Information theory bounds on randomness-based phase transitions

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We introduce a new perspective on the connection between many-body physics and information theory. We study phase transitions in models with randomness, such as localization in disordered systems, or random quantum circuits with measurements. Utilizing information-based arguments regarding probability distribution differentiation, rigorous results for bounds on critical exponents in such phase transitions are obtained with minimal effort. This allows us to rigorously prove bounds which were previously only conjectured for dynamical critical exponents in localization transitions. In addition, we obtain new bounds on critical exponents in many-body Fock space localization transition and localization in Coulomb-interacting models. Somewhat surprisingly, our bounds are not obeyed by previous studies of these systems, indicating inconsistencies in previous results, which we discuss. Finally, we apply our method to measurement-induced phase transition in random quantum circuits, obtaining bounds transcending recent mapping to percolation problems.

I. INTRODUCTION

Over the years different connections have been recognized between information theory and physics, going from the Maxwell demon paradox [1], through using (quantum) information measures to bound renormalization group flows [2], to the black hole information problem [3]. Information theory concepts, such as complexity or entanglement, are used in many-body physics as a marker or a characteristic of phases [4–7] and in the study of the classical representability of many-body states [8–12]. Can one go further and use pure information or algorithmic complexity considerations to deduce the behavior of physical problems? Here only scant few results exist (see, e.g., Ref. [13] for a recent example in the context of the AdS/CFT duality). In our work, we propose a new approach for using classical-information-theoretical arguments to obtain physical results, bounding the behavior of randomness-based phase transitions — seemingly unrelated to classical or quantum information.

The principal idea is as follows: We examine quantum phase transitions which rely on randomness, e.g., localization transitions. A model is defined by a probability distribution $\mathcal{P}_W$, where W is proportional to $\mathcal{P}$'s width. A phase transition occurs when W reaches a critical value W_c. We then ask: What is the required system size for the system to "know" its phase? The answer may be based on the physical properties of the system, depending on the correlation length characterizing the phase. It may also be based on information theory, requiring that the number of samples out of $\mathcal{P}_W$ would be sufficient to statistically determine W. Comparing both viewpoints gives a classical-information-theoretical bound on the critical exponents of the phase transition. Our method generalizes the rigorous proof of the Harris bound [14] as obtained by Chayes et al. [15, 16]. The generalization allows to apply the method to a vast range of models and obtain new bounds, as we demonstrate below.

The rest of this paper is organized as follows: In Sec. II A, we present our method by applying it to a paradigmatic phase transition, the Anderson localization (AL) transition [17–22]. We rederive the Harris bound as discussed above [14–16], using a much more intuitive approach, which allows to extend its validity to any smooth dependence of the disorder $\mathcal{P}_W$ on W, thereby complementing the argument. We discuss the extension of the method to classical temperature-driven phase transitions, the original realm of Harris's work, in Sec. II B. In Sec. II C, we adjust the method to bound dynamical critical exponents, and demonstrate the extension on non-Anderson transitions in Weyl and related systems, obtaining rigorous bounds which were previously only conjectured.

We then move to apply our method to interacting localization models, pointing at inconsistencies in previous results in two separate cases: In Sec. III A we apply our method to many-body localization (MBL) phase transition [23–31], studied in the setting of MBL in Fock space (FS). We find discrepancies between our information-theoretic bounds and numerical results in limited-sized systems. Such discrepancies were already observed in MBL in real space [32, 33], and we identify an additional one in FS. This may be relevant to the ongoing discussion regarding the nature of MBL in the thermodynamic limit [34–56]. In Sec. III B we apply our method to localization transitions with Coulomb interactions and bound a dynamical critical exponent proposed for such a model [57]. Our bound is not obeyed by previous theoretical estimates in Ref. [57], which calls for additional investigation.

In Sec. IV A we study measurement-induced transitions in random quantum circuits [58–61]. Results for such phase transitions have previously been obtained rigorously for a specific setting, namely, when examining the zeroth Rényi entropy (Eq. (21)) as the order parameter, or for infinite local Hilbert space dimension. We obtain a generic bound for all circuit settings, which is obeyed by existing numerical data. In Sec. IV B we address measurement-induced phase transitions in the specific case of error correction threshold in quantum circuits. We compare favorably our bounds to existing numerical results [62–64]. We summarize our conclusions and offer future outlook in Sec. V. Some

technical details are relegated to the Appendixes.

II. INTRODUCTION OF THE METHOD

A. Warmup example: Anderson localization

First, we demonstrate our method by rederiving the Harris bound on Anderson localization transitions. We study a noninteracting tight-binding model with Hamiltonian H_0 with added disorder, i.e., random potentials,

$$H = H_0 + \sum_i \epsilon_i c_i^\dagger c_i, \tag{1}$$

where c_i are annihilation operators, and $\{\epsilon_i\}$ are independent and identically distributed (i.i.d) variables drawn from a probability $\mathcal{P}_W(\epsilon)$. W is called the *disorder parameter*, typically defined to be proportional to $\mathcal{P}$'s width. We focus on lattice systems in spatial dimension d.

When $d > 2$, the eigenstates of the system with energy E undergo a phase transition at a critical disorder $W_c(E)$ [19]. When $W > W_c(E)$, the system is in a localized phase [22]: eigenstates with absolute values of energy $> |E|$ are localized around a small group of lattice sites, their amplitude decaying exponentially with the distance from these sites. This phase is characterized by a localization length ξ_E. When $W < W_c\,(E)$ the system enters a diffusive phase at energy E [22, 65], in which particles with absolute values of energy $< |E|$ scatter across the lattice with distribution similar to that of a classical random walker, with a correlation length ξ_E^{diff}. Assuming that the phase transition is second order, we define the critical exponent ν,

$$\xi_E, \xi_E^{\text{diff}} \sim |W - W_c(E)|^{-\nu}\,, \tag{2}$$

which governs the behavior in both phases due to renormalization group arguments [22].

We now introduce our method. Consider the following hypothetical setting: A classical computer receives an input composed of a model as in Eq. (1) above. The input also contains the values of E, $W_c(E)$, ν, and all the coefficients necessary in order to describe the behavior of the system near criticality, up to any desired accuracy. The source of this prior knowledge is unimportant (e.g., a preliminary numerical computation using any amount of resources). In this setting, the classical computer is given a task: The computer is presented with a blackbox, emitting values distributed by $\mathcal{P}_{W_c(1\pm\delta)}$, where δ is small and given. It is then required to determine the sign, $\pm$.

A possible strategy for solving the task is to determine the physical phase defined by $\mathcal{P}_{W_c(1\pm\delta)}$, by simulating a disordered physical system based on samples from the blackbox. The decision protocol goes as follows. A classical computer simulates a d-dimensional lattice of length $CC'\,\xi_E$. Here, ξ_E is the localization length for energy E and disorder parameter $W_c(1+\delta)$. The coefficients $C, C' \gg 1$ can be chosen arbitrarily large. The total number of sites is $N = (CC'\,\xi_E)^d$. From the blackbox, N values are sampled and used as the local potentials $\{\epsilon_i\}$ at the lattice sites. A clean wire is then attached to the middle of the lattice. A particle with energy arbitrarily close to E is injected through the wire. The simulation tracks the particle's wavefunction over time.

If the sought for sign is positive, the phase is localized: The particle remains within a radius $\sim C\xi_E$ of its starting point. If the sign is negative, the phase is diffusive: the particle will eventually reach sites beyond that radius. The simulation runs long enough for a diffusive particle to move past $C\,\xi_E$, if the phase is diffusive. The value of C' is chosen so that boundary effects do not influence the result. Finally, the particle's position at the end of the simulation is used to determine the system's phase, and in turn, the sign $\pm$. The required system size for determining the phase is $N \sim \xi_E^d \sim \delta^{-\nu d}$. The strategy proposed above is illustrated in Fig. 1. The full protocol, as required for the rigorous proof, is presented in Table I of Appendix A.

We denote by N_{opt} the optimal number of samples required for differentiating the two probability distributions $\mathcal{P}_{W_c(1\pm\delta)}$. The classical-information-theoretical requirement on N_{opt} is

$$N_{\text{opt}} = \Theta\left(\delta^{-2}\right), \tag{3}$$

that is, $\lim_{\delta\to 0} N_{\text{opt}}/\delta^{-2} = \text{const}$. Eq. (3) can be understood intuitively, following Chebyshev's inequality. A more general and formal argument relies on a theorem stating that, as $\delta \to 0$, the required sample size scales with the Hellinger distance d_H between the two probability distributions [66]:

$$\begin{aligned} N_{\text{opt}} &\propto d_H^{-2}\left(\mathcal{P}_{W_c(1+\delta)}, \mathcal{P}_{W_c(1-\delta)}\right), \\ d_H(p,q) &= \sqrt{\int \frac{1}{2}\left(\sqrt{p(x)} - \sqrt{q(x)}\right)^2 dx}. \end{aligned} \tag{4}$$

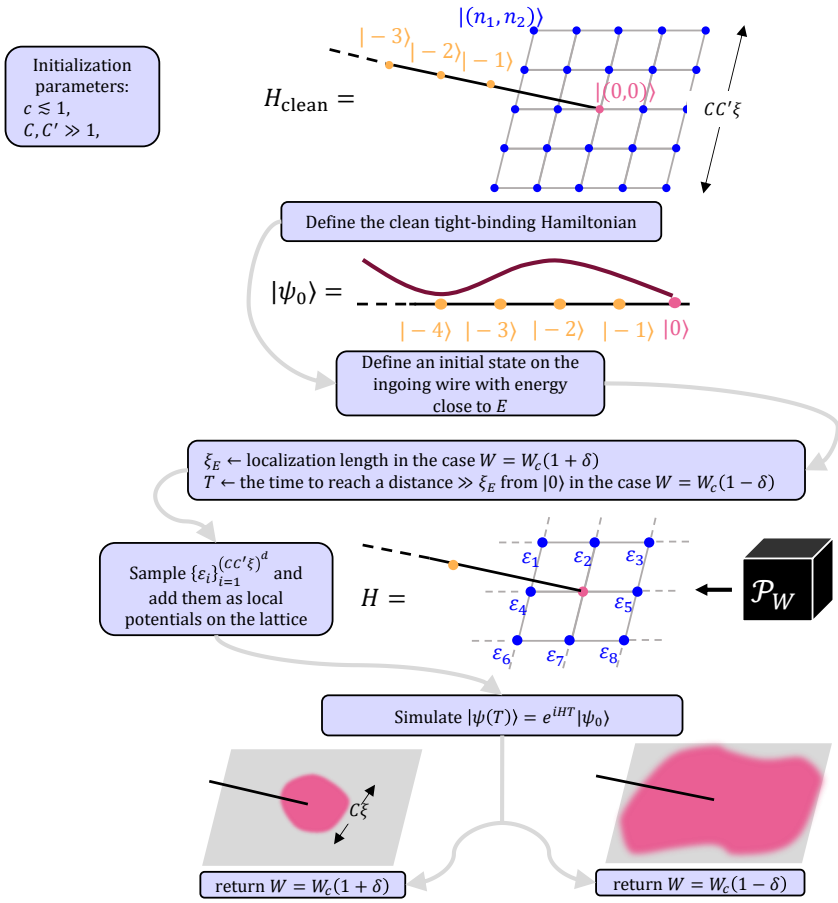

FIG. 1. Illustration of the protocol run (hypothetically) by the classical computer to distinguish the probability distributions $\mathcal{P}_{W_c(1\pm\delta)}$. The top sketch illustrates the simulated graph. A clean wire of length L (orange nodes with negative indices) is coupled to the center (pink node, denoted by $|(0,0)\rangle$) of a d-dimensional lattice (blue nodes with positive indices (n_1, n_2)). An initial state of a free wave with energy very close to E is defined on the wire. Random potentials are added to the lattice sites with positive indices. The evolution of the state in time is simulated, and based on the amplitude of the state after a long time, the algorithm determines whether the system is in a localized ($W = W_c(1+\delta)$) or delocalized ($W = W_c(1-\delta)$) phase. Note that for clarity of the sketch, we use $d = 2$. However, the phase transition only occurs at $d > 2$. Further details can be found in Appendix A.

Expanding the Hellinger distance for small δ one obtains $d_H \sim \delta$, and in turn, $N_{\text{opt}} \sim \delta^{-2}$.

It is required that $N_{\text{opt}} \leq N$. Therefore,

$$\delta^{-2} = O(\delta^{-\nu d}) \quad \Rightarrow \nu \geq \frac{2}{d}. \tag{5}$$

We see that a bound on the critical exponent was obtained based solely on information-theoretical arguments, with minimal use of physical assumptions. We stress that while the setting above might be challenging to obtain, it is technically possible, which is sufficient to obtain a proof on the bound. The argument is based only on comparing number of samples in both hypothetical cases, rather than protocol complexity. The rigorous claim we make is the following: If the phase transition occurs and is described by Eq. (2), then necessarily $\nu \geq 2/d$. If this requirement is not obeyed by some numerical or plausible theoretical argument, one may deduce either that the assumptions regarding the nature of the phase transitions were false, or that the results were inaccurate, due to, e.g., too small system sizes used numerically or the inadequacy of some plausible argument. Note that the method only deals with the limit $\delta \to 0$, which corresponds to the thermodynamic limit, $N \to \infty$. A summary of the approach is presented in Fig. 2.

We note that usually, numerically-studied systems are simulated with a uniform disorder distribution rather than a smooth distribution. A single sample in the range $\pm[W_c(1-\delta), W_c(1+\delta)]$ would be enough for differentiating the

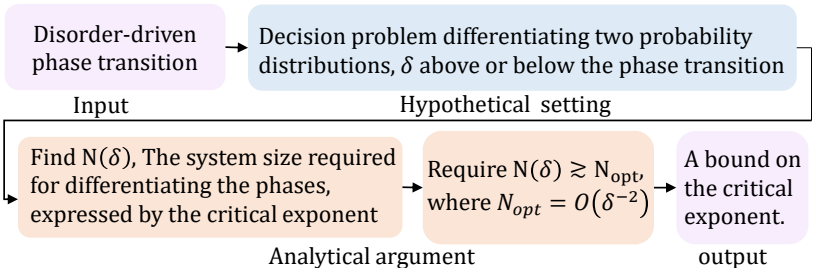

FIG. 2. A brief summary of the approach. We start with a disorder-driven phase transition assumed to be second order. A hypothetical decision problem is then considered, distinguishing between two probability distributions corresponding to two sides of the transition with a distance δ from the critical point. The δ-dependence of the required system size in order to solve the problem is obtained in two ways: from physical considerations, based on the localization/correlation length, and from information theory, based on Eq. (3). The two requirements are compared to obtain a bound on the critical exponent.

distributions, and therefore the number of required samples in this case is $N_{\text{opt}} \sim \frac{W_c(1+\delta)}{2W_c\delta} = O\left(\delta^{-1}\right)$, as the Hellinger distance would indeed show. Thus, the bound obtained by our method would be $\nu \geq 1/d$. Nevertheless, smooth and uniform distributions should behave similarly, since the latter may be considered a limiting case of the former. Indeed, numerical results obey the tighter bound, $\nu \geq 2/d$. We also note that one could use other quantities to distinguish the two phases, e.g., the localization properties of the eigenstate closest in energy to E, see Appendix B. This would parallel the Fock space discussion in Sec. III A below.

Finally let us mention that we assume one of the following to hold: Either the calculated quantity is self-averaging, so it can be deduced with sufficiently high precision from a single realization with dimensions which are larger than ξ by some possibly large constant factor; or the standard deviation of the result for one system does not scale with its size, so that, to obtain sufficient accuracy, one may simulate an ensemble of systems whose number does not scale with the size of each individual system.

B. Harris bound for classical thermal phase transitions

One may also extend our argument to the systems originally considered by Harris [14], namely the classical ferromagnetic Ising model in d space dimensions, with a small random addition ΔJ to the exchange coupling J. Let us briefly recall Harris' argument. If the system is at a dimensionless temperature distance $\delta_T \equiv (T - T_c)/T_c$ from the clean critical temperature T_c, the correlation length is $\xi \sim |\delta_T|^{-\nu}$ in the clean case. Spins in regions of size $\sim \xi^d \sim |\delta_T|^{-\nu d}$ are thus correlated. The average disorder in such a region will be of order $\xi^{-d/2}$, leading to a correspondingly large shift in δ_T. This shift is small with respect to the staring value of δ_T, making the criticality robust to small disorder, only if $\nu \geq 2/d$.

We may rederive this result using our approach: Making the realistic assumption that small disorder shifts T_c with some smooth dependence on the disorder distribution width, one may assume that near criticality, $T_c(1 \pm \delta_T)$ correspond, respectively, to disorder distribution width $W(1 \pm \delta)$ for some $\delta \sim \delta_T$. A decision problem may be defined to distinguish two probability distributions of widths $W(1 \pm \delta)$. A classical computer may then sample $N \sim \delta^{-\nu d} \sim \delta_T^{-\nu d}$ values from the distribution and determine the phase it corresponds to by calculating the partition function to sufficient accuracy, either by brute force or by, e.g., a Monte Carlo simulation running for a long enough time (recall that the only thing that matters to our argument is the number of disorder samples, not the calculation time). From the phase, the computer determines the distribution width and solves the decision problem. Comparing with $N_{\text{opt}} \sim \delta^{-2}$, the Harris bound $\nu \geq 2/d$ is recovered.

C. Dynamical critical exponents

Our approach may also be used to bound dynamical critical exponents, i.e., critical exponents that govern the behavior as a function of energy. We demonstrate it on the non-Anderson disorder-driven transitions in Weyl-like systems [67].

We focus on noninteracting Hamiltonians with short-range correlated random potential, such as the ones studied in Refs. [67–70]. It was shown there that a disorder-driven phase transition occurs in such systems even if localization is forbidden by topological or symmetry constraints. The transition is then manifested, e.g., by a change in the density

of states (DoS) — the DoS at low energies vanishes when $W < W_c$, and becomes finite at larger disorder. The correlation length near the phase transition is governed by a critical exponent $\xi \sim |W - W_c|^{-\nu}$. The Harris bound on ν may be obtained similarly to the bound on the Anderson transition above.

The phase transition may also be approached by tuning the energy E instead of the disorder, around a critical value E_c, which is fixed to the Weyl or Dirac nodes in the corresponding systems. Tuning the energy, we define the dynamical critical exponent z as

$$\xi \sim |E - E_c|^{-z} . \tag{6}$$

Near W_c and approaching E_c, the DoS ρ is expected to behave as

$$\rho \sim |E - E_c|^{\frac{d}{z}-1} . \tag{7}$$

We adjust our approach to bound $1/z$: Consider a disordered model with a critical energy E_c. One may present a hypothetical decision problem differentiating between two probability distributions with the same width, W_c, but different means: $E[\mathcal{P}_0] = 0, E[\mathcal{P}_{-\delta}] = -\delta$. Again, the required number of samples from $\mathcal{P}$ for differentiating the cases is $N_{\text{opt}} = O\left(\delta^{-2}\right)$.

A classical computer may sample L^d values from the probability distribution and use it to define a disordered Hamiltonian. The computer may then calculate, e.g., the finite-size energy gap Δ in the spectrum around E_c. If the probability distribution is $\mathcal{P}_0$ the critical value is E_c, whereas for $\mathcal{P}_{-\delta}$ the spectrum shifts to lower energies and therefore the probed energy has a finite DoS. The expected behavior of Δ in the two phases is

$$\Delta \sim \begin{cases} L^{-z} & E = E_c, \\ L^{-d}\rho^{-1}(\delta) & E = E_c(1+\delta). \end{cases} \tag{8}$$

The first (critical) case in Eq. (8) is derived from the definition of z, while the second is consistent with a finite DoS. The required number of samples, N, may be extracted by comparing the cases above, obtaining a requirement on the minimal L needed to distinguish between the phases. By substituting $N = L^d$ and ρ from Eq. (7), we obtain:

$$L^{-z} \propto L^{-d}\rho^{-1}(\delta) \quad \Rightarrow \quad N \propto \delta^{-\frac{d}{z}}, \tag{9}$$

and from the requirement $N \geq N_{\text{opt}}$, the Harris bound

$$1/z \geq 2/d \tag{10}$$

is obtained, which was conjectured and empirically found to hold in Ref. 70.

Considering the numerical results obtained for z [34, 71–75], an interesting point arises: Unlike the real-space critical exponent ν, z tends to saturate (or come very close to saturating) the bound in Eq. (10). This implies that the (hypothetical) protocol above is equivalent to the optimal way of differentiating distributions. Put differently, it means that no information on $E[\mathcal{P}]$ is "thrown out" when computing the DoS or the energy gap Δ. It would be interesting to characterize what makes this phase transition information-efficient.

III. LOCALIZATION TRANSITIONS IN INTERACTING MODELS

A. Fock space localization

We turn to obtaining new results on MBL transition in Fock space (FS). The bound is not obeyed by previous numerical results [76], and we discuss this discrepancy below.

Adding interaction to Anderson-localized single-particle models may reintroduce ergodicity to the system. However, for a given interaction strength, it is widely believed that there is still a critical disorder value above which the system is in a many-body localized (MBL) phase, at least in 1D [23–31, 77]. The interacting nature of MBL, combined with the fact that it occurs at excited states, limits heavily the accessible system size for numerical simulations of MBL (See Ref. [78] for a review of numerical simulation of MBL and the challenges it faces). The study of MBL is thus required to navigate plausible theoretical arguments suitable for infinite-sized systems, and numerical results, limited to small-sized systems. This gap leads to challenges in the understanding of MBL and fuels a debate about whether such a phase even exists [35–41, 48–56, 79].

Since our argument only relies on the number of samples needed to classically simulate the system and distinguish the phase, but not on the runtime, it applies to interacting systems as well. Thus, defining correlation/localization

length based on the real-space response of the system to a local perturbation, one may derive the same bound as in the noninteracting case, namely Eq. (5). In fact, one may as well adapt the original argument by Chayes et. al [16] to derive the same bound. Interetstingly, numerical simulations suggest that the critical exponents violate this bound [32, 33]. This discrepancy has raised concerns that the phase transition may not follow the expected behavior. Specifically, it appears that the assumed second-order behavior of the transition overlooks the avalanche mechanism [35–41, 48–56, 79], which instead suggests Berezinskii-Kosterlitz-Thouless (BKT) scaling for the real-space correlation/localization length, implying $\nu \to \infty$, in line with Eq. (5). Renormalization group arguments support this interpretation [80].

A recent enlightening angle on MBL originates from studying it in FS: the system can be thought of as a single particle scattered across the graph defined by the disordered Hamiltonian in FS. The study of MBL in FS allows to use single-particle techniques, and may shed light on MBL behavior [81–86]. In FS, the Hamiltonian can be written as a disordered tight-binding Hamiltonian with correlated local potentials: the number of sites in FS is $\mathcal{N} = d_{\text{single}}^N$ where d_{single} is the Hilbert space size of a single site, but their random local potentials depend only on the N parameters in real space.

A measure of MBL for an eigenstate $|\psi\rangle$ studied in FS may be defined by [87–92]:

$$S_q\left(|\psi\rangle\right) = \frac{1}{1-q} \ln\left(\sum_{\alpha=1}^{\mathcal{N}} \left|\langle\psi \,|\, \alpha\rangle\right|^{2q}\right). \tag{11}$$

Using finite-size scaling, the behavior of S_q's disorder average, $\overline{S_q}$, is predicted near the critical point [91–93]:

$$\overline{S_q}\left(|\psi\rangle\right) = \begin{cases} \ln\mathcal{N} + b_{q,\text{erg}} & \text{Ergodic (delocalized)} \\ D_q \ln\mathcal{N} + b_{q,\text{MBL}} & \text{MBL} \end{cases}, \tag{12}$$

where $b_{q,\text{erg}}$ is an emergent scale that goes to zero near the critical point, D_q is a multifractal dimension, which goes to a critical value, $D_{q,c}$, at W_c, and $b_{q,\text{MBL}}$ is an emergent scale which remains constant near the phase transition:

$$\begin{aligned} D_q - D_{q,c} &\sim \delta^{d\beta_q}, \\ b_{q,\text{erg}} &\sim \delta^{-d\alpha_q}, \\ b_{q,\text{MBL}} &\sim \text{const.}, \end{aligned} \tag{13}$$

introducing two critical exponents, α_q, β_q.

In Refs. 91, 94, a FS localization length is defined and assumed to scale with the same critical exponents as the real space localization length, in order to bound the FS behavior using the Harris criterion. We obtain an equivalent bound without relying on any such assumption. Near the critical point, S_q varies continuously in the MBL phase while the ergodic phase displays a jump in D_q, as can be seen, e.g., in Ref. 92. We therefore focus on the requirement on the ergodic size to obtain a Harris-like bound is on the ergodic phase critical exponent α_q:

$$\alpha_q \geq 2/d. \tag{14}$$

As in the Anderson localization case, a classical computer may be hypothetically required to differentiate two probability distributions, $\mathcal{P}_{W_c(1\pm\delta)}(\epsilon)$ (again, in a thought experiment in which all prior knowledge of the model is accessible, disregarding the space and time computational costs). The problem may be solved by performing N samples from the distribution $\mathcal{P}$ and using them to define an N-site interactive disordered Hamiltonian. The Hamiltonian is then diagonalized and S_q is computed. The cases $W_c(1 \pm \delta)$ are distinguished based on the proximity of S_q extracted from the simulated system and its expected value for both cases as in Eq. (12). The protocol is presented more rigorously in Table II in Appendix B.

We analyze the requirements for the success of the protocol. ΔS_q is defined to be the deviation of S_q from $\overline{S_q}$. The protocol is successful if

$$\Delta S_q \ll \left|\overline{S_{q,\text{erg}}} - \overline{S_{q,\text{MBL}}}\right|. \tag{15}$$

Dividing by $\ln\mathcal{N}$ and substituting Eq. (13), we obtain the requirement

$$\begin{aligned} \Delta S_q / \ln\mathcal{N} &\ll \left|\overline{S_{q,\text{erg}}} - \overline{S_{q,\text{MBL}}}\right| / \ln\mathcal{N} \\ &= \left|1 - D_q(\delta) + \frac{b_{q,\text{erg}}(\delta)}{\ln\mathcal{N}} - \frac{b_{q,\text{MBL}}(\delta)}{\ln\mathcal{N}}\right|. \end{aligned} \tag{16}$$

The behavior presented in Eqs. (13),(12) is apparent for large system sizes, that is, $\ln\mathcal{N}/b_{q,\mathrm{erg}} \gg 1, \ln\mathcal{N}/b_{q,\mathrm{MBL}} \gg 1$. Approaching the critical point from above (localized phase), S_q varies continuously to its critical value $S_{q,c}$, while the ergodic (critical) phase one expects a jump in D_q as W is varied towards W_c, see, e.g., in Ref. [92]. The above implies that it is the ergodic phase which defines the size of the system in which the phases would be distinguishable. This leads to the requirement

$$\ln\mathcal{N} \gg b_{q,\mathrm{erg}}. \tag{17}$$

In addition, we need ΔS_q to obey Eq. (16). Since the right hand side of Eq. (16) is constant in the thermodynamic limit, this requirement is met provided $\Delta S_q/\ln\mathcal{N}$ decreases with increasing system size. This is found to be the case numerically [76], as we further discuss below. Following Eqs. (13),(17), the former requirement may be translated to $N \sim \delta^{-d\alpha_q}$. The requirement $N \geq N_{\mathrm{opt}}$ results in a Harris bound on α_q,

$$\alpha_q \geq 2/d. \tag{18}$$

As mentioned above, this bound was already presented in Ref. [91], relying on the assumption that the real space and Fock space phase transition displays the same behavior. Here we obtained the bound without such an assumption.

Interestingly, The FS transition has been studied in Refs. 76, 92, for a one-dimensional 1/2-spin system with a uniformly distributed disorder and $q = 2$. The obtained critical exponent α_2 was $\alpha_2 \approx 0.5$, which violates our rigorous bound (14) (Note that even if the behavior of the uniform disorder distribution is different than that of smooth distributions, the requirement would be $\alpha_2 \geq 1$. The numerical results remain inconsistent with our bound even in this relaxed case). This implies either that the phase transition does not obey the expected form, or that the numerical results suffer from finite size effects. The discrepancy we uncover thus points at an overlooked mechanism, which may explain the inconsistency between the behavior in finite- vs. infinite- sized systems. It is in line with recent real-space numerical works which provide evidence that current numerically accessible system sizes might be too small [95–98], perhaps due to the avalanche mechanism mentioned above. This or another unaccounted-for mechanism might be necessary for a full physical picture of the FS behavior, which may result in a different behavior than the one in Eqs. (12), (13). Indeed, Ref. [99] provides numerical evidence suggesting that the transition in FS also appears to be of the BKT type, at least within the limitations of numerically accessible system sizes, leading to $\alpha_q \to \infty$, in agreement with the bound (14).

Let us finally note that the Fock space of a finite-dimensional disordered system resembles a random regular graph (RRG) [100–103], which is locally similar to a Bethe lattice. When the system resides on an RRG or a Bethe lattice in real-space, our method in its current form does not provide a nontrivial bound, since the system size, and in turn, the number of random potential samples N, becomes exponential in the correlation length ξ. It would be interesting to try to extend our method to this case [80, 104–106], and to try to bound the behavior of $\log\xi$. The generalization can potentially be formulated based on the correlation tree depth, following Ref. [107].

B. Localization transition with Coulomb interaction

We apply our method to disordered models with long-range Coulomb interaction, uncovering an additional inconsistency in a previous result, as discussed below. The phenomenology of localization in this case is different than the standard MBL [57]. Particularly, it is found that in the delocalized regime (but not in the localized one) a dephasing length for excitations L_ϕ appears, which might be shorter than the correlation length ξ. Near and above the critical energy E_c, the dephasing length of an excitation at energy E behaves as

$$L_\phi \sim (E - E_c)^{-\frac{1}{z}}, \tag{19}$$

where z is the dynamical critical exponent. For $E < E_C$, in the localized regime, $L_\phi = \infty$, i.e., there is no inelastic decay. On the other hand, the localization length ξ scales as $\xi_E/\xi_{E=0} \sim (|E_C - |E||/E_C)^{-\nu}$, with the noninteracting exponent ν. For $\nu z > 1$ one has $\xi > L_\phi$. Hence, the localized/delocalized phases are distinguishable already in systems of size $\sim L_\phi$.

We now derive a bound on the dynamical exponent: We introduce a hypothetical decision problem of differentiating two probability distributions with the same width and different means, $\mathbb{E}[\mathcal{P}_\pm] = \pm\delta/2$. The distributions may be distinguished by sampling $N \sim L_\phi^d$ values from the distribution and using them to define a system with Coulomb interaction and random disorder. If $\mathcal{P}_\pm = \mathcal{P}_+$, E is in the delocalized regime and vice versa. The phase of the system may then be determined, e.g., by simulating the time evolution of an excitation of energy E_c. From this the sign of the critical energy shift, and therefore the probability distribution, could be identified. We now substitute Eq. (19) into the requirement $N \geq N_{\mathrm{opt}} \sim \delta^{-2}$ and obtain

$$1/z \geq 2/d, \tag{20}$$

provided that $\nu z > 1$, as explained above.

Intriguingly, the theoretical estimate provided for z_ϕ in Ref. 57 at $d = 3$ obeys the condition $\nu z > 1$ but violates Eq. (20). This may result from the inadequacy of using the Fermi golden rule to estimate the dephasing length in this regime [108].

IV. PHASE TRANSITIONS IN NOISY QUANTUM CIRCUITS

A. Measurement-induced phase transition in random unitary circuits

We proceed to obtain new bounds on measurement-induced phase transition in quantum circuits. Apart from the role played by randomness, these models are unrelated to localization, demonstrating the generality of our approach. Quantum circuits subject to random measurements [58–61] provide a generic model for open systems interacting with an environment, motivated both by quantum technology and many-body physics.

As a basic model, consider the following: The system is composed of a register of L^d qubits, organized as a d-dimensional lattice of length L. t layers of spatially local random 2-qubit unitary gates are then applied to the qubits, where in between each layer, each qubit is measured with probability p in the $\{|0\rangle, |1\rangle\}$ basis. For p close to 1, the qubits decay frequently into pure states, and the resulting state has little entanglement, typically an area-law. For small values of p, the system is close to not being measured at all, with a volume-law entanglement in the average case [58, 59]. A phase transition occurs at a critical value p_c.

The two phases are characterized by the entanglement between two parts of the system, denoted by A and $\overline{A}$. A standard measure of entanglement is the von Neumann entropy, $\mathcal{S}(\rho_A) = -\mathrm{Tr}\left(\rho_A \log \rho_A\right)$, where $\rho_A = \mathrm{Tr}_{\overline{A}}(\rho)$ is the reduced density matrix of A. Due to the numerical and analytical inaccessibility of the von Neumann entropy, Rényi entropies are also introduced,

$$\mathcal{S}_n(\rho_A) = \log\left(\mathrm{Tr}\left(\rho_A^n\right)\right)/(1-n), \tag{21}$$

which obey $\lim_{n\to 1} \mathcal{S}_n(\rho_A) = \mathcal{S}(\rho_A)$. The Rényi entropies are entanglement monotones, and for $n > 0$ they are continuous and reflect small changes in the entanglement accurately.

We follow Ref. 59 and define the characteristic length scale ξ and time (circuit depth) scale τ by

$$\mathcal{S}_n\left(\rho_A(L,t,p)\right) - \mathcal{S}_n\left(\rho_A(L,t,p_c)\right) = f\left(\frac{L^d}{\xi^d}, \frac{t}{\tau}\right), \tag{22}$$

where f is a scaling function. We then assume a second order phase transition:

$$\xi \sim \delta^{-\nu}, \quad \tau \sim \delta^{-z\nu}, \tag{23}$$

where $z = 1$ in the space-time symmetric case.

Under certain conditions, the problem can be described as a graph percolation problem, each measurement becoming a cut in the circuit graph [59, 109]. This allows to leverage on the large body of knowledge on percolation problems [110]. We note in particular that a percolation phase transition must obey the Harris criterion. However, for this mapping to hold, certain conditions must be satisfied: Either one considers $n = 0$, i.e., the Schmidt rank, or the quantum register must consist of qudits with infinite local dimension. As we will see below, our approach does not suffer from these restriction.

The behaviors of Rényi entropies for other values of n near the phase transition were studied numerically for $1+1$ dimensions [59], and they seem to obey the Harris criterion as well, although to the best of our knowledge, so far its validity had not been argued for theoretically. Note that the critical value p_c extracted for $n \neq 0$ is different from the one extracted for $n = 0$ (which matched the percolation critical value, $p_c = 1/2$), suggesting that percolation might indeed not be a good model for the behavior of general Rényi entropies.

We now bound the critical exponents defined in Eqs. (23). The (hypothetical) decision problem may be differentiating two probability distributions over $\{0, 1\}$, where $\mathcal{P}(1) = p_c(1 \pm \delta)$. In order to differentiate the distributions, a classical computer may simulate a random circuit of size L^d and depth t, with $N = tL^d$ possible spontaneous measurements, and compute $\mathcal{S}$ or $\mathcal{S}_n$. It is required that $L \gg \xi$ and $t \gg \tau$, i.e., $N = tL^d \gg \tau\xi^d \sim \delta^{-\nu(d+z)}$. The requirement $N \geq N_{\mathrm{opt}}$ leads to

$$\nu(d+z) \geq 2. \tag{24}$$

The obtained bound is obeyed numerically [59, 111–113]. As opposed to existing analytical results relying on mapping to percolation, our bound is not limited in terms of the local dimension or the order of the Rényi entropy.

Note that entanglement serves as just one of several indicators for the phase transition discussed here. In Ref. [85], the authors investigate this phase transition by examining the participation entropies S_q, as defined in Eq. (11). When applying our method with this alternative phase identifier, the value of N_{opt} remains unchanged. Consequently, the bound established in Eq. (24) is applicable regardless of which phase identifier is used. This is confirmed by the critical exponent reported in Ref. [85], which indeed satisfies our derived bound.

B. Additional phase transitions in noisy quantum circuits

In this subsection, we study additional versions of phase transitions in noisy quantum circuits, used to model various types of errors or dissipation. We consider error resilience phase transitions (error correction threshold) in quantum circuits, as studied in Refs. [62–64]. The setting in these works is as follows: A set of unitary gates is applied, free from noise, to a quantum register of N qubits, representing encoding of an error correction code (typically random). A local error is then applied to each qubit. The effect of the error is then assessed in various ways: In Ref. [63], the reverse unitary gates are applied and the error is quantified by the fidelity of the final and original states; Ref. [62] studies the probability of success of optimal decoding.

The error is quantified by a parameter denoted by e, and a phase transition occurs at a critical value e_c, above which the error is uncorrectable in the thermodynamic limit. We denote the order parameter, i.e., the error resilience, by F. The phase transition is expected to be second order, that is, near the phase transition

$$F = \tilde{f}\left[(e - e_c)\, N^{1/\nu'}\right], \tag{25}$$

where $\tilde{f}$ is a scaling function, and ν' corresponds to νd, as it applies to the dependence on the number of qubits N instead of the linear dimension. The error may be random, in which case e measures the randomness, as is the case in Ref. [62]. However, it may be a depolarizing channel or a coherent phase uniform across the register, as in Ref. [63]. In both cases the error is of constant depth in time, corresponding to $z = 0$.

When dealing with independent random errors, the transition is randomness-induced. Consequently, the bound established in Eq. (24) should apply, which is confirmed in Ref. [62], where the bound is satisfied [114]. In the scenarios of Ref. [63], however, the phase transition does not stem from randomness, making our method inapplicable. This explains why the critical exponent $\nu' = 1$ reported in Ref. [63] for the case where the same incoherent or coherent error channel is applied to each qubit, do not obey the bound in Eq. (24) for $z = 0$ and $d = 1$. Interestingly, that paper also considers the case where the error parameters vary randomly and independently between the qubits, in which case our bound applies, and it is indeed found that thus changes the critical exponent $\nu' = 2$.

A slightly different error model is examined in Ref. [64], where a clean unitary circuit is applied to a quantum register of N qubits. At the edge of the register, a depolarizing channel is applied at each timestep of the circuit, thus representing dissipation at the edge. While this example models a different physical scenario than Refs. [62, 63] above, the resulting system is very similar with an exchange of the temporal and spatial directions; thus, $z = 1$ while d should be replaced by $d-1$ (the dimension of the edge). In that work the depolarizing channel is applied at each time step so our argument does not apply, but it turns out that the critical exponent found does satisfy the bound (24).

To conclude, our bound seems to hold in all cases previously considered in the literature where it should apply. Notably, the bound is actually saturated in all these cases. It would be interesting to test this more systematically, as well as to decipher the significance of this saturation.

V. CONCLUSION AND FUTURE OUTLOOK

We introduced a rigorous approach for obtaining critical exponent bounds in randomness-driven phase transitions, combining information bounds with physical phenomena. Due to its generality and the minimal amount of physical assumptions it requires, the bounds obtained by the approach are robust. By applying the method to several localization transitions, as well as measurement-induced phase transitions, we were able to obtain surprising results in some models, and rigorous support to the previous conjectures in others.

Our approach may be applied to additional phase transitions, such as ones driven by correlated disorder or interdependent networks [115–121]. It can be extended to models with random hopping terms [15], spin glass models [122–124], Hilbert space localization in random circuits [125], variations of the quantum sun model [126] (following the scaling arguments in Ref. [127]), and other aspects of the measurement-induced transition, such as the purification transition [111, 128] or the learnability transition [129]. As discussed in Sec. IV B, it is suited for error-resilience phase transitions as well. As mentioned at the end of Sec. III A, it would be interesting to try and extend the method to models defined on random regular graphs or Bethe lattices [80, 100–106], following Ref. [107].

Hopefully, the minimal effort required for obtaining such bounds, along with their robustness and potential for guiding further discoveries, would encourage wide use of our approach in further studies.

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Appendix A: Re-obtaining the Harris bound for Anderson localization — Rigorous protocol

In this section, we provide the full decision protocol used in the proof made in Sec. II A in the main text, concentrating on the Anderson localization transition as a simple but paradigmatic example. The protocol is run by a classical computer in the hypothetical setting as described in the main text, and its goal is to distinguish between the two probability distributions, $\mathcal{P}_{W_c(1\pm\delta)}$, based on sampling. The computer uses the sampled values to define a disordered system and simulate the evolution of a particle in this system. Based on the simualtion's result, the phase of the system is determined, and in turn, the sign $\pm$. The algorithm is sketched in Fig. 1 in the main text and presented rigorously in Table I.

We recount here some of the established phenomenology of the diffusive phase, observed when $W < W_c(E)$, on which the hypothetical protocol relies [22, 65]. In this phase, an eigenstate with absolute value of energy $< |E|$ divides the lattice into regions of size $\left[\xi_E^{\text{diff}}\right]^d$. Locally it appears as if the state is localized at the center of a region. However, amplitudes of different neighboring regions overlap at their common border, allowing such eigenstate to spread across the lattice. Thus, a wavepacket with energy expectation value $< |E|$ in absolute value (assuming a sufficiently narrow energy spread) may diffuse from one region to another across the entire lattice, with coarse-grained distribution similar to that of a classical random walker. Therefore, the probability distribution on the graph for such a process, starting at site i, is, to a first approximation, the Rayleigh distribution,

$$\mathbb{E}_{\mathcal{P}_W}\left[\left|\langle i|\, e^{-iHt}\,|j\rangle\right|^2\right] \propto \frac{1}{\left(D\left(E\right)t\right)^{d/2}} e^{-\frac{|\vec{r}_j-\vec{r}_i|^2}{4D(E)t}}, \tag{A1}$$

with $D(E) \sim \left[\xi_E^{\text{diff}}\right]^{-(d-2)}$ being the diffusion coefficient, which is proportional to the conductivity, and $\vec{r}_i, \vec{r}_j$ are the positions of sites i, j, respectively. As mentioned in Eq. (2) in the main text, near the phase transition ξ_E^{diff} scales with the same critical exponent ν as in the localized phase, due to renormalization group arguments [22],

$$\xi_E^{\text{diff}} \sim \delta^{-\nu}. \tag{A2}$$

We note that the (hypothetical) protocol suggested in Table I may fail if, by some rare event, the local potentials drawn from $\mathcal{P}_{W(1\pm\delta)}$ results locally in the opposite phase from the one expected, that is, delocalization if $\delta > 0$ or vice versa. However, repeating the process would decrease the probability for it up to an arbitrarily small one, exponentially fast in the number of repetitions, such that the relation $N \sim \delta^{-\nu d}$ remains valid for arbitrarily small probability of failure.

Let us finally note that here, we used the scattering of a wavepacket as the test property for distinguishing between the phases. However, a variety of properties may be used instead, e.g., the localization properties of the eigenstate closest in energy to E. We make use of this in the derivation of the Fock space localization critical exponent in Appendix B below.

Initialization(E,d,δ)**:**	
$0 \leftarrow$ starting_site $c \leftarrow$ small constant $\lesssim 1$ $C \leftarrow$large constant $\gg 1$ $C' \leftarrow$large constant $\gg 1$ $\varepsilon \leftarrow$small constant $\ll C^{-d}$	Choose ancillary constants.
$k \leftarrow \cos^{-1}\frac{E}{2}$ $L \leftarrow \left\lceil \left(\delta^{\nu d}\varepsilon\right)^{-1} \right\rceil$ $\left\|\psi_0\right\rangle \leftarrow \frac{1}{\sqrt{n}}\sum_{j=-L}^{0}\sin(kj)\left\|j\right\rangle$ $H_{\text{wire}} \leftarrow \sum_{j=-L}^{0}\left(\left\|j\right\rangle\left\langle j+1\right\| + \text{h.c}\right)$ $n \leftarrow \sum_{j=-L}^{0}\sin^2(kj)$ $\xi_E \leftarrow$ localization_length (E,δ) $H_{\text{clean}} \leftarrow \sum_{\langle i,j\rangle i,j\in 1\ldots(CC'\xi_E)^d}\left\|i\right\rangle\left\langle j\right\| + \text{h.c.}$ $H \leftarrow H_{\text{wire}} + H_{\text{clean}}$	Construct the Hamiltonian of a clean d-dimensional lattice of size $(C'C\xi)^d$, and attach to it a clean wire which is long enough such that its level spacing near E is smaller. The initial state is confined to the wire, with energy expectation value close to E. The system is illustrated in Fig. 1 in the main text.
$T \leftarrow \min\left\{t \middle\| \mathbb{E}_{\mathcal{P}_{W_c(1-\delta)}}\sum_{j \text{ s.t. } \left\|r_j-r_0\right\|\geq\lceil C\xi_E\rceil}\left[\left\|\left\langle\psi_0\right\| e^{i\left(H+\sum_{i=1}^{(CC'\xi)^d}\epsilon_i\left\|i\right\rangle\left\langle i\right\|\right)t}\left\|j\right\rangle\right\|^2\right] \geq c\right\}$	Choose a time T long enough for a diffusive particle on a d-dimensional lattice to reach a distance of $C\xi_E$ from the 0 site with probability c, assuming the system is diffusive (that is, $W = W_c(1-\delta)$).
$P_{\text{exp},d} = \mathbb{E}_{\mathcal{P}_{W_c(1-\delta)}}\left[\sum_{j \text{ s.t. } \left\|r_j-r_0\right\|\geq\lceil C\xi_E\rceil}\left\|\left\langle\psi_0\right\| e^{i\left(H+\sum_{i=1}^{(CC'\xi)^d}\epsilon_i\left\|i\right\rangle\left\langle i\right\|\right)T}\left\|j\right\rangle\right\|^2\right]$ $P_{\text{exp},l} = \mathbb{E}_{\mathcal{P}_{W_c(1+\delta)}}\left[\sum_{j \text{ s.t. } \left\|r_j-r_0\right\|\geq\lceil C\xi_E\rceil}\left\|\left\langle\psi_0\right\| e^{i\left(H+\sum_{i=1}^{(CC'\xi)^d}\epsilon_i\left\|i\right\rangle\left\langle i\right\|\right)T}\left\|j\right\rangle\right\|^2\right]$	Based on Eq. (A1), Find the expected probability that a particle reached a distance $C\xi_E$ in the localized and delocalized phases, respectively.
return $C, H, T, P_{\text{exp},d}, P_{\text{exp},l}, \xi_E$	

determine_delta_sign$(E,\ d,\ \delta,\ \mathcal{P})$**:**	
$C, H, T, P_{\text{exp},d}, P_{\text{exp},l}, \xi_E \leftarrow$**Initialization**$(E,d,\delta)$	
for i **in** $\left\{1\ldots(CC'\xi)^d\right\}$ **do** $\epsilon_i \leftarrow$ **Sample** $(\mathcal{P})$ $H \leftarrow H + \epsilon_i\left\|i\right\rangle\left\langle i\right\|$	Construct a Hamiltonian of a subsystem of volume $(C')^d = O\left(\delta^{-\nu d}\right)$. Based on Eq. (A1) we see that the amplitude on sites i for which $\left\|r_i - r_0\right\| \geq \sqrt{D(E)T} \sim \delta^{-\nu d}$ decays exponentially, and therefore considering a graph of this size will result in a good separation of the phases.
$\left\|\psi(T)\right\rangle = \exp\left(-iHT\right)\left\|\psi_0\right\rangle$ $P_{C\xi_E} = \sum_{j \text{ s.t. } \left\|r_j-r_0\right\|\geq\lceil C\xi_E\rceil}\left\|\left\langle\psi(T)\right\| j\right\rangle\right\|^2$	Calculate the state $e^{-iHT}\left\|\psi_0\right\rangle$, and from it deduce the probability for a particle to be at a distance larger than $\lceil C\xi\rceil$.
if $\left\|P_{C\xi_E} - P_{\text{exp},d}\right\| > \left\|P_{C\xi_E} - P_{\text{exp},l}\right\|$ **do** **return** $\delta < 0$. **else do** **return** $\delta > 0$.	If the particle has reached a distance of $\lceil C\xi\rceil$ or more with high probability, we conclude that the simulated system in the diffusive phase and $\delta < 0$. Otherwise, we conclude that $\delta > 0$.

TABLE I. The suggested algorithm that utilizes the AL transition in order to differentiate the probability distributions. The algorithm is presented on the left column, and on the right we provide explanations.

Initialization(E, δ, μ)**:**	
$0 \leftarrow$ starting_site $c \leftarrow$ const., $0 < c < 1$ $N \leftarrow \lceil \delta^{-d\alpha_q c} \rceil$ $\mathcal{N} \leftarrow d_{\text{single}}^{N}$ $H \leftarrow H_{\text{clean}}(N)$ $q \leftarrow$ const. > 1	Choose ancillary coefficients. Choose a system size $N = \delta^{-d\alpha_q c}$, where c is a positive constant smaller than 1, which guarantees that in the vicinity of the phase transition, $\frac{b_{q,\text{erg}}}{N} \to 0$. Start with the N-particle disorder-free Hamiltonian. d_{single} denotes the Hilbert space size of a single site. Choose $q > 1$ for the FS-locality measure, S_q.
determine_delta_sign():	

for i **in** $1..N$ **do**
  $h_i \leftarrow$ **Sample** $(\mathcal{P})$
  $H \leftarrow H + h_i \sigma_i^z$
	Add the disorder to the Hamiltonian by performing N samples from the distribution.

$\lvert\psi_E\rangle \leftarrow$ **eigenvector**(H, E) $S_q \leftarrow -\ln\left(\sum_{\alpha=1}^{\mathcal{N}} \lvert\langle\psi_E \mid \alpha\rangle\rvert^{2q}\right)$	Diagonalize H to obtain the eigenvector with eigenvalue closest to E and calculate its S_q.

If $\left\lvert \frac{S_q}{\ln \mathcal{N}} - \left(1 + \frac{b_{q,\mathbf{erg}}(\delta)}{\ln \mathcal{N}}\right)\right\rvert < \left\lvert \frac{S_q}{\ln \mathcal{N}} - \left(D_q(\delta) + \frac{b_{q,\mathbf{MBL}}(\delta)}{\ln \mathcal{N}}\right)\right\rvert$ **do**
  return $\delta < 0$**.**
else do
  return $\delta > 0$**.**
	Determine the phase of the simulated system based on the calculated S_q.

TABLE II. The suggested algorithm that utilizes the MBL transition in FS in order to differentiate the probability distributions. The algorithm is presented on the left column, and on the right we provide explanations.

Appendix B: Harris-like bound for Fock space localization transition - rigorous protocol

Here we bring the full details of the hypothetical decision protocol used in Sec. III A in the main text. The protocol is detailed in Table II.

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