Exact solution of the Izergin-Korepin Gaudin model with periodic and open boundaries

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Abstract

We study the Izergin-Korepin Gaudin models with both periodic and open integrable boundary conditions, which describe novel quantum systems exhibiting long-range interactions. Using the Bethe ansatz approach, we derive the eigenvalues of the Gaudin operators and the corresponding Bethe ansatz equations.

PACS: 03.65.Fd; 04.20.Jb; 05.30.-d; 75.10.Jm
Keywords: Gaudin models, Izergin-Korepin model, Bethe ansatz, $T-Q$ relation

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1 Introduction

The Gaudin model [1] describes an important class of one-dimensional many-body systems with long-range interactions, which have found widespread applications in various branches of physics, from condensed matter to high-energy physics. For example, they are relevant in the simplified BCS theory for small metallic particles [2, 3] and in the Seiberg-Witten theory of supersymmetric gauge theory [4, 5]. Moreover, Gaudin models provide powerful tools for constructing integral representations of solutions to the Knizhnik-Zamolodchikov (KZ) equations [6–9].

The Gaudin operators with integrable boundary conditions are well known to be constructed via a quasi-classical expansion of the inhomogeneous transfer matrix of quantum models [7, 10, 11]. Within this framework, the Gaudin operators can be diagonalized once the exact solutions of the corresponding transfer matrix are derived. As a result, the most well-studied Gaudin models are those with $U(1)$ symmetry [11–15], where the conventional Bethe ansatz can be applied. The lack of exact solutions for many integrable models always presents a significant challenge in studying novel Gaudin models.

Recent advancements in several analytical methodssuch as the (generalized) algebraic Bethe ansatz method [16, 17], the functional $T-Q$ relation [18, 19], and the off-diagonal Bethe ansatz method [20–24]have enabled us to solve various non-trivial integrable models that either lack $U(1)$ symmetry or beyond the A-type Lie algebra [20, 21, 25–27]. This progress has motivated us to explore and analyze new Gaudin models.

In this paper, we focus on the Izergin-Korepin (IK) Gaudin model [28] with periodic and open boundary conditions. The IK model has played a fundamental role in the study of non-A-type integrable models. Exact solutions of the IK model with periodic and generic open boundaries have been constructed using the algebraic Bethe ansatz [29] and the off-diagonal Bethe ansatz [20,27], respectively. Following the approach in Ref. [30], we construct the exactly solvable IK Gaudin operators from the corresponding quantum transfer matrix. With the help of the known exact solutions of the IK model, the eigenvalues of the Gaudin operators are derived through analytical calculations.

The paper is organized as follows. In Section 2, we introduce the periodic IK model and its exact solutions. Section 3 focuses on the construction of the IK Gaudin operators under periodic boundary conditions and provides the solutions for these operators, including their eigenvalues and corresponding Bethe ansatz equations (BAEs). In Section 4, we present the IK model with open boundaries, including its integrability and exact solutions. Section 5 is dedicated to the construction of the IK Gaudin model with generic open boundaries. In Section 6, we derive the eigenvalues of the open Gaudin operators and the corresponding BAEs. A summary is provided in the final section.

2 The IK model with periodic boundaries

2.1 Integrability of periodic IK model

The R-matrix of the IK model [28], associated with the simplest twisted affine algebra $A_2^{(2)}$, is given by

$$R(u)=\left(\begin{array}{ccc|ccc|ccc} c(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & b(u) & 0 & e(u) & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & d(u) & 0 & g(u) & 0 & f(u) & 0 & 0\\ \hline 0 & \bar{e}(u) & 0 & b(u) & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \bar{g}(u) & 0 & a(u) & 0 & g(u) & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & b(u) & 0 & e(u) & 0\\ \hline 0 & 0 & \bar{f}(u) & 0 & \bar{g}(u) & 0 & d(u) & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \bar{e}(u) & 0 & b(u) & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c(u) \end{array}\right). \tag{2.1}$$

The expressions for the functions in Eq. (2.1) are

$$\begin{aligned} &a(u)=\sinh(u-3\eta)-\sinh 5\eta+\sinh 3\eta+\sinh\eta, \quad b(u)=\sinh(u-3\eta)+\sinh 3\eta,\\ &c(u)=\sinh(u-5\eta)+\sinh\eta, \quad d(u)=\sinh(u-\eta)+\sinh\eta,\\ &e(u)=-2\mathrm{e}^{-\frac{u}{2}}\sinh 2\eta\cosh(\tfrac{u}{2}-3\eta), \quad \bar{e}(u)=-2\mathrm{e}^{\frac{u}{2}}\sinh 2\eta\cosh(\tfrac{u}{2}-3\eta),\\ &f(u)=-2\mathrm{e}^{-u+2\eta}\sinh\eta\sinh 2\eta-\mathrm{e}^{-\eta}\sinh 4\eta,\\ &\bar{f}(u)=2\mathrm{e}^{u-2\eta}\sinh\eta\sinh 2\eta-\mathrm{e}^{\eta}\sinh 4\eta,\\ &g(u)=2\mathrm{e}^{-\frac{u}{2}+2\eta}\sinh\tfrac{u}{2}\sinh 2\eta, \quad \bar{g}(u)=-2\mathrm{e}^{\frac{u}{2}-2\eta}\sinh\tfrac{u}{2}\sinh 2\eta. \end{aligned} \tag{2.2}$$

The R-matrix in (2.1) satisfies the quantum Yang-Baxter equation (QYBE)

$$R_{1,2}(u_1-u_2)R_{1,3}(u_1-u_3)R_{2,3}(u_2-u_3)=R_{2,3}(u_2-u_3)R_{1,3}(u_1-u_3)R_{1,2}(u_1-u_2), \tag{2.3}$$

and possesses the following properties

$$\text{Initial condition}: \quad R_{1,2}(0) = \kappa P_{1,2}, \quad \kappa = \sinh\eta - \sinh 5\eta, \tag{2.4}$$

$$\text{Unitarity relation}: \quad R_{1,2}(u)R_{2,1}(-u) = c(u)c(-u) \times \mathrm{id}, \tag{2.5}$$

$$\text{Crossing relation}: \quad R_{1,2}(u) = V_1 R_{1,2}^{t_2}(-u + 6\eta + \mathrm{i}\pi)V_1^{-1}, \tag{2.6}$$

$$\text{Quasi-classical property}: \quad R(u)|_{\eta\to 0} = \sinh u \times \mathrm{id}, \tag{2.7}$$

where $R_{2,1}(u) = P_{1,2}R_{1,2}(u)P_{1,2}$, $P_{1,2}$ is the permutation operator and the superscript t_i indicates the transposition in the i-th space.

From the algebraic Bethe ansatz method, one can construct the quantum transfer matrix

$$t^{(p)}(u) = \mathrm{tr}_0\{R_{0,N}(u-\theta_N)R_{0,N-1}(u-\theta_{N-1})\cdots R_{0,1}(u-\theta_1)\}, \tag{2.8}$$

where $\{\theta_1, \ldots, \theta_N\}$ is a set of inhomogeneous parameters.

By using the QYBE (2.3) repeatedly, one can demonstrate that the transfer matrices with different spectral parameters commute with each other [31] :

$$[t^{(p)}(u),\ t^{(p)}(v)] = 0. \tag{2.9}$$

Therefore, $t^{(p)}(u)$ acts as the generating functional of the conserved quantities of the system and the integrability of the system is proved.

2.2 Exact solutions of periodic IK model

Introduce some functions

$$\tilde{\mathbf{a}}(u) = \prod_{l=1}^{N} c(u-\theta_l), \tag{2.10}$$

$$\tilde{\mathbf{d}}(u) = \prod_{l=1}^{N} d(u-\theta_l), \tag{2.11}$$

$$\tilde{\mathbf{b}}(u) = \prod_{l=1}^{N} b(u-\theta_l). \tag{2.12}$$

From the conventional Bethe ansatz, the eigenvalues of the transfer matrix $t^{(p)}(u)$ are given by the following $T-Q$ relation [29, 32]

$$\Lambda^{(p)}(u) = \tilde{\mathbf{a}}(u)\frac{\widetilde{Q}(u+4\eta)}{\widetilde{Q}(u)} + \tilde{\mathbf{d}}(u)\frac{\widetilde{Q}(u-6\eta+\mathrm{i}\pi)}{\widetilde{Q}(u-2\eta+\mathrm{i}\pi)} + \tilde{\mathbf{b}}(u)\frac{\widetilde{Q}(u-4\eta)\widetilde{Q}(u+2\eta+\mathrm{i}\pi)}{\widetilde{Q}(u-2\eta+\mathrm{i}\pi)\widetilde{Q}(u)}, \tag{2.13}$$

where

$$\widetilde{Q}(u)=\prod_{j=1}^{M}\sinh\Big(\frac{u-\lambda_j-2\eta}{2}\Big), \tag{2.14}$$

and $M=0,1,\ldots,2N$. The Bethe roots $\{\lambda_1,\ldots,\lambda_M\}$ satisfy the folowing Bethe ansatz equations

$$\prod_{l=1}^{N}\frac{\sinh\Big(\frac{\lambda_j-\theta_l-2\eta}{2}\Big)}{\sinh\Big(\frac{\lambda_j-\theta_l+2\eta}{2}\Big)}=-\frac{\widetilde{Q}(\lambda_j-2\eta)\widetilde{Q}(\lambda_j+4\eta+\mathrm{i}\pi)}{\widetilde{Q}(\lambda_j+6\eta)\widetilde{Q}(\lambda_j+\mathrm{i}\pi)},\quad j=1,\ldots,M. \tag{2.15}$$

3 IK Gaudin model with periodic boundaries and its exact solutions

The IK Gaudin operators $\{H_1^{(p)},\ldots,H_N^{(p)}\}$ with periodic boundary conditions can be obtained by expanding the transfer matrix at the point $u=\theta_j$ around $\eta=0$

$$t^{(p)}(\theta_j)=\kappa\Big[\mathsf{t}_0^{(p)}(\theta_j)+\eta H_j^{(p)}+\cdots\Big],\quad j=1,\ldots,N, \tag{3.1}$$

$$H_j^{(p)}=\frac{\partial t^{(p)}(\theta_j)}{\kappa\,\partial\eta}\Big|_{\eta=0}. \tag{3.2}$$

From Refs. [20, 24], we know that

$$t^{(p)}(\theta_j)=\kappa R_{j,j-1}(\theta_j-\theta_{j-1})\cdots R_{j,1}(\theta_j-\theta_1)R_{j,N}(\theta_j-\theta_N)\cdots R_{j,j+1}(\theta_j-\theta_{j+1}). \tag{3.3}$$

The quasi-classical properties of the R-matrix (2.7) allow us introduce the corresponding classical r-matrix $r(u)$

$$\begin{aligned}R(u)&=\sinh u\times\mathrm{id}+\eta\,r(u)+o(\eta^2),\qquad\text{when}\ \ \eta\to 0,\\ r(u)&=\frac{\partial R(u)}{\partial\eta}\Big|_{\eta=0}.\end{aligned} \tag{3.4}$$

The matrix representation of $r(u)$ is

$$r(u) = \left(\begin{array}{ccc|ccc|ccc} c'(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b'(u) & 0 & e'(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d'(u) & 0 & g'(u) & 0 & f'(u) & 0 & 0 \\ \hline 0 & \bar{e}'(u) & 0 & b'(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{g}'(u) & 0 & a'(u) & 0 & g'(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(u) & 0 & e'(u) & 0 \\ \hline 0 & 0 & \bar{f}'(u) & 0 & \bar{g}'(u) & 0 & d'(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{e}'(u) & 0 & b'(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c'(u) \end{array}\right). \tag{3.5}$$

The expressions for the functions in Eq. (3.5) read

$$\begin{aligned} &a'(u) = -3\cosh u - 1, \quad b'(u) = 3 - 3\cosh u, \quad c'(u) = 1 - 5\cosh u, \\ &d'(u) = 1 - \cosh u, \quad e'(u) = -4\mathrm{e}^{-\frac{u}{2}}\cosh\tfrac{u}{2}, \quad \bar{e}'(u) = -4\mathrm{e}^{\frac{u}{2}}\cosh\tfrac{u}{2}, \\ &f'(u) = -4, \quad \bar{f}'(u) = -4, \quad g'(u) = 4\mathrm{e}^{-\frac{u}{2}}\sinh\tfrac{u}{2}, \quad \bar{g}'(u) = -4\mathrm{e}^{\frac{u}{2}}\sinh\tfrac{u}{2}. \end{aligned} \tag{3.6}$$

With the help of Eq. (2.7), we can derive the expression of $\mathsf{t}_0^{(p)}(\theta_j)$ and the Gaudin operators $H_j^{(p)}$

$$\mathsf{t}_0^{(p)}(\theta_j) = \prod_{l\neq j}^{N}\sinh(\theta_j - \theta_l) \times \mathrm{id}, \tag{3.7}$$

$$H_j^{(p)} = \prod_{k\neq j}^{N}\sinh(\theta_j - \theta_k)\sum_{l\neq j}^{N}\Gamma_{j,l}(\theta_j,\theta_l), \quad \Gamma_{j,l}(\theta_j,\theta_l) = \frac{r_{l,j}(\theta_j - \theta_l)}{\sinh(\theta_j - \theta_l)}. \tag{3.8}$$

Here $\Gamma_{j,l}(\theta_j,\theta_l)$ is a long-range two-site interactions between site j and site l (with $l \neq j$), which only depends on the homogeneous parameters θ_j and θ_l.

Based on the expansion of $t^{(p)}(\theta_j)$ with respect to η (3.1) and the commutation relation of the transfer matrix under different parameters (2.9), we can prove that $H_j^{(p)}$ commute with each other. The proof is as follows.

Proof. For convenience, we omit the symbol (p) in $\mathsf{t}_0^{(p)}(\theta_j)$ and $H_j^{(p)}$ in the proof. From the commutation relation $[t(\theta_j),\, t(\theta_l)]$, we get

$$\begin{aligned} &[\mathsf{t}_0(\theta_j) + \eta H_j + \eta^2 H_j^{(2)} + \cdots,\ \mathsf{t}_0(\theta_l) + \eta H_l + \eta^2 H_l^{(2)} + \cdots] \\ &= [\mathsf{t}_0(\theta_j), \mathsf{t}_0(\theta_l)] + \eta\Big\{[\mathsf{t}_0(\theta_j), H_l] + [H_j, \mathsf{t}_0(\theta_l)]\Big\} \\ &\quad + \eta^2\Big\{[H_j^{(2)}, \mathsf{t}_0(\theta_l)] + [\mathsf{t}_0(\theta_j), H_l^{(2)}] + [H_j, H_l]\Big\} + \cdots = 0, \end{aligned} \tag{3.9}$$

Since η is arbitrary, the coefficients of each power of η in (3.9) must be zero, i.e.

$$[\mathsf{t}_0(\theta_j), \mathsf{t}_0(\theta_l)] = 0, \tag{3.10}$$

$$[\mathsf{t}_0(\theta_j), H_l] + [H_j, \mathsf{t}_0(\theta_l)] = 0, \tag{3.11}$$

$$[H_j, H_l^{(2)}] + [H_j^{(2)}, H_l] + [H_j, H_l] = 0, \tag{3.12}$$

$$\cdots .$$

We see that $\mathsf{t}_0(\theta_j)$ is proportional to the identity matrix and commutes with any operator. Then we have

$$\begin{aligned} &[\mathsf{t}_0(\theta_j), H_l] = 0, \qquad [H_j, \mathsf{t}_0(\theta_l)] = 0, \\ &[\mathsf{t}_0(\theta_j), H_l^{(2)}] = 0, \qquad [H_j^{(2)}, \mathsf{t}_0(\theta_l)] = 0. \end{aligned} \tag{3.13}$$

Combining Eqs. (3.11) and (3.13), we derive

$$[H_j, H_l] = 0, \qquad j, l = 1, \ldots, N. \tag{3.14}$$

□

The aforementioned proof is also valid for the open system. It should be remarked that we require $\lim_{\eta\to 0} \kappa^{-1}\, t(\theta_j)$ to be proportional to the identity operator. This condition is automatically satisfied in the periodic system. However, for the open system, the model parameters must satisfy certain constraints for this condition to hold (see Eq. (5.3)).

Therefore, the Gaudin model defined by (3.8) is exactly solvable. We can deirive the eigenvalues of the Gaudin operators from the $T-Q$ relation of the transfer matrix (2.13). The Bethe roots in the $T-Q$ relation are also related to the parameter η. Therefore, we can expand the Bethe roots $\{\lambda_j|_{j=1,\ldots,M}\}$ in terms of η as follows:

$$\lambda_j = \lambda_j^{(0)} + \eta\lambda_j^{(1)} + o(\eta^2). \tag{3.15}$$

We can get the eigenvalues of the periodic IK Gaudin operators $E_j^{(p)}$ by setting $u = \theta_j$ in the $T-Q$ relation (2.13) and taking the first derivative of $\kappa^{-1}\Lambda^{(p)}(\theta_j)$ with respect to η at $\eta = 0$. After some analytical calculations, we arrive at

$$\begin{aligned} E_j^{(p)} &= \kappa^{-1} \left. \frac{\partial \Lambda^{(p)}(\theta_j)}{\partial \eta} \right|_{\eta=0} \\ &= \prod_{l\neq j}^{N} \sinh(\theta_j - \theta_l) \left\{ \frac{1 - 5\cosh(\theta_j - \theta_l)}{\sinh(\theta_j - \theta_l)} + 2\sum_{k=1}^{M} \coth\left(\frac{\theta_j - \lambda_k^{(0)}}{2}\right) \right\}. \end{aligned} \tag{3.16}$$

With the help of Eq. (2.15), the corresponding BAEs for $\{\lambda_j^{(0)}\}$ can be obtained

$$\sum_{l\neq j}^{M}\left[-2\coth\Big(\frac{\lambda_j^{(0)}-\lambda_l^{(0)}}{2}\Big)+\tanh\Big(\frac{\lambda_j^{(0)}-\lambda_l^{(0)}}{2}\Big)\right]$$
$$+\sum_{l=1}^{N}\coth\Big(\frac{\lambda_j^{(0)}-\theta_l}{2}\Big)=0,\qquad j=1,\ldots,M. \tag{3.17}$$

4 The IK model with open boundaries

4.1 Integrability of open IK model

For an integrable open system, in addition to the R-matrix, we also require the boundary-related K-matrices [31]. In this paper, we consider the type II generic non-diagonal K-matrices in Ref. [33]

$$K^-(u)=\begin{pmatrix} 1+2\mathrm{e}^{-u-\epsilon}\sinh\eta & 0 & 2\mathrm{e}^{-\epsilon+\sigma}\sinh u \\ 0 & 1-2\mathrm{e}^{-\epsilon}\sinh(u-\eta) & 0 \\ 2\mathrm{e}^{-\epsilon-\sigma}\sinh u & 0 & 1+2\mathrm{e}^{u-\epsilon}\sinh\eta \end{pmatrix}, \tag{4.1}$$

$$K^+(u)=\mathcal{M}K^-(-u+6\eta+\mathrm{i}\pi)\Big|_{(\epsilon,\sigma)\to(\epsilon',\sigma')}, \tag{4.2}$$

where $\mathcal{M}$ is a constant diagonal matrix

$$\mathcal{M}=\begin{pmatrix} \mathrm{e}^{2\eta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{-2\eta} \end{pmatrix}. \tag{4.3}$$

The matrices $K^-(u)$ and $K^+(u)$ satisfy the reflection equation (RE) and the dual RE respectively [34,35], as follows

$$R_{1,2}(u_1-u_2)K_1^-(u_1)R_{2,1}(u_1+u_2)K_2^-(u_2)$$
$$=K_2^-(u_2)R_{1,2}(u_1+u_2)K_1^-(u_1)R_{2,1}(u_1-u_2), \tag{4.4}$$
$$R_{1,2}(u_2-u_1)K_1^+(u_1)\mathcal{M}_1^{-1}R_{2,1}(-u_1-u_2+12\eta)\mathcal{M}_1K_2^+(u_2)$$
$$=K_2^+(u_2)\mathcal{M}_2^{-1}R_{1,2}(-u_1-u_2+12\eta)\mathcal{M}_2K_1^+(u_1)R_{2,1}(u_2-u_1). \tag{4.5}$$

Then the double-row transfer matrix of the IK model is constructed

$$t(u)=\mathrm{tr}_0\{K_0^+(u)R_{0,N}(u-\theta_N)R_{0,N-1}(u-\theta_{N-1})\cdots R_{0,1}(u-\theta_1)$$
$$\times K_0^-(u)R_{1,0}(u+\theta_1)R_{2,0}(u+\theta_2)\cdots R_{N,0}(u+\theta_N)\}. \tag{4.6}$$

With the help of QYBE (2.3) and (dual) REs (4.4) and (4.5), one can prove that the transfer matrices with different spectral parameters commute with each other [31] :

$$[t(u),\, t(v)] = 0. \tag{4.7}$$

This ensures the integrability of the inhomogeneous IK model with open boundaries. It should be noted that the transfer matrix (4.6) indeed does depend the inhomogeneous parameters $\{\theta_j\}$ and four free boundary parameters $\{\epsilon,\, \sigma,\, \epsilon',\, \sigma'\}$.

4.2 Exact solutions of open IK model

In Refs. [20, 27], the transfer matrix $t(u)$ in (4.6) has been exactly diagonalized by the off-diagonal Bethe ansatz method. Let us recall the $T-Q$ relation.

First, introduce some functions

$$\begin{aligned}\mathbf{a}(u) =& \prod_{l=1}^{N} c(u-\theta_l)c(u+\theta_l)(1-2\mathrm{e}^{-\epsilon}\sinh(u-\eta))(1-2\mathrm{e}^{-\epsilon'}\sinh(u-\eta)) \\ &\times \frac{\sinh(u-6\eta)\cosh(u-\eta)}{\sinh(u-2\eta)\cosh(u-3\eta)},\end{aligned} \tag{4.8}$$

$$\begin{aligned}\mathbf{d}(u) =& \prod_{l=1}^{N} d(u-\theta_l)d(u+\theta_l)(1-2\mathrm{e}^{-\epsilon}\sinh(u-5\eta))(1-2\mathrm{e}^{-\epsilon'}\sinh(u-5\eta)) \\ &\times \frac{\sinh u\cosh(u-5\eta)}{\sinh(u-4\eta)\cosh(u-3\eta)},\end{aligned} \tag{4.9}$$

$$\begin{aligned}\mathbf{b}(u) =& \prod_{l=1}^{N} b(u-\theta_l)b(u+\theta_l)(1+2\mathrm{e}^{-\epsilon}\sinh(u-3\eta))(1+2\mathrm{e}^{-\epsilon'}\sinh(u-3\eta)) \\ &\times \frac{\sinh u\sinh(u-6\eta)}{\sinh(u-2\eta)\sinh(u-4\eta)},\end{aligned} \tag{4.10}$$

$$\mathbf{c}(u) = 4^{1-N}c_0\sinh u\sinh(u-6\eta)\prod_{l=1}^{N} c(u-\theta_l)c(u+\theta_l)d(u-\theta_l)d(u+\theta_l). \tag{4.11}$$

The eigenvalue of the transfer matrix $t(u)$, denoted as $\Lambda(u)$, can be parameterized by the following $T-Q$ relation [20, 27]

$$\begin{aligned}\Lambda(u) =& \mathbf{a}(u)\frac{Q_1(u+4\eta)}{Q_2(u)} + \mathbf{d}(u)\frac{Q_2(u-6\eta+\mathrm{i}\pi)}{Q_1(u-2\eta+\mathrm{i}\pi)} + \mathbf{b}(u)\frac{Q_1(u+2\eta+\mathrm{i}\pi)Q_2(u-4\eta)}{Q_2(u-2\eta+\mathrm{i}\pi)Q_1(u)} \\ &+ \frac{1}{\cosh(u-3\eta)}\left[\frac{\mathbf{c}(u)Q_1(u+2\eta+\mathrm{i}\pi)}{Q_1(u)Q_2(u)} - \frac{\mathbf{c}(-u+6\eta+\mathrm{i}\pi)Q_2(u-4\eta)}{Q_1(u-2\eta+\mathrm{i}\pi)Q_2(u-2\eta+\mathrm{i}\pi)}\right].\end{aligned} \tag{4.12}$$

The function $Q_i(u)$ depends on $\bar{N}$ parameters $\{\lambda_j|j=1,\ldots,\bar{N}\}$

$$Q_1(u)=\prod_{k=1}^{\bar{N}}\sinh\Big(\frac{u-\lambda_k-2\eta}{2}\Big),\quad \bar{N}=4N-2, \tag{4.13}$$

$$Q_2(u)=\prod_{k=1}^{\bar{N}}\sinh\Big(\frac{u+\lambda_k-2\eta}{2}\Big), \tag{4.14}$$

and the functions $c(u)$, $b(u)$ and $d(u)$ are the non-zero elements of the R-matrix given by (2.2) and the constant c_0 is specified as follows

$$c_0=-2\mathrm{e}^{-\epsilon-\epsilon'}\left\{\frac{\cosh(\sigma'-\sigma+2\eta)-\cosh(\bar{N}\eta-\sum_{j=1}^{\bar{N}}\lambda_j)}{\cosh(\frac{\bar{N}\eta}{2}-\frac{1}{2}\sum_{j=1}^{\bar{N}}\lambda_j)}\right\}. \tag{4.15}$$

It is easy to verify that the functions $Q_i(u)$ possess the following properties

$$Q_i(u+2\mathrm{i}\pi)=Q_i(u),\quad \text{for } i=1,2,\quad \text{and}\quad Q_2(u)=Q_1(-u+4\eta). \tag{4.16}$$

The analyticity of $\Lambda(u)$ requires the apparent and simple poles $u=\lambda_j+2\eta$, $j=1\ldots,\bar{N}$ are not real poles. Therefore, the residues of $\Lambda(u)$ at these points must vanish, which leads to the following Bethe asnatz equations

$$\begin{aligned}&\frac{(1+2\mathrm{e}^{-\epsilon}\sinh(\lambda_j-\eta))(1+2\mathrm{e}^{-\epsilon'}\sinh(\lambda_j-\eta))\cosh(\lambda_j-\eta)}{4\sinh\lambda_j\sinh(\lambda_j-2\eta)}\\&=-\prod_{l=1}^{N}\sinh\Big(\frac{\lambda_j-\theta_l-2\eta}{2}\Big)\sinh\Big(\frac{\lambda_j+\theta_l-2\eta}{2}\Big)\cosh\Big(\frac{\lambda_j-\theta_l}{2}\Big)\cosh\Big(\frac{\lambda_j+\theta_l}{2}\Big)\\&\quad\times\frac{c_0\,Q_2(\lambda_j+\mathrm{i}\pi)}{Q_2(\lambda_j-2\eta)Q_2(\lambda_j+2\eta)},\qquad j=1,\ldots,\bar{N}.\end{aligned} \tag{4.17}$$

Remark 4.1. *The $T-Q$ relation (4.12) is constructed by analyzing the analytic properties of the polynomial $\Lambda(u)$, which is obtained from a sufficient set of operator identities related to the transfer matrix $t(u)$. Consequently, the function $\Lambda(u)$ given by the $T-Q$ relation satisfies any equations that the eigenvalue of the transfer matrix satisfies, although some of these equations are not easily derived from the $T-Q$ relation.*

Compared with the $T-Q$ relation of the periodic IK model in Eq. (2.13), the one in (4.12) contains an inhomogeneous term due to the lack of $U(1)$ symmetry. Under certain conditions, the inhomogeneous $T-Q$ relation will reduce to a homogeneous one, which are discussed in Section 6.

5 IK Gaudin model with generic open boundaries

Following the approach outlined in Refs. [10,13,36], one can construct the associated Gaudin operators $\{H_j\}$ by expanding the transfer matrix $t(\theta_j)$ around $\eta=0$, specially as follows

$$\begin{aligned} t(\theta_j) &= \kappa(\mathsf{t}_0(\theta_j) + \eta H_j + \cdots), \quad j=1,\ldots,N, \\ H_j &= \left.\frac{\partial t(\theta_j)}{\kappa\,\partial\eta}\right|_{\eta=0}. \end{aligned} \tag{5.1}$$

Equation (2.7) implies that

$$\begin{aligned} \mathsf{t}_0(\theta_j) &= \lim_{\eta\to 0}\mathrm{tr}_0\left\{\prod_{l\neq j}^{N}\sinh(\theta_j-\theta_l)\prod_{l=1}^{N}\sinh(\theta_j+\theta_l)K_0^+(\theta_j)P_{0,j}K_0^-(\theta_j)\right\} \\ &= \prod_{l\neq j}^{N}\sinh(\theta_j-\theta_l)\prod_{l=1}^{N}\sinh(\theta_j+\theta_l)\,\lim_{\eta\to 0}\{K_j^-(\theta_j)K_j^+(\theta_j)\}. \end{aligned} \tag{5.2}$$

To ensure that the resulting Gaudin operators form a commuting family

$$[H_i,\, H_j]=0, \qquad i,j=1,2,\ldots,N,$$

which is essential for the integrability of the corresponding Gaudin model [1], we require that $\mathsf{t}_0(\theta_j)$ be proportional to the identity operator (see Section 3)., i.e,

$$\lim_{\eta\to 0}\left\{K_j^-(\theta_j)K_j^+(\theta_j)\right\} \propto \mathrm{id}. \tag{5.3}$$

Equation (5.3) gives rise to the following restrictions for the boundary parameters

$$\lim_{\eta\to 0}\mathrm{e}^{\sigma} = \mathrm{e}^{\sigma'}, \quad \lim_{\eta\to 0}\mathrm{e}^{\epsilon'} = -\mathrm{e}^{\epsilon}. \tag{5.4}$$

Without loss of generality, we assume that the boundary parameters ϵ, ϵ' and σ do not depend on the crossing parameter η, while σ' does. Then, we get

$$\sigma' = \sigma + \bar{\sigma}\eta, \quad \epsilon' = \epsilon + \mathrm{i}\pi. \tag{5.5}$$

As a consequence, the following equation can be derived

$$\mathsf{t}_0(\theta_j) = \prod_{l\neq j}^{N}\sinh(\theta_j-\theta_l)\prod_{i=1}^{N}\sinh(\theta_j+\theta_l)\,(1-4\mathrm{e}^{-2\epsilon}\sinh^2\theta_j)\times\mathrm{id}. \tag{5.6}$$

Using the initial condition of R-matrix (2.4) and the QYBE (2.3), the double row transfer matrix at the point $u = \theta_j$ can be expressed as [20]

$$
\begin{aligned}
t(\theta_j) = &\kappa R_{j,j-1}(\theta_j - \theta_{j-1}) \cdots R_{j,1}(\theta_j - \theta_1) K_j^-(\theta_j) R_{1,j}(\theta_j + \theta_1) \cdots R_{j-1,j}(\theta_j + \theta_{j-1}) \\
&\times R_{j+1,j}(\theta_j + \theta_{j+1}) \cdots R_{N,j}(\theta_j + \theta_N) \mathrm{tr}_0\{K_0^+(\theta_j) P_{0,j} R_{j,0}(2\theta_j)\} \\
&\times R_{j,N}(\theta_j - \theta_N) \cdots R_{j,j+1}(\theta_j - \theta_{j+1}).
\end{aligned}
\tag{5.7}
$$

Then, we can derive the expression of the Gaudin operator

$$
H_j = \prod_{l \neq j}^{N} \sinh(\theta_j - \theta_l) \prod_{l=1}^{N} \sinh(\theta_j + \theta_l) \left\{ \Gamma_j(\theta_j) + \sum_{l \neq j}^{N} \Gamma'_{j,l}(\theta_j, \theta_l) \right\},
\tag{5.8}
$$

where the operator $\Gamma_j(\theta_j)$ and $\Gamma'_{j,l}(\theta_j, \theta_l)$ are given by

$$
\Gamma_j(\theta_j) = \frac{1}{\sinh(2\theta_j)} \lim_{\eta \to 0} \frac{\partial}{\partial \eta} \Big[K_j^-(\theta_j)\, \mathrm{tr}_0\{K_0^+(\theta_j) P_{0,j} R_{j,0}(2\theta_j)\} \Big],
\tag{5.9}
$$

$$
\Gamma'_{j,l}(\theta_j, \theta_l) = \left. \frac{r_{j,l}(\theta_j - \theta_l) K_j^-(\theta_j) K_j^+(\theta_j)}{\sinh(\theta_j - \theta_l)} \right|_{\eta \to 0} + \left. \frac{K_j^-(\theta_j) r_{l,j}(\theta_j + \theta_l) K_j^+(\theta_j)}{\sinh(\theta_j + \theta_l)} \right|_{\eta \to 0},
\tag{5.10}
$$

and the operator $r_{j,l}$ is defined in (3.4). Here $\Gamma_j(\theta_j)$ describes the on-site potential, while $\Gamma' j, l(\theta_j, \theta_l)$ represents a site-dependent, long-range two-site interaction. Unlike the two-site interaction $\Gamma j, l(\theta_j, \theta_l)$ in the periodic system (given in Eq. (3.8)), $\Gamma'_{j,l}(\theta_j, \theta_l)$ in the open system depends not only on the inhomogeneous parameters θ_j and θ_l, but also on the boundary parameters $\sigma, \bar{\sigma}$, and ϵ.

The Gaudin operator defined by (5.8) is exactly solvable. In the next section, we will derive the eigenvalues of the Gaudin operators and give the corresponding BAEs.

6 Exact solution of the IK Gaudin model with open boundaries

6.1 Generic open boundary

Equation (5.1) allows us to extract the eigenvalues of the Gaudin operators and the corresponding Bethe ansatz equations from the exact spectrum of the transfer matrix $t(u)$ at $u = \theta_j$, which are demonstrated in Eqs. (4.12)-(4.17).

To achieve this, let us evaluate the function, let us evaluate the function $\Lambda(u)$ at $\eta = 0$

$$\begin{aligned}\Lambda(\theta_j) =&\mathbf{a}(\theta_j)\frac{Q_1(\theta_j+4\eta)}{Q_2(\theta_j)}\\ =&\kappa\left[\Lambda^{(0)}(\theta_j)+\eta\,E_j+\cdots\right], \qquad j=1,\ldots,N.\end{aligned} \tag{6.1}$$

From the operator identity (5.6), we can easily get the equation

$$\Lambda^{(0)}(\theta_j) = \prod_{l\neq j}^{N}\sinh(\theta_j-\theta_l)\prod_{i=1}^{N}\sinh(\theta_j+\theta_l)\,(1-4\mathrm{e}^{-2\epsilon}\sinh^2\theta_j). \tag{6.2}$$

Expand the Bethe roots $\{\lambda_j|_{j=1,\cdots,\bar{N}}\}$ with respect to η as follows:

$$\lambda_j = \lambda_j^{(0)} + \eta\lambda_j^{(1)} + o(\eta^2). \tag{6.3}$$

Then one can also derive the expression of $\Lambda^{(0)}(\theta_j)$ from the $T-Q$ relation (4.12), specifically as follows

$$\Lambda^{(0)}(\theta_j) = \prod_{l\neq j}^{N}\sinh(\theta_j-\theta_l)\prod_{l=1}^{N}\sinh(\theta_j+\theta_l)\,(1-4\mathrm{e}^{-2\epsilon}\sinh^2\theta_j)\prod_{l=1}^{\bar{N}}\frac{\sinh\left(\frac{\theta_j-\lambda_l^{(0)}}{2}\right)}{\sinh\left(\frac{\theta_j+\lambda_l^{(0)}}{2}\right)}. \tag{6.4}$$

Comparing Eqs. (6.2) and (6.4), we conclude the following identities

$$\prod_{l=1}^{\bar{N}}\frac{\sinh\left(\frac{\theta_j-\lambda_l^{(0)}}{2}\right)}{\sinh\left(\frac{\theta_j+\lambda_l^{(0)}}{2}\right)} = 1, \quad j=1,\ldots,N. \tag{6.5}$$

As explained in Remark 4.1, the validity of Eq. (6.5) is guaranteed by the $T-Q$ relation and the corresponding Bethe ansatz equations.

Using Eqs. (5.1) and (6.5), the eigenvalue E_j of the Gaudin operators H_j can be derived as follows

$$\begin{aligned}E_j =&\kappa^{-1}\left.\frac{\partial\Lambda(\theta_j)}{\partial\eta}\right|_{\eta=0}\\ =&\Lambda^{(0)}(\theta_j)\Bigg\{-\tanh\theta_j-6\coth\theta_j+\sum_{l\neq j}^{N}\left[\frac{1-5\cosh(\theta_j-\theta_l)}{\sinh(\theta_j-\theta_l)}+\frac{1-5\cosh(\theta_j+\theta_l)}{\sinh(\theta_j+\theta_l)}\right]\\ &+\frac{4\sinh(2\theta_j)\mathrm{e}^{-2\epsilon}}{1-4\mathrm{e}^{-2\epsilon}\sinh^2\theta_j}+\sum_{k=1}^{\bar{N}}\left[\coth\frac{\theta_j+\lambda_k^{(0)}}{2}+\coth\frac{\theta_j-\lambda_k^{(0)}}{2}\right]\left(1-\frac{\lambda_k^{(1)}}{2}\right)\Bigg\}.\end{aligned} \tag{6.6}$$

From the above expressions, we can see that the eigenvalues of the Gaudin operators E_j depend on the zero-order and first-order expansion coefficients of the Bethe roots with respect to η, namely $\{\lambda_j^{(0)}\}$ and $\{\lambda_j^{(1)}\}$. Therefore, we need to obtain the equations for these two sets of parameters. By expanding the left and right sides of BAEs (4.17) by η and comparing the zero-order and first-order coefficients, we finally obtain the following two sets of equations

$$\begin{aligned}&\left(\mathrm{e}^{2\epsilon}-4\sinh^2\lambda_j^{(0)}\right)\cosh\lambda_j^{(0)}\prod_{l=1}^{\bar{N}}\sinh^2\left(\frac{\lambda_j^{(0)}+\lambda_l^{(0)}}{2}\right)\cosh\left(\frac{1}{2}\sum_{j=1}^{\bar{N}}\lambda_j^{(0)}\right)\\&=8\sinh^2\lambda_j^{(0)}\left[1-\cosh\left(\sum_{j=1}^{\bar{N}}\lambda_j^{(0)}\right)\right]\prod_{l=1}^{\bar{N}}\cosh\left(\frac{\lambda_j^{(0)}+\lambda_l^{(0)}}{2}\right)\\&\quad\times\prod_{l=1}^{N}\left[\frac{1}{4}\sinh(\lambda_j^{(0)}-\theta_l)\sinh(\lambda_j^{(0)}+\theta_l)\right],\qquad j=1,\ldots,\bar{N},\end{aligned}\tag{6.7}$$

$$\begin{aligned}&\left(\lambda_j^{(1)}-1\right)\tanh\lambda_j^{(0)}-\frac{2\left(\lambda_j^{(1)}-1\right)\coth\lambda_j^{(0)}}{1-4\mathrm{e}^{-2\epsilon}\sinh^2\lambda_j^{(0)}}-\frac{1}{2}\left(\bar{N}-\sum_{j=1}^{\bar{N}}\lambda_j^{(1)}\right)\tanh\left(\frac{1}{2}\sum_{j=1}^{\bar{N}}\lambda_j^{(0)}\right)\\&=\sum_{l=1}^{N}\frac{2\sinh\lambda_j^{(0)}\cosh\theta_l-(\lambda_j^{(1)}-1)\sinh(2\lambda_j^{(0)})}{\sinh(\theta_l-\lambda_j^{(0)})\sinh(\theta_l+\lambda_j^{(0)})}+\frac{\left(\bar{N}-\sum_{j=1}^{\bar{N}}\lambda_j^{(1)}\right)\sinh\left(\sum_{j=1}^{\bar{N}}\lambda_j^{(0)}\right)}{1-\cosh\left(\sum_{j=1}^{\bar{N}}\lambda_j^{(0)}\right)}\\&\quad+\sum_{l=1}^{\bar{N}}\frac{\left[\cosh\left(\lambda_j^{(0)}+\lambda_l^{(0)}\right)+3\right]\left(2-\lambda_j^{(1)}-\lambda_l^{(1)}\right)}{2\sinh\left(\lambda_j^{(0)}+\lambda_l^{(0)}\right)},\qquad j=1,\ldots,\bar{N}.\end{aligned}\tag{6.8}$$

It should be noted that in deriving the BAEs presented in Eq. (6.8), we used the equations in (6.7). These $2\bar{N}$ equations will completely determine the $2\bar{N}$ unknown Bethe roots $\{\lambda_j^{(0)}\}$ and $\{\lambda_j^{(1)}\}$. We observe that the BAEs in (6.7) and (6.8) are independent of σ and $\bar{\sigma}$. Consequently, σ and $\bar{\sigma}$ do not contribute to the energy of the Gaudin operator.

6.2 Constrained open boundaries

In the generic case, the $T-Q$ relation (4.12) includes an inhomogeneous term. However, under certain constraints, the inhomogeneous term will vanish, leading to simpler expressions for the exact solution of the transfer matrix and the corresponding Gaudin operator.

If the boundary parameters σ and σ' satisfy the following constraint [27]

$$\mathrm{e}^{\sigma-\sigma'}=\mathrm{e}^{-4\mathbf{k}\eta},\quad \mathbf{k}\in Z,\tag{6.9}$$

the eigenvalue of the transfer matrix can be parameterized by a homogeneous $T-Q$ relation

$$\Lambda(u)=\mathbf{a}(u)\frac{Q(u+4\eta)}{Q(u)}+\mathbf{d}(u)\frac{Q(u-6\eta+\mathrm{i}\pi)}{Q(u-2\eta+\mathrm{i}\pi)}+\mathbf{b}(u)\frac{Q(u+2\eta+\mathrm{i}\pi)Q(u-4\eta)}{Q(u-2\eta+\mathrm{i}\pi)Q(u)},\tag{6.10}$$

where the resulting function $Q(u)$ is

$$Q(u)=\prod_{j=1}^{M}\sinh\Big(\frac{u-\lambda_j-2\eta}{2}\Big)\sinh\Big(\frac{u+\lambda_j-2\eta}{2}\Big). \tag{6.11}$$

Here M is a non-negative integer and takes the following values

$$M=\begin{cases}N-\mathbf{k}, & \mathbf{k}\leq -N,\\ N+\mathbf{k}+1, & \mathbf{k}\geq N+1,\\ N-\mathbf{k}, & 1-N\leq \mathbf{k}\leq N,\\ N+\mathbf{k}-1, & 1-N\leq \mathbf{k}\leq N.\end{cases} \tag{6.12}$$

The resulting BAEs now read

$$\begin{aligned}&\prod_{l=1}^{N}\frac{\sinh\Big(\frac{\lambda_j-\theta_l-2\eta}{2}\Big)\sinh\Big(\frac{\lambda_j+\theta_l-2\eta}{2}\Big)}{\sinh\Big(\frac{\lambda_j-\theta_l+2\eta}{2}\Big)\sinh\Big(\frac{\lambda_j+\theta_l+2\eta}{2}\Big)}\frac{(1-2\mathrm{e}^{-\epsilon}\sinh(\lambda_j+\eta))(1-2\mathrm{e}^{-\epsilon'}\sinh(\lambda_j+\eta))}{(1+2\mathrm{e}^{-\epsilon}\sinh(\lambda_j-\eta))(1+2\mathrm{e}^{-\epsilon'}\sinh(\lambda_j-\eta))}\\ &=-\frac{\sinh(\lambda_j+2\eta)\cosh(\lambda_j-\eta)}{\sinh(\lambda_j-2\eta)\cosh(\lambda_j+\eta)}\frac{Q(\lambda_j-2\eta)Q(\lambda_j+4\eta+\mathrm{i}\pi)}{Q(\lambda_j+6\eta)Q(\lambda_j+\mathrm{i}\pi)},\quad j=1,\ldots,M.\end{aligned} \tag{6.13}$$

Following Eqs. (5.4) and (6.9), one can construct the corresponding Gaudin operator by letting $\bar{\sigma}=\mathbf{k}$ and $\mathrm{e}^{\epsilon'}=-\mathrm{e}^{\epsilon}$. The eigenvalues of the IK Gaudin operators become

$$\begin{aligned}E_j=&\Lambda^{(0)}(\theta_j)\Bigg\{-6\coth\theta_j-\tanh\theta_j+\sum_{l=1}^{M}\frac{4\sinh\theta_j}{\cosh\theta_j-\cosh\lambda_l^{(0)}}\\ &+\sum_{k\neq j}^{N}\Big[\frac{1-5\cosh(\theta_j-\theta_k)}{\sinh(\theta_j-\theta_k)}+\frac{1-5\cosh(\theta_j+\theta_k)}{\sinh(\theta_j+\theta_k)}\Big]+\frac{4\mathrm{e}^{-2\epsilon}\sinh(2\theta_j)}{1-4\mathrm{e}^{-2\epsilon}\sinh^2(2\theta_j)}\Bigg\},\end{aligned} \tag{6.14}$$

and the Bethe ansatz equations are

$$\begin{aligned}&\sum_{k\neq j}^{M}\Big[\frac{2}{\cosh\lambda_j^{(0)}-\cosh\lambda_k^{(0)}}-\frac{1}{\cosh\lambda_j^{(0)}+\cosh\lambda_k^{(0)}}\Big]\\ &+\sum_{l=1}^{N}\frac{1}{\cosh\lambda_j^{(0)}-\cosh\theta_l}+\frac{4\cosh\lambda_j^{(0)}}{\mathrm{e}^{2\epsilon}+4-4\cosh^2\lambda_j^{(0)}}=0,\qquad j=1,\ldots,M.\end{aligned} \tag{6.15}$$

We observe that both the eigenvalue of the Gaudin operator H_j and the BAEs (6.15) depend on the set $\Big\{\cosh\lambda_j^{(0)}\Big\}$.

Remark 6.1. *Under the constrained boundary condition (6.9), the $U(1)$ symmetry of the system is still broken; however, the system possesses certain symmetries. In this case, the* *integer M is fixed by the system parameters. One can then find a proper "local vacuum" to proceed with the conventional Bethe ansatz and study the physical quantities of the model [16, 37]. When $M \geq 2N$, the $T-Q$ relation (6.10) with M Bethe roots may provide the complete set of eigenvalues of the transfer matrix. On the other hand, when $0 \leq M < 2N$, two $T-Q$ relations may be needed to parameterize the full spectrum of the transfer matrix, with the number of Bethe roots being M and $2N-M-1$, respectively. Such degenerate points exist in various integrable models, e.g., the anisotropic spin-$\frac{1}{2}$ chains with non-diagonal boundary fields [16, 18, 24, 37, 38]).*

6.3 Diagonal open boundaries

As the boundary parameter ϵ approaches infinity $\epsilon \to +\infty$, the resulting K-matrices become diagonal

$$K^-(u) = \mathrm{id}, \quad K^+(u) = \mathcal{M}, \tag{6.16}$$

where the matrix $\mathcal{M}$ is defined in (4.3). In this case, the K-matrices automatically satisfy the operator relation

$$\lim_{\eta \to 0}\{K^+(u)K^-(u)\} = \mathrm{id}. \tag{6.17}$$

The $U(1)$-symmetry of the model is now recovered, and one can also use a homogeneous $T-Q$ relation to parameterize the spectrum of the transfer matrix. The functions $\mathbf{a}(u)$, $\mathbf{b}(u)$ and $\mathbf{d}(u)$ given by Eqs. (4.8)-(4.10) reduce to [39]

$$\bar{\mathbf{a}}(u) = \prod_{l=1}^{N} c(u-\theta_l)c(u+\theta_l)\,\frac{\sinh(u-6\eta)\cosh(u-\eta)}{\sinh(u-2\eta)\cosh(u-3\eta)}, \tag{6.18}$$

$$\bar{\mathbf{d}}(u) = \prod_{l=1}^{N} d(u-\theta_l)d(u+\theta_l)\,\frac{\sinh u\cosh(u-5\eta)}{\sinh(u-4\eta)\cosh(u-3\eta)}, \tag{6.19}$$

$$\bar{\mathbf{b}}(u) = \prod_{l=1}^{N} b(u-\theta_l)b(u+\theta_l)\,\frac{\sinh u\sinh(u-6\eta)}{\sinh(u-2\eta)\sinh(u-4\eta)}. \tag{6.20}$$

The $T-Q$ relation (4.12) is now a homogeneous one [39]

$$\Lambda(u) = \bar{\mathbf{a}}(u)\frac{Q(u+4\eta)}{Q(u)} + \bar{\mathbf{d}}(u)\frac{Q(u-6\eta+\mathrm{i}\pi)}{Q(u-2\eta+\mathrm{i}\pi)} + \bar{\mathbf{b}}(u)\frac{Q(u-4\eta)Q(u+2\eta+\mathrm{i}\pi)}{Q(u-2\eta+\mathrm{i}\pi)Q(u)}, \tag{6.21}$$

where the function $Q(u)$ is defined as

$$Q(u)=\prod_{j=1}^{M}\sinh\Big(\frac{u-\lambda_j-2\eta}{2}\Big)\sinh\Big(\frac{u+\lambda_j-2\eta}{2}\Big). \tag{6.22}$$

In this case, the integer M is adjustable and can take the following values

$$M=0,1,\ldots,2N. \tag{6.23}$$

For each of these permissble values, the resulting BAEs are

$$\prod_{l=1}^{N}\frac{\sinh\Big(\frac{\lambda_j-\theta_l-2\eta}{2}\Big)\sinh\Big(\frac{\lambda_j+\theta_l-2\eta}{2}\Big)}{\sinh\Big(\frac{\lambda_j-\theta_l+2\eta}{2}\Big)\sinh\Big(\frac{\lambda_j+\theta_l+2\eta}{2}\Big)}\frac{\sinh(\lambda_j-2\eta)\cosh(\lambda_j+\eta)}{\sinh(\lambda_j+2\eta)\cosh(\lambda_j-\eta)}$$
$$=-\frac{Q(\lambda_j-2\eta)Q(\lambda_j+4\eta+\mathrm{i}\pi)}{Q(\lambda_j+6\eta)Q(\lambda_j+\mathrm{i}\pi)},\qquad j=1,\ldots,M. \tag{6.24}$$

Thanks to the conditions (2.7) and (6.17), the IK Gaudin operators with diagonal boundary K-matrices can be constructed by taking the first-order derivative of the transfer matrix with respect to η as $\eta\to 0$. Performing a Taylor series expansion in Eqs. (6.21)C(6.24) around $\eta=0$ and taking the first-order coefficient, the corresponding eigenvalues of the IK Gaudin operators are derived as follows

$$E_j=\prod_{l\neq j}^{N}\sinh(\theta_j-\theta_l)\prod_{l=1}^{N}\sinh(\theta_j+\theta_l)\Bigg\{\sum_{k\neq j}^{N}\Bigg[\frac{1-5\cosh(\theta_j-\theta_k)}{\sinh(\theta_j-\theta_k)}+\frac{1-5\cosh(\theta_j+\theta_k)}{\sinh(\theta_j+\theta_k)}\Bigg]$$
$$+\sum_{k=1}^{M}\frac{4\sinh\theta_j}{\cosh\theta_j-\cosh\lambda_k^{(0)}}-\tanh\theta_j-6\coth\theta_j\Bigg\}, \tag{6.25}$$

where the Bethe roots $\{\lambda_j^{(0)}|j=1,\ldots,M\}$ satisfy

$$\sum_{l=1}^{N}\frac{1}{\cosh\lambda_j^{(0)}-\cosh\theta_l}+\sum_{k\neq j}^{M}\Bigg[\frac{2}{\cosh\lambda_j^{(0)}-\cosh\lambda_k^{(0)}}$$
$$-\frac{1}{\cosh\lambda_j^{(0)}+\cosh\lambda_k^{(0)}}\Bigg]=0,\qquad j=1,\ldots,M. \tag{6.26}$$

7 Conclusions

In this paper, we study the IK Gaudin model with both periodic and open boundary conditions. We derive the Gaudin operator from the inhomogeneous quantum transfer matrix at the point $u = \theta_j$, which ensures the solvability of the Gaudin operator. Thanks to the exact solutions of the IK model obtained via the Bethe ansatz method, we derive the exact spectrum of the Gaudin operator in terms of the Bethe roots, which are determined by the corresponding Bethe ansatz equations.

The open system requires further explanation. In this paper, we focus on the non-diagonal K-matrices in Eqs. (4.1) and (4.2). It is important to note that our approach is also valid for other types of K-matrices discussed in Refs. [34,35]. Since the Gaudin operator depends on the boundary parameters, different choices of K-matrices will lead to a different Gaudin operator.

Subsequent research is analyzing the solutions of the Bethe ansatz equations, and it will be interesting to explore the existence of infinite Bethe roots and string solutions in the IK Gaudin model. Another open question is the construction of the eigenstates of the Gaudin operator.

Acknowledgments

Financial support from the National Natural Science Foundation of China (Grant Nos. 12105221, 12247103, 12074410, 12047502), the Strategic Priority Research Program of the Chinese Academy of Sciences (Grant No. XDB33000000), Shaanxi Fundamental Science Research Project for Mathematics and Physics (Grant Nos. 22JSZ005), the Scientific Research Program Funded by Shaanxi Provincial Education Department (Grant No. 21JK0946), Beijing National Laboratory for Condensed Matter Physics (Grant No. 202162100001), and the Double First-Class University Construction Project of Northwest University is gratefully acknowledged.

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