{(m)}, \quad (19)$$

$$\simeq -\frac{\mathbf{F} \cdot \boldsymbol{\lambda}^*}{(\sum_\alpha |V_{\alpha m}|^2)^{1/2}} \varphi_j^{(m)}, \quad (20)$$

215 where, as one might expect, the spatial profile in the region \mathcal{R} is dictated by the selected
216 eigenstate $\varphi_j^{(m)}$. We notice that the strict absence of γ in Eq. (20) is a consequence of the limit
217 $\gamma \rightarrow 0$ we used to get some simple closed expressions. The full solution contains also some
218 attenuation due to the decay of $g_{ij}^{\mathcal{R}}(\tilde{\omega}_d)$ with the distance between sites i and j , that arise
219 from the contribution of the neglected modes $n \neq m$.

220 In the particular case of an arbitrary 1D array, driven on sites 0 and N and coupled to the
221 inner region by a coupling J , Eq. (20) can be further simplified to give

$$\begin{aligned} \bar{\phi}_j &\simeq -\frac{\mathbf{F} \cdot \boldsymbol{\lambda}^*}{J} \frac{\varphi_j^{(m)}}{\sqrt{|\varphi_1^{(m)}|^2 + |\varphi_{N-1}^{(m)}|^2}}, \\ &\simeq -\frac{F_0}{J} \frac{\varphi_j^{(m)}}{\varphi_1^{(m)}} = -\frac{F_0}{J_{\text{eff}}^{(m)}} \varphi_j^{(m)}. \end{aligned} \quad (21)$$

222 Note that the amplitude of $\bar{\phi}_1 = -F_0/J$ is fixed by the driving. Similarly $\bar{\phi}_{N-1} = -F_N/J$.
223 Here, $J_{\text{eff}}^{(m)}$ is the effective coupling between the driven sites and the eigenmode $|\varphi_m\rangle$. The

224 homogeneous linear array discussed in the previous section is obtained from here by using:
 225 $\varphi_j^{(m)} = \sqrt{2/N} \sin(k_m j)$, $\mathbf{F} \cdot \boldsymbol{\lambda}^* = \sqrt{2}F_0$ and $k_m = m\pi/N$ with $m = 1, \dots, N-1$.

226 The results presented in this section provide a recipe (Eqs. (15) and (17)) for the design
 227 of localized polariton fields on arbitrary lattices and might then serve as a guide for future
 228 experiments. Alternatively, one could use particular lattices with directional propagation at
 229 given frequencies to generate different patterns as discussed in Ref. [26].

230 4 Nonlinear regime

231 When interactions are taken into account Eq. (1) becomes nonlinear as the site energy changes
 232 $\hbar\omega_0 \mapsto \hbar\omega_0 + \hbar U|\phi_j|^2$. Nevertheless, Eq. (3) remains valid as far as the driving has a single
 233 Fourier component and provided a constant amplitude solution is possible [27]. In that case,
 234 Eq. (3) must be understood as a self-consistent equation for $\bar{\phi}$. Namely,

$$\bar{\phi} = \mathbf{G}(\tilde{\omega}_d, \bar{\phi}) \cdot \mathbf{F}. \quad (22)$$

235 There are a number of numerical protocols to find solutions for such fixed point equation. Their
 236 implementation, stability and computation requirements depend very much on the geometry
 237 and dimensionality of the lattice and the distribution of the driving sites. The numerical so-
 238 lutions can be very complex and present hysteric behavior—the stable/unstable solutions will
 239 depend on the way the driving amplitude is set up (ramping up or down). For the same rea-
 240 sons, analytical solutions or approximations are in general rather difficult to find. It should
 241 be noted though, that some of the formal relations found for the linear case, such as Eqs. (4),
 242 (13) and (14), are still valid (as self-consistent equations) as well as some of the approxima-
 243 tions that followed them. Of course, one can still argue that if the condition for localization is
 244 met ($|\bar{\phi}_\alpha| \propto \gamma/|J| \ll 1$) then $\hbar\omega_d$ should match an eigenenergy ($\hbar\Omega_m$) of the self-consistent
 245 Hamiltonian $H_{\mathcal{R}}$ —provided its eigenenergies are well resolved. Of course, since now $\varphi_j^{(m)}$ is
 246 self-determined, so is $\boldsymbol{\lambda}$ and one cannot longer guarantee that \mathbf{F} (externally fixed) is propor-
 247 tional to it. Nevertheless, the analytical results of the previous section might serve as a guide
 248 to interpret the fully numerical results. There are, however, special cases that admit some
 249 treatment as the 1D systems we now discuss.

250 4.1 Nonlinear linear array

251 Let us consider the nonlinear version of the linear array discussed in Sec. 3.1. Figure 3 shows
 252 the numerical results obtained using Eq. (22) for $N = 10$ and for four different values of
 253 $U/|J| = 0, 0.1, 0.2, 0.5$ with even driving ($F_0 = F_{10}$). The ‘internal’ region \mathcal{R} is then defined
 254 by sites 1 to 9. The interaction term is included only from sites -9 through 19, a justified
 255 approximation when looking at states localized inside \mathcal{R} , that allows us to include the rest
 256 of the (infinite) sites as a self-energy on the boundary sites (-9 and 19). The self-consistent
 257 procedure is initiated with $\bar{\phi} = 0$ and it is carried out until $|\Delta\bar{\phi}|/|\bar{\phi}| < 10^{-6}$. There are several
 258 important aspects of Fig. 3 to point out: (i) the colormaps of $|\bar{\phi}_j|^2$ (left panels) shows that
 259 in all cases the resonantly driven localization persist for the appropriate value of ω_d ; (ii) the
 260 interaction blue shifts the resonant condition, as it is clearly evident on the right panels where
 261 we plot $|\bar{\phi}_5|^2$ (center of \mathcal{R}) and $|\bar{\phi}_{-5}|^2$ (outside \mathcal{R}), see discussion below; (iii) the extend of the
 262 localization is not substantially affected by U in the range presented here; (iv) the noisy data
 263 near the band edges reflect the presence of instabilities where convergence is not possible.

264 To better understand these numerical results, we can take advantage of the analytical
 265 expressions found before. In this case one has that $\varphi_1^{(m)} = \pm\varphi_{N-1}^{(m)}$, that is, the even/odd
 266 symmetry is preserved and so the choice $F_0 = \pm F_N$ guarantees that $\mathbf{F} \parallel \boldsymbol{\lambda}$. Therefore, one can

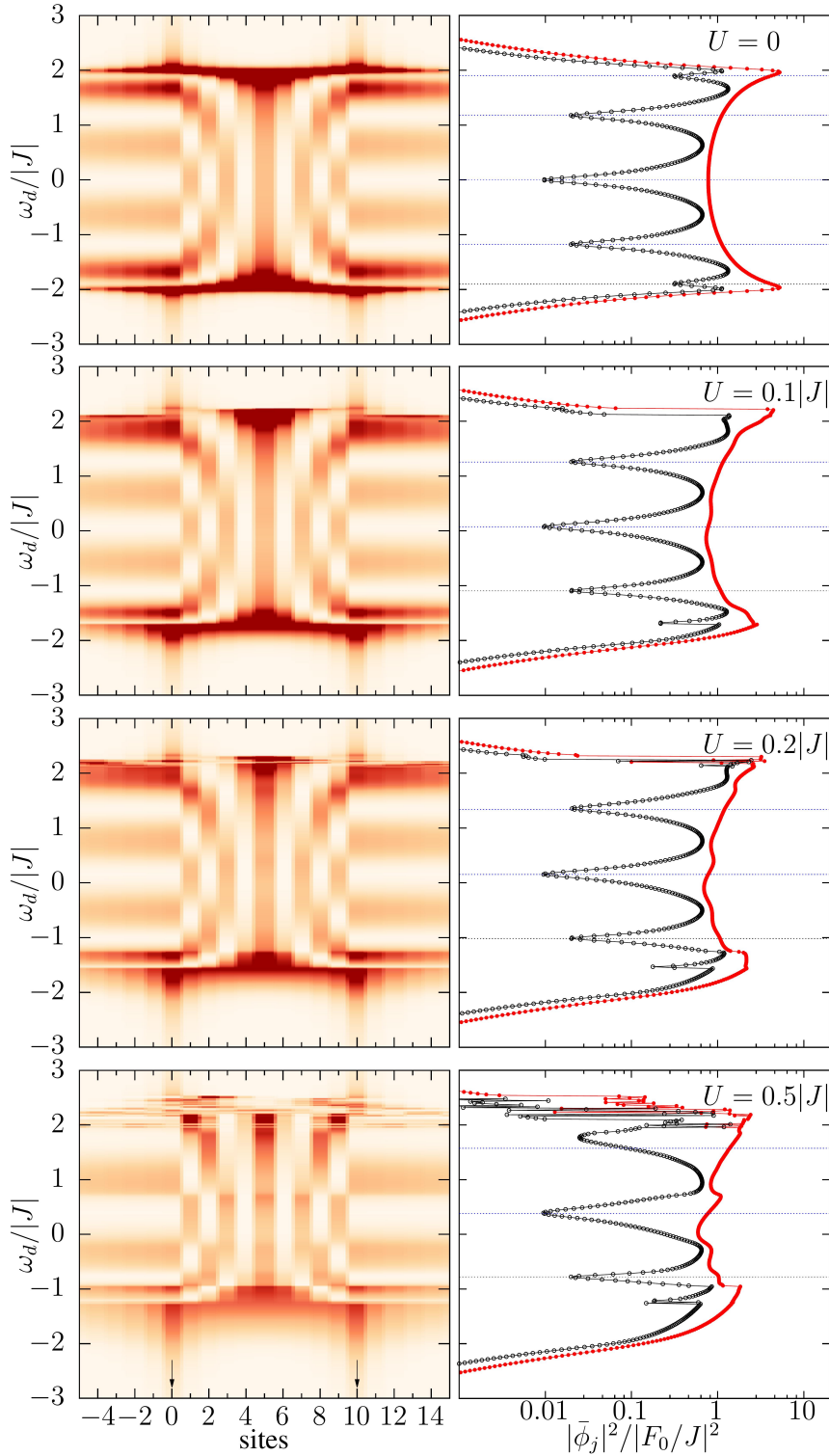


Figure 3: $|\bar{\phi}_j|^2$ as a function of the driving frequency ω_d for the interacting linear array described in the text with $N = 10$ (sites 0 and 10 are driven, indicated by the black arrows). Left panels shown a colormaps for sites -5 to 15 while in the right panels only the central site amplitude, $|\bar{\phi}_5|^2$ (solid symbols), and $|\bar{\phi}_{-5}|^2$ (open symbols) are shown. The dashed horizontal lines in the later correspond to the estimated shifts, Eq. (24), including the factor $(1 - 2\mu_m)$, see text. Here $U/|J| = 0, 0.1, 0.2$ and 0.5 , from top to bottom, and $\gamma/|J| = 0.05$.

267 make use of Eq. (21) to express the local interaction potential in terms of the self-consistent
268 mode,

$$U|\bar{\phi}_j|^2 = \frac{U|F_0|^2}{|\varphi_1^{(m)}|^2 J^2} |\varphi_j^{(m)}|^2 = \frac{U_{\text{eff}}^{(m)} |F_0|^2}{J^2} |\varphi_j^{(m)}|^2, \quad (23)$$

269 where we introduced an effective $U_{\text{eff}}^{(m)} > U$ instead of $J_{\text{eff}}^{(m)}$. This leads to an interesting and
270 non-trivial consistency problem whose full solution is beyond the scope of the present work.
271 Here we only consider some limiting scenarios. First, we notice that $U_{\text{eff}}^{(m)}$ near the band center
272 is smaller than towards the band edges, so one expect the interaction effects to be stronger
273 in the latter case. In the limit $U/|J| \ll 1$ we can applied first order perturbation theory to
274 calculate the shift of the localization condition due to the interaction, namely

$$\begin{aligned} \omega_d &= \bar{\Omega}_m, \\ \omega_d &\simeq \Omega_m + \frac{U_{\text{eff}}^{(m)} |F_0|^2}{J^2} \sum_{j=1}^{N-1} |\varphi_j^{(m)}|^4, \\ &\simeq \Omega_m + \frac{2}{N} \frac{U|F_0|^2}{J^2 \sin^2(k_m)} \sum_{j=1}^{N-1} \sin^4(k_m j). \end{aligned} \quad (24)$$

275 When N is even, which corresponds to an odd number of sites between the driven sites, the
276 state in the band center has $k_m = \pi/2$, given $\omega_d = \omega_0 + U|F_0|^2/J^2$ as the condition for optimal
277 localization (this agrees with the numerical results obtained in [24] for $N = 2$ and $N = 6$).
278 In fact, this is a result valid beyond perturbation theory as $\varphi_j^{(N/2)} = \sqrt{2/N} \sin(j\pi/2)$ is an
279 exact self-consistent eigenfunction of $H_{\mathcal{R}}$ in the presence of an uniform energy shift—though
280 not necessarily stable for large U . This is so because in that mode all sites in \mathcal{R} have zero or
281 equal amplitude. We note however, that this is valid under the simplifications used to derive
282 Eq. (21).

283 In order to compare the shift predicted by Eq. (24) with the numerical results, and since the
284 latter where obtained using a small but finite value of γ , it is important to retain in Eqs. (18)
285 and (20) corrections up to order $(\gamma/J)^2$ and $\gamma/|J|$, respectively. This adds a factor $(1 - \mu_m)$ to
286 Eq. (23) and $(1 - 2\mu_m)$ to the shift in Eq. (24),

$$\begin{aligned} \delta\omega_d &= \omega_d - \Omega_m, \\ &\simeq \frac{2}{N} \frac{U|F_0|^2(1 - 2\mu_m)}{J^2 \sin^2(k_m)} \sum_{j=1}^{N-1} \sin^4(k_m j). \end{aligned} \quad (25)$$

287 with $\mu_m = \gamma N / (4|J| \sin(k_m))$. This renormalized frequency shift is indicated in Fig. 3 with
288 dashed lines, showing a remarkable agreement despite the fact that $U/|J|$ is not necessarily
289 small. A similar comparison, and agreement, is presented in Fig. 4 for $N = 2$ up to a even
290 higher value of $U/|J|$. Here the shift is given by

$$\omega_d \simeq \omega_0 + \frac{U|F_0|^2}{J^2} \left(1 - \frac{\gamma}{|J|}\right), \quad (26)$$

291 and it is indicated in panel (a) for each value of U by the corresponding arrows. In panel (b)
292 we subtracted the shift to show that there is a small enhancement of the localization (width of
293 minimum of $|\bar{\phi}_{-1}|^2$) with increasing U . This shift induced by the interactions serves as another
294 knob to tune the localization as now both ω_d and F_0 determine the optimal condition [24].

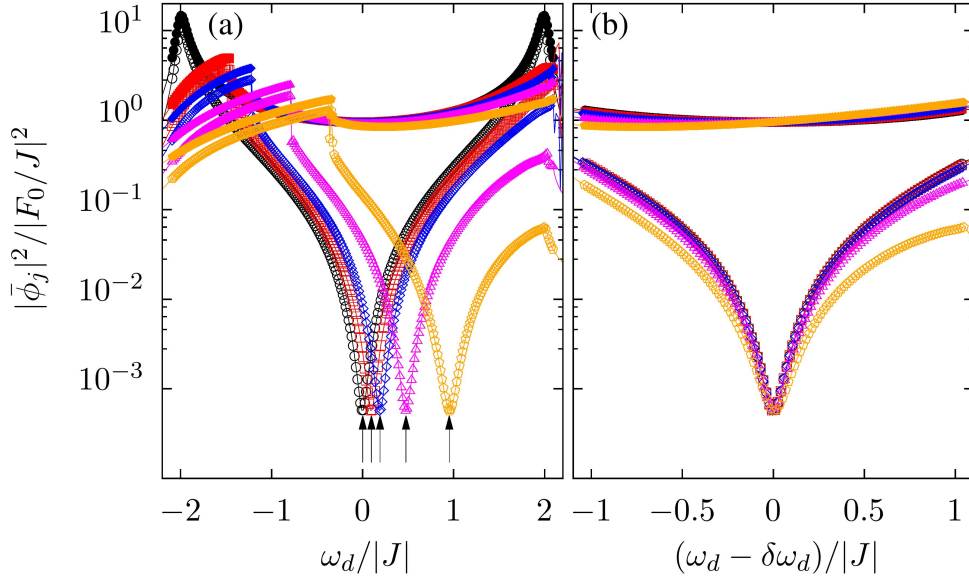


Figure 4: (a) Plot of $|\bar{\phi}_1|^2$ (solid symbols) and $|\bar{\phi}_{-1}|^2$ (open symbols) as a function of the driving frequency ω_d for the linear array described in the text with $N = 2$ (sites 0 and 2 are driven). Here $U/|J| = 0$ (circles), 0.1 (squares), 0.2 (diamonds), 0.5 (triangles) and 1 (pentagons). The arrows indicate the corresponding predictions of Eq. (26). Here we took $\omega_0 = 0$ and $\gamma/|J| = 0.05$. (b) Same as before but with the predicted shift $\delta\omega_d$ subtracted.

295 4.2 Nonlinear SSH chain

296 Another simple but experimentally relevant case [20] corresponds to the SSH chain where two
 297 consecutive sites, say 1 and 2, are driven. We assume they are linked by the strongest coupling
 298 J , though some of the following results are independent of that. Note that here there is no
 299 ‘internal’ region \mathcal{R} so some of the previous results do not directly apply.

300 The two driven sites constitute an effective molecule embedded on a lattice. The exact
 301 effective Green function for the driven sites can be cast as,

$$\begin{aligned} \tilde{\mathbf{G}}(\omega) &= \begin{pmatrix} \tilde{G}_{11}(\omega) & \tilde{G}_{12}(\omega) \\ \tilde{G}_{21}(\omega) & \tilde{G}_{22}(\omega) \end{pmatrix}, \\ &= \frac{1}{\mathcal{D}(\omega)} \begin{pmatrix} \omega - \omega_2 - \Sigma_R(\omega) & J \\ J & \omega - \omega_1 - \Sigma_L(\omega) \end{pmatrix}, \end{aligned} \quad (27)$$

302 where $\omega_{1,2} = \omega_0 + U|\bar{\phi}_{1,2}|^2$, $\mathcal{D}(\omega) = \text{Det}(\tilde{\mathbf{G}}(\omega))^{-1}$ and $\hbar\Sigma_{L,R}(\omega)$ are the boundary self-
 303 energies due to the semi-infinite chain attached to each of the driven sites (Σ_L to site 1 and Σ_R
 304 to site 2). They can be written as $\Sigma_L(\omega) = J'^2 g_{33}(\omega)$ and $\Sigma_R(\omega) = J'^2 g_{00}(\omega)$ where $g_{33}(\omega)$
 305 and $g_{00}(\omega)$ are the local surface Green functions of the respective semi-infinite chain, which
 306 do not explicitly depend on $\bar{\phi}_1$ and $\bar{\phi}_2$. The later are given by,

$$\begin{aligned} \bar{\phi}_1 &= \tilde{G}_{11}(\tilde{\omega}_d)F_1 + \tilde{G}_{12}(\tilde{\omega}_d)F_2, \\ \bar{\phi}_2 &= \tilde{G}_{21}(\tilde{\omega}_d)F_1 + \tilde{G}_{22}(\tilde{\omega}_d)F_2. \end{aligned} \quad (28)$$

307 *Symmetric solution*–. We start by searching for symmetric solutions with respect to the two
 308 driven sites for the case $F_1 = F_2 = F$. In that case, $\tilde{G}_{11} = \tilde{G}_{22}$ and $\bar{\phi}_1 = \bar{\phi}_2 = \bar{\phi}$ so that Eq. (28)
 309 reduces to

$$\bar{\phi} = \frac{F}{\delta\omega' + i\gamma'/2 - U|\bar{\phi}|^2 - J}, \quad (29)$$

310 with $\delta\omega' = \delta\omega - \text{Re}[\Sigma(\tilde{\omega}_d)]$, $\delta\omega = \omega_d - \omega_0$, and $\gamma' = \gamma - 2\text{Im}[\Sigma(\tilde{\omega}_d)]$. Eq. (29) has the form
 311 of the standard bistability equation for a driven non-linear resonator [27], except for the fact
 312 that $\Sigma(\omega)$ depends on the amplitude of the other sites through the local nonlinear terms. To
 313 proceed, we assume that ω_d lies inside the SSH gap, that is $|\delta\omega| < |J - J'|$. We then expect
 314 that the amplitudes $\bar{\phi}_j$ will be strongly localized around the driven sites. If the localization is
 315 strong enough, as a first approximation, one can ignore the nonlinear terms on the non-driven
 316 sites so that $\Sigma(\omega)$ has a closed analytical form (not shown). It can be easily checked that for
 317 $\delta\omega = 0$ the exact self-energy is purely imaginary, while near the center of the gap it takes the
 318 form

$$\Sigma(\tilde{\omega}_d) \sim -\left(\delta\omega + i\frac{\gamma}{2}\right) \frac{J'^2}{J^2 - J'^2}. \quad (30)$$

319 where $|\delta\omega|, \gamma \ll |J - J'|$ and $|J| > |J'|$. It is then clear that Eq. (29) corresponds to an
 320 effectively isolated dimer with slightly renormalized parameters, the only role of the rest of
 321 the SSH chain in determining the stationary values of $\bar{\phi}$. Specifically, $\delta\omega' = \epsilon\delta\omega$ and $\gamma' = \epsilon\gamma$,
 322 with $\epsilon = J^2/(J^2 - J'^2)$. A standard stability analysis of Eq. (29) yields two unstable thresholds,
 323 up to corrections of order $\gamma'^2/|J|$,

$$\begin{aligned} (U|\bar{\phi}_+|^2, F_+) &= \left(\frac{-J + \delta\omega'}{3}, \frac{2}{3\sqrt{3}} \sqrt{\frac{(-J + \delta\omega')^3}{U}} \right), \\ (U|\bar{\phi}_-|^2, F_-) &= \left(-J + \delta\omega', \frac{\gamma'}{2} \sqrt{\frac{-J + \delta\omega'}{U}} \right), \end{aligned} \quad (31)$$

324 that correspond to the situation where F is increased (+) or decreased (−). Notice that the
 325 sign of J , which we take to be negative, is important for the symmetric solution to be possible
 326 (when $J > 0$ the symmetric solution requires $F_1 = -F_2$).

327 The exact numerical solution of Eq. (22) for an infinite lattice but where interactions are
 328 included only on seven unit cells (14 sites), three on each side of the one being driven, is
 329 shown in Fig. 5. Panel (a) shows a colormap of $|\bar{\phi}_j|^2$ as a function of F (ramped up) while
 330 panel (b) shows the behavior of $|\bar{\phi}_2|^2$, $|\bar{\phi}_3|^2$ and $|\bar{\phi}_4|^2$. The alternating low power spatial
 331 profile can be understood, as discussed in the previous cases, in terms of the surface lattice
 332 Green function in the absence interactions. Note that there is an excellent agreement with
 333 the threshold estimated by Eq. (31) as indicated in the figure by the pentagon symbol at the
 334 lowest value of F/F_{th} .

335 Figure 5(a-b) also shows, in agreement with the experimental data of Ref. [20], a second
 336 transition at a higher power. This is a signal that the assumption of negligible interaction effects
 337 for the undriven sites is not longer valid. In fact, it is clear from the figure that neighboring
 338 sites acquire a significant occupation after the first transition. To take this into account we can
 339 use the fact that $\bar{\phi}_j = g_{j3}(\tilde{\omega}_d)J'\bar{\phi}$ with $j \geq 3$ (cf. Eq. (13)). If interactions are only included
 340 in the two neighboring sites, 3 and 4 (sites 0 and −1 are taken into account by the assumed
 341 symmetry of the solution), this can be viewed as an additional effective molecule ‘driven’ by the
 342 field $J'\bar{\phi}$ but with a SSH chain attached to site 4 only. Following a similar procedure as before,
 343 for the case $\delta\omega = 0$ and for increasing power, one finds that the second transition takes place
 344 when $U|\bar{\phi}_2|^2 = -(64/81\sqrt{3})J^3/J'^2$, $U|\bar{\phi}_3|^2 = -J/3\sqrt{3}$, and $U|\bar{\phi}_4|^2 = -J/\sqrt{3}$. The threshold
 345 value for F can be obtained from Eq. (29) taking into account that the contribution to $\delta\omega'$
 346 from the self-energy is the shift $(3\sqrt{3}/8)J'^2/J$ —once again, this is the only effect of the rest
 347 of the SSH chain. These estimates, also shown in Fig. 5(b), are in very good agreement with
 348 the full numerical data. Note that the blue shift of the driven sites can be quite significant as
 349 $|J/J'| > 1$.

350 *Asymmetric solution*–. We now look for the conditions to obtain an asymmetric solution
 351 where only the sites on one side of the driven sites are significantly populated. For that,

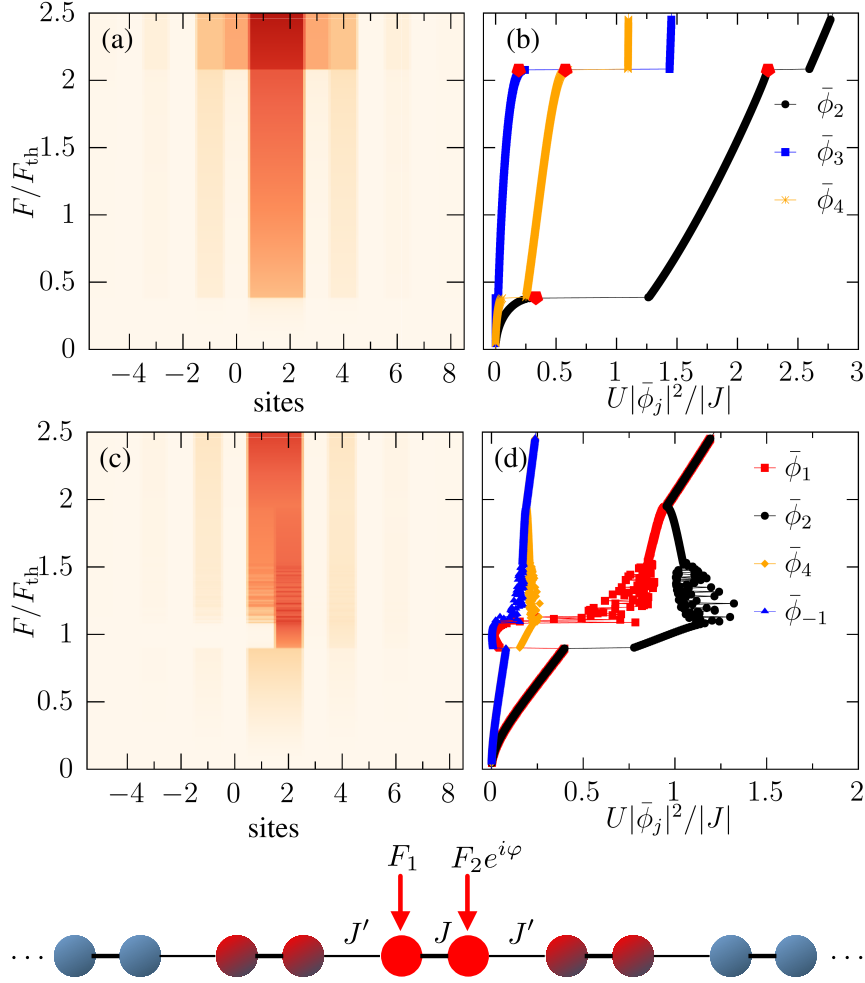


Figure 5: Left panels: colormap of $|\bar{\phi}_j|^2$ as a function of the driving field amplitude $F = |F_1| = |F_2|$ (ramped up) for the SSH chain (see scheme at the bottom) for two different relative phases $\varphi = 0$ (a) and $\varphi = \pi$ (c). Other parameters are: $J' = 0.45J$, $U = 0.1|J|$, $\gamma = 0.1|J|$, $F_{\text{th}}^2 = |J|^3/U$, $\delta\omega = 0$ and $J < 0$. Right panels: corresponding amplitudes for some relevant sites. Red pentagons in (b) indicate transition points obtained from Eq. (31).

352 we impose that one of the driven sites, say site 1, has zero (or very small) amplitude—this
 353 immediately implies that $\bar{\phi}_j = g_{j0}(\tilde{\omega}_d)J'\bar{\phi}_1$ with $j \leq 0$ are also zero (very small) as $g_{j0}(\tilde{\omega}_d)$
 354 has no poles (here we assume $|J| > |J'|$ as before and ω_d to be inside the SSH gap). Making
 355 $\bar{\phi}_1 = 0$ in Eq. (28) leads to

$$\begin{aligned}\bar{\phi}_2 &= \left(\tilde{G}_{21}(\tilde{\omega}_d) - \tilde{G}_{22}(\tilde{\omega}_d) \frac{\tilde{G}_{11}(\tilde{\omega}_d)}{\tilde{G}_{12}(\tilde{\omega}_d)} \right) F_1, \\ &= -\frac{\text{Det}(\tilde{\mathbf{G}}(\tilde{\omega}_d))}{\tilde{G}_{12}(\tilde{\omega}_d)} F_1 = -\frac{F_1}{J},\end{aligned}\quad (32)$$

356 which is valid even in the presence of non-linear terms in all sites. In addition,

$$\begin{aligned}\frac{F_2}{F_1} &= -\frac{\tilde{G}_{11}(\tilde{\omega}_d)}{\tilde{G}_{12}(\tilde{\omega}_d)}, \\ &= \frac{\delta\omega' + i\gamma'/2 - U|\bar{\phi}_2|^2}{-J}.\end{aligned}\quad (33)$$

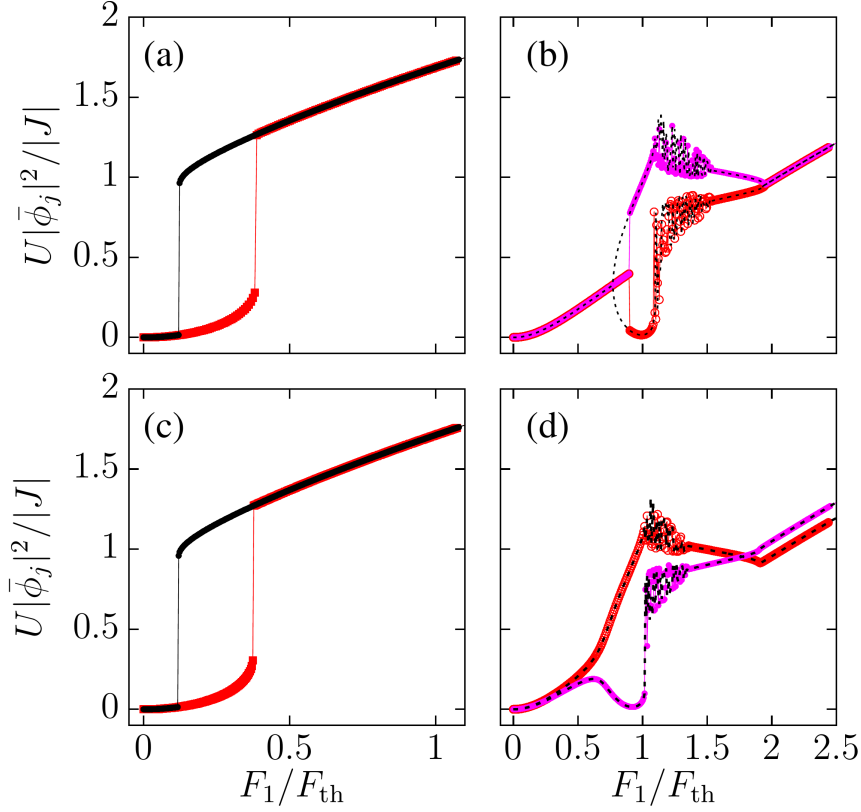


Figure 6: Low power hysteresis loop for the SSH chain. Only the amplitudes of the two driven sites are shown. Red/magenta (black) symbols/lines correspond to increasing (decreasing) driving amplitude. On the top panels we use $|F_2| = |F_1|$ while in the bottom ones $|F_2| = 1.05|F_1|$. Panels (a) and (c), where $\varphi = 0$, show a standard hysteresis loop. Panels (b) and (d), where $\varphi = \pi$, show a more complex behavior with dynamical instabilities (noisy data) in both cases but only hysteric behavior on (b). Open and closed symbols correspond to sites 1 and 2, respectively. The rest of the parameters are: $\omega_d = \omega_0$, $U = 0.1|J|$, $\gamma = 0.1|J|$, $J' = 0.45J$ and $J < 0$.

357 This imposes a precise requirement for the phase relation between the two driving lasers, as
 358 in the polariton molecule discussed before. For small detuning, $|\delta\omega'| < U|\bar{\phi}_2|^2$, as it is the
 359 case considered here, this requires a phase difference $\varphi \simeq \pi$ (0) for $J < 0$ ($J > 0$). If both
 360 lasers have the same or similar amplitudes, $|F_2/F_1| \simeq 1$, that is the experimental situation in
 361 Ref. [20], Eq. (33) cannot be satisfied exactly, which also means that $\bar{\phi}_1 \neq 0$. Nevertheless,
 362 since $\gamma'/|J| \ll 1$, and if we take $F_2 = -F_1$, then $|\bar{\phi}_1|$ gets minimized (to the lowest order in
 363 $\gamma'/|J|$) when the real part of Eq. (33) is -1 . That is,

$$U|\bar{\phi}_2|^2 = -J + \delta\omega', \quad |F_1|^2 = F_{\text{th}}^2 = (-J + \delta\omega')J^2/U, \quad (34)$$

364 which implies $|\bar{\phi}_1| \sim (\gamma'/2)F_{\text{th}}/J^2$. The amplitude of the sites to the right of site 2 follow the
 365 spatial dependence dictated by $\bar{\phi}_j = g_{j3}(\tilde{\omega}_d)J'\bar{\phi}_2$ with $j \geq 3$, which has a large amplitude
 366 only on the same sublattice sites. This explains the experimental results of Ref. [20]. The
 367 numerical results for this case are presented in Figs. 5 (c) and (d) where a strong asymmetric
 368 solution develops for $|F_1|^2 = -J^3/U$ —here $\omega_d = \omega_0$. The noisy data in Figs. 5(d) reflects the
 369 presence of dynamical instabilities.

370 *Hysteresis*—Figure 6 presents the low power hysteresis loop for both the symmetric $\varphi = 0$
 371 [(a)-(c)] and anti-symmetric $\varphi = \pi$ [(b)-(d)] cases obtained from the numerical solution of
 372 Eq. (22). On the top panel both driven sites have the same driving field amplitude, $|F_1| = |F_2|$,

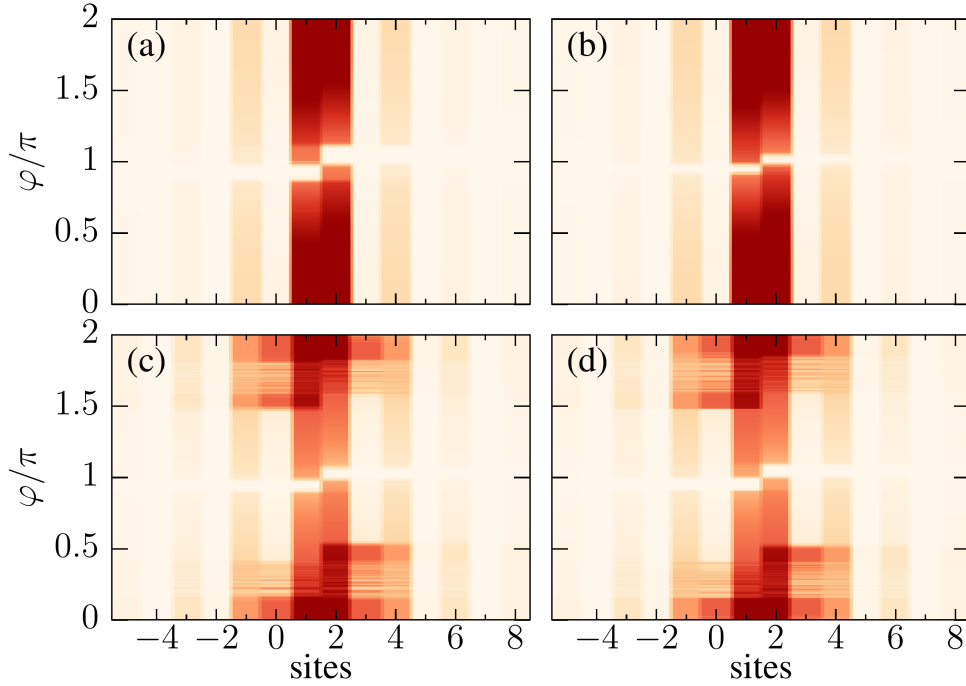


Figure 7: Phase dependence of a two-sites driven SSH chain (see scheme of Fig. 5) for $\omega_d = \omega_0$, $U = 0.1|J|$, $\gamma = 0.1|J|$ and $J < 0$. Top panels: interactions are included in 14 sites, $J' = 0.45J$, $|F_2| = 1.02|F_1|$; (a) $F_1 = 0.96F_{\text{th}}$; (b) $F_1 = 1.04F_{\text{th}}$. Bottom panels: interactions are included in 18 sites, $J' = 0.55J$, $F_1 = F_{\text{th}}$; (c) $|F_2| = 1.02|F_1|$; (d) $|F_2| = (1 + \delta f)|F_1|$ with $\delta f \in [-0.02, 0.02]$ a random variable. In (a), (b) and (c) the data was averaged over ten realizations of an added small random phase fluctuation $\delta\varphi \in [-\pi/50, \pi/50]$.

373 while in the bottom ones a small asymmetry was introduced, $|F_2| = 1.05|F_1|$, both to mimics
 374 more realistic scenarios and to favor the spontaneous spatial symmetry breaking. The sym-
 375 metric solutions shows always an hysteric behavior with thresholds that are in agreement with
 376 Eq. (31). The asymmetric case, on the other hand, is more involved probably due to the singu-
 377 lar behavior of the surface Green function due to the SSH edge state. In particular, hysteric
 378 behavior is only found for $|F_1| = |F_2|$ (panel (b)). Noisy data signaling the presence of dynam-
 379 ical instabilities is always present after the transition near F_{th} .

380 Figure 7 shows the dependence on the relative phase of the laser, that determines the side
 381 of the chain that is populated near $\varphi = \pi$ (symmetry breaking point), for two different values
 382 of the SSH gap—the later controls the extend of the edge mode and hence the relevance of
 383 the interaction on the neighboring sites. The self-consistency iteration is started with $\bar{\phi} = 0$.
 384 The top panel correspond to the case of a larger SSH gap and hence a stronger localization of
 385 the soliton mode. To test the resilience of the effect and mimic possible experimental effect
 386 we added a small fluctuation both in phase and amplitude of the driving field as indicated
 387 in the caption. In all cases the emergence of the asymmetric soliton mode near $\varphi = \pi$ is
 388 apparent from the figure. It is worth pointing out the resemblance of our numerics with the
 389 experimental data of Ref. [20], as for instance the appearance of asymmetric thresholds at
 390 higher phase differences.

391 5 Summary and conclusions

392 Using lattice Green function techniques we have analyzed how resonant driving engineering
393 can be used in polariton or photonic arrays to tailor spatially localized states.

394 In the linear regime, our approach is exact and allowed us to determine the precise condi-
395 tions on the laser field to obtain the maximum localization of the polariton field on the region
396 delimited by the driving lasers. We believe this would be very helpful both to design particular
397 driving setups and, hopefully, to better understand their experimental outcomes. Generaliza-
398 tions to the case of spatially periodic driving is straightforward. Additionally, a similar scheme
399 as the one described in Ref. [26] could be also used in the present context to generate differ-
400 ent patterns on particular lattices. In another interesting direction, slow time modulations of
401 the driving field, either the phase and/or the amplitude, could also be addressed within this
402 method by using appropriate adiabatic expansions of the Green function, hence allowing to
403 properly account for the presence of topological pumping effects—work on this direction is
404 underway and will be presented elsewhere.

405 The treatment of nonlinear effects, on the other hand, is in general more involved as they
406 usually lead to some complex dynamical behaviour with the presence of hysteresis loops and
407 instabilities. Yet, the approach presented here, when thought as a self-consistent fixed point
408 equation, allows a clear understanding of several features of the solutions and their experi-
409 mental implementations. In fact, for simple 1D cases, we were able to obtain analytical ex-
410 pressions for the energy shifts, mode amplitudes and power thresholds that agree quit well
411 with the full numerics. In more complicated situations or in higher dimensions, we expect
412 that our approach will be also helpful, providing theoretical insights and/or appropriate ini-
413 tial configurations for full time dependent calculations.

414

415 *Note:* Upon completion of this work we learned of Ref. [24] where localization in polari-
416 ton lattices in the presence of optical Kerr non-linearities was studied with some overlapping
417 results for the linear array case.

418

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425 References

- 426 [1] H. Deng, H. Haug and Y. Yamamoto, *Exciton-polariton Bose-Einstein condensation*, *Rev.*
427 *Mod. Phys.* **82**(2), 1489 (2010), doi:[10.1103/RevModPhys.82.1489](https://doi.org/10.1103/RevModPhys.82.1489).
- 428 [2] I. Carusotto and C. Ciuti, *Quantum fluids of light*, *Rev. Mod. Phys.* **85**(1), 299 (2013),
429 doi:[10.1103/RevModPhys.85.299](https://doi.org/10.1103/RevModPhys.85.299).

- 430 [3] A. Amo, J. Lefrère, S. Pigeon, C. Adrados, C. Ciuti, I. Carusotto, R. Houdré, E. Giacobino
431 and A. Bramati, *Superfluidity of polaritons in semiconductor microcavities*, Nat. Phys.
432 **5**(11), 805 (2009), doi:[10.1038/nphys1364](https://doi.org/10.1038/nphys1364).
- 433 [4] A. Amo, S. Pigeon, D. Sanvitto, V. G. Sala, R. Hivet, I. Carusotto, F. Pisanello,
434 G. Leménager, R. Houdré, E. Giacobino, C. Ciuti and A. Bramati, *Polariton Su-
435 perfluids Reveal Quantum Hydrodynamic Solitons*, Science **332**(6034), 1167 (2011),
436 doi:[10.1126/science.1202307](https://doi.org/10.1126/science.1202307).
- 437 [5] P. St-Jean, V. Goblot, E. Galopin, A. Lemaître, T. Ozawa, L. Le Gratiet, I. Sagnes, J. Bloch
438 and A. Amo, *Lasing in topological edge states of a one-dimensional lattice*, Nat. Photonics
439 **11**(10), 651 (2017), doi:[10.1038/s41566-017-0006-2](https://doi.org/10.1038/s41566-017-0006-2).
- 440 [6] S. Klembt, T. H. Harder, O. A. Egorov, K. Winkler, R. Ge, M. A. Bandres, M. Emmerling,
441 L. Worschech, T. C. H. Liew, M. Segev, C. Schneider and S. Höfling, *Exciton-polariton
442 topological insulator*, Nature **562**(7728), 552 (2018), doi:[10.1038/s41586-018-0601-5](https://doi.org/10.1038/s41586-018-0601-5).
- 443 [7] D. D. Solnyshkov, G. Malpuech, P. St-Jean, S. Ravets, J. Bloch and A. Amo, *Micro-
444 cavity polaritons for topological photonics*, Opt. Mater. Express **11**(4), 1119 (2021),
445 doi:[10.1364/OME.414890](https://doi.org/10.1364/OME.414890).
- 446 [8] V. Sala, D. Solnyshkov, I. Carusotto, T. Jacqmin, A. Lemaître, H. Terças, A. Nalitov,
447 M. Abbarchi, E. Galopin, I. Sagnes, J. Bloch, G. Malpuech *et al.*, *Spin-Orbit Cou-
448 pling for Photons and Polaritons in Microstructures*, Phys. Rev. X **5**(1), 011034 (2015),
449 doi:[10.1103/PhysRevX.5.011034](https://doi.org/10.1103/PhysRevX.5.011034).
- 450 [9] K. G. Lagoudakis, M. Wouters, M. Richard, A. Baas, I. Carusotto, R. André, L. S. Dang and
451 B. Deveaud-Plédran, *Quantized vortices in an exciton-polariton condensate*, Nat. Phys.
452 **4**(9), 706 (2008), doi:[10.1038/nphys1051](https://doi.org/10.1038/nphys1051).
- 453 [10] A. A. Reynoso, G. Usaj, D. L. Chafatinos, F. Mangussi, A. E. Bruchhausen, A. S.
454 Kuznetsov, K. Biermann, P. V. Santos and A. Fainstein, *Optomechanical parametric
455 oscillation of a quantum light-fluid lattice*, Phys. Rev. B **105**(19), 195310 (2022),
456 doi:[10.1103/PhysRevB.105.195310](https://doi.org/10.1103/PhysRevB.105.195310).
- 457 [11] N. Carlon Zambon, Z. Denis, R. De Oliveira, S. Ravets, C. Ciuti, I. Favero and J. Bloch,
458 *Enhanced Cavity Optomechanics with Quantum-Well Exciton Polaritons*, Phys. Rev. Lett.
459 **129**(9), 093603 (2022), doi:[10.1103/PhysRevLett.129.093603](https://doi.org/10.1103/PhysRevLett.129.093603).
- 460 [12] D. L. Chafatinos, A. S. Kuznetsov, A. A. Reynoso, G. Usaj, P. Sesin, I. Papuccio, A. E.
461 Bruchhausen, K. Biermann, P. V. Santos and A. Fainstein, *Asynchronous locking in meta-
462 materials of fluids of light and sound*, Nature Communications **14**(1), 3485 (2023),
463 doi:[10.1038/s41467-023-38788-9](https://doi.org/10.1038/s41467-023-38788-9).
- 464 [13] I. Carraro-Haddad, D. L. Chafatinos, A. S. Kuznetsov, I. A. Papuccio-Fernández, A. A.
465 Reynoso, A. Bruchhausen, K. Biermann, P. V. Santos, G. Usaj and A. Fainstein, *Solid-state
466 continuous time crystal in a polariton condensate with a built-in mechanical clock*, Science
467 **384**, 995 (2024), doi:[10.1126/science.adn7087](https://doi.org/10.1126/science.adn7087).
- 468 [14] Q. Fontaine, D. Squizzato, F. Baboux, I. Amelio, A. Lemaître, M. Morassi, I. Sagnes,
469 L. Le Gratiet, A. Harouri, M. Wouters, I. Carusotto, A. Amo *et al.*, *Kardar-Parisi-Zhang
470 universality in a one-dimensional polariton condensate*, Nature **608**(7924), 687 (2022),
471 doi:[10.1038/s41586-022-05001-8](https://doi.org/10.1038/s41586-022-05001-8).

- 472 [15] T. Byrnes, N. Y. Kim and Y. Yamamoto, *Exciton–polariton condensates*, Nat. Phys. **10**(11),
473 803 (2014), doi:[10.1038/nphys3143](https://doi.org/10.1038/nphys3143).
- 474 [16] C. Schneider, K. Winkler, M. D. Fraser, M. Kamp, Y. Yamamoto, E. A. Ostrovskaya and
475 S. Höfling, *Exciton-polariton trapping and potential landscape engineering*, Rep. Prog.
476 Phys. **80**(1), 016503 (2016), doi:[10.1088/0034-4885/80/1/016503](https://doi.org/10.1088/0034-4885/80/1/016503).
- 477 [17] T. Boulier, M. J. Jacquet, A. Maître, G. Lerario, F. Claude, S. Pigeon, Q. Glorieux, A. Amo,
478 J. Bloch, A. Bramati and E. Giacobino, *Microcavity Polaritons for Quantum Simulation*,
479 Adv. Quantum Technol. **3**(11), 2000052 (2020), doi:[10.1002/qute.202000052](https://doi.org/10.1002/qute.202000052).
- 480 [18] J. Bloch, I. Carusotto and M. Wouters, *Non-equilibrium Bose–Einstein condensation in*
481 *photonic systems*, Nat. Rev. Phys. **4**(7), 470 (2022), doi:[10.1038/s42254-022-00464-0](https://doi.org/10.1038/s42254-022-00464-0).
- 482 [19] A. Kavokin, T. C. H. Liew, C. Schneider, P. G. Lagoudakis, S. Klembt and S. Hoeftling, *Po-*
483 *lariton condensates for classical and quantum computing*, Nat Rev Phys **4**(7), 435 (2022),
484 doi:[10.1038/s42254-022-00447-1](https://doi.org/10.1038/s42254-022-00447-1).
- 485 [20] N. Pernet, P. St-Jean, D. D. Solnyshkov, G. Malpuech, N. Carlon Zambon, Q. Fontaine,
486 B. Real, O. Jamadi, A. Lemaître, M. Morassi, L. Le Gratiet, T. Baptiste *et al.*, *Gap solitons*
487 *in a one-dimensional driven-dissipative topological lattice*, Nat. Phys. **18**(6), 678 (2022),
488 doi:[10.1038/s41567-022-01599-8](https://doi.org/10.1038/s41567-022-01599-8).
- 489 [21] O. Jamadi, B. Real, K. Sawicki, C. Hainaut, A. González-Tudela, N. Pernet, I. Sagnes,
490 M. Morassi, A. Lemaître, L. Le Gratiet, A. Harouri, S. Ravets *et al.*, *Reconfigurable photon*
491 *localization by coherent drive and dissipation in photonic lattices*, Optica **9**(7), 706 (2022),
492 doi:[10.1364/OPTICA.452624](https://doi.org/10.1364/OPTICA.452624).
- 493 [22] N. Carlon Zambon, S. R. K. Rodriguez, A. Lemaître, A. Harouri, L. Le Gratiet, I. Sagnes,
494 P. St-Jean, S. Ravets, A. Amo and J. Bloch, *Parametric instability in coupled nonlinear*
495 *microcavities*, Phys. Rev. A **102**(2), 023526 (2020), doi:[10.1103/PhysRevA.102.023526](https://doi.org/10.1103/PhysRevA.102.023526).
- 496 [23] M. Wouters and I. Carusotto, *Excitations in a Nonequilibrium Bose-Einstein*
497 *Condensate of Exciton Polaritons*, Phys. Rev. Lett. **99**(14), 140402 (2007),
498 doi:[10.1103/PhysRevLett.99.140402](https://doi.org/10.1103/PhysRevLett.99.140402).
- 499 [24] A. Muñoz de las Heras, A. Amo and A. González-Tudela, *Nonlinearity-enabled lo-*
500 *calization in driven-dissipative photonic lattices*, Phys. Rev. A **109**, 063523 (2024),
501 doi:[10.1103/PhysRevA.109.063523](https://doi.org/10.1103/PhysRevA.109.063523).
- 502 [25] F. Mangussi, M. Milićević, I. Sagnes, L. L. Gratiet, A. Harouri, A. Lemaître, J. Bloch,
503 A. Amo and G. Usaj, *Multi-orbital tight binding model for cavity-polariton lattices*,
504 Journal of Physics: Condensed Matter **32**(31), 315402 (2020), doi:[10.1088/1361-](https://doi.org/10.1088/1361-648x/ab8524)
505 [648x/ab8524](https://doi.org/10.1088/1361-648x/ab8524).
- 506 [26] A. González-Tudela, *Connecting steady-states of driven-dissipative photonic lattices with*
507 *spontaneous collective emission phenomena*, New J. Phys. **24**(4), 043001 (2022),
508 doi:[10.1088/1367-2630/ac58b7](https://doi.org/10.1088/1367-2630/ac58b7).
- 509 [27] D. Sarchi, I. Carusotto, M. Wouters and V. Savona, *Coherent dynamics and parametric*
510 *instabilities of microcavity polaritons in double-well systems*, Phys. Rev. B **77**(12), 125324
511 (2008), doi:[10.1103/PhysRevB.77.125324](https://doi.org/10.1103/PhysRevB.77.125324).