

The fermionic double smeared null energy condition

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Abstract

Energy conditions are crucial for understanding why exotic phenomena such as traversable wormholes and closed timelike curves remain elusive. In this paper, we prove the Double Smeared Null Energy Condition (DSNEC) for the fermionic free theory in 4-dimensional flat Minkowski spacetime, extending previous work on the same energy condition for the bosonic case [1] [2] by adapting Fewster and Mistry's method [3] to the energy-momentum tensor T_{++} . A notable difference from previous works lies in the presence of the $\gamma_0\gamma_+$ matrix in T_{++} , causing a loss of symmetry. This challenge is addressed by making use of its square-root matrix. We provide explicit analytic results for the massless case as well as numerical insights for the mass-dependence of the bound in the case of Gaussian smearing.

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1 Introduction

In general relativity, Einstein's equation itself doesn't impose any restrictions on the form of the energy-momentum tensor $T_{\mu\nu}$. This freedom allows the existence of solutions $G_{\mu\nu}$ that may lead to surprising phenomena, such as macroscopic traversable wormholes [4], closed timelike curves [5] or other causality violations. Energy conditions are essential for explaining why these phenomena have never been observed. The Null Energy Condition (NEC) is particularly important since it is essential in the proof of Penrose's singularity theorem [6] and the second law of black hole thermodynamics (or the Area Theorem) [7] [8] [9].

Previous work has been done for new types of energy conditions namely the Smeared Null Energy Condition (SNEC) [10] and the Double-Smeared Null Energy Condition (DSNEC) [1] [2] which were used to deal with problems that arise when generalizing the NEC to a quantum setup [11–19]. Since this was only studied for the free bosonic theory, in this paper, we will focus on the SNEC and DSNEC for the fermionic theory. Our derivation will closely follow the reasoning of Fewster and Mistry [3] on the Quantum Weak Energy Inequalities for the Dirac field, who deduced a bound for the T_{00} component of the massive fermionic free theory in four-dimensional flat Minkowski spacetime, and Wei-Wing *et al* [20], who generalized this result for Minkowski spacetime of arbitrary dimensions.

We introduce operators $\mathcal{O}_{\mu i}$ that enable us to express the smeared energy-momentum tensor as the difference between a positive semi-definite operator and a c-number. The primary challenge in defining these operators arises from the presence of the $\gamma_0\gamma_+$ matrix in T_{++} , which reduces the symmetry of the problem. This obstacle is overcome by incorporating the square-root matrix of $\gamma_0\gamma_+$ in the definition of $\mathcal{O}_{\mu i}$.

In brief, the structure of the paper is as follows: In Section 2, we undertake the derivation outlined above and obtain an inequality for the once-smeared T_{++} . However, this inequality is completely trivial, i.e. the lower bound obtained is $-\infty$. We address this issue in Section 3.1 by applying the smearing in two directions, providing a new energy condition:

$$\langle T_{f_+f_-} \rangle \geq -\frac{1}{2\pi^4} \int_0^\infty du \int_{\frac{m^2}{u}}^\infty dv \left(\frac{vu^3}{6} - \frac{m^2u^2}{2} + \frac{m^4u}{2v} - \frac{m^6}{6v^2} \right) |\hat{g}_+(u)|^2 |\hat{g}_-(v)|^2, \quad (1)$$

where $f_\pm$ are smearing functions in spacetime coordinates and $\hat{g}_\pm$ denotes the Fourier transform of $\sqrt{f_\pm}$. Additionally, we present explicit results for the massless case in Section 3.2, where we employ a Gaussian distribution as the smearing function and derive a bound that depends rationally on the standard deviations,

$$\langle T_{f_+f_-} \rangle \geq -\frac{1}{48\pi^4\sigma_+^3\sigma_-}. \quad (2)$$

Finally, in Section 3.3, we provide numerical results concerning the mass-dependence of the bound. In particular, we observe that for large masses, the bound asymptotically tends to zero.

2 Derivation of the smearing null energy condition

In this section, we will derive a bound for the T_{++} component of the energy-momentum tensor when smeared over the x^+ -direction¹. The quantum field theory considered is the free fermion in Minkowski flat spacetime. Note that, despite the bound derived being trivial, the idea can and will be used to deduce a non-trivial bound in Section 3.1.

¹The light-cone variables x^+ and x^- are defined in Appendix A, as well as the light-cone momentum coordinates k^+ and k^- .

First, let us write the symmetrized version of the energy-momentum tensor for the free fermion (the Belinfante tensor):

$$T_{\mu\nu} = \frac{i}{4}(\bar{\psi}\gamma_\mu\partial_\nu\psi - \partial_\nu\bar{\psi}\gamma_\mu\psi + \bar{\psi}\gamma_\nu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma_\nu\psi). \quad (3)$$

In particular, we are interested in the light-cone component,

$$T_{++} = \frac{i}{2}(\psi^\dagger A \partial_+ \psi - \partial_+ \psi^\dagger A \psi), \quad (4)$$

where we define $A = \gamma_0 \gamma_+$.

The decomposition of the fermionic quantum field into Fourier modes yields the following:

$$\psi(x) = \sum_{k,\alpha} b_\alpha(k) u^\alpha(k) e^{-ik \cdot x} + d_\alpha^\dagger(k) v^\alpha(k) e^{ik \cdot x}, \quad (5)$$

where here we are considering discrete Dirac quantization in a box of side L . The spinors $u^\alpha(k)$ and $v^\alpha(k)$, where $\alpha = 1, 2$ labels the two independent spin states, are given by

$$u^\alpha(k) = \begin{bmatrix} \sqrt{\frac{\omega_k+m}{2\omega_k V}} \phi^\alpha \\ \frac{\vec{\sigma} \cdot \vec{k}}{\sqrt{2\omega_k(\omega_k+m)V}} \phi^\alpha \end{bmatrix} \text{ and } v^\alpha(k) = \begin{bmatrix} \frac{\vec{\sigma} \cdot \vec{k}}{\sqrt{2\omega_k(\omega_k+m)V}} \phi^\alpha \\ \sqrt{\frac{\omega_k+m}{2\omega_k V}} \phi^\alpha \end{bmatrix}, \quad (6)$$

in which the two dimensional column vectors ϕ^α are $\phi^{1\dagger} = (1, 0)$, $\phi^{2\dagger} = (0, 1)$ and $V = L^3$. The normalization has been chosen so that

$$\sum_\alpha ||u^\alpha(k)||^2 = \sum_\alpha ||v^\alpha(k)||^2 = \frac{2}{V} \quad (7)$$

At the end of the derivation, we will take the continuous limit at $L \rightarrow +\infty$.

Now, with these expansions, we can expand the first term of T_{++} ,

$$\begin{aligned} \frac{i}{2}(\psi^\dagger A \partial_+ \psi) &= \frac{1}{2} \sum_{k, \tilde{k}, \alpha, \alpha'} \tilde{k}_+ b_\alpha^\dagger(k) b_{\alpha'}(\tilde{k}) u_\alpha^\dagger(k) A u_{\alpha'}^\dagger(\tilde{k}) e^{i(k-\tilde{k}) \cdot x} \\ &\quad - \tilde{k}_+ b_\alpha^\dagger(k) d_{\alpha'}^\dagger(\tilde{k}) u_\alpha^\dagger(k) A v_{\alpha'}^\dagger(\tilde{k}) e^{i(k+\tilde{k}) \cdot x} \\ &\quad + \tilde{k}_+ d_\alpha(k) b_{\alpha'}(\tilde{k}) v_\alpha^\dagger(k) A u_{\alpha'}^\dagger(\tilde{k}) e^{-i(k+\tilde{k}) \cdot x} \\ &\quad - \tilde{k}_+ d_\alpha(k) d_{\alpha'}^\dagger(\tilde{k}) v_\alpha^\dagger(k) A v_{\alpha'}^\dagger(\tilde{k}) e^{-i(k-\tilde{k}) \cdot x}, \end{aligned} \quad (8)$$

and similarly for the second term. Normal ordering will switch $d_\alpha(k)$ with $d_{\alpha'}^\dagger(\tilde{k})$ providing an additional minus sign:

$$\begin{aligned} : T_{++} : &:= \frac{1}{2} \sum_{k, \tilde{k}, \alpha, \alpha'} (k_+ + \tilde{k}_+) [b_\alpha^\dagger(k) b_\alpha(\tilde{k}) u_\alpha^\dagger(k) A u_{\alpha'}^\dagger(\tilde{k}) e^{i(k-\tilde{k}) \cdot x} \\ &\quad + d_{\alpha'}^\dagger(\tilde{k}) d_\alpha^\dagger(k) v_\alpha^\dagger(k) A v_{\alpha'}^\dagger(\tilde{k}) e^{-i(k-\tilde{k}) \cdot x}] \\ &\quad + (k_+ - \tilde{k}_+) [d_\alpha(k) b_\alpha(\tilde{k}) v_\alpha^\dagger(k) A u_{\alpha'}^\dagger(\tilde{k}) e^{-i(k+\tilde{k}) \cdot x} \\ &\quad - b_\alpha^\dagger(k) d_{\alpha'}(\tilde{k}) u_\alpha^\dagger(k) A v_{\alpha'}^\dagger(\tilde{k}) e^{i(k+\tilde{k}) \cdot x}]. \end{aligned} \quad (9)$$

We are interested in smearing T_{++} in the x^+ -direction. So, let us put all the other inputs to zero:

$$\begin{aligned} : T_{++} : (x^+, 0) &= \frac{1}{2} \sum_{k, \tilde{k}, \alpha, \alpha'} (k_+ + \tilde{k}_+) [b_\alpha^\dagger(k) b_\alpha(\tilde{k}) u_\alpha^\dagger(k) A u_{\alpha'}^\dagger(\tilde{k}) e^{i(k_+ - \tilde{k}_+) \cdot x^+} \\ &\quad + d_{\alpha'}^\dagger(\tilde{k}) d_\alpha^\dagger(k) v_\alpha^\dagger(k) A v_{\alpha'}^\dagger(\tilde{k}) e^{-i(k_+ - \tilde{k}_+) \cdot x^+}] \\ &\quad + (k_+ - \tilde{k}_+) [d_\alpha(k) b_\alpha(\tilde{k}) v_\alpha^\dagger(k) A u_{\alpha'}^\dagger(\tilde{k}) e^{-i(k_+ + \tilde{k}_+) \cdot x^+} \\ &\quad - b_\alpha^\dagger(k) d_{\alpha'}(\tilde{k}) u_\alpha^\dagger(k) A v_{\alpha'}^\dagger(\tilde{k}) e^{i(k_+ + \tilde{k}_+) \cdot x^+}]. \end{aligned} \quad (10)$$

For general configurations, the expression above is point-wise unbounded from below, so we have to introduce a smearing. Define, for a smearing function f for which we assume to have the positivity condition $f = g^2$ for some other real function g , the smeared energy-momentum tensor component:

$$T_f = \int_{-\infty}^{+\infty} dx^+ : T_{++} : (x^+, 0) f(x^+). \quad (11)$$

By the definition of the fourier transform, $\hat{f}(k) = \int_{-\infty}^{+\infty} dx f(x) e^{-ikx}$, we get the following expression:

$$\begin{aligned} T_f = \frac{1}{2} \sum_{k, \tilde{k}, \alpha, \alpha'} (k_+ + \tilde{k}_+) [& b_\alpha^\dagger(k) b_\alpha(\tilde{k}) u_\alpha^\dagger(k) A u_{\alpha'}(\tilde{k}) \hat{f}(\tilde{k}_+ - k_+) \\ & + d_{\alpha'}^\dagger(\tilde{k}) d_\alpha^\dagger(k) v_\alpha^\dagger(k) A v_{\alpha'}(\tilde{k}) \hat{f}(k_+ - \tilde{k}_+)] \\ & + (k_+ - \tilde{k}_+) [d_\alpha(k) b_\alpha(\tilde{k}) v_\alpha^\dagger(k) A u_{\alpha'}(\tilde{k}) \hat{f}(k_+ + \tilde{k}_+) \\ & - b_\alpha^\dagger(k) d_{\alpha'}(\tilde{k}) u_\alpha^\dagger(k) A v_{\alpha'}(\tilde{k}) \hat{f}(-k_+ - \tilde{k}_+)]. \end{aligned} \quad (12)$$

Note that the matrix A has eigenvalues $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = 2$, so it is positive semi-definite. Then it is possible to find the matrix B such that $B^\dagger B = B^\dagger B = A$, i.e. B is the square root matrix of A . It is easy to obtain B explicitly but we will only use its existence.

Define the following family of operators for $i \in \{1, 2, 3, 4\}$ and $\mu \in \mathbb{R}$:

$$\mathcal{O}_{\mu i} = \sum_{k, \alpha} \hat{g}(-k_+ + \mu) \overline{b_\alpha(k) (B u_\alpha(k))_i} + \hat{g}(k_+ + \mu) d_\alpha^\dagger(k) (B v_\alpha(k))_i, \quad (13)$$

$$\mathcal{O}_{\mu i}^\dagger = \sum_{k, \alpha} \hat{g}(-k_+ + \mu) b_\alpha^\dagger(k) (u_\alpha^\dagger(k) B^\dagger)_i + \hat{g}(k_+ + \mu) d_\alpha(k) (v_\alpha^\dagger(k) B^\dagger)_i. \quad (14)$$

So that $\mathcal{O}_\mu$ for a fixed $\mu \in \mathbb{R}$ is a four-dimensional vector of operators, and $\mathcal{O}_\mu^\dagger$ a co-vector of the same type. Using the anti-commutation relations of the fields, one finds

$$\begin{aligned} \mathcal{O}_\mu^\dagger \mathcal{O}_\mu = S_\mu^v \mathbb{1} + \sum_{k, \tilde{k}, \alpha, \alpha'} & \hat{g}(-k_+ + \mu) \overline{\hat{g}(-\tilde{k}_+ + \mu) b_\alpha^\dagger(k) b_{\alpha'}(\tilde{k}) u_\alpha^\dagger(k) A u_{\alpha'}(\tilde{k})} \\ & - \hat{g}(k_+ + \mu) \overline{\hat{g}(\tilde{k}_+ + \mu) d_{\alpha'}^\dagger(k) d_\alpha(\tilde{k}) v_\alpha^\dagger(k) A v_{\alpha'}(\tilde{k})} \\ & + \hat{g}(k_+ + \mu) \overline{\hat{g}(-\tilde{k}_+ + \mu) d_\alpha(k) b_{\alpha'}(\tilde{k}) v_\alpha^\dagger(k) A u_{\alpha'}(\tilde{k})} \\ & + \hat{g}(-k_+ + \mu) \overline{\hat{g}(\tilde{k}_+ + \mu) b_\alpha^\dagger(k) d_{\alpha'}^\dagger(\tilde{k}) u_\alpha^\dagger(k) A v_{\alpha'}(\tilde{k})}, \end{aligned}$$

where we have defined

$$S_\mu^v \equiv \sum_{k, \alpha} \hat{g}(k_+ + \mu) \overline{\hat{g}(\tilde{k}_+ + \mu) \delta_{\alpha, \alpha'} \delta_{k, \tilde{k}} v_\alpha^\dagger(k) A v_{\alpha'}(\tilde{k})} = \sum_{k, \alpha} \hat{g}(k_+ + \mu) \overline{\hat{g}(k_+ + \mu) v_\alpha^\dagger(k) A v_\alpha(k)}. \quad (15)$$

Proven in the literature [3], the following lemma allows us to recover T_f .

Lemma 1 *Let $f = g^2$ with g a real, smooth, compactly-supported² function. Then the following identity holds:*

$$(k_+ + \tilde{k}_+) \hat{f}(k_+ - \tilde{k}_+) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\mu \mu \hat{g}(k_+ - \mu) \overline{\hat{g}(\tilde{k}_+ - \mu)}.$$

²Note that the assumption of compact support is stronger than necessary. For example, the lemma still holds for g a Gaussian distribution, since the rapid decay of the function secures convergence of the integral.

81 Using this lemma,

$$T_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mu \mu (\mathcal{O}_{\mu i}^{\dagger} \mathcal{O}_{\mu i} - S_{\mu}^v \mathbb{1}). \quad (16)$$

82 One can then compute the anti-commutator of the operator $\mathcal{O}$,

$$\begin{aligned} \{\mathcal{O}_{\mu i}^{\dagger}, \mathcal{O}_{\mu i}\} &= \sum_{k, \tilde{k}, \alpha, \alpha'} (\hat{g}(-k_+ + \mu) \overline{\hat{g}(-\tilde{k}_+ + \mu)} \delta_{\alpha, \alpha'} \delta_{k, \tilde{k}} u_{\alpha}^{\dagger}(k) A u_{\alpha'}(\tilde{k}) \\ &\quad + \hat{g}(k_+ + \mu) \overline{\hat{g}(\tilde{k}_+ + \mu)} \delta_{\alpha, \alpha'} \delta_{k, \tilde{k}} v_{\alpha}^{\dagger}(k) A v_{\alpha'}(\tilde{k})) \mathbb{1} \\ &= (S_{-\mu}^u + S_{\mu}^v) \mathbb{1}. \end{aligned} \quad (17)$$

83 Note that since g is real valued, $|\hat{g}|$ is even.

84 Using the anti-commutation relation obtained above, we can split the integral,

$$\begin{aligned} T_f &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mu \mu (\mathcal{O}_{\mu i}^{\dagger} \mathcal{O}_{\mu i} - S_{\mu}^v \mathbb{1}) \\ &= \frac{1}{2\pi} \int_0^{\infty} d\mu \mu (\mathcal{O}_{\mu i}^{\dagger} \mathcal{O}_{\mu i} - S_{\mu}^v \mathbb{1}) + \frac{1}{2\pi} \int_{-\infty}^0 d\mu \mu (S_{-\mu}^u \mathbb{1} - \mathcal{O}_{\mu i} \mathcal{O}_{\mu i}^{\dagger}). \end{aligned} \quad (18)$$

85 Notice that if $\mu \geq 0$, then $\mu \langle \mathcal{O}_{\mu i}^{\dagger} \mathcal{O}_{\mu i} \rangle_{\psi} \geq 0$, and similarly if $\mu \leq 0$, then $-\mu \langle \mathcal{O}_{\mu i} \mathcal{O}_{\mu i}^{\dagger} \rangle_{\psi} \geq 0$, for
86 any state $|\psi\rangle$. Hence,

$$\begin{aligned} \langle T_f \rangle_{\psi} &\geq -\frac{1}{2\pi} \int_0^{\infty} d\mu \mu S_{\mu}^v + \frac{1}{2\pi} \int_{-\infty}^0 d\mu \mu S_{-\mu}^u \\ &= -\frac{1}{2\pi} \int_0^{\infty} d\mu \mu (S_{\mu}^v + S_{\mu}^u). \end{aligned} \quad (19)$$

87 The computation preformed in appendix B shows that

$$\sum_{\alpha} u_{\alpha}^{\dagger}(k) A u_{\alpha}(k) = \frac{2}{V} \left(1 - \frac{k^1}{\omega_k} \right), \quad (20)$$

$$\sum_{\alpha} v_{\alpha}^{\dagger}(k) A v_{\alpha}(k) = \frac{2}{V} \left(1 - \frac{k^1}{\omega_k} \right). \quad (21)$$

88 Finally, plugging it in the expression for S^u and S^v and taking the continuous limit $\frac{1}{V} \sum_{\vec{k}} \rightarrow \int \frac{d^3 \vec{k}}{(2\pi)^3}$,
89 we come to the conclusion that

$$\langle T_f \rangle_{\psi} \geq -\frac{2}{\pi} \int_0^{\infty} d\mu \mu \int \frac{d^3 k}{(2\pi)^3} |\hat{g}(k_+ + \mu)|^2 \left(1 - \frac{k^1}{\omega_k} \right). \quad (22)$$

90 We will denote this bound as $\mathcal{B}_1$, where the subscript 1 represents the number of smearing
91 directions, and the dependence on the smearing function is implicit.

92 Unfortunately, the integral obtained in equation (22) diverges. By definition, $k_+ = \frac{1}{2}(\omega_k + k_1)$
93 $= \frac{1}{2}(\omega_k - k^1) = \frac{1}{2}(\sqrt{k_1^2 + k_2^2 + k_3^2 + m^2} - k^1)$, so we can change the integral variable accord-
94 ingly. Using that the measure transforms as $dk_+ = \frac{1}{2}(\frac{k^1}{\omega_k} - 1)dk^1$, we have that the bound is
95 proportional to

$$\mathcal{B}_1 \propto - \int_0^{\infty} d\mu \int dk_+ \mu |\hat{g}(k_+ + \mu)|^2 \int dk^2 dk^3. \quad (23)$$

We can then note that the integrals in k_2 and k_3 are decoupled and they will contribute with the volume of the space in those directions. Since the integral in μ and k_+ does not vanish for non-trivial smearing functions, the expression above diverges, which means the bound is completely trivial.

This outcome is clearly unsatisfactory since our aim was to derive a non-trivial lower bound. Such a bound would allow us to explore the extent to which the Null Energy Condition (NEC) is violated within the framework of free fermionic quantum field theory.

However, this divergence is not unexpected, drawing an analogy with the divergence of the bound of the free bosonic theory [10], when the UV cut-off approaches zero. The main issue is that, in order to obtain a convergent integral, it is necessary to fully smear it in the time direction. Note that t is linearly dependent on x^+ and x^- due to $x^+ + x^- = t$. Hence, we expect that, if we smear T_{++} in both light-cone directions, we will obtain a convergent lower bound. An earlier treatment can be found in [21], which also gives general reasons for the lack of any quantum energy inequalities over a finite null segment in dimensions higher than 2.

3 Double smeared null energy condition

3.1 Derivation of the non-trivial bound

In this section, we will prove a convergent lower bound, $\mathcal{B}_2$, by smearing T_{++} in both the x^+ and x^- -direction. In general, the smearing function can be of the form $f(x^+, x^-)$. For practical purposes, we restrict our argument to the case where f is separable, i.e. the function factors multiplicatively $f(x^+, x^-) = f_+(x^+)f_-(x^-)$. In other words, we are now interested in obtaining a lower bound for

$$T_{f_+f_-} = \int dx^+ \int dx^- : T_{++} : (x^+, x^-, 0) f_+(x^+) f_-(x^-). \quad (24)$$

Keeping in mind that $e^{ik \cdot x} = e^{ik_+ x^+} e^{ik_- x^-} e^{-ik_\perp \cdot x^\perp}$ and using the definition of Fourier transform, we can carry out the same procedure as before to write

$$\begin{aligned} T_{f_+f_-} = & \frac{1}{2} \sum_{k, \tilde{k}, \alpha, \alpha'} (k_+ + \tilde{k}_+) [b_\alpha^\dagger(k) b_\alpha(\tilde{k}) u_\alpha^\dagger(k) A u_{\alpha'}(\tilde{k}) \hat{f}_+(\tilde{k}_+ - k_+) \hat{f}_-(\tilde{k}_- - k_-) + d_{\alpha'}^\dagger(\tilde{k}) d_\alpha^\dagger(k) \\ & v_\alpha^\dagger(k) A v_{\alpha'}(\tilde{k}) \hat{f}_+(k_+ - \tilde{k}_+) \hat{f}_-(k_- - \tilde{k}_-)] + (k_+ - \tilde{k}_+) [d_\alpha(k) b_\alpha(\tilde{k}) v_\alpha^\dagger(k) A u_{\alpha'}(\tilde{k}) \\ & \hat{f}_+(k_+ + \tilde{k}_+) \hat{f}_-(k_- + \tilde{k}_-) - b_\alpha^\dagger(k) d_{\alpha'}(\tilde{k}) u_\alpha^\dagger(k) A v_{\alpha'}(\tilde{k}) \hat{f}_+(-k_+ - \tilde{k}_+) \hat{f}_-(-k_- - \tilde{k}_-)]. \end{aligned} \quad (25)$$

Denoting $g_\pm = \sqrt{f_\pm}$, we define the new operators:

$$\mathcal{O}_{\mu i} = \sum_{k, \alpha} \hat{G}(-k + \mu) b_\alpha(k) (B u_\alpha(k))_i + \overline{\hat{G}(k + \mu)} d_\alpha^\dagger(k) (B v_\alpha(k))_i \quad (26)$$

$$\mathcal{O}_{\mu i}^\dagger = \sum_{k, \alpha} \hat{G}(-k + \mu) b_\alpha^\dagger(k) (u_\alpha^\dagger(k) B^\dagger)_i + \hat{G}(k + \mu) d_\alpha^\dagger(k) (v_\alpha^\dagger(k) B^\dagger)_i, \quad (27)$$

where $\hat{G}(k + \mu) = \hat{g}_+(k_+ + \mu_+) \hat{g}_-(k_- + \mu_-)$ and $\mu_\pm$ are two dummy variables that shall be integrated out in the end. We will denote $\mu = (\mu_+, \mu_-)$.

123 Since the Fourier transform of the product is the convolution, $f = g^2$ implies that its Fourier
 124 transform $\hat{f} = \frac{1}{2\pi} \int d\mu \hat{g}(\mu) \hat{g}(k - \mu)$, so we have

$$\hat{f}_-(k_- - \tilde{k}_-) = \frac{1}{2\pi} \int d\mu \hat{g}_-(\mu) \overline{\hat{g}_-(\mu - (k_- - \tilde{k}_-))} \quad (28)$$

$$= \frac{1}{2\pi} \int d\mu_- \hat{g}_-(k_- - \mu_-) \overline{\hat{g}_-(\tilde{k}_- - \mu_-)}, \quad (29)$$

125 where we changed the variable $\mu = k_- - \mu_-$ and used that since g_- is real, $\overline{\hat{g}_-(x)} = \hat{g}_-(-x)$.
 126 Applying lemma 1 to f_+ and g_+ , one obtains

$$(k_+ + \tilde{k}_+) \hat{f}_+(k_+ - \tilde{k}_+) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\mu_+ \mu_+ \hat{g}_+(k_+ - \mu_+) \overline{\hat{g}_+(\tilde{k}_+ - \mu_+)}. \quad (30)$$

127 Then for the double smearing case, applying lemma 1 again, we have

$$(k_+ + \tilde{k}_+) \hat{f}_+(k_+ - \tilde{k}_+) \hat{f}_-(k_- - \tilde{k}_-) = \frac{1}{2\pi^2} \int d\mu_+ d\mu_- \mu_+ \hat{g}_+(k_+ - \mu_+) \overline{\hat{g}_+(\tilde{k}_+ - \mu_+)} \overline{\hat{g}_-(k_- - \mu_-)} \overline{\hat{g}_-(\tilde{k}_- - \mu_-)}. \quad (31)$$

128 Taking advantage of the modified lemma and symmetry,

$$\begin{aligned} & \frac{1}{2\pi^2} \int_{-\infty}^{+\infty} d\mu_+ \int_0^{+\infty} d\mu_- \mu_+ (\mathcal{O}_\mu^\dagger \mathcal{O}_\mu - S_\mu^\nu \mathbb{1}) \\ &= \sum_{k, \tilde{k}, \alpha, \alpha'} \frac{1}{2\pi^2} \int_{-\infty}^{+\infty} d\mu_+ \mu_+ \int_0^{+\infty} d\mu_- \frac{1}{2} (\hat{g}(-k_- + \mu_-) \hat{g}(\tilde{k}_- - \mu_-) + \hat{g}(-k_- - \mu_-) \hat{g}(\tilde{k}_- + \mu_-)) \\ & \quad (\hat{g}(-k_+ + \mu_+) \hat{g}(\tilde{k}_+ - \mu_+) b_\alpha^\dagger(k) b_{\alpha'}(\tilde{k}) u_\alpha^\dagger(k) A u_{\alpha'}(\tilde{k}) - \hat{g}(k_+ + \mu_+) \hat{g}(-\tilde{k}_+ - \mu_+) d_{\alpha'}^\dagger(k) d_\alpha(\tilde{k}) v_\alpha^\dagger(k) A v_{\alpha'}(\tilde{k})) \\ & \quad + \frac{1}{2} (\hat{g}(k_- + \mu_-) \hat{g}(\tilde{k}_- - \mu_-) + \hat{g}(k_- - \mu_-) \hat{g}(\tilde{k}_- + \mu_-)) \\ & \quad (\hat{g}(k_+ + \mu_+) \hat{g}(\tilde{k}_+ - \mu_+) d_\alpha(k) b_{\alpha'}(\tilde{k}) v_\alpha^\dagger(k) A u_{\alpha'}(\tilde{k}) + \hat{g}(-k_+ + \mu_+) \hat{g}(\tilde{k}_+ + \mu_+) b_\alpha^\dagger(k) d_{\alpha'}^\dagger(\tilde{k}) u_\alpha^\dagger(k) A v_{\alpha'}(\tilde{k})) \\ &= \frac{1}{4\pi^2} \sum_{k, \tilde{k}, \alpha, \alpha'} \int_{-\infty}^{+\infty} d\mu_+ \mu_+ \int_{-\infty}^{+\infty} d\mu_- \hat{g}(-k_- + \mu_-) \hat{g}(\tilde{k}_- - \mu_-) (\hat{g}(-k_+ + \mu_+) \hat{g}(\tilde{k}_+ - \mu_+) b_\alpha^\dagger(k) b_{\alpha'}(\tilde{k}) u_\alpha^\dagger(k) A u_{\alpha'}(\tilde{k}) \\ & \quad - \hat{g}(k_+ + \mu_+) \hat{g}(-\tilde{k}_+ - \mu_+) d_{\alpha'}^\dagger(k) d_\alpha(\tilde{k}) v_\alpha^\dagger(k) A v_{\alpha'}(\tilde{k})) + \hat{g}(k_- + \mu_-) \hat{g}(\tilde{k}_- - \mu_-) \\ & \quad (\hat{g}(k_+ + \mu_+) \hat{g}(\tilde{k}_+ - \mu_+) d_\alpha(k) b_{\alpha'}(\tilde{k}) v_\alpha^\dagger(k) A u_{\alpha'}(\tilde{k}) + \hat{g}(-k_+ + \mu_+) \hat{g}(\tilde{k}_+ + \mu_+) b_\alpha^\dagger(k) d_{\alpha'}^\dagger(\tilde{k}) u_\alpha^\dagger(k) A v_{\alpha'}(\tilde{k})) \\ & \quad (33) \end{aligned}$$

129 where now the definition of S_μ^ν is different from the once-smeared case:

$$S_\mu^\nu = \sum_{k, \alpha} |\hat{g}_+(k_+ + \mu_+)|^2 |\hat{g}_-(k_- + \mu_-)|^2 v_\alpha^\dagger(k) A v_\alpha(k) \quad (35)$$

$$= \frac{2}{V} \sum_k |\hat{g}_+(k_+ + \mu_+)|^2 |\hat{g}_-(k_- + \mu_-)|^2 (1 - \frac{k^1}{\omega_k}) \quad (36)$$

130 and the anti-commutator is what we expect, with the new definitions of S_μ^ν and S_μ^u :

$$\{\mathcal{O}_{\mu i}^\dagger, \mathcal{O}_{\mu i}\} = (S_\mu^u + S_\mu^\nu) \mathbb{1}. \quad (37)$$

131 Comparing with the expression of (25), in an analogous way as before, we can prove that

$$\langle T_{f_+ f_-} \rangle \geq -\frac{1}{2\pi^2} \int_0^{+\infty} d\mu_+ \int_0^{+\infty} d\mu_- \int \frac{d^3 \vec{k}}{(2\pi)^3} \mu_+ |\hat{g}_+(k_+ + \mu_+)|^2 |\hat{g}_-(k_- + \mu_-)|^2 (1 - \frac{k^1}{\omega_k}). \quad (38)$$

132 We know that

$$k_+ = \frac{1}{2}(\omega_k + k_1), \quad (39)$$

$$k_- = \frac{1}{2}(\omega_k - k_1), \quad (40)$$

$$\omega_k^2 = k_1^2 + k_2^2 + k_3^2 + m^2. \quad (41)$$

133 So setting $k_\perp := \sqrt{k_2^2 + k_3^2}$ we obtain,

$$4k_+k_- = k_2^2 + k_3^2 + m^2 = k_\perp^2 + m^2, \quad (42)$$

134 and it's straightforward to find that

$$d(k_+k_-) = \frac{1}{2}k_\perp dk_\perp. \quad (43)$$

135 Since $dk_1 \wedge dk_2 \wedge dk_3 = dk_1 \wedge dk_\perp \wedge k_\perp d\theta$, where θ is the angular variable in polar co-
 136 ordinates, we can rewrite part of our integral measure in terms of dk^+ , dk^- and $d\theta$, i.e.
 137 $dk_1 \wedge dk_2 \wedge dk_3 = 2(k_+ + k_-)dk_- \wedge dk_+ \wedge d\theta$.

138 With all the considerations discussed above, one can change the variable of the integral in
 139 the double-smear bound (38),

$$\begin{aligned} \langle T_{f_+f_-} \rangle &\geq -\frac{1}{2\pi^2} \int_0^{+\infty} d\mu_+ \int_0^{+\infty} d\mu_- \int_{\mathcal{D}} 2(k_- + k_+) dk_- dk_+ 2\pi \frac{1}{(2\pi)^3} \mu_+ |\hat{g}_+(k_+ + \mu_+)|^2 |\hat{g}_-(k_- + \mu_-)|^2 \left(\frac{2k_+}{k_- + k_+} \right) \\ &= -\frac{1}{2\pi^4} \int_0^{+\infty} d\mu_+ \int_0^{+\infty} d\mu_- \int_{\mathcal{D}} dk_+ dk_- \mu_+ k_+ |\hat{g}_+(k_+ + \mu_+)|^2 |\hat{g}_-(k_- + \mu_-)|^2 \end{aligned} \quad (44)$$

140 where the integration domain is $\mathcal{D} = \{k_\pm \geq 0 | k_+k_- \geq m^2\}$.

141 Equation (44) can be further simplified by changing variables. Setting $u = k_+ + \mu_+$ and
 142 $v = k_- + \mu_-$,

$$\langle T_{f_+f_-} \rangle \geq -\frac{1}{2\pi^4} \int_0^\infty du \int_{\frac{m^2}{u}}^\infty dv \int_{\frac{m^2}{v}}^u dk_+ \int_{\frac{m^2}{k_+}}^v dk_- (u - k_+) k_+ |\hat{g}_+(u)|^2 |\hat{g}_-(v)|^2. \quad (45)$$

143 Performing the k^- and k^+ integrals, we can present our main result.

$$\boxed{\langle T_{f_+f_-} \rangle \geq -\frac{1}{2\pi^4} \int_0^\infty du \int_{\frac{m^2}{u}}^\infty dv \left(\frac{vu^3}{6} - \frac{m^2u^2}{2} + \frac{m^4u}{2v} - \frac{m^6}{6v^2} \right) |\hat{g}_+(u)|^2 |\hat{g}_-(v)|^2} \quad (46)$$

144 It's worth mentioning that the form of our bound looks simpler than the result for the
 145 bosonic case in [2]. What appears in our expression is simply an integral of polynomial with
 146 smearing function while the result (equation (50)) in [2] involves a parameter η remained to
 147 be fixed in the parameter space for a minimum value.

148 The result can't be further simplified without assuming the exact form of the smearing
 149 function. However, we can still analyze whether this integral converges qualitatively. The
 150 potential sources of divergence can be divided into two categories $u \rightarrow 0$ and $u, v \rightarrow \infty$. For
 151 the first category, as $u \rightarrow 0$, the lower limit of v approaches ∞ . The domain of integration
 152 in v shifts towards ∞ , where $|\hat{g}_-(v)|^2$ decays very rapidly. Additionally, for small u , all terms
 153 in the integrand vanish, so there are no divergences as $u \rightarrow 0$. For the second category, the
 154 dominant term in the integrand is $\frac{vu^3}{6}$. Since for large u and v , the decay of our smearing
 155 function is faster than any polynomial, this rapid decay regulates any polynomial growth in u
 156 and v . In conclusion, we reach a converged bound by smearing in two directions.

157 In the next subsections, we will explore this bound in more specific circumstances, where
 158 we can find simpler analytic expressions or numerical results.

3.2 Massless, Gaussian-smeared bound

Let us first investigate the massless case, where we can compute some analytic results for specific smearing functions. Take the Gaussian function $|\hat{g}_+(u)|^2 = \sigma_+ e^{-(\sigma_+ u)^2}$ (similarly $|\hat{g}_-(v)|^2 = \sigma_- e^{-(\sigma_- v)^2}$) ³ as a particular example of smearing function. By changing variables ($\tilde{u} = \sigma_+ u$ and $\tilde{v} = \sigma_- v$), from equation (46) we obtain the expression for the bound,

$$\langle T_{f,f_-} \rangle \geq -\frac{1}{12\pi^4} \int_0^\infty d\tilde{u} \int_0^\infty d\tilde{v} \frac{1}{\sigma_+^3 \sigma_-} \tilde{u}^3 \tilde{v} e^{-\tilde{u}^2} e^{-\tilde{v}^2} \quad (47)$$

$$= -\frac{1}{48\pi^4 \sigma_+^3 \sigma_-}, \quad (48)$$

which turns out to be a satisfactory finite negative number. So, we obtained a non-trivial lower bound for the doubled-smeared T_{++} for the simple case where the smearing is Gaussian. Moreover, σ_+ has a larger effect on the bound comparatively to σ_- . This asymmetry of the dependence on the deviations is expected since the energy-momentum tensor component considered has, by definition, a preferred spacetime direction.

Since large $\sigma_\pm$ correspond to a wide smearing in spacetime, we expect the bound to approach zero. This is indeed in agreement with the well-studied averaged null energy condition (ANEC) [22]. On the other hand, in the $\sigma_\pm \rightarrow 0$ case, i.e. there is no smearing in spacetime, we obtain a trivial bound. This is expected since the expected value of the energy-momentum evaluated at a particular spacetime point is generally unbounded.

Comparing with the result of [2] in the massless limit, we find that our fermionic result shares similar features with the bosonic case. Taking $n = 4$ and the limit of $m \rightarrow 0$, equation (1) in [2] turns into

$$\langle T_{--}^{\text{smear}} \rangle \geq -\frac{\mathcal{N}_2}{(\delta^+)(\delta^-)^3}, \quad (49)$$

where $\mathcal{N}_2$ is simply a constant since $m \rightarrow 0$. Now that we are evaluating T_{++}^{smear} instead of T_{--}^{smear} , the role of δ^+ and δ^- will exchange as expected and our pre-factor $\frac{1}{48\pi^4}$ takes place of the previous constant $\mathcal{N}_2$.

As a consistency check, let us take $\sigma_- \rightarrow 0$ while keeping σ_+ fixed. In this limit, the right-hand side of equation (48) diverges to $-\infty$, once again yielding a trivial bound when smearing in a single direction.

3.3 Mass dependence of the Gaussian-smeared bound

One can also wonder about how this bound, which will now denote by $\mathcal{B}_2$, depends on the mass. Let us choose the two smearing functions to be Gaussians with standard deviation $\sigma = 1$, i.e. $|\hat{g}_+(x)|^2 = |\hat{g}_-(x)|^2 = e^{-x^2}$. By dimensional analysis, we have $[\sigma] = -1$. Now the bound takes the following form:

$$\mathcal{B}_2 = -\frac{1}{2\pi^4} \int_0^\infty du \int_{\frac{m^2}{u}}^\infty dv \left(\frac{vu^3}{6} - \frac{m^2 u^2}{2} + \frac{m^4 u}{2v} - \frac{m^6}{6v^2} \right) e^{-(u^2+v^2)}. \quad (50)$$

We can numerically integrate the expression above to obtain the following relation between $\mathcal{B}_2$ and the mass, which is shown in Figure 1. Since we work with natural units, u and v are in the same unit as m , so $[\mathcal{B}_2] = 4$. It's worth mentioning that the features of Figure 1 are close to the blue line in Figure 1 of [2], which again shows the similarity between fermionic and bosonic case.

³We are defining our Gaussian function slightly different from the usual form $\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{u}{\sigma})^2}$ here. This way, $\sigma_\pm$ are the smearing lengths in spacetime. It's convenient for the later comparison with the previous result because the variables for our smearing function in the final expression are the wave number after the Fourier transform.

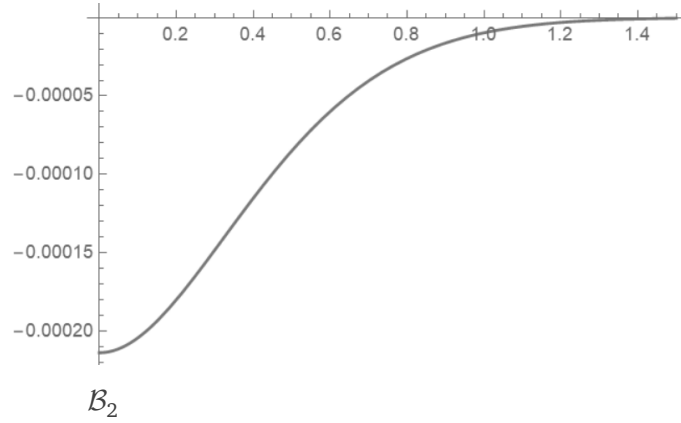


Figure 1: The bound $\mathcal{B}_2$ as a function of the mass m .

193 Note that in the highly massive region, the lower bound approaches zero, which also ap-
 194 pears in the bosonic case [2]. We can understand this result in the following qualitative way.
 195 Roughly speaking, quantum effects are relevant when the de Broglie wavelength of the parti-
 196 cle, $(\frac{h}{mv})$, is much greater than the characteristic size of the system, d . In our case, we simply
 197 take this d to be the smearing length. For small m , quantum behavior becomes prominent, but
 198 as m increases, classical behavior dominates. Given that the classical case satisfies the Null
 199 Energy Condition (NEC), the bound is anticipated to approach zero as m becomes large, which
 200 is verified numerically in the figure above.

201 4 Conclusion

202 In this work, we investigated the (Double) Smeared Null Energy Condition for the fermionic
 203 free theory in 4-dimensional flat Minkowski spacetime.

204 We first obtained an inequality for the once-smeared T_{++} . This inequality holds for all
 205 physically reasonable states. Actually, a fully rigorous formulation would require it to hold for
 206 all Hadamard states, as the claim in [3], since we are using the similar procedure. However,
 207 unlike the result in [3], we illustrated that our derived lower bound diverges. The reason for
 208 this is that instead of smearing T_{00} directly in time direction in [3], time is not fully smeared
 209 in our case when we only smear in direction x_+ for T_{++} .

210 We addressed the triviality of this once-smeared T_{++} later by applying the smearing in
 211 two directions, providing a new energy condition. We offered explicit analytic results for the
 212 massless case and numerical insights for the mass-dependence of the latter bound in the case
 213 of Gaussian smearing.

214 Regarding the outlook of this research, as mentioned in [1], understanding the behavior
 215 of DSNEC in interacting field theories remains still an open question. Though we are using
 216 different methods than those used in [1] to derive the DSNEC for fermions, our approach still
 217 specifically relies on the canonical quantization of the Dirac field and the form of the basis
 218 wavefunctions satisfying the Dirac equation. It remains to be further investigated whether the
 219 stress-tensor can be exactly expressed by terms involving an operator of the type $\mathcal{O}\mathcal{O}^\dagger$ in an
 220 interacting theory.

221 Moreover, our discussion is limited to Minkowski spacetime. Extending the DSNEC to
 222 curved spaces is a highly significant direction, given that it is crucial for its application in
 223 semiclassical gravity. It would then be interesting to explore a generalized version of our
 224 results in different curved spacetimes.

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A Conventions

In this paper, we work in 4 dimensional Minkowski spacetime with the “mostly minus” signature in natural units ($c = \hbar = 1$).

The position light-cone coordinates are given by $x^\pm = t \pm x^1$. The corresponding metric tensor for the coordinates (x^+, x^-, x^2, x^3) is,

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (\text{A.1})$$

In momentum space we will denote $k_\pm = \frac{1}{2}(\omega_k \pm k_1)$ such that $k \cdot x = k_+ x^+ + k_- x^- + x_i \cdot k^i$.

In our convention, the 4-by-4 gamma matrices in the position space are defined as

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (\text{A.2})$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \quad (\text{A.3})$$

Since we are working with the mostly-minus metric we obtain

$$\gamma_0 = \gamma^0, \quad (\text{A.4})$$

$$\gamma_i = -\gamma^i. \quad (\text{A.5})$$

In particular,

$$\gamma_+ = \gamma_0 + \gamma_1 = \gamma_0 - \gamma^1 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad (\text{A.6})$$

and we define

$$A = \gamma_0 \gamma_+ = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} - \mathbb{L}, \quad (\text{A.7})$$

where $\mathbb{1}$ is the identity matrix, and $\mathbb{L}$ is the exchange matrix.

244 B Explicit computations for $\sum_{\alpha} u_{\alpha}^{\dagger}(k) A u_{\alpha}(k)$

245 In Appendix B, the explicit computations for $\sum_{\alpha} u_{\alpha}^{\dagger}(k) A u_{\alpha}(k)$ will be made.

246 Note that $\sum_{\alpha} u_{\alpha}^{\dagger}(k) A u_{\alpha}(k) = \frac{2}{V} - \sum_{\alpha} u_{\alpha}^{\dagger}(k) \mathbb{L} u_{\alpha}(k)$, using the decomposition $A = \mathbb{1} - \mathbb{L}$ and
 247 the normalization of $u_{\alpha}(k)$. Then, we can write

$$u_1(k) = \begin{bmatrix} a_1 \\ C b_1 \end{bmatrix}, \quad (\text{B.1})$$

248 where we define

$$a_1 = \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (\text{B.2})$$

$$b_1 = \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (\text{B.3})$$

$$C = \vec{\sigma} \cdot \vec{k}. \quad (\text{B.4})$$

249 Using this notation we obtain that

$$u_1^{\dagger}(k) \mathbb{L} u_1(k) = a_1^{\dagger} \sigma_1 C b_1 + b_1^{\dagger} C^{\dagger} \sigma_1 a_1. \quad (\text{B.5})$$

250 The matrix σ_1 appears in the non-zero blocks of $\mathbb{L}$.

251 Now,

$$a_1^{\dagger} \sigma_1 C b_1 = \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 & 0 \end{bmatrix} k^1 \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \mathbb{1}_{2 \times 2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\text{B.6})$$

$$+ \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 & 0 \end{bmatrix} k^2 \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\text{B.7})$$

$$+ \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 & 0 \end{bmatrix} k^3 \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\text{B.8})$$

$$= \frac{1}{V} \frac{1}{2\omega_k} (k^1 + i k^2). \quad (\text{B.9})$$

252 For the second term, we have something similar:

$$a_1^{\dagger} \sigma_1 C b_1 = \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 & 0 \end{bmatrix} k^1 \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \mathbb{1}_{2 \times 2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\text{B.10})$$

$$+ \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 & 0 \end{bmatrix} k^2 \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\text{B.11})$$

$$+ \sqrt{\frac{\omega_k + m}{2\omega_k V}} \begin{bmatrix} 1 & 0 \end{bmatrix} k^3 \frac{1}{\sqrt{2\omega_k(\omega_k + m)V}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\text{B.12})$$

$$= \frac{1}{V} \frac{1}{2\omega_k} (k^1 - i k^2). \quad (\text{B.13})$$

253 With this we conclude that

$$u_1^{\dagger}(k) \mathbb{L} u_1(k) = \frac{1}{V} \frac{1}{\omega_k} k^1. \quad (\text{B.14})$$

254 The analogous computation for $u_2(k)$ yields the same result. For $u_2(k)$,

$$u_2^{\dagger}(k) \mathbb{L} u_2(k) = \frac{1}{V} \frac{1}{\omega_k} k^1. \quad (\text{B.15})$$

Summing both, we obtain:

$$u_\alpha(k)^\dagger(k)\mathbb{L}u_\alpha(k) = \frac{2}{V} \frac{1}{\omega_k} k^1. \quad (\text{B.16})$$

In the same way, we obtain the exact same result for $v_\alpha(k)^\dagger(k)\mathbb{L}v_\alpha(k)$.

$$v_\alpha(k)^\dagger(k)\mathbb{L}v_\alpha(k) = \frac{2}{V} \frac{1}{\omega_k} k^1. \quad (\text{B.17})$$

References

- [1] J. R. Fliss and B. Freivogel, *Semi-local bounds on null energy in QFT*, SciPost Physics **12**(3), 084 (2022), doi:[10.21468/SciPostPhys.12.3.084](https://doi.org/10.21468/SciPostPhys.12.3.084).
- [2] J. R. Fliss, B. Freivogel and E.-A. Kontou, *The double smeared null energy condition*, SciPost Physics **14**(2), 024 (2023), doi:[10.21468/SciPostPhys.14.2.024](https://doi.org/10.21468/SciPostPhys.14.2.024).
- [3] C. Fewster and B. Mistry, *Quantum weak energy inequalities for the Dirac field in flat spacetime*, Physical Review D **68**, 105010 (2003), doi:[10.1103/PhysRevD.68.105010](https://doi.org/10.1103/PhysRevD.68.105010).
- [4] L. Ford and T. A. Roman, *Quantum field theory constrains traversable wormhole geometries*, Physical Review D **53**(10), 5496 (1996), doi:[10.1103/PhysRevD.53.5496](https://doi.org/10.1103/PhysRevD.53.5496).
- [5] F. Lobo and P. Crawford, *Time, closed timelike curves and causality*, The nature of time: Geometry, physics and perception pp. 289–296 (2003), doi:[10.1007/978-94-010-0155-7_30](https://doi.org/10.1007/978-94-010-0155-7_30).
- [6] R. Penrose, *Gravitational collapse and space-time singularities*, Physical Review Letters **14**(3), 57 (1965), doi:[10.1103/PhysRevLett.14.57](https://doi.org/10.1103/PhysRevLett.14.57).
- [7] S. W. Hawking, *Black holes in general relativity*, Communications in Mathematical Physics **25**, 152 (1972), doi:[10.1007/BF01877517](https://doi.org/10.1007/BF01877517).
- [8] J. M. Bardeen, B. Carter and S. W. Hawking, *The four laws of black hole mechanics*, Communications in mathematical physics **31**, 161 (1973), doi:[10.1007/BF01645742](https://doi.org/10.1007/BF01645742).
- [9] R. M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago press, ISBN 978-0-226-87027-4 (1994).
- [10] B. Freivogel and D. Krommydas, *The smeared null energy condition*, Journal of High Energy Physics **2018**(12), 1 (2018), doi:[10.1007/JHEP12\(2018\)067](https://doi.org/10.1007/JHEP12(2018)067).
- [11] N. Graham and K. D. Olum, *Achronal averaged null energy condition*, Physical Review D **76**(6), 064001 (2007), doi:[10.1103/PhysRevD.76.064001](https://doi.org/10.1103/PhysRevD.76.064001).
- [12] A. C. Wall, *Proving the achronal averaged null energy condition from the generalized second law*, Physical Review D **81**(2), 024038 (2010), doi:[10.1103/PhysRevD.81.024038](https://doi.org/10.1103/PhysRevD.81.024038).
- [13] D. Urban and K. D. Olum, *Averaged null energy condition violation in a conformally flat spacetime*, Physical Review D **81**(2), 024039 (2010), doi:[10.1103/PhysRevD.81.024039](https://doi.org/10.1103/PhysRevD.81.024039).
- [14] E.-A. Kontou and K. D. Olum, *Proof of the averaged null energy condition in a classical curved spacetime using a null-projected quantum inequality*, Physical Review D **92**(12), 124009 (2015), doi:[10.1103/PhysRevD.92.124009](https://doi.org/10.1103/PhysRevD.92.124009).

- 288 [15] T. Faulkner, R. G. Leigh, O. Parrikar and H. Wang, *Modular Hamiltonians for deformed half-*
289 *spaces and the averaged null energy condition*, Journal of High Energy Physics **2016**(9),
290 1 (2016), doi:[10.1007/JHEP09\(2016\)038](https://doi.org/10.1007/JHEP09(2016)038).
- 291 [16] R. Bousso, Z. Fisher, S. Leichenauer and A. C. Wall, *Quantum focusing conjecture*, Physical
292 Review D **93**(6), 064044 (2016), doi:[10.1103/PhysRevD.93.064044](https://doi.org/10.1103/PhysRevD.93.064044).
- 293 [17] R. Bousso, Z. Fisher, J. Koeller, S. Leichenauer and A. C. Wall, *Proof of*
294 *the quantum null energy condition*, Physical Review D **93**(2), 024017 (2016),
295 doi:[10.1103/PhysRevD.93.024017](https://doi.org/10.1103/PhysRevD.93.024017).
- 296 [18] S. Balakrishnan, T. Faulkner, Z. U. Khandker and H. Wang, *A general proof of the*
297 *quantum null energy condition*, Journal of High Energy Physics **2019**(9), 1 (2019),
298 doi:[10.1007/JHEP09\(2019\)020](https://doi.org/10.1007/JHEP09(2019)020).
- 299 [19] F. Ceyhan and T. Faulkner, *Recovering the QNEC from the ANEC*, Communications in
300 Mathematical Physics **377**, 999 (2020), doi:[10.1007/s00220-020-03751-y](https://doi.org/10.1007/s00220-020-03751-y).
- 301 [20] S. Wei-Xing, Y. Hong-Wei, L. Fei, W. Pu-Xun and R. Zhong-Zhou, *Restrictions on nega-*
302 *tive energy density for the Dirac field in flat spacetime*, Chinese Physics **15**, 934 (2006),
303 doi:[10.1088/1009-1963/15/5/011](https://doi.org/10.1088/1009-1963/15/5/011).
- 304 [21] C. J. Fewster and T. A. Roman, *Null energy conditions in quantum field theory*, Phys. Rev.
305 D **67**, 044003 (2003), doi:[10.1103/PhysRevD.67.044003](https://doi.org/10.1103/PhysRevD.67.044003), [Erratum: Phys.Rev.D 80,
306 069903 (2009)], [gr-qc/0209036](https://arxiv.org/abs/gr-qc/0209036).
- 307 [22] E. Curiel, *A primer on energy conditions*, Towards a theory of spacetime theories pp.
308 43–104 (2017), doi:[10.1007/978-1-4939-3210-8_3](https://doi.org/10.1007/978-1-4939-3210-8_3).