

Two-Point Superstring Tree Amplitudes Using the Pure Spinor Formalism

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Abstract

We provide a prescription for computing two-point tree-level amplitudes in the pure spinor formalism that provides finite results that agree with the corresponding expression in the field theories. In [1, 2], the same result was discovered in the bosonic strings with indications of generalization to superstrings in the Ramond-Neveu-Schwarz formalism. Pure spinor formalism is the unique super-Poincare covariant approach to quantizing superstrings [3]. Since the pure spinor formalism is equivalent to other superstring formalisms, it verifies the above claim. We introduce a *mostly BRST exact operator* to achieve this.

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1 Introduction

The two-point tree-level bosonic string amplitudes in flat spacetime, hitherto known to vanish, are finite and in agreement with the corresponding free particle expression in the quantum field theories [1, 2]. We anticipate the same holds in the superstrings as well. Further, [1] suggested that this analysis can be carried over identically to the $NS - NS$ sector of superstrings. Also, spacetime supersymmetry would ensure a similar story would repeat for the other sectors of superstrings. It is desirable to see if this claim can be explicitly verified and check if the results hold in other formalisms of superstrings. Here, we shall focus on the pure spinor formalism that keeps the Poincare and spacetime supersymmetry manifest [3]. A naive application of the pure spinor amplitude prescription gives a vanishing two-point tree-level amplitude. We shall see why this is the case and how to rectify it. Before delving further into the superstring case, in what follows, we briefly recall why the two-point tree-level amplitudes in the bosonic strings were thought to vanish.

In bosonic strings, we have a $\text{diff} \times \text{Weyl}$ symmetry - a local gauge symmetry. As is usual in gauge theories, the path integrals for various physical quantities in string theory also get divided by the volume of this local gauge group. Following a gauge fixing procedure such as the Faddeev-Popov method usually produces a factor proportional to the volume of the gauge group that usually cancels the factor in the denominator. Schematically

$$\langle f(\Phi) \rangle = \frac{\int [d\Phi] \exp[-S[\Phi]] f[\Phi]}{V_G} \xrightarrow{\text{Faddeev-Popov}} \frac{V_G \int [d\hat{\Phi}] \exp[-\hat{S}[\hat{\Phi}]] \hat{f}[\hat{\Phi}]}{V_G}$$

where Φ stands for all the fields, S is the action and V_G represents the volume of the gauge group G . All

the hatted symbols denote gauge fixed quantities.

For the bosonic strings, $G = \text{diffeomorphism} \times \text{Weyl}$, which can be fixed locally by fixing the components of the worldsheet metric. The choice of metric, however, does not fully fix all the symmetries. These residual unfixed symmetries do not affect the gauged fixed metric. These are called the conformal killing group (CKG). Here, we are interested in the string amplitudes on a disk/sphere. On a disk/sphere, the CKG is three-dimensional and is non-compact. For amplitudes involving three or more strings, the CKG gets completely fixed by fixing the positions of three vertex operators. In the two-point amplitude, after fixing the position of both vertex operators, the residual conformal killing group still has infinite volume. This volume appears in the denominator of the corresponding path integral, which naively implies that the path integral vanishes. Thus, these amplitudes were assumed to vanish. This understanding, however, relies on the assumption that the numerator of the corresponding path integral is finite. [1] noticed for the first time that the numerator is also ∞ , and hence one has to make sense of an expression of the form $\frac{\infty}{\infty}$. Closer examination reveals that this expression is finite and gives rise to the expected two-point amplitudes. Further, the same arguments may be repeated to the NS-NS sector of the RNS superstrings.

The authors of [2] revisited these amplitudes using the operator formalism of bosonic strings. The tree-level amplitudes in this formalism vanish unless they involve three or more vertex operators since otherwise, the saturation of c ghost zero modes is impossible [4]. The saturation of c ghost zero mode requires three c ghosts on the disk. Consequently, the two-point amplitudes vanish on the disk/sphere as the vertex operators supply two c ghosts. Hence, to agree with the path integral method [1], the amplitudes involving fewer than three strings at the tree level must be modified.

The authors of [2] arrived at this modification by introducing a novel vertex operator that saturates the ghost zero modes in these amplitudes and provides the desired two-point answer. This vertex operator is *mostly BRST exact*. In this work, we introduce a mostly BRST exact operator in the pure spinor formalism that provides non-vanishing two-point tree-level amplitudes in the pure spinor formalism. Explicitly,

$$V_0(z) \equiv \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} dq (\lambda\gamma^0\theta) e^{iqX^0(z)} \quad (1)$$

Notice that the integrand of the above operator for $q \neq 0$ can be re-written as $[Q, \star]$, justifying the nomenclature- *mostly BRST exact* (see equation (28) for explicit form). The mostly BRST exact operators appeared earlier but in a different context in [5].

Since V_0 isolates the time index "0", Lorentz invariance of the amplitudes involving these operators must be explicitly checked. We must also ensure that V_0 insertion does not violate the super-Poincare and conformal invariance, as is required of these amplitudes. We verify that these symmetries are preserved¹, in the appendix D. Consequently, we shall freely use the consequences of these symmetries to reach various conclusions.

¹Up to BRST exact terms, which give a vanishing contribution.

The rest of the paper is organized as follows. In section 2, we first address why the two-point amplitudes in the pure spinor formalism vanish on a naive use of the standard prescription. Next, we introduce the mostly BRST operator and show that its use gives rise to the expected two-point result. We end with a discussion in 3 and defer some details and computations to the appendices.

2 Two point amplitudes in the pure spinor formalism

In this section, we begin by addressing the problem with two-point amplitudes in the pure spinor formalism. We shall consider open strings for simplicity as the generalization to other string theories is straightforward - we make some comments on this in the discussion section 3. The amplitude prescription at tree level in the pure spinor formalism is [3]

$$\mathcal{A}_n = \int \prod_{j=4}^n dz_j \langle V_1(z_1) V_2(z_2) V_3(z_3) U_j(z_j) \rangle_{D^2} \quad (2)$$

where, V are the unintegrated vertex operators and U are the integrated vertex operators. We shall not present all details of the pure spinor formalism, but give a quick review in A containing the important ingredients required in this work - see [6, 7, 8, 9] for detailed reviews. We shall only require the use of unintegrated vertex operators. In the plane wave basis, these are given by

$$V(z) = \hat{V} e^{ik.X} \equiv \lambda^\alpha O_\alpha e^{ik.X}, \quad QV = 0, \quad k^2 = -\frac{n}{\alpha'} \quad (3)$$

where O_α are conformal weight n , composite operators constructed out of the basic world-sheet fields $\Pi^m, d_\alpha, \theta^a, N^{mn}, J$. Here, λ^α is the pure spinor, a Grassmann even spinor satisfying

$$\lambda^\alpha (\gamma^m)_{\alpha\beta} \lambda^\beta = 0, \forall m.$$

γ^m are the Chiral-Gamma matrices in 10 dimensions. Also, n stands for the n -th excited level of the string and Q denotes the BRST charge. The $e^{ik.X}$ cancels the conformal weight of O_α so that V has zero conformal dimension.

We should note that there is so far no genuine derivation of amplitude prescription (2) due to the absence of the underlying gauge theory for pure spinor formalism whose gauge fixing gives (2)². A justification for this amplitude prescription relies on the fact that the pure spinor formalism in its non-minimal version is $N = 2$ topological strings [11], whose amplitude prescription is same as that of the bosonic strings³.

All the non-trivial amplitudes in the pure spinor formalism can be brought to a form that contains

²Recently a gauge theory behind the pure spinor formalism was proposed in [10]. Perhaps one can arrive at the amplitude prescription using this.

³This amplitude prescription was studied in [12] by coupling the standard pure spinor formalism to topological gravity and performing a BRST quantization.

three λ and five θ zero modes in the corresponding correlator. We choose to normalize all the amplitudes with the following correlator⁴⁵

$$\langle (\lambda\gamma^m\theta) (\lambda\gamma^m\theta) (\lambda\gamma^p\theta) (\theta\gamma_{mnp}\theta) \rangle = 1 \quad (4)$$

For $n = 2$, a naive application of (2) gives

$$\mathcal{A}_2 = \langle V_1(z_1)V_2(z_2) \rangle_{D^2} \propto \langle (\lambda^\alpha O_\alpha^1)(z_1) (\lambda^\beta O_\beta^2)(z_2) \rangle_{D^2} \quad (5)$$

Since the above correlator has only two λ , it vanishes identically, implying that the above prescription gives a trivial two-point amplitude. We must find the correlator that gives rise to the correct two-point scattering amplitude. It must involve an extra *vertex operator* to be non-vanishing and at the same time get rid of $\delta(k_1^0 - k_2^0)$. This suggests that the extra piece must have one λ^α . The operator proposed in (1) fulfills all these requirements.

Let us begin by calculating the following amplitude

$$A \equiv \langle V_0(z)V_1(z_1)V_2(z_2) \rangle = \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} dq \langle [(\lambda\gamma^0\theta)(z)e^{iqX^0(z)}] V_1(z_1)V_2(z_2) \rangle \quad (6)$$

where, V_0 is the operator introduced in (1) which we fix at z , while V_1 and V_2 are the unintegrated vertex operators (fixed at z_1 and z_2 respectively). To calculate this, we split the integral into three parts as follows

$$A = \frac{1}{2\pi\alpha'} \left(\int_{-\infty}^{-\epsilon} dq \langle [(\lambda\gamma^0\theta)(z)e^{iqX^0(z)}] V_1(z_1)V_2(z_2) \rangle + \int_{-\epsilon}^{\epsilon} dq \langle [(\lambda\gamma^0\theta)(z)e^{iqX^0(z)}] V_1(z_1)V_2(z_2) \rangle + \int_{\epsilon}^{\infty} dq \langle [(\lambda\gamma^0\theta)(z)e^{iqX^0(z)}] V_1(z_1)V_2(z_2) \rangle \right)$$

In the above equation, $\epsilon > 0$ is an infinitesimal parameter. This allows us to replace the operator in the square brackets of the first and the third term with the BRST expression given in (28). This replacement implies that first and the third terms of the above equation vanish individually. This can be seen by unwrapping the contour in the definition of the mBRST exact operator (see equation(1)) and writing it in terms of the following three contours - C_{z_1} : contour around z_1 , C_{z_2} : contour z_2 , and C_{z,z_1,z_2} : contour containing z, z_1, z_2 . Schematically

$$\oint_{C_z} = \oint_{C_{z,z_1,z_2}} - \oint_{C_{z_1}} - \oint_{C_{z_2}}$$

⁴Normalizing this correlation function is sufficient since there is only one scalar present in tensor product of three λ and five θ .

⁵There is an alternative zero mode normalization for λ and θ given by $\langle \mathbf{1} \rangle_0 = 1$ [13]. We shall not be working with this. See [14] for an application of this prescription to compute one-point closed string amplitudes on a disk.

The contour integrals around z_1 and z_2 vanish due to $QV_1 = 0$ and $QV_2 = 0$, while the remaining one can be deformed to include point at infinity which vanishes as well since there are no non-trivial operators there. Thus, we are left with

$$A = \frac{1}{2\pi\alpha'} \int_{-\epsilon}^{\epsilon} dq \left\langle \left[(\lambda\gamma^0\theta)(z) e^{iqX^0(z)} \right] V_1(z_1) V_2(z_2) \right\rangle \quad (7)$$

On substituting the form of V_i given in (3), we can factor the amplitude as

$$A \equiv \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} dq \left\langle \left\langle (\lambda\gamma^0\theta)(z) \hat{V}_1(z_1) \hat{V}_2(z_2) \right\rangle \right\rangle \left\langle e^{iqX^0(z)} e^{ik_1 \cdot X(z_1)} e^{-ik_2 \cdot X(z_2)} \right\rangle \quad (8)$$

where we use the notation $\langle\langle \dots \rangle\rangle$ as a shorthand to denote that the necessary OPEs have been taken among the operators inside the bracket. We have taken the momentum k_1 to be incoming and k_2 to be outgoing. The Koba-Nielsen factor of the above expression reduces to

$$\begin{aligned} & \left\langle e^{iqX^0(z)} e^{ik_1 \cdot X(z_1)} e^{-ik_2 \cdot X(z_2)} \right\rangle \\ &= iC_{D_2}^X (2\pi)^{10} \delta(q + k_1^0 - k_2^0) \delta^{(9)}(\vec{k}_1 - \vec{k}_2) |z - z_1|^{2\alpha' q k_1^0} |z - z_2|^{-2\alpha' q k_2^0} |z_1 - z_2|^{-2\alpha' k_1 \cdot k_2} \end{aligned} \quad (9)$$

On substituting the above result in (8), we find

$$\begin{aligned} A &= \frac{i}{\alpha'} C_{D_2}^X \int_{-\infty}^{\infty} dq \delta(q + k_1^0 - k_2^0) (2\pi)^9 \delta^{(9)}(\vec{k}_1 - \vec{k}_2) \left\langle \left\langle (\lambda\gamma^0\theta)(z) \hat{V}_1(z_1) \hat{V}_2(z_2) \right\rangle \right\rangle \\ &\quad \times |z - z_1|^{2\alpha' q k_1^0} |z - z_2|^{-2\alpha' q k_2^0} |z_1 - z_2|^{-2\alpha' k_1 \cdot k_2} \end{aligned} \quad (10)$$

We are interested in the on-shell amplitudes for which $k^0 = \sqrt{|\vec{k}|^2 + m^2}$, for a particle carrying momentum \vec{k} and mass m . The space Dirac-delta function in the above sets $\vec{k}_1 = \vec{k}_2$. Further, let us assume that $m_2 - m_1 = \delta$, where m_1 and m_2 are the masses of the string 1 and 2. If $m_1 = m$ and $m_2 = m + \delta$, we find that on-shell

$$k_2^0 - k_1^0 = \sqrt{|\vec{k}|^2 + m^2} \left[\sqrt{1 + \frac{\delta^2 + 2\delta m}{|\vec{k}|^2 + m^2}} - 1 \right]$$

However, since $-\epsilon < q < \epsilon$, for getting support from the energy Dirac-delta we must have

$$-\epsilon < \sqrt{|\vec{k}|^2 + m^2} \left[\sqrt{1 + \frac{\delta^2 + 2\delta m}{|\vec{k}|^2 + m^2}} - 1 \right] < \epsilon.$$

Since, $\epsilon \rightarrow 0$, we must have $\delta \rightarrow 0$ for energy Dirac-delta to provide a non-trivial contribution. This means unless the masses of the strings are the same, the amplitude vanishes on-shell. Thus, we find that

$$A = \frac{i}{\alpha'} C_{D_2}^X (2\pi)^9 \delta^{(9)}(\vec{k}_1 - \vec{k}_2) \left\langle \left\langle (\lambda\gamma^0\theta)(z) \hat{V}_1(z_1) \hat{V}_2(z_2) \right\rangle \right\rangle |z_1 - z_2|^{2\alpha' m_1^2} \delta_{m_1, m_2} \quad (11)$$

where we used $k_1 \cdot k_2 = -k_1^0 k_2^0 + \vec{k}_1 \cdot \vec{k}_2 = -(k_1^0)^2 + |\vec{k}_1|^2 = -m_1^2$. Notice the factor of $|z_1 - z_2|^{-2\alpha m_1^2}$. This factor cancels with a similar factor coming from pure spinor superspace, namely $\langle\langle (\lambda\gamma^0\theta) \hat{V}_1(z_1) \hat{V}_2(z_2) \rangle\rangle$. Recall that at the n^{th} level of open superstring we have $(\text{mass})^2 = \frac{n}{\alpha'}$ and also that the conformal dimensions of \hat{V}_i for $i = 1, 2$ are n . Furthermore, $(\lambda\gamma^0\theta)$ has zero conformal weight. Thus, upon using the standard result for a 3-point function in a CFT, we find

$$\langle\langle (\lambda\gamma^0\theta)(z) \hat{V}_1(z_1) \hat{V}_2(z_2) \rangle\rangle \propto |z_1 - z_2|^{-2n} = |z_1 - z_2|^{-2\alpha' m_1^2} \quad (12)$$

which, cancels the coordinate dependence coming from the Koba-Nielsen factor. Thus, the amplitude is coordinate invariant.

Having fixed the coordinate dependence, let us further elaborate on the dependence of $\langle\langle \dots \rangle\rangle$ on the kinematic data, namely the polarizations and momenta. We argue that it must be of the form

$$\langle\langle (\lambda\gamma^0\theta)(z) \hat{V}_1(z_1) \hat{V}_2(z_2) \rangle\rangle \propto f^0(\epsilon_1, \epsilon_2; k) \quad (13)$$

where ϵ_i are the polarizations and k is momentum of the state represented by vertex operators V_1 and V_2 . Since our theory is supersymmetric, giving the argument for purely bosonic states is sufficient. The polarizations for the bosonic states are specified by using the Lorentz vector indices. Further, the polarizations satisfy $k_m \epsilon_i^{m\dots} = 0$. Let us assume that the 0 index is supplied by ϵ_1 . For a non-zero answer, we contract the rest of the indices of ϵ_1 with only ϵ_2 . But, this leaves a free index on ϵ_2 , which must be contracted by k , giving a vanishing contribution. Hence, there is a unique choice - the polarization tensors contract among themselves, and the 0 index is supplied by k^0 . In appendix C, we explicitly verify this for all the states at the massless level. Thus, we find⁶

$$\langle\langle (\lambda\gamma^0\theta) \hat{V}_1(z_1) \hat{V}_2(z_2) \rangle\rangle \propto k^0 \delta_{jj'} \quad (14)$$

where we have used j and j' in $\delta_{jj'}$ to distinguish between states with degenerate masses like gluon and gluino. Hence, the final result

$$A \propto (2\pi)^9 \delta^{(9)}(\vec{k}_1 - \vec{k}_2) k^0 \delta_{m_1, m_2} \delta_{jj'} \quad (15)$$

reproduces the expected two-point amplitude in a field theory in D dimensions given by

$$\mathcal{A}_2 = 2k^0 (2\pi)^{D-1} \delta^{D-1}(\vec{k}_1 - \vec{k}_2), \quad k^0 \equiv \sqrt{m^2 + \vec{k}^2} \quad (16)$$

⁶There are other factors containing the contribution of non-zero modes of various worldsheet fields, normalization of polarizations, and pure spinor superspace computations. These are all non-zero.

up to a constant of proportionality, which can be determined through unitarity

$$A_2(k_1, k_2) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} A_2(k_1, k) A_2(k, k_2) \quad (17)$$

In appendix C, we explicitly verify the above result for some amplitudes. Thus, the two-point amplitudes in the pure spinor formalism using operator V_0 behave as is expected.

3 Discussion

The non-vanishing of two-point tree-level amplitudes in string theory is desirable for various consistencies—see [15] for a discussion and detailed set of references. We provide a brief reminder of S matrix and its properties in the context of quantum field theories in appendix E. On the physical grounds, it is reasonable to demand that these properties carry over to scattering amplitudes in string theories. In particular string scattering amplitudes must follow the cluster decomposition principle, which requires a finite two-point tree-level amplitude. Thus, a crucial requirement of the scattering amplitudes of string theory is that they have a non-vanishing two-point tree-level amplitude. In particular these must have a form that is given by (16). Using a *mostly BRST-exact* vertex operator, we found non-zero two-point tree amplitudes in the pure spinor formalism in open strings that has same form as equation (16). Further, we have checked that this amplitude are super-Poincare and conformally invariant (see appendix D). Generalization to the closed strings is straightforward by adding a right-moving sector. Generalization to Heterotic strings follows by using the pure spinor prescription of this work for the supersymmetric side, and the analysis of [1, 2] for the bosonic side.

To conclude, we have identified the correlation functions that give rise to correct two-point amplitude in the pure spinor formalism. However, we do not know from a fundamental point of view why the additional vertex operator is of this form⁷. It is important to explore for a fundamental origin of the mostly BRST exact operator we used in this work, perhaps by making use of the gauge invariant action presented in [10] (see also [16, 17] which gave important insights that lead to [10]). This investigation we leave for future work.

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⁷Perhaps these are related to the zero momentum states as pointed out by Renann Lipinski Jusinkas. We note that the additional operators in [1, 2] too resemble zero momentum states. The understanding of their precise role we leave for future work.

A Review of the Pure Spinor Formalism

In this appendix, we very briefly review the pure spinor formalism that makes the discussion in the main text coherent. The world sheet action in a conformal gauge for strings in a flat $10D$ spacetime takes the form

$$S = \frac{1}{\pi\alpha'} \int d^2z \left(\frac{1}{2} \partial X^m \bar{\partial} X_m + p_\alpha \bar{\partial} \theta^\alpha - w_\alpha \bar{\partial} \lambda^\alpha \right) \quad (18)$$

where, $m \in \{0, 1, \dots, 9\}$ and $\alpha \in \{1, \dots, 16\}$. X^m are the spacetime coordinates, θ^α, w_β are anti-commuting Majorana-Weyl spinors while λ^α are commuting Weyl Spinors. $\{X^m, \theta^\alpha, \lambda^\alpha\}$ are scalars on the worldsheet, while p_α, w_α , the conjugate momenta fields of θ^α and w_α respectively, carry weight 1. Further, λ^α satisfy the *pure spinor*

$$\lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta = 0 \quad \forall m \quad (19)$$

where, $\gamma_{\alpha\beta}^m$ are symmetric 16×16 Gamma matrices in 10 dimensional spacetime. To keep the supersymmetry manifest, instead of working with p_α and ∂X^m , we work with the supersymmetric combinations

$$\begin{aligned} d_\alpha &= p_\alpha - \frac{1}{2} \gamma_{\alpha\beta}^m \theta^\beta \partial X_m - \frac{1}{8} \gamma_{\alpha\beta}^m \gamma_{m\sigma\delta} \theta^\beta \theta^\sigma \partial \theta^\delta \\ \Pi^m &= \partial X^m + \frac{1}{2} \gamma_{\alpha\beta}^m \theta^\alpha \partial \theta^\beta \end{aligned} \quad (20)$$

The BRST operator is *postulated* to be

$$Q = \oint dz \lambda^\alpha(z) d_\alpha(z) \quad (21)$$

Due to the pure spinor constraint, w_α remains defined up to a gauge transformation

$$w_\alpha = \Lambda_m (\gamma^m \lambda)_\alpha \quad (22)$$

with Λ_m playing the role of gauge parameters. To take care of this gauge symmetry, we always work with the following gauge invariant combinations

$$N_{mn} = \frac{1}{2} w_\alpha (\gamma_{mn})^\alpha_\beta \lambda^\beta \quad , \quad J = w_\alpha \lambda^\alpha \quad , \quad T = w_\alpha \partial \lambda^\alpha$$

J is the ghost-number current and sets the ghost number of λ^α to 1.

The physical states lie in the BRST cohomology with ghost number 1. The vertex operators are

constructed out of $\{\Pi^m, d_\alpha, \theta^\alpha, N^{mn}, J, \lambda^\alpha\}$. The only non-trivial OPE we shall require is given by

$$d_\alpha(z)V(w) = \frac{\alpha'}{2(z-w)}D_\alpha V(w) + \dots \quad (23)$$

where, V denotes an arbitrary superfield while D_α is the super-covariant derivative given by

$$D_\alpha \equiv \partial_\alpha + \gamma_{\alpha\beta}^m \theta^\beta \partial_m \quad \implies \quad \{D_\alpha, D_\beta\} = 2(\gamma^m)_{\alpha\beta} \partial_m \quad (24)$$

where, $\partial_m = \frac{\partial}{\partial X^m}$ and $\partial_\alpha = \frac{\partial}{\partial \theta^\alpha}$. The above describes the *minimal* version of the pure spinor formalism. It will be sufficient for this work.

To simplify the notation, we shall be implicit about normal orderings. The normal ordered operator $:AB:(z)$ is defined as

$$:AB:(z) \equiv \frac{1}{2\pi i} \oint_z \frac{dw}{w-z} A(w)B(z) \quad (25)$$

where, A and B are arbitrary operators.

B V_0 as a mostly BRST exact operator

In this appendix, we demonstrate that the V_0 introduced in the main body is a *mostly BRST-exact* operator. Let us begin by noticing

$$[Q, e^{iqX^0(z)}] \equiv \oint_z dw (\lambda^\alpha d_\alpha)(w) e^{iqX^0(z)} = \frac{\alpha'}{2} \oint_z dw \lambda^\alpha(w) \left[\frac{D_\alpha e^{iqX^0(z)}}{w-z} + \dots \right] \quad (26)$$

where, we used the standard OPE $d_\alpha(z)V(w) \simeq \frac{\alpha'}{2} \frac{D_\alpha V}{z-w}$. On recalling that $D_\alpha = \partial_\alpha + (\gamma^m \theta)_\alpha \partial_m$, we find that

$$[Q, e^{iqX^0(z)}] = \frac{iq\alpha'}{2} (\lambda\gamma^0\theta) e^{iqX^0} \quad (27)$$

Hence, for $q \neq 0$ we have

$$(\lambda\gamma^0\theta) e^{iqX^0} = -\frac{1}{q} \left[Q, \left(\frac{2i}{\alpha'} e^{iqX^0(z)} \right) \right] \quad (28)$$

showing that the integrand of V_0 is BRST-exact for $q \neq 0$ and thus V_0 is mostly BRST exact.

C Some explicit examples

In this appendix, we evaluate the two-point massless open superstring amplitudes (on a disk) with the new prescription given equation (6) in this paper. The goal is to substantiate the claim in (14) by providing explicit examples. For this we essentially need to compute the $\langle\langle(\lambda\gamma^0\theta)\hat{V}_1\hat{V}_2\rangle\rangle$ where $\hat{V}_i = \lambda^\alpha A_{i\alpha}$ and show it is proportional to k^0 . For the massless case, the task is trivial as everything inside the bracket is conformal weight zero and hence there are no non-trivial OPEs. Consequently, we can use the pure spinor-superspace method to perform the computation. The relevant theta expansion is given by (we follow the notation and conventions used in [18])

$$A_\alpha = a_m (\gamma^m)_\alpha - \frac{2}{3} (\gamma^m \theta)_\alpha (\theta \gamma_m \chi) + \dots \quad (29)$$

where we have not shown the higher order θ terms as they will not be required. Also, a_m represents the gluon field and χ^α the gluino field. Consequently, we find

$$\langle\langle(\lambda\gamma^0\theta)\hat{V}_1\hat{V}_2\rangle\rangle = \langle(\lambda\gamma^0\theta)(\lambda^\alpha A_{1\alpha})(\lambda^\alpha A_{2\alpha})\rangle = \frac{i}{180} k^0 (a_r^m a_{r'm}) + \frac{1}{360} (\chi_s \gamma^0 \chi_{s'}) \quad (30)$$

where, r, r' and s, s' denote polarizations and helicities of gluons and gluinos respectively. We made use of the following pure spinor superspace identities [19], for performing the gluon calculation

$$\langle(\lambda\theta\gamma^m\rho)(\lambda\theta\gamma^n\rho)(\lambda\theta\gamma^p\rho)(\lambda\theta\gamma_{stu}\rho)\rangle = \frac{1}{120} \delta_{stu}^{mnp} \quad (31)$$

and

$$\begin{aligned} & \langle(\lambda\gamma^u\theta)(\theta\gamma_{fgh}\theta)(\theta\gamma_{jkl}\theta)(\lambda\gamma_{mnpqr}\lambda)\rangle \\ &= -\frac{4}{35} \left[\delta_{[j}^{[m} \delta_k^n \delta_l^p \delta_{[f}^q \delta_g^r] \delta_h^u] + \delta_{[f}^{[m} \delta_g^n \delta_h^p \delta_{[j}^q \delta_k^r] \delta_l^u] - \frac{1}{2} \delta_{[j}^{[m} \delta_k^n \eta_{l][f} \delta_g^p \delta_h^q \eta^{r]u]} - \frac{1}{2} \delta_{[f}^{[m} \delta_g^n \eta_{h][j} \delta_k^p \delta_l^q \eta^{r]u]} \right] \\ & \quad - \frac{1}{1050} \epsilon^{mnpqr} \epsilon_{abcde} \left[\delta_{[j}^{[a} \delta_k^b \delta_l^c \delta_{[f}^d \delta_g^e] \delta_h^u] + \delta_{[f}^{[a} \delta_g^b \delta_h^c \delta_{[j}^d \delta_k^e] \delta_l^u] - \frac{1}{2} \delta_{[j}^{[a} \delta_k^b \eta_{l][f} \delta_g^c \delta_h^d \eta^{e]u]} - \frac{1}{2} \delta_{[f}^{[a} \delta_g^b \eta_{h][j} \delta_k^c \delta_l^d \eta^{e]u]} \right] \end{aligned} \quad (32)$$

for gluino calculation⁸. We further note that the polarizations are normalized as (see for example [22])

$$a_r^m a_{r'm} = \delta_{rr'} , \quad (\chi_s \gamma^0 \chi_{s'}) = k^0 \delta_{ss'} \quad (33)$$

Thus, we find that

$$\langle\langle(\lambda\gamma^0\theta)\hat{V}_{1j}\hat{V}_{2j'}\rangle\rangle \propto k^0 \delta_{jj'} \delta_{ss'} \quad (34)$$

⁸We acknowledge the use of [20, 21] for performing the calculations.

where, j, j' stand for the particle species and s, s' denote the corresponding polarizations. We note that we get the same relative factor in gluon and gluino amplitude as in [18] and that the gluon-gluino amplitudes vanish automatically as we expect them to. For the massive case, the computations can be repeated using [18, 23], though we expect them to be involved.

D Some consistency checks for V_0 insertion

In this appendix, we verify that the insertion of V_0 does not spoil the super-Poincare and conformal invariance. First note that the expression of V_0 is not Lorentz covariant as it isolates $0th$ spacetime component. Superstring theory in the flat background is super-Poincare invariant. So, let us first see if the expression that we have arrived has this symmetry. Notice that

$$\delta\langle V_0 V_1 V_2 \rangle = \langle \delta V_0 V_1 V_2 \rangle + \underbrace{\langle V_0 \delta V_1 V_2 \rangle + \langle V_0 V_1 \delta V_2 \rangle}_{\text{automatically invariant}} = \langle \delta V_0 V_1 V_2 \rangle \quad (35)$$

where δ denotes the change after applying some symmetry transformation. The vertex operators V_1 and V_2 are invariant under δ . Hence, we only need to evaluate δV_0 . To facilitate the discussion, let us note that we can write V_0 as

$$V_0 = \int_{-\infty}^{\infty} \frac{dq}{i\pi\alpha'^2} \frac{1}{q} \left[Q, e^{iqX^0} \right] \quad (36)$$

Now, we can write the variations as (on noticing that $\delta Q = 0$)

$$\delta V_0 = \int_{-\infty}^{\infty} \frac{dq}{i\pi\alpha'^2} \frac{1}{q} \left[Q, \delta e^{iqX^0} \right] \quad (37)$$

Let us now consider all the transformations one by one. To distinguish one transformation from another, we shall provide a subscript on δ . Under translations

$$X^m \rightarrow X^m + a^m \implies \delta_a X^m = a^m \implies \delta_a e^{iqX^0} = iqa^0 e^{iqX^0} \quad (38)$$

Thus, we see that

$$\delta_a V_0 = \int_{-\infty}^{\infty} \frac{dq}{i\pi\alpha'^2} \frac{1}{q} \left[Q, iqa^0 e^{iqX^0} \right] = \left[Q, \int_{-\infty}^{\infty} \frac{dq}{\pi\alpha'^2} a^0 e^{iqX^0} \right] \quad (39)$$

On using $QV_1 = 0 = QV_2$, we can easily see that the two-point amplitude is translationally invariant. Similarly, under Lorentz transformations

$$X^m \rightarrow X^m + \Lambda^m_n X^n \implies \delta_\Lambda V_0 = \left[Q, \int_{-\infty}^{\infty} \frac{dq}{\pi\alpha'^2} \Lambda^0_m X^m e^{iqX^0} \right] \quad (40)$$

and under supersymmetry transformation

$$X^m \rightarrow X^m + (\eta\gamma^m\theta) \implies \delta_\eta V_0 = \left[Q, \int_\infty^\infty \frac{dq}{\pi\alpha'^2} (\eta\gamma^m\theta) e^{iqX^0} \right] \quad (41)$$

Thus, we see that the amplitude is super-Poincare invariant. Finally under conformal transformations $z \rightarrow z + \epsilon(z)z \implies \delta X^m = \epsilon(z)\partial X^m(z)$, we have

$$\delta_\epsilon V_0 = \left[Q, \int_\infty^\infty \frac{dq}{\pi\alpha'^2} \epsilon(z) e^{iqX^0} \right] \quad (42)$$

showing that the two-point amplitude is conformally invariant.

E Requirement of a non-vanishing two point string amplitude

What do the scattering amplitudes in string theory correspond to? Do they represent the full S -matrix or just the interaction part? Unlike the earlier view, the non-vanishing two-point tree-level string amplitude, as found in [1], demonstrates that string amplitudes calculate the full S -matrix. Here, we briefly recall the scattering matrix, S and some of its properties in a quantum (field) theory. For more details, and references, see [15]. The S -matrix takes an incoming set of particles from the far past, represented by $|in\rangle$ state, to an $|out\rangle$ state, the set of all outgoing particles in the far future. It is separated as

$$S = I + iT \quad (43)$$

where I denotes the processes in which the particles in the $|in\rangle$ state go to $|out\rangle$ without undergoing any interaction, while iT represents their effect on each other. Scattering amplitudes denoted $A_n(p_1, \dots, p_n)$, for n particles with momenta $\{p_i\}$, correspond to processes in which particles from the far past come near, interact, and then move away from each other into the far future. They correspond to the matrix elements of $S - iT$ and are related to $G_n(p_1, \dots, p_n)$, the Green's functions, via the LSZ procedure

$$A_n = G_n(p_1, p_2, \dots, p_n) \prod_{i=1}^n (p_i^2 + m_i^2) \quad (44)$$

The Green's functions are calculated using the Feynmann rules and are proportional to momentum conserving delta functions i.e.

$$G_n(p_1, p_2, \dots, p_n) \propto \delta^D(p_1, p_2, \dots, p_n) \quad (45)$$

For the connected processes, for $n \geq 3$, the identity part of the S -matrix vanishes and $S_c = iT_c$. For $n = 2$, however, $A_2 = 0$. To see this consider a scalar field with mass m . Then

$$iT_c^2 = G_2(p_1, p_2)(p_1^2 + m^2)^2 \propto \frac{\delta^D(p_1 + p_2)}{(p_1^2 + m^2)}(p_1^2 + m^2)^2 \xrightarrow{p_1^2 \rightarrow -m^2} 0 \quad (46)$$

This can be shown for other fields as well. Consequently, we have $S_2^c = I_2$ and this is understood to capture the following:

1. The recursive definition of scattering amplitudes, consistent with cluster decomposition, requires a non-zero two-point tree-level amplitude.
2. The two-point amplitude can be thought of as corresponding to the normalization of single-particle states.
3. A single particle coming from the far past and going to the far future without interaction is a connected physical process.
4. The unitarity of 2 point amplitudes [1] requires a non-zero two point amplitude.

Thus, a self-consistent definition of scattering amplitudes requires a non-zero two-point scattering amplitude. The usual LSZ procedure gives a zero answer because it assumes a zero overlap between the in and the out state, which is not the case for a two-point amplitude. The above requirements suggest the following form for the complete two-point tree-level amplitude

$$\mathcal{A}_2(p, p') = 2p^0(2\pi)^{D-1}\delta^{D-1}(\vec{p} - \vec{p}')$$

Notice that we denote it by \mathcal{A}_2 to distinguish it from A_2 which is zero (and represents only the $S - iT$ part).

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