The fermionic double smeared null energy condition

Duarte Fragoso[1⋆] **and Lihan Guo**[1,2†]

1 ITFA, Universiteit van Amsterdam,
Science Park 904, 1098 XH Amsterdam, the Netherlands
2 Dipartimento di Matematica e Fisica, Universit'a Cattolica del Sacro Cuore,
Via della Garzetta 48, 25133 Brescia, Italy

⋆ duarte.fragoso@student.uva.nl , † lihan.guo01@icatt.it

Abstract

Energy conditions are crucial for understanding why exotic phenomena such as traversable wormholes and closed timelike curves remain elusive. In this paper, we prove the Double Smeared Null Energy Condition (DSNEC) for the fermionic free theory in 4-dimensional flat Minkowski space-time, extending previous work on the same energy condition for the bosonic case [1] [2] by adapting Fewster and Mistry's method [3] to the energy-momentum tensor T_{++}. A notable difference from previous works lies in the presence of the $\gamma_0\gamma_+$ matrix in T_{++}, causing a loss of symmetry. This challenge is addressed by making use of its square-root matrix. We provide explicit analytic results for the massless case as well as numerical insights for the mass-dependence of the bound in the case of Gaussian smearing.

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1 Introduction

In general relativity, Einstein's equation itself doesn't impose any restrictions on the form of the energy-momentum tensor $T_{\mu\nu}$. This freedom allows the existence of solutions $G_{\mu\nu}$ that may lead to surprising phenomena, such as macroscopic traversable wormholes [4], closed timelike curves [5] or other causality violations. Energy conditions are essential for explaining why these phenomena have never been observed. The Null Energy Condition (NEC) is particularly important since it is essential in the proof of Penrose's singularity theorem [6] and the second law of black hole thermodynamics (or the Area Theorem) [7] [8] [9].

Previous work has been done for new types of energy conditions namely the Smeared Null Energy Condition (SNEC) [10] and the Double-Smeared Null Energy Condition (DSNEC) [1] which were used to deal with problems that arise when generalizing the NEC to a quantum setup [11–19]. Since this was only studied for the free bosonic theory, in this paper, we will focus on the SNEC and DSNEC for the fermionic theory. Our derivation will closely follow the reasoning of Fewster and Mistry [3] on the Quantum Weak Energy Inequalities for the Dirac field, who deduced a bound for the T_{00} component of the massive fermionic free theory in four-dimensional flat Minkowski space-time, and Wei-Wing *et al* [20], who generalized this result for Minkowski space-time of arbitrary dimensions.

We introduce operators $\mathcal{O}_{\mu i}$ that enable us to express the smeared energy-momentum tensor as the difference between a positive semi-definite operator and a c-number. The primary challenge in defining these operators arises from the presence of the $\gamma_0\gamma_+$ matrix in T_{++}, which reduces the symmetry of the problem. This obstacle is overcome by incorporating the square-root matrix of $\gamma_0\gamma_+$ in the definition of $\mathcal{O}_{\mu i}$.

In brief, the structure of the paper is as follows: In Section 2, we undertake the derivation outlined above and obtain an inequality for the once-smeared T_{++}. However, this inequality is completely trivial, i.e. the lower bound obtained is $-\infty$. We address this issue in Section 3.1 by applying the smearing in two directions, providing a new energy condition:

$$\langle T_{f_+f_-}\rangle \geq -\frac{2}{\pi^3}\int_0^\infty du \int_{\frac{m^2}{u}}^\infty dv \left(\frac{vu^3}{6} - \frac{m^2u^2}{2} + \frac{m^4 u}{2v} - \frac{m^6}{6v^2}\right)|\hat{g}_+(u)|^2|\hat{g}_-(v)|^2, \tag{1}$$

where $f_\pm$ are smearing functions in space-time coordinates and $\hat{g}_\pm$ denotes the Fourier transform of $\sqrt{f_\pm}$. Additionally, we present explicit results for the massless case in Section 3.2, where we employ a Gaussian distribution as the smearing function and derive a bound that depends rationally on the standard deviations,

$$\langle T_{f_+f_-}\rangle \geq -\frac{1}{12\pi^3\sigma_+^3\sigma_-}. \tag{2}$$

Finally, in Section 3.3, we provide numerical results concerning the mass-dependence of the bound. In particular, we observe that for large masses, the bound asymptotically tends to zero.

2 Derivation of the smearing null energy condition

In this section, we will derive a bound for the T_{++} component of the energy-momentum tensor when smeared over the x^+-direction[1]. The quantum field theory considered is the free fermion in Minkowski flat space-time. Note that, despite the bound derived being trivial, the idea can and will be used to deduce a non-trivial bound in Section 3.1.

[1] The light-cone variables x^+ and x^- are defined in Appendix A, as well as the light-cone momentum coordinates k^+ and k^-.

First, let us write the symmetrized version of the energy-momentum tensor for the free fermion (the Belinfante tensor):

$$T_{\mu\nu} = \frac{i}{4}(\bar{\psi}\gamma_\mu\partial_\nu\psi - \partial_\nu\bar{\psi}\gamma_\mu\psi + \bar{\psi}\gamma_\nu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma_\nu\psi). \tag{3}$$

In particular, we are interested in the light-cone component,

$$T_{++} = \frac{i}{2}(\psi^\dagger A\partial_+\psi - \partial_+\psi^\dagger A\psi), \tag{4}$$

where we define $A = \gamma_0\gamma_+$.

The decomposition of the fermionic quantum field into Fourier modes yields the following:

$$\psi(x) = \sum_{k,\alpha} b_\alpha(k)u^\alpha(k)e^{-ik\cdot x} + d_\alpha^\dagger(k)v^\alpha(k)e^{ik\cdot x}, \tag{5}$$

where here we are considering discrete Dirac quantization in a box of side L. At the end of the derivation, we will take the continuous limit at $L \longrightarrow +\infty$.

Now, with these expansions, we can expand the first term of T_{++},

$$\begin{aligned}\frac{i}{2}(\psi^\dagger A\partial_+\psi) = \frac{1}{2}\sum_{k,\tilde{k},\alpha,\alpha'} & \tilde{k}_+ b_\alpha^\dagger(k)b_{\alpha'}(\tilde{k})u_\alpha^\dagger(k)Au_{\alpha'}^\dagger(\tilde{k})e^{i(k-\tilde{k})\cdot x} \\ & -\tilde{k}_+ b_\alpha^\dagger(k)d_{\alpha'}(\tilde{k})u_\alpha^\dagger(k)Av_{\alpha'}(\tilde{k})e^{i(k+\tilde{k})\cdot x} \\ & +\tilde{k}_+ d_\alpha(k)b_{\alpha'}(\tilde{k})v_\alpha^\dagger(k)Au_{\alpha'}(\tilde{k})e^{-i(k+\tilde{k})\cdot x} \\ & -\tilde{k}_+ d_\alpha(k)d_{\alpha'}^\dagger(\tilde{k})v_\alpha^\dagger(k)Av_{\alpha'}^\dagger(\tilde{k})e^{-i(k-\tilde{k})\cdot x},\end{aligned} \tag{6}$$

and similarly for the second term. Normal ordering will switch $d_\alpha(k)$ with $d_{\alpha'}^\dagger(\tilde{k})$ providing an additional minus sign:

$$\begin{aligned}:T_{++}: = \frac{1}{2}\sum_{k,\tilde{k},\alpha,\alpha'} & (k_+ + \tilde{k}_+)[b_\alpha^\dagger(k)b_\alpha(\tilde{k})u_\alpha^\dagger(k)Au_{\alpha'}(\tilde{k})e^{i(k-\tilde{k})\cdot x} \\ & +d_{\alpha'}^\dagger(\tilde{k})d_\alpha(k)v_\alpha^\dagger(k)Av_{\alpha'}(\tilde{k})e^{-i(k-\tilde{k})\cdot x}] \\ & +(k_+ - \tilde{k}_+)[d_\alpha(k)b_\alpha(\tilde{k})v_\alpha^\dagger(k)Au_{\alpha'}(\tilde{k})e^{-i(k+\tilde{k})\cdot x} \\ & -b_\alpha^\dagger(k)d_{\alpha'}(\tilde{k})u_\alpha^\dagger(k)Av_{\alpha'}(\tilde{k})e^{i(k+\tilde{k})\cdot x}].\end{aligned} \tag{7}$$

We are interested in smear T_{++} in the x^+-direction. So, let us put all the other inputs to zero:

$$\begin{aligned}:T_{++}:(x^+,0) = \frac{1}{2}\sum_{k,\tilde{k},\alpha,\alpha'} & (k_+ + \tilde{k}_+)[b_\alpha^\dagger(k)b_\alpha(\tilde{k})u_\alpha^\dagger(k)Au_{\alpha'}(\tilde{k})e^{i(k_+-\tilde{k}_+)\cdot x^+} \\ & +d_{\alpha'}^\dagger(\tilde{k})d_\alpha(k)v_\alpha^\dagger(k)Av_{\alpha'}(\tilde{k})e^{-i(k_+-\tilde{k}_+)\cdot x^+}] \\ & +(k_+ - \tilde{k}_+)[d_\alpha(k)b_\alpha(\tilde{k})v_\alpha^\dagger(k)Au_{\alpha'}(\tilde{k})e^{-i(k_++\tilde{k}_+)\cdot x^+} \\ & -b_\alpha^\dagger(k)d_{\alpha'}(\tilde{k})u_\alpha^\dagger(k)Av_{\alpha'}(\tilde{k})e^{i(k_++\tilde{k}_+)\cdot x^+}].\end{aligned} \tag{8}$$

For general configurations, the expression above is point-wise unbounded from below, so we have to introduce a smearing. Define, for a smearing function f for which we assume to have the positivity condition $f = g^2$ for some other real function g, the smeared energy-momentum tensor component:

$$T_f = \int_{-\infty}^{+\infty} dx^+ :T_{++}:(x^+,0)f(x^+). \tag{9}$$

By the definition of the fourier transform, $\hat{f}(k) = \int_{-\infty}^{+\infty} dx f(x) e^{-ikx}$, we get the following expression:

$$
\begin{aligned}
T_f = \frac{1}{2} \sum_{k,\tilde{k},\alpha,\alpha'} & (k_+ + \tilde{k}_+)[b^\dagger_\alpha(k) b_\alpha(\tilde{k}) u^\dagger_\alpha(k) A u_{\alpha'}(\tilde{k}) \hat{f}(\tilde{k}_+ - k_+) \\
& + d^\dagger_{\alpha'}(\tilde{k}) d_\alpha(k) v^\dagger_\alpha(k) A v_{\alpha'}(\tilde{k}) \hat{f}(k_+ - \tilde{k}_+)] \\
& + (k_+ - \tilde{k}_+)[d_\alpha(k) b_\alpha(\tilde{k}) v^\dagger_\alpha(k) A u_{\alpha'}(\tilde{k}) \hat{f}(\tilde{k}_+ - k_+) \\
& - b^\dagger_\alpha(k) d_{\alpha'}(\tilde{k}) u^\dagger_\alpha(k) A v_{\alpha'}(\tilde{k}) \hat{f}(-k_+ - \tilde{k}_+)].
\end{aligned} \tag{10}
$$

Note that the matrix A has eigenvalues $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = 2$, so it is positive semi-definite. Then it is possible to find the matrix B such that $B^\dagger B = B^\dagger B = A$, i.e. B is the square root matrix of A. It is easy to obtain B explicitly but we will only use its existence.

Define the following family of operators for $i \in \{1,2,3,4\}$ and $\mu \in \mathbb{R}$:

$$
\mathcal{O}_{\mu i} = \sum_{k,\alpha} \overline{\hat{g}(-k_+ + \mu)} b_\alpha(k) (B u_\alpha(k))_i + \overline{\hat{g}(k_+ + \mu)} d^\dagger_\alpha(k) (B v \alpha(k))_i, \tag{11}
$$

$$
\mathcal{O}^\dagger_{\mu i} = \sum_{k,\alpha} \hat{g}(-k_+ + \mu) b^\dagger_\alpha(k) (u^\dagger_\alpha(k) B^\dagger)_i + \hat{g}(k_+ + \mu) d^\dagger_\alpha(k) (v^\dagger_\alpha(k) B^\dagger)_i. \tag{12}
$$

So that $\mathcal{O}_\mu$ for a fixed $\mu \in \mathbb{R}$ is a four-dimensional vector of operators, and $\mathcal{O}^\dagger_\mu$ a co-vector of the same type. Using the anti-commutation relations of the fields, one finds

$$
\begin{aligned}
\mathcal{O}^\dagger_\mu \mathcal{O}_\mu = S^v_\mu \mathbb{1} + \sum_{k,\tilde{k},\alpha,\alpha'} & \hat{g}(-k_+ + \mu) \overline{\hat{g}(-\tilde{k}_+ + \mu)} b^\dagger_\alpha(k) b_{\alpha'}(\tilde{k}) u^\dagger_\alpha(k) A u_{\alpha'}(\tilde{k}) \\
& - \hat{g}(k_+ + \mu) \overline{\hat{g}(\tilde{k}_+ + \mu)} d^\dagger_{\alpha'} d_\alpha(k)(\tilde{k}) v^\dagger_\alpha(k) A v_{\alpha'}(\tilde{k}) \\
& + \hat{g}(k_+ + \mu) \overline{\hat{g}(-\tilde{k}_+ + \mu)} d_\alpha(k) b_{\alpha'}(\tilde{k}) v^\dagger_\alpha(k) A u_{\alpha'}(\tilde{k}) \\
& + \hat{g}(-k_+ + \mu) \overline{\hat{g}(\tilde{k}_+ + \mu)} b^\dagger_\alpha(k) d^\dagger_{\alpha'}(\tilde{k}) u^\dagger_\alpha(k) A v_{\alpha'}(\tilde{k}),
\end{aligned}
$$

where we have defined

$$
S^v_\mu \equiv \sum_{k,\alpha} \hat{g}(k_+ + \mu) \overline{\hat{g}(\tilde{k}_+ + \mu)} \delta_{\alpha,\alpha'} \delta_{k,\tilde{k}} v^\dagger_\alpha(k) A v_{\alpha'}(\tilde{k}) = \sum_{k,\alpha} \hat{g}(k_+ + \mu) \overline{\hat{g}(k_+ + \mu)} v^\dagger_\alpha(k) A v_\alpha(k). \tag{13}
$$

Proven in the literature [3], the following lemma alows us to recover T_f.

Lemma 1 *Let $f = g^2$ with g a real, smooth, compactly-supported*[2] *function. Then the following identity holds:*

$$
(k_+ + \tilde{k}_+) \hat{f}(k_+ - \tilde{k}_+) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\mu \mu \hat{g}(k_+ - \mu) \overline{\hat{g}(\tilde{k}_+ - \mu)}.
$$

Using this lemma,

$$
T_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mu \mu (\mathcal{O}^\dagger_\mu \mathcal{O}_\mu - S^v_\mu \mathbb{1}). \tag{14}
$$

One can then compute the anti-commutator of the operator $\mathcal{O}$,

$$
\begin{aligned}
\{\mathcal{O}^\dagger_{\mu i}, \mathcal{O}_{\mu i}\} = \sum_{k,\tilde{k},\alpha,\alpha'} & (\hat{g}(-k_+ + \mu) \overline{\hat{g}(-\tilde{k}_+ + \mu)} \delta_{\alpha,\alpha'} \delta_{k,\tilde{k}} u^\dagger_\alpha(k) A u_{\alpha'}(\tilde{k}) \\
& + \hat{g}(k_+ + \mu) \overline{\hat{g}(\tilde{k}_+ + \mu)} \delta_{\alpha,\alpha'} \delta_{k,\tilde{k}} v^\dagger_\alpha(k) A v_{\alpha'}(\tilde{k})) \mathbb{1} \\
= & (S^u_{-\mu} + S^v_\mu) \mathbb{1}.
\end{aligned} \tag{15}
$$

[2] Note that the assumption of compact support is stronger than necessary. For example, the lemma still holds for g a Gaussian distribution, since the rapid decay of the function secures convergence of the integral.

Note that since g is real valued, $|\hat{g}|$ is even.

Using the anti-commutation relation obtained above, we can split the integral,

$$\begin{aligned} T_f &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu\mu(\mathcal{O}_{\mu i}^{\dagger}\mathcal{O}_{\mu i} - S_{\mu}^{v}\mathbb{1}) \\ &= \frac{1}{2\pi}\int_{0}^{\infty} d\mu\mu(\mathcal{O}_{\mu i}^{\dagger}\mathcal{O}_{\mu i} - S_{\mu}^{v}\mathbb{1}) + \frac{1}{2\pi}\int_{-\infty}^{0} d\mu\mu(S_{-\mu}^{u}\mathbb{1} - \mathcal{O}_{\mu i}\mathcal{O}_{\mu i}^{\dagger}). \end{aligned} \tag{16}$$

Notice that if $\mu \geq 0$, then $\mu\langle\mathcal{O}_{\mu i}^{\dagger}\mathcal{O}_{\mu i}\rangle_{\psi} \geq 0$, and similarly if $\mu \leq 0$, then $-\mu\langle\mathcal{O}_{\mu i}\mathcal{O}_{\mu i}^{\dagger}\rangle_{\psi} \geq 0$, for any state $|\psi\rangle$. Hence,

$$\begin{aligned} \langle T_f\rangle_{\psi} &\geq -\frac{1}{2\pi}\int_{0}^{\infty} d\mu\mu S_{\mu}^{v} + \frac{1}{2\pi}\int_{-\infty}^{0} d\mu\mu S_{-\mu}^{u} \\ &= -\frac{1}{2\pi}\int_{0}^{\infty} d\mu\mu(S_{\mu}^{v} + S_{\mu}^{u}). \end{aligned} \tag{17}$$

The computation preformed in appendix B shows that

$$\sum_{\alpha} u_{\alpha}^{\dagger}(k)Au_{\alpha}(k) = \frac{2}{V}\left(1 - \frac{k^1}{\omega_k}\right), \tag{18}$$

$$\sum_{\alpha} v_{\alpha}^{\dagger}(k)Av_{\alpha}(k) = \frac{2}{V}\left(1 - \frac{k^1}{\omega_k}\right). \tag{19}$$

Finally, plugging it in the expression for S^u and S^v and taking the continuous limit $\frac{1}{V}\sum_{\vec{k}} \to \int \frac{d^3\vec{k}}{(2\pi)^3}$, we come to the conclusion that

$$\langle T_f\rangle_{\psi} \geq -\frac{2}{\pi}\int_{0}^{\infty} d\mu\mu \int \frac{d^3k}{(2\pi)^3}|\hat{g}(k_+ + \mu)|^2\left(1 - \frac{k^1}{\omega_k}\right). \tag{20}$$

We will denote this bound as $\mathcal{B}_1$, where the subscript 1 represents the number of smearing directions, and the dependence on the smearing function is implicit.

Unfortunately, the integral obtained in equation (20) diverges. By definition, $k_+ = \frac{1}{2}(\omega_k + k_1)$ $= \frac{1}{2}(\omega_k - k^1) = \frac{1}{2}(\sqrt{k_1^2 + k_2^2 + k_3^2 + m^2} - k^1)$, so we can change the integral variable accordingly. Using that the measure transforms as $dk_+ = \frac{1}{2}(\frac{k^1}{\omega_k} - 1)dk^1$, we have that the bound is proportional to

$$\mathcal{B}_1 \propto -\int_{0}^{\infty} d\mu \int dk_+\mu|\hat{g}(k_+ + \mu)|^2 \int dk^2 dk^3. \tag{21}$$

We can then note that the integrals in k_2 and k_3 are decoupled and they will contribute with the volume of the space in those directions. Since the integral in μ and k_+ does not vanish for non-trivial smearing functions, the expression above diverges, which means the bound is completely trivial.

This outcome is clearly unsatisfactory since our aim was to derive a non-trivial lower bound. Such a bound would allow us to explore the extent to which the Null Energy Condition (NEC) is violated within the framework of free fermionic quantum field theory.

However, this divergence is not unexpected, drawing an analogy with the divergence of the bound of the free bosonic theory [10], when the UV cut-off approaches zero. The main issue is that, in order to obtain a convergent integral, it is necessary to fully smear it in the time direction. Note that t is linearly dependent on x^+ and x^- due to $x^+ + x^- = t$. Hence, we expect that, if we smear T_{++} in both light-cone directions, we will obtain a convergent lower bound.

3 Double smeared null energy condition

3.1 Derivation of the non-trivial bound

In this section, we will prove a convergent lower bound, $\mathcal{B}_2$, by smearing T_{++} in both the x^+ and x^--direction. In general, the smearing function can be of the form $f(x^+, x^-)$. For practical purposes, we restrict our argument to the case where f is separable, i.e. the function factors multiplicatively $f(x^+, x^-) = f_+(x^+)f_-(x^-)$. In other words, we are now interested in obtaining an upper bound for

$$T_{f_+f_-} = \int dx^+ \int dx^- : T_{++} : (x^+, x^-, 0) f_+(x^+) f_-(x^-). \tag{22}$$

Keeping in mind that $e^{ik\cdot x} = e^{ik_+x^+}e^{ik_-x^-}e^{-ik_\perp\cdot x^\perp}$ and using the definition of Fourier transform, we can carry out the same procedure as before to write

$$\begin{aligned}T_{f_+f_-} =&\frac{1}{2}\sum_{k,\tilde{k},\alpha,\alpha'}(k_+ + \tilde{k}_+)[b_\alpha^\dagger(k)b_\alpha(\tilde{k})u_\alpha^\dagger(k)Au_{\alpha'}(\tilde{k})\hat{f}_+(\tilde{k}_+ - k_+)\hat{f}_-(\tilde{k}_- - k_-) + d_{\alpha'}^\dagger(\tilde{k})d_\alpha(k)\\&v_\alpha^\dagger(k)Av_{\alpha'}(\tilde{k})\hat{f}_+(k_+ - \tilde{k}_+)\hat{f}_-(k_- - \tilde{k}_-)] + (k^- - \tilde{k}^-)[d_\alpha(k)b_\alpha(\tilde{k})v_\alpha^\dagger(k)Au_{\alpha'}(\tilde{k})\\&\hat{f}_+(k_+ - \tilde{k}_+)\hat{f}_-(k_- - \tilde{k}_-) - b_\alpha^\dagger(k)d_{\alpha'}(\tilde{k})u_\alpha^\dagger(k)Av_{\alpha'}(\tilde{k})\hat{f}_+(-k_+ - \tilde{k}_+)\hat{f}_-(-k_- - \tilde{k}_-)].\end{aligned} \tag{23}$$

Denoting $g_\pm = \sqrt{f_\pm}$, we define the new operators:

$$\mathcal{O}_{\mu i} = \sum_{k,\alpha} \overline{\hat{G}(-k+\mu)} b_\alpha(k)(Bu_\alpha(k))_i + \overline{\hat{G}(k+\mu)} d_\alpha^\dagger(k)(Bv\alpha(k))_i \tag{24}$$

$$\mathcal{O}_{\mu i}^\dagger = \sum_{k,\alpha} \hat{G}(-k+\mu) b_\alpha^\dagger(k)(u_\alpha^\dagger(k)B^\dagger)_i + \hat{G}(k+\mu) d_\alpha^\dagger(k)(v_\alpha^\dagger(k)B^\dagger)_i, \tag{25}$$

where $\hat{G}(k+\mu) = \hat{g}_+(k_+ + \mu_+)\hat{g}_-(k_- + \mu_-)$ and $\mu_\pm$ are two dummy variables that shall be integrated out at the end. We will denote $\mu = (\mu_+, \mu_-)$.

Since the Fourier transform of the product is the convolution, $f = g^2$ implies that $\hat{f} = \hat{g} * \hat{g}$ $= \int d\mu \hat{g}(\mu)\hat{g}(k-\mu)$, so we have

$$\begin{aligned}\hat{f}_-(k_- - \tilde{k}_-) &= \int d\mu \hat{g}_-(\mu)\overline{\hat{g}_-(\mu - (k_- - \tilde{k}_-))} &(26)\\ &= \int d\mu_- \hat{g}_-(k_- - \mu_-)\overline{\hat{g}_-(\tilde{k}_- - \mu_-)}, &(27)\end{aligned}$$

where we changed the variable $\mu = k_- - \mu_-$ and used that since g_- is real, $\overline{\hat{g}_-(x)} = \hat{g}_-(-x)$. Applying lemma 1 to f_+ and g_+, one obtains

$$(k_+ + \tilde{k}_+)\hat{f}_+(k_+ - \tilde{k}_+) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\mu_+ \mu_+ \hat{g}_+(k_+ - \mu_+)\overline{\hat{g}_+(\tilde{k}_+ - \mu_+)}. \tag{28}$$

Then for the double smearing case, applying lemma 1 again, we have

$$(k_+ + \tilde{k}_+)\hat{f}_+(k_+ - \tilde{k}_+)\hat{f}_-(k_- - \tilde{k}_-) = \int d\mu_+ d\mu_- \mu_+ \hat{g}_+(k_+ - \mu_+)\overline{\hat{g}_+(\tilde{k}_+ - \mu_+)}\hat{g}_-(k_- - \mu_-)\overline{\hat{g}_-(\tilde{k}_- - \mu_-)}. \tag{29}$$

In a analogous way as before, we can prove that

$$T_{f_+f_-} = \int_0^{+\infty} d\mu_+ \int_0^{+\infty} d\mu_- \mu_+ (\mathcal{O}_\mu^\dagger \mathcal{O}_\mu - S_\mu^v \mathbb{1}), \tag{30}$$

where now the definition of S_μ^v is different from the once-smeared case:

$$S_\mu^v = \sum_{k,\alpha} |\hat{g}_+(k_+ + \mu_+)|^2 |\hat{g}_-(k_- + \mu_-)|^2 v_\alpha^\dagger(k) A v_\alpha(k) \tag{31}$$

$$= \frac{2}{V} \sum_k |\hat{g}_+(k_+ + \mu_+)|^2 |\hat{g}_-(k_- + \mu_-)|^2 (1 - \frac{k^1}{\omega_k}) \tag{32}$$

and the anti-commutator is what we expect, with the new definitions of S_μ^v and S_μ^u:

$$\{\mathcal{O}_{\mu i}^\dagger, \mathcal{O}_{\mu i}\} = (S_{-\mu}^u + S_\mu^v) \mathbb{1}. \tag{33}$$

In the end, we get the following bound,

$$\langle T_{f_+f_-} \rangle \geq -\frac{2}{\pi} \int_0^{+\infty} d\mu_+ \int_0^{+\infty} d\mu_- \int \frac{d^3\vec{k}}{(2\pi)^3} \mu_+ |\hat{g}_+(k_+ + \mu_+)|^2 |\hat{g}_-(k_- + \mu_-)|^2 (1 - \frac{k^1}{\omega_k}). \tag{34}$$

We know that

$$k_+ = \frac{1}{2}(\omega_k + k_1), \tag{35}$$

$$k_- = \frac{1}{2}(\omega_k - k_1), \tag{36}$$

$$\omega_k^2 = k_1^2 + k_2^2 + k_3^2 + m^2. \tag{37}$$

So setting $k_\perp := \sqrt{k_2^2 + k_3^2}$ we obtain,

$$4k_+k_- = k_2^2 + k_3^2 + m^2 = k_\perp^2 + m^2, \tag{38}$$

and it's straightforward to find that

$$d(k_+k_-) = \frac{1}{2} k_\perp dk_\perp. \tag{39}$$

Since $dk_1 \wedge dk_2 \wedge dk_3 = dk_1 \wedge dk_\perp \wedge k_\perp d\theta$, we can rewrite part of our integral measure in terms of dk^+, dk^- and $d\theta$, i.e. $dk_1 \wedge dk_2 \wedge dk_3 = 2(k_+ + k_-) dk_- \wedge dk_+ \wedge d\theta$.

With all the considerations discussed above, one can change the variable of the integral in the double-smeared bound (34),

$$\langle T_{f_+f_-} \rangle \geq -\frac{2}{\pi} \int_0^{+\infty} d\mu_+ \int_0^{+\infty} d\mu_- \int_{\mathcal{D}} 2(k_- + k_+) dk_- dk_+ 2\pi \frac{1}{(2\pi)^3} \mu_+ |\hat{g}_+(k_+ + \mu_+)|^2 |\hat{g}_-(k_- + \mu_-)|^2 \left(\frac{2k_+}{k_- + k_+}\right)$$

$$= -\frac{2}{\pi^3} \int_0^{+\infty} d\mu_+ \int_0^{+\infty} d\mu_- \int_{\mathcal{D}} dk_+ dk_- \mu_+ k_+ |\hat{g}_+(k_+ + \mu_+)|^2 |\hat{g}_-(k_- + \mu_-)|^2 \tag{40}$$

where the integration domain is $\mathcal{D} = \{k_\pm \geq 0 | k_+k_- \geq m^2\}$.

Equation (40) can be further simplified by changing variables. Setting $u = k_+ + \mu_+$ and $v = k_- + \mu_-$,

$$\langle T_{f_+f_-} \rangle \geq -\frac{2}{\pi^3} \int_0^\infty du \int_{\frac{m^2}{u}}^\infty dv \int_{\frac{m^2}{v}}^u dk_+ \int_{\frac{m^2}{k_+}}^v dk_- (u - k_+) k_+ |\hat{g}_+(u)|^2 |\hat{g}_-(v)|^2. \tag{41}$$

Performing the k^- and k^+ integrals, we can present our main result.

$$\boxed{\langle T_{f_+f_-}\rangle \geq -\frac{2}{\pi^3}\int_0^\infty du \int_{\frac{m^2}{u}}^\infty dv \left(\frac{vu^3}{6} - \frac{m^2u^2}{2} + \frac{m^4 u}{2v} - \frac{m^6}{6v^2}\right)|\hat{g}_+(u)|^2|\hat{g}_-(v)|^2} \tag{42}$$

It's worth mentioning that the form of our bound looks simpler than the result for the bosonic case in [1] and [2]. In the next subsections, we will explore this bound in more specific circumstances, where we can find simpler analytic expressions or numerical results.

3.2 Massless, Gaussian-smeared bound

Let us first investigate the massless case, where we can compute some analytic results for specific smearing functions. Take the Gaussian function $|\hat{g}_+(u)|^2 = \sigma_+ e^{-(\sigma_+ u)^2}$ (similarly $|\hat{g}_-(v)|^2 = \sigma_- e^{-(\sigma_- v)^2}$) [3] as a particular example of smearing function. By changing variables ($\tilde{u} = \sigma_+ u$ and $\tilde{v} = \sigma_- v$), from equation (42) we obtain the expression for the bound,

$$\langle T_{f_+f_-}\rangle \geq -\frac{1}{3\pi^3}\int_0^\infty d\tilde{u}\int_0^\infty d\tilde{v}\frac{1}{\sigma_+^3\sigma_-}\tilde{u}^3\tilde{v}e^{-\tilde{u}^2}e^{-\tilde{v}^2} \tag{43}$$

$$= -\frac{1}{12\pi^3\sigma_+^3\sigma_-}, \tag{44}$$

which turns out to be a satisfactory finite negative number. So, we obtained a non-trivial lower bound for the doubled-smeared T_{++} for the simple case where the smearing is Gaussian. Moreover, σ_+ has a larger effect on the bound comparatively to σ_-. This asymmetry of the dependence on the deviations is expected since the energy-momentum tensor component considered has, by definition, a preferred space-time direction.

Since large $\sigma_\pm$ correspond to a wide smearing in space-time, we expect the bound to approach zero. This is indeed in agreement with the well-studied null energy condition (NEC) [21]. On the other hand, in the $\sigma_\pm \to 0$ case, i.e. there is no smearing in space-time, we obtain a trivial bound. This is expected since the expected value of the energy-momentum evaluated at a particular space-time point is generally unbounded.

3.3 Mass dependence of the Gaussian-smeared bound

One can also wonder about how this bound, which will now denote by $\mathcal{B}_2$, depends on the mass. Let us choose the two smearing functions to be Gaussians with standard deviation $\sigma = 1$, i.e. $|\hat{g}_+(x)|^2 = |\hat{g}_-(x)|^2 = e^{-x^2}$. By dimensional analysis, we have [σ]=-1. Now the bound takes the following form:

$$\mathcal{B}_2 = -\frac{2}{\pi^3}\int_0^\infty du \int_{\frac{m^2}{u}}^\infty dv \left(\frac{vu^3}{6} - \frac{m^2u^2}{2} + \frac{m^4 u}{2v} - \frac{m^6}{6v^2}\right)e^{-(u^2+v^2)}. \tag{45}$$

We can numerically integrate the expression above to obtain the following relation between $\mathcal{B}_2$ and the mass, which is shown in Figure 1. Since we work with natural units, u and v are in the same unit as m, so [$\mathcal{B}_2$]= 4.

Note that in the highly massive region, the lower bound approaches zero. We can understand this result in the following qualitative way. Roughly speaking, quantum effects are relevant when the de Broglie wavelength of the particle, ($\frac{h}{mv}$), is much greater than the characteristic size of the system, d. In our case, we simply take this d to be the smearing length.

[3]We are defining our Gaussian function slightly different from the usual form $\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(\frac{u}{\sigma})^2}$ here. This way, $\sigma_\pm$ are the smearing lengths in space-time since the Fourier transform of a Gaussian function with .

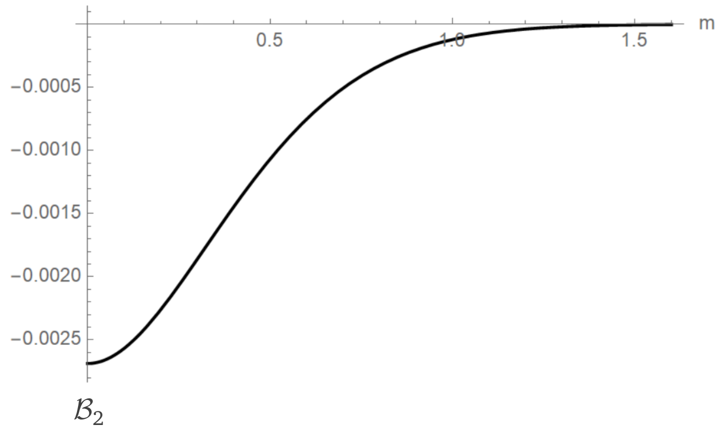

Figure 1: The bound $\mathcal{B}_2$ as a function of the mass m.

For small m, quantum behavior becomes prominent, but as m increases, classical behavior dominates. Given that the classical case satisfies the Null Energy Condition (NEC), the bound is anticipated to approach zero as m becomes large, which is verified numerically in the figure above.

4 Conclusion

In this work, we investigated the Double Smeared Null Energy Condition for the fermionic free theory in 4-dimensional flat Minkowski space-time. We first obtained an inequality for the once-smeared T_{++}. We addressed its triviality later by applying the smearing in two directions, providing a new energy condition. We offered explicit analytic results for the massless case and numerical insights for the mass-dependence of the later bound in the case of Gaussian smearing.

Regarding the outlook of this research, as mentioned in [1], understanding the behavior of DSNEC in interacting field theories is generally still an open question. Since we are using different methods than those used in [1] to derive the DSNEC for fermions, our approach may be promising for interacting field theories. This is because our techniques are not as specifically tailored to treat free field theories.

Moreover, our discussion is limited to Minkowski spacetime. Extending the DSNEC to curved spaces is a highly significant direction, given that it is crucial for its application in semiclassical gravity. It would then be interesting to explore a generalized version of our results in different curved spacetimes.

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A Conventions

In this paper, we work in 4 dimensional Minkowski spacetime with the "mostly minus" signature in natural units ($c = \hbar = 1$).

The position light-cone coordinates are given by $x^{\pm} = t \pm x^1$. The corresponding metric tensor for the coordinates (x^+, x^-, x^2, x^3) is,

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.1}$$

In momentum space we will denote $k_{\pm} = \frac{1}{2}(\omega_k \pm k_1)$ such that $k \cdot x = k_+ x^+ + k\_x^- + x_i \cdot k^i$.

In our convention, the 4-by-4 gamma matrices are defined as

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A.2}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \tag{A.3}$$

Since we are working with the mostly-minus metric we obtain

$$\gamma_0 = \gamma^0, \tag{A.4}$$

$$\gamma_i = -\gamma^i. \tag{A.5}$$

In particular,

$$\gamma_+ = \gamma_0 + \gamma_1 = \gamma_0 - \gamma^1 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \tag{A.6}$$

and we define

$$A = \gamma_0 \gamma_+ = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} - \mathbb{L}, \tag{A.7}$$

where $\mathbb{1}$ is the identity matrix, and $\mathbb{L}$ is the exchange matrix.

B Explicit computations for $\sum_\alpha u_\alpha^\dagger(k) A u_\alpha(k)$

In Appendix B, the explicit computations for $\sum_\alpha u_\alpha^\dagger(k) A u_\alpha(k)$ will be made.

Note that $\sum_\alpha u_\alpha^\dagger(k) A u_\alpha(k) = \frac{2}{V} - \sum_\alpha u_\alpha^\dagger(k) \mathbb{L} u_\alpha(k)$, using the decomposition $A = \mathbb{1} - \mathbb{L}$ and the normalization of $u_\alpha(k)$. Then, we can write

$$u_1(k) = \begin{bmatrix} a \\ Cb \end{bmatrix}, \tag{B.1}$$

where we define

$$a = \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{B.2}$$

$$b = \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{B.3}$$

$$C = \vec{\sigma} \cdot \vec{k}. \tag{B.4}$$

Using this notation we obtain that

$$u_1^\dagger(k)\mathbb{L}u_1(k) = a^\dagger \sigma_1 C b + b^\dagger C^\dagger \sigma_1 a. \tag{B.5}$$

The matrix σ_1 appears in the non-zero blocks of $\mathbb{L}$.

Now,

$$a^\dagger \sigma_1 C b = \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 & 0 \end{bmatrix} k^1 \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \mathbb{1}_{2\times 2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{B.6}$$

$$+ \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 & 0 \end{bmatrix} k^2 \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{B.7}$$

$$+ \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 & 0 \end{bmatrix} k^3 \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{B.8}$$

$$= \frac{1}{V} \frac{1}{2\omega_k} (k^1 + ik^2). \tag{B.9}$$

For the second term, we have something similar:

$$a^\dagger \sigma_1 C b = \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 & 0 \end{bmatrix} k^1 \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \mathbb{1}_{2\times 2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{B.10}$$

$$+ \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 & 0 \end{bmatrix} k^2 \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{B.11}$$

$$+ \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 & 0 \end{bmatrix} k^3 \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{B.12}$$

$$= \frac{1}{V} \frac{1}{2\omega_k} (k^1 - ik^2). \tag{B.13}$$

With this we conclude that

$$u_1^\dagger(k)\mathbb{L}u_1(k) = \frac{1}{V} \frac{1}{2\omega_k} k^1. \tag{B.14}$$

The analogous computation for $u_2(k)$ yields the same result. Summing both, we obtain:

$$u_\alpha(k)^\dagger(k)\mathbb{L}u_\alpha(k) = \frac{1}{V} \frac{1}{\omega_k} k^1. \tag{B.15}$$

In the same way, we obtain the exact same result for $v_\alpha(k)^\dagger(k)\mathbb{L}v_\alpha(k)$.

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